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THE NOTION OF A POLYNOMIAL IN THE SECONDARY-TERTIARY TRANSITION

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The notion of a polynomial, as one of the most fundamental concepts in all areas of mathematics, has a prominent role in pre-tertiary and tertiary education. Due to the repeated encounter of students with this notion, it makes sense to study its role in the transition to university mathematics. In this paper, the secondary-tertiary transition of mathematics students at the University of Split concerning the notion of a polynomial is analysed using tools of the Anthropological Theory of the Didactic.

Keywords: Transition to, across and from university mathematics; Teaching and learning of linear and abstract algebra; Curricular and institutional issues concerning the teaching of mathematics at university level; Polynomial; Anthropological Theory of the Didactic.

THE POSITION OF POLYNOMIALS IN MATHEMATICS AND RESEARCH IN MATHEMATICAL EDUCATION

The notion of a polynomial is significant in almost every area of mathematics as a discipline. The short route through algebra shows a long and fundamental role of polynomials: from classical algebra problems such as solving algebraic equations, through Galois theory and ring theory in modern algebra and beyond. On the other hand, the notion of a polynomial includes polynomials as the simplest functions that are easy to evaluate. Therefore, polynomials play an important role in the analysis and theory of approximations. It is sufficient to mention the Taylor polynomial or the Weierstrass approximation theorem. An attempt to position polynomials in somewhat younger areas of mathematics than those previously mentioned leads to the same conclusion. The close connection of some areas of mathematics with computers, such as coding theory and cryptography, leads to the importance of polynomials over finite fields. As the notion of a polynomial can be expected to be significant for both pre-tertiary and tertiary education, it makes sense to observe a secondary-tertiary transition in terms of this notion. Expected repeated encounters with polynomials will be observed in the Croatian context, more precisely, through the secondary-tertiary transition that students of the first year of Mathematics at the Faculty of Science, University of Split are going through. The motivation to study this topic comes from the difficulties of first-year students observed through teaching courses in linear algebra. According to our knowledge, many studies mention polynomials, but most often as examples in a topic that includes polynomials. Although the cumulative effect of such studies contributes significantly to the knowledge about the realization of the notion of a polynomial in different phases of education, there is a lack of research

dedicated exclusively to this notion at the tertiary level. Bolondi, Ferretti & Maffia's (2020) analysis of some Italian, Spanish, and North American high school textbooks came across different schemes for defining the notion of a polynomial. Sultan and Artzt (2011), in a book aimed at bridging the gap between mathematics that high school teachers learn at university and mathematics that they teach in high school, write the following:

Before getting into a deep discussion of finding roots of polynomials, we review the definition of a polynomial. This is probably the most misunderstood word in secondary school mathematics. (p. 71)

Therefore, the question arises about the cause of the observed phenomena in secondary education and its possible effects on undergraduate mathematics education.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

Secondary-tertiary transition is a common research topic in mathematics education (Vleeschouwer, 2010). As it includes institutional transition, the Anthropological Theory of the Didactic (ATD) is a suitable theoretical framework for the topic as it observes mathematical and didactic activities in the complexity of social institutions. Dependence of knowledge on the institution, which breaks the *illusion of transparency* of knowledge, is explained by the process of *didactic transposition* (Bosch & Gascón, 2006), which observes knowledge in four phases: *scholarly knowledge*, *knowledge to be taught* (curricula), *taught knowledge* by teachers in the classroom and *learned knowledge* by students. For an object of knowledge o in an institution I in which p is a position, the relation, in the notation $R_I(p, o)$, to o with respect to p in I can be observed (Bosch et. al., 2019). It remains to explain how o can be described by the tools of ATD. According to ATD, every human activity, including mathematical ones, can be described in terms of *praxeology* (Bosch & Gascón, 2006). Praxeology $[T/\tau/\theta/\Theta]$ consists of four components: type of tasks T ; a *set of techniques* τ that can solve T ; a *technology* θ that explains and justifies τ ; and a *theory* Θ that is a formal argument and thus justifies θ . The ordered pair $[T/\tau]$ of *type of tasks* T and the *set of techniques* τ is called the *praxis* block of $[T/\tau/\theta/\Theta]$ and is associated with *know-how*; while the ordered pair $[\theta/\Theta]$ of *technology* θ and *theory* Θ is called the *logos* block of praxeology and is associated with *know-why* (Bosch et al., 2019). *Mathematical praxeologies* or *mathematical organizations* (MO) are classified as follows: *point praxeology* (generated by a single type of tasks), *local praxeology* (several point praxeologies with common technological discourse), *regional praxeology* (several local praxeologies with common theory). Every mathematical praxeology is "activated through the manipulation of ostensives" (Arzarello et al., 2008, p. 181), perceptual objects; whereby this manipulation is guided by *non-ostensives*, that is, concepts.

Before addressing research questions, let us introduce the necessary notations to apply the ATD tools to the topic of this paper. A relation of the high school student hs to the knowledge P of the notion of a polynomial at the end of the high school HS in Croatia will be denoted by $R_{HS}(hs, P)$. The relation of the student s_1 of the first year of the

undergraduate study program Mathematics at the Faculty of Science, University of Split (designation U) to the P will be denoted by $R_U(s_1, P)$. The transition from secondary education HS to the first year of U is denoted by $R_{HS}(hs, P) \rightarrow R_U(s_1, P)$.

RQ1: How do polynomial-related praxeologies develop and differentiate through Croatian high schools? What can be said about $R_{HS}(hs, P)$?

RQ2: How do polynomial-related praxeologies develop and differentiate through the first year of U ? What phenomena can be observed in the transition $R_{HS}(hs, P) \rightarrow R_U(s_1, P)$?

METHODOLOGY

The research methodology follows the phases of didactic transposition of the notion of a polynomial both in HS and U . Croatian high school mathematics curricula and textbooks are analysed to assess the prior knowledge of first-year mathematics students about polynomials, i.e., $R_{HS}(hs, P)$. Analogously, to assess $R_U(s_1, P)$, the study program, syllabi, exams and materials of first-year courses at U are analysed. This part provides information about *knowledge to be taught* in both institutions, HS and U, and the way in which the *taught* and *learned knowledge* are assessed is explained below. $R_{HS}(hs, P)$ are supplemented by the results of the questionnaire conducted among students at the beginning of the first semester of U in the academic year 2019/2020. Questionnaires were also conducted among students at the beginning of, and in the middle of the second semester to answer RQ2 more precisely. All questionnaires, with polynomials as the main topic, were conducted unannounced among mathematics students in attendance in classes of one of the first-year courses. All the above, supplemented by interviews with two high school mathematics teachers and seven university teachers at U , will enable the detection of the phenomenon of the transition $R_{HS}(hs, P) \rightarrow R_U(s_1, P)$.

KNOWLEDGE TO BE TAUGHT

High school curricula and textbooks

In this article, only the gymnasium¹ mathematics curricula, which were in force in Croatia from 1994 to 2019, will be presented². The observed curricula prescribed the following contents: the notion of a polynomial, the algebra of polynomials, rational functions (first grade - age 15); the notion, graph and zero points of a second-degree polynomial, problems with extremes, quadratic inequalities, intersection of lines and parabolas (second grade - age 16). In addition to the previous contents, the curriculum of mathematical gymnasiums³ prescribed additional and more advanced contents such

¹ In Croatia, a gymnasium is a type of high school (lasting four years) whose goal is to prepare students for further university education.

² The U is enrolled almost only by students who have completed some of the gymnasium programs, and the generation of students who finished high school according to the new curricula is still not enrolled.

³ Gymnasiums with the maximum number of mathematics classes.

as reducible and irreducible polynomials, Fundamental Theorem of Algebra, the ring of polynomials of two variables. Polynomials are no longer explicitly mentioned anywhere in curricula, but a direct connection with the following contents is evident: linear equations and inequalities, linear and affine functions, the graph of a linear function (first grade); second-order curves (third grade - age 17); functions, derivation and integral (fourth grade - age 18). Although curricula seem to prescribe a functional approach to a polynomial, an analysis of high school textbook sets used in the last twenty years, shows that most authors introduce this notion in the first grade of high school in a section about algebraic expressions, with noticeable differences in discourse and its level. The observed textbooks for first grade differ in the following issues.

In which way the notion of a variable is introduced? What does it mean? What is the relationship between notions of exponentiation, algebraic expression, and polynomial? Are operations between polynomials explained in some way, or are they introduced as rules? Is the algebra of polynomials related to the properties of real numbers? Has the notion of equality of polynomials been introduced, and if so, in what way? What is the null polynomial? Has a connection been established between the notion of a first-degree polynomial and the notion of a linear function? If so, in what way?

Comparing even the sets of textbooks of the same group of authors, a gradual decrease in the representation of logos blocks in textbooks of the observed period is noticeable. The changes in the observed period will be briefly illustrated through the example of the most used sets of textbooks intended for mathematical gymnasiums. The most used textbook from 2005 explicitly explains the difference between a constant ("predetermined number") and a variable ("any real number", "number not predetermined"). An algebraic expression is defined as "any expression that consists of variables and constants, obtained by four basic algebraic operations and parentheses", while a polynomial is defined as "an algebraic expression obtained only by addition and multiplication operations", and in the lesson on polynomial algebra is written:

The general form of a polynomial of one variable is

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \quad (1)$$

The exponent n is called the degree of the polynomial. Coefficients of the polynomial are the real numbers a_0, a_1, \dots, a_n . The coefficient $a_n \neq 0$ is called the leading coefficient.

In the 2015 textbook of the same group of authors, it is commented that the constants are "special numbers", and the variables are "general numbers". An algebraic expression is defined as "any expression that consists of variables and constants related to basic algebraic operations", a monomial as "a product of a constant and a variable", a binomial as a "sum of monomials", and a polynomial as a "multi-member algebraic expression". The algebra of polynomials has been transferred to the second-grade textbook (when a polynomial is defined as a real function), and in the first-grade textbook is only briefly mentioned that "polynomials are calculated using known properties of arithmetic operations". Definitions of a polynomial in Croatian textbooks

for first grade, but also in textbooks in other countries for students of similar age (Bolondi et al., 2020), are an obvious example of *didactic transposition*, i.e., adapting mathematical *scholarly knowledge* to *knowledge to be taught* in high school. For example, for the logos of the presented textbook from 2005, we certainly cannot say that it corresponds to the academic approach to a polynomial as a function, in the notation P_F ; but it obviously does not define a polynomial as a completely algebraic object, in the notation P_A , because x in (1) is not a formal variable (indeterminate, placeholder) but x is an arbitrary real number.

When looking at praxeologies whose object of knowledge is a polynomial, two *local praxeologies* can be noticed in Croatian textbooks: algebraic-functional, in the notation $MO_{P_{AF}}$, and functional, in the notation MO_{P_F} . Moving away from the functional approach is noticeable when considering the most common types of problems in a section about algebraic expressions ("simplify algebraic expression", "reduce algebraic fraction") and presented techniques for solving them, which are reduced to manipulation of symbols based on polynomial algebra for P_A . Considering the organization of almost all observed textbooks, praxeologies whose types of tasks are solving linear equations also belong to $MO_{P_{AF}}$ because the lesson on a linear equation precedes the lesson on a linear function. The same is true for praxeologies whose objects of knowledge are quadratic functions in textbooks for the second grade of high school. On the other hand, MO_{P_F} consists of praxeologies about linear function in the first grade, quadratic function in the second grade, and polynomials as one of the elementary functions in the fourth grade. Attempts to unite $MO_{P_{AF}}$ and MO_{P_F} are visible in some textbooks, such as in the presented textbook from 2005. For the same purpose, in some other textbooks, a lesson on the value of algebraic expression has been included, but none of this is enough to be able to say that $MO_{P_{AF}}$ and MO_{P_F} form a regional praxeology. Finally, the reduction of the polynomial-related logos, especially in first-grade textbooks, separated the praxeologies that were associated with it.

Undergraduate study program, syllabi, and course materials

In terms of content, the first year of U has hardly changed in the last twenty years, and since the introduction of the Bologna System in 2005, it has not changed structurally. The first year, as in most undergraduate study programs in mathematics in the world, is dominated by two *modules* (Bosch et al., 2021): algebraic and analytical. The module of analysis includes the first-semester course Introduction to Mathematical Analysis (IMA), and the second-semester course Mathematical Analysis I (MAI). The algebra module includes the first-semester course Linear Algebra I (LAI) and the second-semester course Linear Algebra II (LAI).

Introduction to Mathematics (IM) course is a first-semester course aimed at bridging the gap between high-school mathematics and university mathematics. In IM, among the elementary functions, polynomials are treated as real functions of one real variable. The course covers the knowledge of polynomials prescribed by the curriculum for mathematical gymnasium, only with a more rigorous approach. The characterization

of the equality of polynomials and the characterization of the null polynomial, in high school usually introduced only as rules without explanation, are proved. The supplementary literature mentions that the null polynomial theorem need not hold over finite fields, and provides counterexample: „the polynomial $f(x) = x^3 + 2x$ is a null polynomial over \mathbb{Z}_3 , and not all its coefficients are equal to zero“. Also, after the theorem on the equality of polynomials, it was noted that a polynomial can be identified⁴ with a series of its coefficients, so "the variable x can be understood as $x = (0,1,0,0, \dots)$ ". In courses belonging to the analysis module a polynomial is defined as a real function of one real variable. Students in the IMA learn about continuity, and in the MAI about differentiation (Taylor's theorem) and integrability. Thus, through the IM, IMA, and MAI MO_{P_F} praxeologies are formalized and supplemented, and students are equipped with new praxeologies whose object of knowledge is P_F .

In the algebra module, the approach to the notion of a polynomial can be twofold: formal and functional. LAI is a first-semester course in which, among other things, basic algebraic structures are introduced: group, ring, field, and vector space. Although students are introduced to some substructures (normal subgroup, quotient subgroup) and homomorphisms, the tasks are reduced to checking the structure, or homomorphism between structures. Therefore, it can be concluded that the development of structuralist praxeologies (Hausberger, 2018) begins in this course. Polynomials over a field of real numbers with standard operations are treated as an example of a commutative unitary ring and real vector space. Unlike other courses of the first semester in LAI, a polynomial is introduced formally, as the finite formal sum of powers, i.e., polynomial in variable x is an expression of the form (1), where $n \in \mathbb{N}_0$ and $a_k \in \mathbb{R}$, for each $k \in \{0,1, \dots, n\}$. For some students, this will be a re-encounter with the definition because, in some high school textbooks, a polynomial is defined in the same way; while in others (such as observed one from 2005), some kind of hybrid definition is introduced between this formal and functional approach because $p(x) \in \mathbb{R}$. This redefines or upgrades $MO_{P_{AF}}$ praxeologies. Polynomials over an arbitrary field are also mentioned in LAI materials (and finite fields are introduced in the section dedicated to fields), but the tasks are focused exclusively on polynomials over a field of real numbers. Polynomials with coefficients from an arbitrary ring, and the functions induced by polynomials, are part of the fourth-semester course Algebraic Structures. LAII is a course dedicated to linear operators on the finite-dimensional vector (unitary) spaces, so polynomials in this course have an important role because of the characteristic and minimal polynomials of a linear operator, but also because vector spaces of polynomials to a certain degree (including null polynomial) are one of the most common examples for the domain or codomain of linear operators. Although the theory is given in full generality (over an arbitrary field), the tasks are limited to vector spaces over fields of real or complex numbers. As LAI, LAII is marked by general logos blocks and specific praxis blocks. Both approaches to a polynomial can be found

⁴ Identification in this course for a student can only mean establishing a 1-1 mapping.

in the prescribed literature (example of literature in English: Friedberg et al., 2018, p. 10) for LAII, even though in some of it the explicit definition of a polynomial is missing (e.g., Hefferon, 2020). The authors seem to refer to this notion as already known (*illusion of transparency*), as evidenced by the frequent use of the phrase "usual operation" (e.g., Hefferon, 2020, p. 88-89) which refers to operations with polynomials. This may mean that the students sometimes have to conclude from the context which approach it is. Let us give one such example from the course materials (similar examples can be found in, e.g., Nicholson, 2013, p. 336, 355), in which it was necessary to examine whether the mapping $f: \mathbb{R}_2[x] \rightarrow \mathbb{R}_2[x]$ given by the rule

$$f(p(x)) = 2p(-x) \quad (2)$$

is linear operator, where it was previously written that $\mathbb{R}_2[x]$ denotes the set of all polynomials with real coefficients in the variable x of degree less than or equal to 2. If $p(x)$ is a formal polynomial (an expression of the form ax^2+bx+c , where $a, b, c \in \mathbb{R}$; like in, e.g., Nicholson, 2013, p. 546), then students may wonder what $p(-x)$ is. Namely, in this case, x is a formal variable, it is not a real number, and in none of the prescribed textbooks the composition between formal polynomials was introduced. On the other hand, the IM, IMA, and MAI courses insist on the difference between a function and the value of a function at a point, so in the case that p is a real function of a real variable, the ostensive (2) can also be confusing for students. In that case, some of the ostensives $(f(p))(x) = 2p(-x)$ or $f(p)(x) = 2p(-x)$ (as can be found in some prescribed textbooks) are more appropriate. Praxeologies, whose objects of knowledge are operators of derivation or integration, are important factors in connecting the two dominant modules. In the algebra module, these operators are introduced formally (without the use of limits) on the vector space of polynomials.

TAUGHT AND LEARNED KNOWLEDGE

The interviewed high school teachers had almost the same experiences and thoughts on the issue of textbooks and equipping students with praxeologies whose object of knowledge is P . Teachers believe that textbooks are important in the teaching process, but because they consider them deficient, inaccurate, and poorly structured, they do not rely heavily on textbooks, especially in terms of explanation and lessons order. They mentioned that 30 years ago there was a school subject in mathematics gymnasiums that was dedicated only to polynomials and that this notion was gradually marginalized, so according to newer textbooks, students learn almost only about second-degree polynomials. The experiences and thoughts of the interviewed teachers are in line with the results of a questionnaire conducted at the very beginning of the first semester among 45 students. When asked to define the notion of a polynomial, about 44% of students did not provide an answer, but they often mentioned binomials and trinomials. When they were asked to provide an example of a polynomial, these students mostly answered by giving an example of a quadratic equation. Around 13% of students explicitly wrote that a polynomial is an equation, while almost 16% of students wrote that a polynomial is an expression, but they wrote it as an algebraic equation. Only one

of these students knew the general form of the algebraic equation, while the others cited the general form of the quadratic equation. Around 22% of students defined a polynomial as an expression of a certain form. Only two students of that 22% wrote a general form of a polynomial in one variable, and the rest wrote a general form of a second-degree polynomial. Only 2 of 45 students defined a polynomial as a real function of one real variable, and these students knew the rule of mapping for polynomials of arbitrary degrees. Immediately after the first two questions, each student was individually asked to explain the meaning of the symbols in the definition and examples he/she had written. Almost all students used the ostensive x for the variable in their definitions and examples. When asked what x is, all but four students who knew the general form of a polynomial (whether they defined it as an algebraic expression or defined it as a function) answered that x is unknown. In interviews⁵ with university teachers, we learned that teachers who teach IM are well aware that students do not distinguish between polynomials and algebraic equations. They pointed out that first-semester students generally do not see the need for characterization of equality of polynomials and characterization of null polynomial. However, the situation changed after the first semester. At the beginning of the second semester, 2 of the 30 students wrote that polynomial is an expression, but they wrote it as an algebraic equation. 60% of students defined a polynomial as a real function of a real variable. About 23% wrote the definition of a polynomial from LAI, but when they were asked what x is in that definition, all students answered that x is a variable with $x \in \mathbb{R}$. 3 out of 30 students did not answer the question at all.

Teachers who teach LAI said that the exams in which students should check whether a mapping, whose domain or codomain is a set of polynomials, is a homomorphism of groups or rings are arguably the worst solved in exams. One of the teachers said: "Students simply do not know what to do with these tasks, and they manage to pass the exam thanks to other tasks." Mentioned praxeologies appear and complement at the very beginning of the LAII course through tasks in which it must be checked whether the given mapping is a linear operator. For example, if a mapping is given with (2), students are generally not sure of the linear combination of what the mapping has to keep, so they often write $f(p(\alpha x + \beta y))$, where $\alpha, \beta \in \mathbb{R}$. Also, when students have to determine a kernel of a linear operator, whose domain is a vector space of polynomials to a certain degree, they often solve algebraic equations. It is also interesting how students sometimes unknowingly generalize the results proved for polynomials of one variable (polynomial equality theorem and null polynomial theorem). One such example is the problem from the third questionnaire in which three linear functionals $F, G, H: \mathbb{R}^3 \rightarrow \mathbb{R}$ were given by the rules $F(x, y, z) = 2x + z$, $G(x, y, z) = 2y$ and $H(x, y, z) = x + y + z$, and it was necessary to check whether

⁵ The first part of the interview with the university teachers consisted of a general question asking them, in case they noticed, to list the difficulties that students have related to the notion of a polynomial, which spans several generations within the first-year courses they teach. For the second part of the interview, questions were asked regarding the course they teach and the answers they had provided in the first part of the interview.

$\{F, G, H\}$ is the basis for the dual vector space of \mathbb{R}^3 . Students either didn't know how to solve this task or set out to test linear independence the way they did assume that $\alpha, \beta, \gamma \in \mathbb{R}$ are such that it holds

$$\alpha F(x, y, z) + \beta G(x, y, z) + \gamma H(x, y, z) = 0$$

from which they got $x(2\alpha + \gamma) + y(2\beta + \gamma) + z(\alpha + \gamma) = 0$ and then the system

$$\begin{cases} 2\alpha + \gamma = 0 \\ 2\beta + \gamma = 0 \\ \alpha + \gamma = 0 \end{cases}$$

but they did not know how to correctly argue how they came to the system. Usually, they only described the technique they used, and the extremely common conclusion was: "This is how it is done." Given that students have difficulty with the previous example, which is one of the simplest types of tasks when it comes to dual vector spaces, recognized in the literature as a notion that carries obstacles (Vleeschouwer, 2010), it can be assumed what happens when the environment is dual vector space of polynomials to some degree.

CONCLUSION

While Croatian high school textbooks have similar praxis blocks, they differ in logos blocks about the notion of a polynomial P , especially in first-grade textbooks. Two local praxeologies are observed: $MO_{P_{AF}}$, which leans towards an algebraic approach to a polynomial, and in some textbooks may contain some parts of a functional logos; and MO_{P_F} based on a functional polynomial approach. It is certainly difficult for a high school teacher to unite $MO_{P_{AF}}$ and MO_{P_F} in some way, so depending on how much the teacher relies on the given textbook, students can enroll in the undergraduate study U equipped with praxeologies with very different logos blocks. However, it seems that $R_{HS}(hS, P)$ is marked by the dominant type of tasks in high school whose object of knowledge is P - by solving the quadratic equation. Given the traditional organization of mathematical knowledge, followed by the organization of study U , the theory of P consists of the theory of a polynomial introduced as a function P_F and the theory of a polynomial introduced as a formal algebraic object P_A . Praxeologies whose logos are those theories do not unite until the course of Algebraic Structure in the fourth semester. From the obtained results we can conclude that $R_U(s_1, P)$ is characterized by the definition of P as a function, and techniques and technologies characteristic of P_A , which fail to be justified by functional logos. In the first year of U , the functional logos is probably still underdeveloped, as a result of the transition from high school, which is an institution dominated by the praxis blocks (Winsløw et al., 2014). In addition to the presented didactic obstacles, the results indicate that the formal approach to P could carry some epistemological obstacles, and the role of *non-ostensives* (polynomial, polynomial function, variable, formal variable, $p(x)$) should not be neglected. All the above are assumptions that still need to be explored and considered in further transitions through undergraduate and graduate study of mathematics. The departments of mathematics in Croatia must not ignore the fact that

the first generation of high school students will graduate in the school year 2021/2022 according to new curricula, which can bring new phenomena in the secondary-tertiary transition.

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