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Potential conflict factors in learning exact differential equations:

An impact of institutional practices

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Many engineering students find university mathematics courses challenging and tend to adopt an instrumental approach to their studies. One of the difficulties pertains to students' inability to relate new material to their existing knowledge and skills. We employ anthropological theory of the didactic and the construct of the concept image for the analysis of two institutional praxeologies in Calculus and Differential Equations indicating potential conflict factors in students' understanding of the concept of an exact differential equation.

Keywords: teaching and learning of mathematics for engineers, curricular and institutional issues concerning the teaching of mathematics at university level, exact differential equations, anthropological theory of the didactic, concept image.

INTRODUCTION

Engineering plays a significant role in the modern society; the demand for engineers with better interdisciplinary and specialist skills is continuously increasing (Kent & Nossum, 2003). Future engineers need a wide spectrum of mathematical competencies and should comfortably use mathematics as a medium for communicating and developing ideas and concepts – “we need engineers who are at ease with it [mathematics] and who can take advantage of new ideas and use them appropriately even if they are expressed using advanced mathematics” (Blockley & Woodman 2002, p. 15). However, educational research acknowledges difficulties with students' conceptual understanding of mathematical disciplines which are often viewed as obstacles on the way to the engineering degree (Ditcher, 2001; Harris et al., 2015).

University courses on differential equations (DEs) are included in most engineering programmes, but related educational research is scarce, with “fewer than two dozen empirical studies published in top journals [in mathematics education]”, which is quite surprising “given the centrality of differential equations (DEs) in the undergraduate curriculum, as well as the move away from a “cookbook” course to one that emphasizes modelling, qualitative, graphical and numerical methods of analysis” (Rasmussen & Wawro, 2017, p. 555). Exact differential equations (EDEs) is a classical topic; it is present in all traditional DEs courses and connects to many important concepts and methods in mathematics and physics including, for instance, integration of first-order linear DEs with variable coefficients, integrating factors, and first integrals. Rezvanifard et al. (2022) discussed difficulties with the learning of EDEs and a positive impact of a puzzle-based learning on students' conceptual understanding of this topic.

Recently, anthropological theory of the didactic (ATD) was employed by González-Martín and Hernandez-Gomes (2018, 2019) to analyse differences in mathematics and engineering courses. Hochmuth and Peters (2021) combined ATD with Weber's construct of ideal type to address variations in institutional praxeologies and individual student work in mathematics and engineering discourses. In this paper, we combine ATD (Chevallard, 2019) with the constructs of concept definition and concept image (Tall & Vinner, 1981) to explore potential conflict factors in the learning and teaching of EDEs. In contrast with the previous research on engineering education employing ATD, both institutional praxeologies in our case are within mathematics domain.

THEORETICAL FRAMEWORK / METHODOLOGY

The anthropological theory of the didactic

ATD furnishes an epistemological framework to describe mathematical knowledge as a human activity. A didactic system $S(X, Y, k)$ includes a class of students X , a team of teachers Y , and a piece of knowledge k from a body of knowledge K (a discipline D), in our case, mathematics. The theory of didactic transposition raises “the question of the precise nature of the piece of knowledge k which is the “didactic stake” – the thing to be taught and learned – in $S(X, Y, k)$ ” (Chevallard, 2019, p. 72). Importantly, this theory views knowledge as “a changing reality, which adapts to its institutional habitat where it occupies a more or less narrow niche” (Chevallard, 2007, p. 132).

ATD “hinges on an essential and founding notion: that of praxeology” posing that “all “knowledge” can be modelled in terms of praxeologies” (Chevallard et al., 2016, pp. 2615-6) used as building blocks for didactic systems. A praxeology consists of a task T , a technique τ (tau), a technology θ (theta), and a theory Θ (big theta). In ATD, the task T is performed using the technique τ . The technology θ is “a way of explaining and justifying or even of “designing” the aforesaid technique τ .” The theory Θ “should explain, justify, or generate whatever part of technology θ may sound unobvious or missing” (Chevallard & Sensevy, 2014, p. 40). A praxeology is construed as the union of two “blocks,” the *praxis part* $\Pi = [T/\tau]$ and the *logos part* $\Lambda=[\Xi/\theta]$. Notably, “it is the theoretical block that makes it possible to preserve the activity as a practice and communicate it to others, so that they, too, can participate in it” (Hardy, 2009, p. 344).

Didactic systems live in institutions understood as “any created reality of which people can be members” (Chevallard & Bosch, 2019, p. xxxi). For example, “a class, with its students and teachers, is an institution” (Chevallard & Sensevy, 2014, p. 2615). Institutional approaches significantly impact student inducting into mathematical practices (Hardy, 2009; Winsløw et al., 2014). Furthermore,

It often happens that an object O lives permanently in an institution J and remains lengthily ignored by another institution I not unconnected with J , while being simultaneously unknown to the overwhelming majority of the persons subjected to I Consequently, for many teachers “the notion that inhabits “my” institution is exactly what this notion is, so that I can ignore all other institutions’ definitions of it.” (Chevallard, 2019, pp. 82-83)

To tackle this problem, it is important to compare the description of praxeologies for the same object adopted by different institutions.

An institutional reference model of praxeologies involving a mathematical notion can be built to describe the practices and knowledge that an institution aims for students to develop – with interconnections between subjects, themes, sectors and domains related to the notion in question. (Winslow et al., 2014, p. 103)

Concept definition and concept image

One of the learning difficulties acknowledged by ATD stems from “a universal belief that any notion has a unique definition, independent of the institution that uses it” (Chevallard, 2019, p. 82). Introducing the constructs of concept definition and concept image, Tall and Vinner (1981) also recognised that this expectation is not justified.

Compared with other fields of human endeavor, mathematics is usually regarded as a subject of great precision in which concepts can be defined accurately to provide a firm foundation for the mathematical theory. The psychological realities are somewhat different. Many concepts we meet in mathematics have been encountered in some form or other before they are formally defined and a complex cognitive structure exists in the mind of every individual, yielding variety of personal mental images when a concept is evoked. (p. 151).

The concept definition is usually regarded as a description of the mathematical notion accepted by the professional community, and this is what mathematics teachers strive to teach to students. However, “for each individual a concept definition generates its own concept image” (Tall & Vinner, 1981, pp.152-153) understood as “the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes.” The concept image “is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures.” Distinct parts of the concept image, termed evoked concept images, can be activated at particular times, including images that may appear conflicting – “if “conflicting” parts of the concept image are called at the same time then a sense of confusion, or conflict may appear.” In this case, a potential conflict factor is defined as “a part of the concept image or concept definition which may conflict with another part of the concept image or concept definition” (Tall & Vinner, 1981, pp.152-153). “The pre-eminence of the concept image is clear when it is time to act or to solve a concrete problem” (Gascón, 2003, p. 47); it often replaces the concept definition.

In general, it is postulated that in informal learning of concepts (which is the most common), the concept image is utilised instead of the concept definition and also when the concept definition has been constructed (parting from the terms of the definitions, if these have already been introduced), this will tend to stay inactive in the mind of the person and may even be forgotten. (Gascón, 2003, p. 47)

The construct of concept image complements ATD in our analysis. ATD acknowledges possibilities for different institutional praxeologies built for the same mathematical

concept and the concept image framework supports the evolution of concepts within institutions that creates potential conflict factors. Combining ATD with the concept image paradigm, we address “the challenge of empowering students with autonomy and insight into the *raisons d’être* and *rationales* of such [mathematical] work” (Winslow et al., 2014, p. 100). Focusing on praxeologies rooted in two mathematical institutions, a Calculus (C) class, and a Differential Equations (DE) class, we analyse two approaches to the concept of an EDE.

The research question addressed in this paper is: *What similarities and distinctions characterising C- and DE- praxeologies create potential conflict factors?*

TWO INSTITUTIONAL VIEWS OF EXACT DIFFERENTIAL EQUATIONS

Multiple mathematical organisations in the engineering curriculum

We explore mathematical organisations MO1 and MO2 in two courses, Mathematics 2, and Mathematics for Mechatronics. Mathematics 2 is taught to first-year engineering students and is based on the text by Adams and Essex (2018). The module includes, among other topics, functions of several variables, vector calculus, and line integrals of vector fields. Mathematics for Mechatronics uses the textbook by Boyce and DiPrima (2013) and is taught in the first semester of a master’s program. The course focuses on the methods for the solution and analysis of DEs, stability, and applications.

Vignette MO1 Conservative vector fields in Calculus (Adams & Essex, 2018, pp. 874-880). A vector field $\vec{F}(x, y)$ in two dimensions defined by

$$\vec{F}(x, y) = F_1(x, y)\vec{i} + F_2(x, y)\vec{j} = \nabla\varphi(x, y) = \frac{\partial\varphi}{\partial x}\vec{i} + \frac{\partial\varphi}{\partial y}\vec{j}$$

is called *conservative*, and the function $\varphi(x, y)$ is called a (scalar) potential of \vec{F} .

The equation $F_1(x, y)dx + F_2(x, y)dy = 0$ is called an *exact differential equation* if its left-hand side is the differential of a scalar function $\varphi(x, y)$.

A necessary condition for a conservative vector field. If $\vec{F}(x, y) = F_1(x, y)\vec{i} + F_2(x, y)\vec{j}$ is a conservative vector field in a domain D of the xy -plane, then

$$\frac{\partial}{\partial y}F_1(x, y) = \frac{\partial}{\partial x}F_2(x, y) \quad \text{in } D.$$

If $\varphi(x, y)$ is a potential function for a conservative field $\vec{F}(x, y)$, the *level curves* $\varphi(x, y) = C$ of $\varphi(x, y)$ are called *equipotential curves* of $\vec{F}(x, y)$.

Example 1 (Adams and Essex, 2018, p. 877). Show that the vector field $\vec{F}(x, y) = x\vec{i} - y\vec{j}$ is conservative, find a potential function and describe the equipotential curves.

Solution Since $\frac{\partial}{\partial y}F_1(x, y) = 0 = \frac{\partial}{\partial x}F_2(x, y)$ in R^2 , \vec{F} is conservative. For any potential function $\varphi(x, y)$, one should have $\frac{\partial\varphi}{\partial x} = F_1(x, y) = x$ and $\frac{\partial\varphi}{\partial y} = F_2(x, y) = -y$

$-y$. Integrating $\frac{\partial \varphi}{\partial x} = F_1(x, y) = x$ with respect to x , we obtain $\varphi(x, y) = \int x dx = \frac{1}{2}x^2 + C_1(y)$, where the “constant” of integration can depend on the variable y . Using $\frac{\partial \varphi}{\partial y} = F_2(x, y) = -y$, taking the derivative $\frac{\partial \varphi}{\partial y} = \frac{\partial}{\partial y} \left(\frac{1}{2}x^2 + C_1(y) \right) = C'_1(y)$ and equating it to $-y$, we deduce that $C_1(y) = -\frac{1}{2}y^2 + C_2$. Therefore, for any constant C_2 , $\varphi(x, y) = \frac{x^2 - y^2}{2} + C_2$ is a potential function for a vector field $\vec{F}(x, y)$. Equipotential curves defined by $x^2 - y^2 = C$ represent a family of rectangular hyperbolas.

Remark 1 A necessary condition is not formulated as a theorem and no proof is provided. However, restrictions on the topology of the domain are mentioned warning that a vector field may not be conservative in a domain that has holes (Adams & Essex, 2018, p. 879). The following example is provided as an illustration of this possibility.

Example 2 (Adams and Essex, 2018, p. 879-880). Verify that a vector field $\vec{F}(x, y)$ defined for $(x, y) \neq (0, 0)$ by $\vec{F}(x, y) = \left(\frac{-y}{x^2 + y^2} \right) \vec{i} + \left(\frac{x}{x^2 + y^2} \right) \vec{j}$ is not conservative on the whole real plane including the origin.

Vignette MO2 EDE in Differential Equations (Boyce & DiPrima, 2013, pp. 95-100).

Given a DE $M(x, y) + N(x, y)y' = 0$, suppose that we can identify a function $\Psi(x, y)$, such that $\frac{\partial \Psi}{\partial x}(x, y) = M(x, y)$, $\frac{\partial \Psi}{\partial y}(x, y) = N(x, y)$, and such that $\Psi(x, y) = c$ defines $y = \varphi(x)$ implicitly as a differentiable function of x . Then

$$M(x, y) + N(x, y)y' = \frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y} \frac{dy}{dx} = \frac{d}{dx} \Psi[x, \varphi(x)],$$

and the DE assumes the form $\frac{d}{dx} \Psi[x, \varphi(x)] = 0$. In this case, the DE $M(x, y) + N(x, y)y' = 0$ is said to be an *exact differential equation*. Solutions are given implicitly by the equation $\Psi(x, y) = c$, where c is an arbitrary constant.

Example 3 (Boyce & DiPrima, 2013, pp. 95). Solve the DE $2x + y^2 + 2xyy' = 0$.

Solution One can guess that the function $\Psi(x, y) = x^2 + xy^2$ has the property that

$$\frac{\partial \Psi}{\partial x}(x, y) = 2x + y^2 = M(x, y), \quad \frac{\partial \Psi}{\partial y}(x, y) = 2xy = N(x, y),$$

and the given DE can be written as

$$\frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y} \frac{dy}{dx} = 0, \quad \text{or} \quad \frac{d}{dx} \Psi(x, y) = \frac{d}{dx} \Psi(x^2 + xy^2) = 0.$$

Thus, solutions to this equation are defined implicitly by $\Psi(x, y) = x^2 + xy^2 = c$.

Theorem 1 (Boyce & DiPrima, 2013, p. 96). Let the functions M, N, M_y and N_x where subscripts denote partial derivatives, be continuous in the rectangular region $R: \alpha < x < \beta, \gamma < y < \delta$. Then the equation $M(x, y) + N(x, y)y' = 0$ is an exact differential

equation in R if and only if $M_y(x, y) = N_x(x, y)$ at each point of R . That is, there exists a function ψ satisfying the equations $\psi_x(x, y) = M(x, y)$, $\psi_y(x, y) = N(x, y)$, if and only if M and N satisfy the equation $M_y(x, y) = N_x(x, y)$.

Remark 2 The necessity part of the proof (Boyce & DiPrima, 2013, pp. 96-98) is constructive; it provides a method for finding the function $\psi(x, y)$. A footnote to the theorem explains that the region may not necessarily be rectangular, but should be simply connected (that is, with no holes in its interior).

Example 4 (Boyce & DiPrima, 2013, pp. 98). Solve the DE $(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0$.

Solution Observe that $M_y(x, y) = \cos x + 2xe^y = N_x(x, y)$, so the given equation is exact, and there should exist a function $\psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x}(x, y) = y\cos x + 2xe^y = M(x, y), \quad \frac{\partial \psi}{\partial y}(x, y) = \sin x + x^2e^y - 1 = N(x, y).$$

Integrating the first equation, one has $\psi(x, y) = y\sin x + x^2e^y + h(y)$. Differentiation of the latter equation yields $\sin x + x^2e^y + h'(y) = \sin x + x^2e^y - 1$, or $h'(y) = -1$, $h(y) = -y$. Then $\psi(x, y) = y\sin x + x^2e^y - y$, and solutions are defined implicitly by the equation $y\sin x + x^2e^y - y = c$.

Praxeological analysis

Vignette MO1 Goal: Calculate the potential of a conservative vector field.

Raison d'être: demonstrate independence of a line integral of a conservative vector field on a path and use it for finding the potential and describing equipotential curves.

Steps in the concept definition for EDE in MO1.

- 1) Using the gradient of a scalar field, define a conservative vector field and its potential.
- 2) Using the differential of a scalar field, define an EDE in the 3D-space.
- 3) Provide necessary conditions for a conservative vector in 2D and 3D-spaces.
- 4) Define solutions of an EDE as equipotential curves.
- 5) Use the procedure of partial integration to find the potential.
- 6) Write the answer in the form of equipotential curves.

Vignette MO2 Goal: Develop an integration method for solving an EDE.

Raison d'être: use the total derivative for developing the integrating factor technique for solving first-order linear DEs and obtaining first integrals.

Steps in the concept definition for EDE in MO2.

- 1) Solve an EDE in Example 3 by rewriting its left-hand side as a total derivative.
- 2) Define solutions implicitly by an algebraic equation $\psi(x, y) = c$ and interpret the level curves of $\psi(x, y) = c$ as integral curves of the given DE.

- 3) Consider the general case of EDEs and give the formal definition.
- 4) Formulate necessary and sufficient condition for a DE to be exact (Theorem 1).
- 5) Prove Theorem 1.
- 6) Use the constructive procedure in the necessity part of Theorem 1 to develop a solution guideline.
- 7) Solve an EDE in Example 4 using the procedure developed in step 6) obtaining solutions defined implicitly by an algebraic equation.

Both the object, an EDE, and the technique (solution method) are similar in MO1 and MO2, but the concept definitions differ. We argue that the concept images for EDE induced by two mathematical organisations are even more distinct. This signals possible conflict situations when students do not see important connections between mathematical notions and ideas. Note that the praxis parts in both mathematical organisations are well defined, $\Pi_1 = [T_1/\tau_1, \tau_2]$ and $\Pi_2 = [T_2/\tau_3, \tau_2]$. Partial integration technique τ_2 is exactly the same in both praxeologies, with minor variations in explanations. The technologies θ_1, θ_2 and θ_3 used in the two logos blocks for MO1 and MO2 justify the same mathematical procedure differently. This is due to the fact that MO1 only postulates a necessary condition for conservative vector fields without proving it and the solution method is introduced in Example 1, whereas Theorem 1 in MO2 furnishes both the exactness test and the justification for the solution method through a constructive proof. Finally, we observe that both logos blocks are incomplete and the theory θ is missing in logos parts, $\Lambda_1 = [\emptyset/\theta_1, \theta_2]$ and $\Lambda_2 = [\emptyset/\theta_3]$. Therefore, the search for the theory takes us beyond these two mathematical organisations. Two theoretical results that can be used to fill in the gaps follow.

Theorem 3 (vector fields; Protter & Morrey, 2012, p. 478). Suppose that \vec{v} is a continuously differentiable vector field with $\text{curl } \vec{v} = \vec{0}$ in some rectangular parallelepiped D in space. Then there exists a continuously differentiable scalar field f in D such that $\nabla f = \vec{v}$. Any two such fields differ by a constant.

Theorem 4 (mixed partials; Young, 1908-09, pp. 163-164). Suppose $f(x, y)$ is defined in a neighborhood of a point (a, b) . Suppose the partial derivatives f_x, f_y are defined in a neighborhood of (a, b) and are differentiable at (a, b) . Then $(f_x)_y(a, b) = (f_y)_x(a, b)$, sometimes stated as $f_{xy}(a, b) = f_{yx}(a, b)$.

An elegant example due to Peano illustrates that Theorem 4 does not provide a sufficient condition (cf. Example 2).

Example 5 (Apostol, 1965, p. 358). Second-order mixed partial derivatives of the function $f(x, y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & (x, y) \neq (0,0), \\ 0, & (x, y) = (0,0), \end{cases}$ are distinct, that is, $f_{yx}(0,0) = 1$, $f_{xy}(0,0) = -1$.

We summarise our praxeological analysis in the following table.

	Task, T	Technique, τ	Technology, θ	Theory, θ
MO1	Find a potential of a conservative vector field T_1 .	Test for conservative vector fields τ_1 . Partial integration τ_2 .	Necessary condition θ_1 for conservative fields without proof. Partial integration procedure θ_2 introduced in Example 1.	Theorem 3
MO2	Solve an exact DE T_2 .	Exactness test τ_3 . Partial integration τ_2 .	The exactness test and partial integration procedure θ_3 in a constructive proof of Theorem 1.	Theorem 4

Table 1: Praxeologies associated with mathematical organisations MO1 and MO2

CONCLUSIONS

Both praxeologies serve their goals but praxeological analysis reveals two main reasons generating conflicting parts of the concept image. Firstly, the technique of partial integration τ_2 in both mathematical organisations is the same, but technologies θ_1 and θ_2 justify the same solution method differently. Secondly, since the theory θ is missing in both mathematical organisations, the technology lacks justification. It is known that incomplete logos part makes the preservation and communication of the practice difficult (Hardy, 2009). Our list of potential conflict factors in the definition of an EDE in MO1 and MO2 includes four contrasting items.

- 1) Defining an EDE, a DE-praxeology uses a derivative form under a default assumption that $y = y(x)$. A C-praxeology uses a more flexible differential form where any of two variables can be viewed as independent.
- 2) Solution of an EDE in a DE-praxeology is viewed as an implicitly defined function describing all solutions (integral curves). A C-praxeology defines them as equipotential surfaces or curves.
- 3) An EDE is considered in a rectangular domain in a DE-praxeology, but it is mentioned that the region has to be simply connected. A C-praxeology emphasises that the existence of a potential for a vector field depends both on the topology of the domain of the field and on the structure of the components of the field itself.
- 4) A DE-praxeology does not consider extensions to higher dimensions at all. A C-praxeology allows an easy extension of the notion an EDE and the formulation of an exactness test to a 3D case (thanks to differential form used for a DE).

These potential conflict factors induced by two institutional praxeologies may lead to significant variations in the construction and evolution of students' own concept images for an EDE. Gascón (2003) pointed out that students often use the concept image instead of the concept definition; the latter tends to stay inactive and may be

even forgotten. It is likely that students' individual concept images of an EDE formed in a Calculus course will refer to one or more of the following: (i) conservative vector fields, potentials, and equipotential curves and surfaces, (ii) equal roles played by both variables and easy extension to higher dimensions, (iii) topological restrictions on the domain and components of a vector field. When students meet an EDE once again in an MS course on DEs, they may not recognise it because of quite significant differences in the two logos blocks $\Lambda_1 = [\emptyset/\theta_1, \theta_2]$ and $\Lambda_2 = [\emptyset/\theta_3]$.

The increasing demands for advanced mathematical thinking of engineering graduates require both the high-quality teaching and agreement between mathematics disciplines in the study curricula. This paper exposes hidden conflict factors in learning differential equations pointing toward the need for the harmonisation of mathematics courses in engineering programmes. We hope that our contribution will stimulate further interest of mathematicians and mathematics education researchers to this important issue.

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