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# Learning two-variable functions using 3D dynamic geometry

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*In this study, we redesigned successful paper-and-pencil activities and implemented them in an introductory multivariate calculus course supported by 3D dynamic geometry software. We used semi-structured interviews and students' written productions during the semester to analyze the use of technology in supporting students' learning. Considering the work of two students, we find that the software has transformative potential but that it did not promote students' learning as expected. We discuss possible reasons for this.*

*Keywords: Functions of two variables, APOS theory, 3D dynamic geometry, University mathematics.*

## INTRODUCTION

The modelling of natural phenomena generally does not depend on a single variable. Over the past decade, several researchers have studied students' understanding of two-variable functions (for a review see Martínez-Planell & Trigueros, 2021). Among these studies, there are some results exploring the use of physical manipulatives for the teaching of two-variable functions. For example, McGee et al. (2012) developed a set of tangible manipulatives and support materials for visualizing concepts related to points, vectors, surfaces, curves, and contours in 3D space and report their positive effect on student learning. Martínez-Planell and Trigueros (2019) used these manipulatives in their teaching and then researched student understanding, also reporting positive effects. Wangberg (2020) observed improved student understanding when using a different manipulative. Some articles consider digital technologies as a means to support visualization in multivariable calculus (e.g., Alves, 2012).

The use of tangible and virtual manipulators in teaching and learning situations is attracting growing interest due to the new possibilities offered by digital technologies (Soury-Lavergne, 2021). One of the possibilities offered by the digital is the passage from static supports (for example, paper and pencil), which allow us fixed figures, to dynamic supports, which would enable us to experiment with mathematical ideas in dynamic figures (Roschelle et al., 2017). Computer algebra systems (CAS), spreadsheets, and dynamic geometry environments (DGE) are the technologies most used in mathematics classrooms. According to Soury-Lavergne, "dynamic geometry is a generic term for a type of software that allows the construction on the screen of dynamic figures that can be deformed while retaining the geometric properties used at the time of their construction" (2020, p. 7). DGEs such as Cabri 3D or GeoGebra 3D can represent three-dimensional surfaces and thus can potentially support student learning of two-variable functions. However, if the student's activity is reduced only to typing algebraic expressions and seeing their representation on the screen, the dynamic

aspect of dragging their free elements and observing what geometric properties of the figure are preserved is missed, thus limiting the technology's didactic potential. In this study we report on results obtained while using GeoGebra 3D together with didactic activities in the learning of two-variable functions.

Our research questions are: What mental constructions do college students show when using graphing activities of functions of two variables supported by 3D dynamic geometry? How do these constructions compare to those shown by students who worked with the same activities on paper and pencil?

## **THEORETICAL BACKGROUND**

In APOS theory (Arnon et al., 2014), an Action transforms a previously constructed mathematical object and is perceived as external, i.e., it will be relatively isolated from the individual's other mathematical knowledge. Performing Actions does not allow per se the individual to justify them. Actions may correspond to mechanically executing a procedure by following explicitly available or memorized instructions. When an Action is repeated, and the individual reflects on it, he/she might interiorize it into a Process. A Process is perceived as internal, which means that the individual can think about the result of its application without following all the necessary steps and without recurring to external support. Different Processes can be coordinated into new Processes and can be reversed. These coordinations allow the individual to justify, imagine, and generate dynamic imagery of the Process. When an individual can think of a Process as a whole and can do or imagine doing Actions on it, the Process is encapsulated into an Object. The essential aspect of the Object structure is that Actions can be performed on it. We will not need to refer to Schemas in this article. APOS structures and mechanisms can be used to design a model of how students might construct a mathematical notion; such a model is called a genetic decomposition (GD). A GD is not unique, multiple GDs of the same concept can coexist, or different researchers can propose other models. What is important is that it is tested by data obtained from students. In general, a GD is used to design teaching activities that help students achieve the constructions conjectured in the GD. These activities are used in the classroom, then research is conducted with students, and, depending on the results obtained from the analysis of data, it may turn out that it needs to be revised and thus also the designed teaching activities. This opens the door to further research using the newly revised GD. One can continue doing research cycles until a GD is stable, that is, until the analysis of the data obtained reflects what the GD predicts about the construction of the mathematical notion at stake.

From its beginning, APOS theory considered technology as part of its teaching methodology by promoting programming as an instrument to encourage exploration, reflection, and concept building. Today, various tools offer new possibilities to promote student reflection. Drijvers (2015) distinguished three didactic functionalities for digital technology: (a) to do mathematics, (b) to practice skills, and (c) to develop conceptual understanding. To examine the different forms in which technology can be used in the classroom, Hughes (2005) developed three categories: technology as (a)

Replacement, (b) Amplification, and (c) Transformation. She defined technology as a replacement when "the technology serves as a different means to the same instructional end"; Technology as an amplifier when it "capitalizes on technology's ability to accomplish tasks more efficiently and effectively, yet the tasks remain the same"; and technology as transformation when it can change "students' learning routines, including content, cognitive processes, and problem-solving" (p. 281).

## METHODOLOGY

Two groups of approximately 30 students each from a Mexican public university participated in this study in synchronous teaching, due to the Covid-19 pandemic, in the fall of 2021. Both groups worked with the activities based on the GD designed for the third cycle of the study by Martínez-Planell and Trigueros (2019). The activities were redesigned to enable the use of a 3D dynamic geometry environment (GeoGebra). The professor who taught this course is one of the authors of this article. Students first worked individually on each activity, then discussed it in teams of four students; during these discussions, the teacher visited the teams –these discussions were video recorded but have not been analyzed in this paper– and finally, a teacher-led whole group discussion was held. At the end of the semester, one student from each of the eleven teams was chosen to be interviewed. The researchers designed an instrument consisting of seven multitask questions to conduct semi-structured interviews with eleven students to test their understanding of the different components of GD. All the students were engineering students who had just completed this introductory multivariate calculus course. Each of the interviews lasted between 60 and 80 minutes and was video recorded. All the data were independently analysed by the researchers and conclusions were negotiated. The written work from the GeoGebra-based activities produced during the semester was obtained from all students and was used in the analysis.

The seven interview questions focused on functions of two variables. For the purpose of this report, only four of these questions are addressed. All of these questions were to be worked with paper and pencil, except where otherwise stated.

1. Draw in three-dimensional space the collection of points in space that satisfy the equation  $y = 2$  and that are also on the graph of the surface  $z = x^2 + x^3(y - 2) + y^2$ .

2. Let  $f(x, y) = x^2$ .

a. Represent in three-dimensional space the intersection of the plane  $y = 1$  with the graph of  $f$ .

b. Draw the intersection of the plane  $z = 1$  with the graph of  $f$ .

c. Draw the graph of  $f$ .

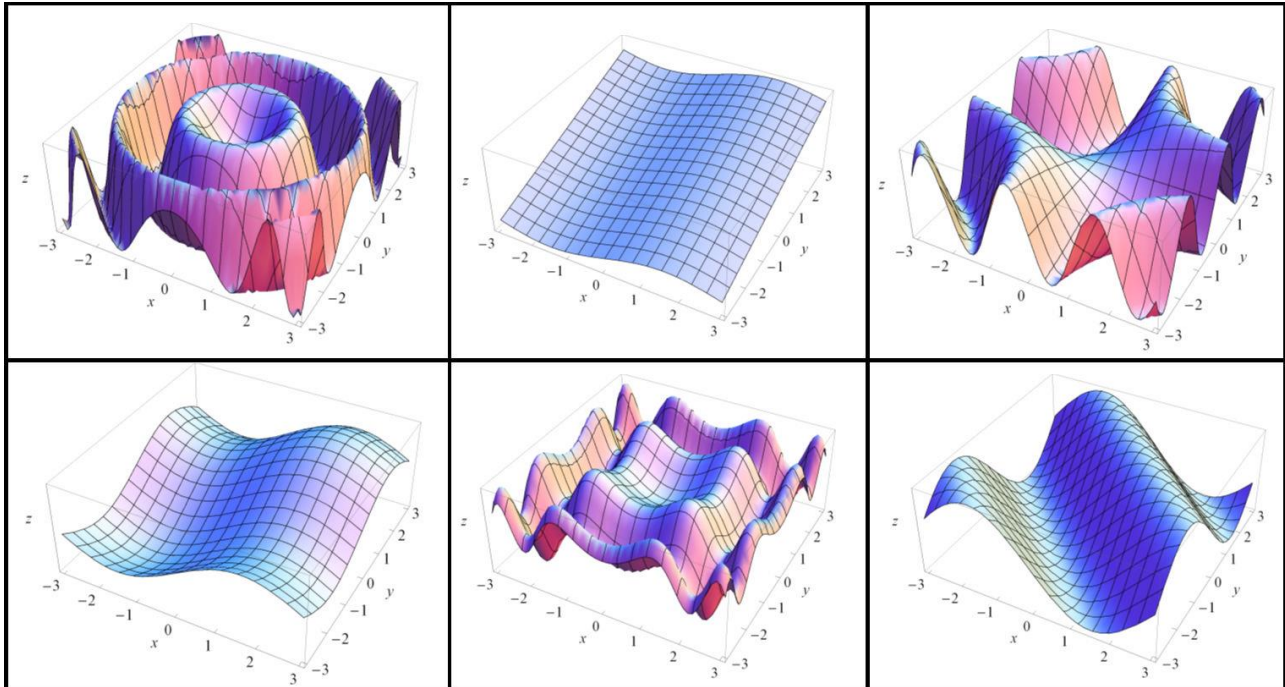
3. Let  $f(x, y) = x \sin(y)$ .

a. What can you say about the intersection of the plane  $x = 0$  with the graph of the function  $f(x, y) = x \sin(y)$ ? Represent the intersection in three-dimensional space.

b. Draw the graph of  $f(x, y) = x\sin(y)$ .

c. You can use the GeoGebra scenario to graph [one was provided, as in Figure 2b].

4. State which figure corresponds to  $f(x, y) = \sin(x) + y$ . Carefully justify your answer (see the figures below).



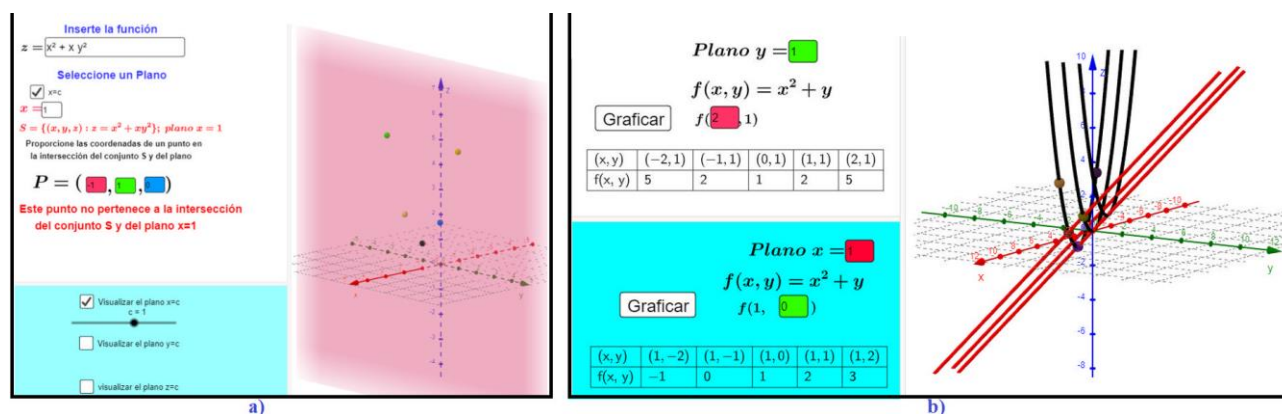
**Figure 1: Surfaces for problem 4, the graph appears on the first row, second column.**

Problems 1, 2a, 2b, and 3a directly deal with students' understanding of "fundamental planes" (planes of the form variable=constant) and the geometric meaning of substituting a number for a variable. Problems 2b, 2c, 3a allow to obtain information on students' understanding of "free variables" (situations represented by an equation with fewer variables than its geometric context). Problem 2c, 3b, 3c, and 4 may give information on students' graphing of two-variable functions. They may also give information on students' use of transversal sections, and thus, of fundamental planes.

### **Design of the Interactive Math Environment (IME)**

We developed an IME based on the DGE, GeoGebra 3D. For the design, we considered the first five sets of activities used in the third research cycle of Martínez-Planell and Trigueros (2019). These had shown to be successful in helping students construct functions of two variables in a paper-and-pencil environment. IMEs consist of two or three Views (see Figures 2a and 2b). In the first activity, students are asked to make point-by-point constructions in which they intersect fundamental planes with surfaces, for example, students are given the set  $S = \{(x, y, z): z = x^2 + xy^2\}$  and are asked to draw its intersection with the plane  $x = 1$ ; IME helps the student observe their Actions on the screen, allowing for an automatic response that helps them identify whether a point they enter belongs to both the plane and the surface. Students are expected to interiorize those Actions into a Process where they can imagine the relation between

the equation of a fundamental plane, its geometric representation, and placement in space.



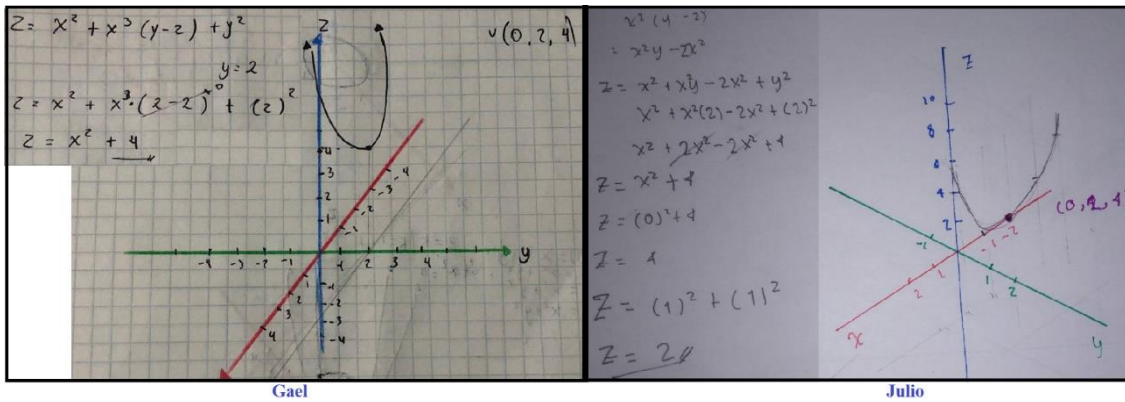
**Figure 2: A scenario where students do a) Actions to construct a fundamental plane and b) Actions on fundamental planes.**

Another activity guides the student to plot the graph of  $f(x, y) = x^2 + y$  by drawing and placing a few curves using specific transversal sections, leading to reflection on the effect of giving different values to different variables. The student chooses the variable and value (e.g.,  $y = -1$ ,  $y = 0$ ,  $y = 1$ ,  $x = 0$ ), GeoGebra plots the curve, and the student may verify by plotting points if desired. The dynamic imagery necessary to make sense of the situation was expected to help students construct a graphing Process. The need to do Actions on fundamental planes was expected to help then encapsulate fundamental planes into an Object (see Figure 2b).

## RESULTS

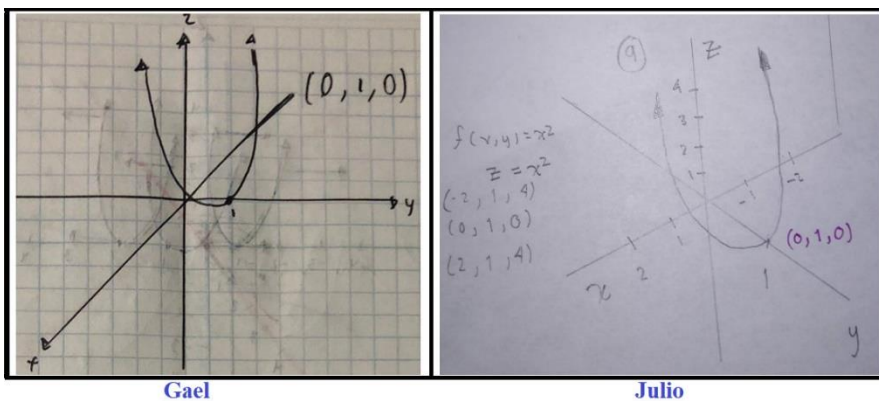
The results obtained in this study were not as good as those of the third research cycle of Martínez-Planell et al. (2019) or the reproducibility study of Borji et al. (2022) in which the original GD-based activities were done with paper-and-pencil. In this report, we consider the results of two of the best students. This will enable us to discuss why activities with GeoGebra were successful in promoting some of their constructions but failed for other students. In doing that, we also give a more detailed account of specific student difficulties with free variables.

In question 1, both Julio and Gael showed the Process of relating graphical and algebraic representations of fundamental planes, as well as imagining their position in 3D space. The results suggest they had constructed the Process on the geometric meaning of substituting a number for a variable. This is consistent with a Process conception of fundamental plane. This construction seems to have been fomented by the use of GeoGebra during the in-class activities, as suggested by the fact that they graphed using the same colours for the axes as the GeoGebra activity (Figure 3).



**Figure 3. Graphical and algebraic representations of the fundamental planes.**

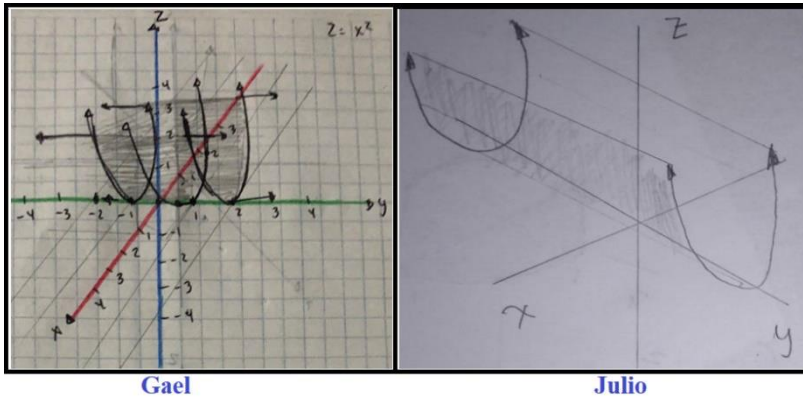
In problem 2a, Julio and Gael could locate the intersection of  $y = 1$  with the graph of  $f(x, y) = x^2$  correctly in space (see Figure 4). Although this question involves the free variable  $y$ , all variables are explicit: students set  $z = x^2$  and are told that  $y = 1$ . This gives further evidence of their understanding of fundamental planes, this time in the case of a “cylinder,” meaning the graph of a two-variable equation that is to be interpreted in the 3D context.



**Figure 4. Intersection of the fundamental plane  $y = 1$  with the graph of  $f(x, y) = x^2$ .**

In problem 2c, both students used transversal sections to draw the graph of  $f(x, y) = x^2$  (see Figure 5). They drew parabolas resulting from giving positive and negative values to  $y$ , then connected these parabolas with straight lines (see Figure 5). Based on the interview and in-class work where they graphed cylinders and justified their reasoning, it seems that they could generate the needed dynamic imagery to join the curves. This is consistent with a Process conception of graphing two-variable functions. Moreover, as they did Actions on such planes in order to graph the cylinder, we considered they had constructed an Object conception of fundamental planes. Students’ work on the GeoGebra scenario used during in-class activities to graph cylinders gives evidence of their use of technology as a Replacement since they did Actions of intersecting fundamental planes with surfaces when plotting point-to-point graphs. The evidence also shows its use as Amplification since it allowed them to automatically validate their calculations and display the points and/or curves in the 3D view (Hughes, 2005), enabling them to imagine the entire surface. However, this was not the case for most other students. Even though the instructions for the activities were careful to request

justifications, most students did not construct a graphing Process. They tended to take the output of the computer in itself as a valid justification, and used technology as replacement, sidestepping the necessary reflection to interiorize Actions into Processes. In this case, we also considered that as lectures and group discussions were conducted virtually, because of the pandemic, it was difficult to develop a culture of discussion and justification in the classroom.



**Figure 5. Graphs of  $f(x, y) = x^2$  during the interview.**

Responses to question 2b suggest that some situations involving free variables are treated differently by students. This difficulty had been observed before (Martínez-Planell & Trigueros, 2019). Both Julio and Gael wrote  $z = x^2$  and substituted  $z = 1$  to obtain  $1 = x^2$ . It can be observed that contrary to the situation of problem 2a, the variable  $y$  does not appear in their computations. So, both of them seemed to do the Action of setting  $y = 0$ , which led them to consider two points rather than two lines. The same behaviour regarding free variables was also observed in question 3a. Both Julio and Gael did the Action of substituting  $x = 0$  in  $z = x \sin(y)$  to obtain  $z = 0$ . It seems that, from their perspective,  $y$  disappeared and so it again seems they set  $y = 0$ .

Julio: It tells us that in the plane  $x = 0$ , we substitute in the function and we get 0,  $z$  is equal to 0, in three-dimensional space, ..., I guess it will be just a point [He drew the point (0,0,0)].

Students had worked on exactly the same problem in the in-class activities. Examining their Actions, we see that the GeoGebra scenario required them to enter different points satisfying both equations, and then the scenario would show their graph; then students connected the resulting  $y$ -axis points by performing Actions, apparently without reasoning algebraically why they could connect them. That is, the GeoGebra scenario did not induce students to reflect on the fact that  $0 = 0 \sin(y)$  is true regardless of  $y$ . Students used technology in this activity as replacement (Hughes, 2005) as they could have performed the same point-by-point graphing Actions on paper-and-pencil. A similar response was observed in most other students. By considering their written response to the GeoGebra-based activities, it becomes clear that some students went beyond the scenarios provided and produced surface graphs using GeoGebra in a way that did not foster their reflection on fundamental planes and the geometric



interpretation of the meaning of holding a variable fixed, thus partially explaining their lack of success.

In problem 3b, when asked to graph  $z = x\sin(y)$ , it became apparent that all students had not constructed the prerequisite Processes of trigonometry to coordinate it with a Process of fundamental plane. Both Julio and Gael stated that they would use transversal sections as their strategy, both knew they were expecting a wave-like surface, but neither of them interpreted  $x$  as an amplitude nor took the sign of  $x$  into account:

Interviewer: how do you plan to draw the graph?

Gael: by transversal sections, ..., that would be the sine graph, which is basic trigonometry, which are like waves, ..., but I'm trying to remember how it behaves when it's multiplied by  $x$ ? I know that if  $x$  weren't there, it would be represented as a galvanized sheet, but when you multiply  $x$ , it behaves differently.

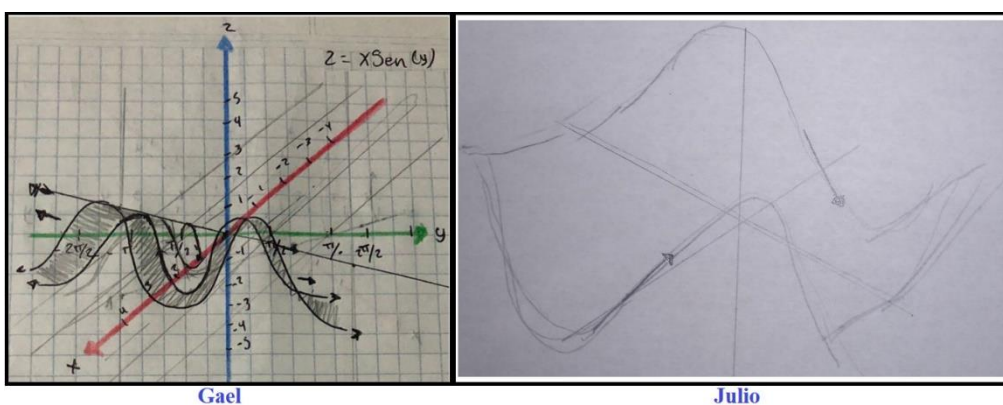
Julio: What I would do is give values to  $x$  and  $y$ , ..., like a wave ...

Most other students showed difficulty with trigonometry. This seems to be an institutional issue. It was not observed in the paper-and-pencil studies mentioned before. Julio and Gael succeeded in graphing the function using a GeoGebra scenario, to do Actions involved in completing a few transversal sections to obtain the surface (see Figure 6). The scenario seemed to also help Julio make sense of the intersection of  $x = 0$  with  $z = x\sin(y)$ :

Interviewer: Ok ... in the first part when  $x = 0$ , do you remember the first part? ... you told me that the answer was a point ... Why do you think, you get a line, Julio?

Julio: Because ... we are only giving values to  $y$ .

It seems that now Julio realizes that  $x = 0, z = 0$ , but that  $y$  can take any value.



**Figure 6. Graph of the function  $f(x, y) = x\sin(y)$  supported by the IME.**

The above discussion underscores that even though students might have constructed a graphing Process, they need to coordinate the Process of fundamental plane with one-variable function graphing Processes, which may not have been constructed due to the pandemic or other institutional reasons. It also shows how technology's capacity to

generate graphs can potentially support students understanding of free variables when used as transformation.

In question 4, when choosing the graph of  $f(x, y) = \sin(x) + y$ , Julio chose the correct graph by doing an Action on fundamental plane. He mentioned that if  $x = 0$ , the function reduces to  $z = y$ , and therefore should result in a line with positive slope. There is only one such option. Gael gave signs of construction of dynamical imagery:

Gael: ... on the  $x$ -axis [meaning the  $x$  direction] it is going to be the graph of sine ... then plus  $y$ ,  $y$  would be a line on the  $y$ -axis [meaning the  $y$  direction] ... when  $y = 0$  I'll be left with only that, its wave in  $z$  ... and it keeps on increasing.

Julio and Gael seem to imagine intersecting fundamental planes with surfaces as a technique for graphing surfaces, they seem able to do Actions on fundamental planes in order to form transversal sections, and show some evidence of generating dynamical imagery when graphing cylinders and functions. While this shows the potential of activities with GeoGebra, for most students it did not encourage the necessary reflection to interiorize their Actions into Processes.

## DISCUSSION AND CONCLUSIONS

We observed that the activities worked in class helped Julio and Gael to construct the geometrical meaning of substituting a number for a variable in a 3D context, to recognize that the graphs of cylinders can be obtained by intersection with fundamental planes corresponding to the missing variable, and to use transversal sections as their chosen graphing strategy. The use of GeoGebra seems to have contributed to this understanding of fundamental planes and cylinders. Although this shows GeoGebra activities have the potential to be used as transformation, most students used the technology as replacement. We found that there are situations involving free variables, particularly situations when during a computation one of the variables “disappears,” in which the GeoGebra activities did not foster the needed student reflection. Thus, some GeoGebra activities need to be redesigned to accomplish this goal.

Considering the entire student population, the results obtained in this study were not as good as those obtained in other studies based on the equivalent paper-and-pencil activities (Martínez-Planell et al., 2019; Borji et al., 2022), which rendered very positive results. We conjecture that one of the reasons for this difference was that classes for this experience were taught virtually due to COVID-19 pandemic. In this context, more attention to classroom culture and management issues are needed, particularly as it regards justification in a DGE. Also, there were other institutional factors, like students' knowledge of trigonometry, that affected outcomes.

When redesigning the activities, the purpose was that the IME would have a didactic functionality to develop conceptual understanding (Drijvers, 2015); however, the results show that this was not achieved in this teaching experiment. The shift from paper-and-pencil activities to 3D dynamic geometry technology is not a direct translation; activities have to be redesigned and institutional conditions have to be

taken into account to allow both for the possibility to use the technology as transformation and for promoting students' reflection.

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