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# Modes of description and indicators of formalism in students' descriptions of eigenvectors

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*Abstract.* In the study presented in this paper, we investigate students' concepts of eigenvectors in an early stage of their education on linear algebra. The different descriptions used by the students for eigenvectors are analysed with respect to both their chosen representations of the mathematical objects (algebraic, geometric, or abstract), and the indicators of formalism used in these descriptions. We find that while the modes of description presented to them seem to influence their own choice of description, students still show their ability to switch between different representations and descriptions and provide individual concept images. However, some shortcomings concerning formalism and preciseness of their descriptions indicate that some mathematical properties and logical relations in the context of learning about eigentheory require particular attention in teaching and learning activities.

*Keywords:* Teaching and learning of linear and abstract algebra, teaching and learning of specific topics in university mathematics, eigentheory, concept image, modes of description.

## INTRODUCTION

Linear algebra is of great use in many fields such as science and mathematics (Wawro et al., 2018). Over the last few decades, the problems in teaching and learning of linear algebra have received increasing attention by researchers in mathematics education (Dorier & Sierpinska, 2001). Eigentheory, the domain of mathematics concerning eigenvectors, eigenvalues and eigenspaces, is often described as a useful set of concepts across disciplines (Wawro et al., 2018). However, as students need to work with several key ideas simultaneously, eigentheory can be conceptually complex. In  $\mathbb{R}^2$  and  $\mathbb{R}^3$ , eigenvectors can be understood geometrically as arrows that are scaled by the transformation or algebraically as the solutions to the eigenequation, but students may not be able to understand these interpretations from the start (Hillel, 2000; Wawro et al., 2019). Dorier and Sierpinska (2001) suggest that the many representations might contribute to the difficulties faced by students learning linear algebra. Wawro et al. (2018, p. 275) claim that research on the teaching and learning of eigentheory is “a fairly recent endeavour and is far from exhausted”. In accordance with that, this study aims to contribute to the research on students' understanding of the concepts of eigenvectors and eigenvalues, by investigating which characterisation of eigenvectors and eigenvalues the participants chose in an early stage of their education on linear algebra. We work with the following overarching research question:

*What characterises the students' conceptions of eigenvectors and eigenvalues?*

## THEORY

### Concept image and concept definition

To describe our insight into the conceptions that the participants of our study had in the field of eigentheory, we make use of the terms *concept image* and *concept definition*, as introduced by Tall and Vinner (1981). The concept definition is a verbal definition that explains the concept in a precise and non-circular way (Vinner, 1983). According to Tall and Vinner (1981), it may be the result of rote learning of a formal concept definition, that is, a definition accepted by the mathematical community, often presented in lectures and textbooks. Alternatively, it can be the students' own reconstruction of it, that is, his or her personal concept definition (Tall & Vinner, 1981). For many people, there is also the concept image (Vinner, 1983). Tall and Vinner (1981) describe the concept image as consisting of all the cognitive structures associated with a concept. It can be non-verbal, but it might be translated into words. Thus, the concept image may consist of various representations as well as examples and non-examples associated with a concept. It is individual and dynamic, in contrast to a formal concept definition, which can be considered objective and constant (Tall & Vinner, 1981). One's concept image and concept definition may be more or less overlapping, contradictory or for some people, the concept image may be non-existent. According to Vinner (2002, p. 69), having a concept image is a necessary condition for understanding: «To understand, so we believe, means to have a concept image.» Given this, we argue that describing the students' concept image can, to some extent, provide information of their understanding of these concepts. Using this terminology, the overarching research question could be rephrased as follows:

*What characterises the students' concept images and concept definitions of eigenvectors and eigenvalues?*

However, as an individual's concept image may be vast and multi-faceted, it is our perspective that it cannot be described in full detail in the scope of this study. Thus, we find it necessary to restrict our inquiry of students' concept images to specific aspects of them. In the following, we will explain our interpretation of Hillel's modes of description, the aspect of formalism, and how these ideas have helped in shaping two supporting research questions.

### Modes of description

Hillel (2000) explains that a typical course in linear algebra applies several modes of description to objects and operations, as well as the transfers between them. These include the *abstract*, the *algebraic* and the *geometric* mode, and they can be applied to vector spaces of all dimensions. Within them, vectors and transformations have different terminology, notation, and representations associated with them. The abstract mode of description uses formal language and concepts from the general  $n$ -space like dimension, kernel and vector space. The algebraic mode concerns the concepts from the more specific theory of  $\mathbb{R}^n$ . Here, vectors are  $n$ -tuples and key topics include matrices, rank and solving linear systems. In the geometric mode, vectors can be considered as arrows, directed line segments or points, and transformations can be understood as

corresponding to spatial actions, like rotations and translations. In this mode, key concepts like orthogonality can be visualised in 2- and 3-space but are used metaphorically in the general part of theory (Hillel, 2000).

The modes are different but not entirely disjoint. According to Hillel (2000), teachers make shifts within and between modes easily and frequently during lectures. However, several researchers have suggested that students struggle to work with these transfers (e.g. Lapp et al., 2010; Sierpinska, 2000; Stewart 2018). In particular, when working with  $\mathbb{R}^n$ , moving from the abstract to the algebraic mode can be a particularly confusing shift for students (Dorier & Sierpinska, 2001; Hillel, 2000). Hillel (2000) notes that the ability to understand how vectors and transformations can be represented differently within and between modes is key to understanding linear algebra. To further explore this aspect of students' concept images, identifying possible preferences and challenges they may have with these modes and transfers, we add a supporting research question:

1. *What modes of description do the students use to explain the concepts of eigenvector and eigenvalue?*

### **The aspect of formalism**

Another great challenge for students learning linear algebra is its formal character (Dorier, 2017). According to Dorier (2017), Robert and Robinet conducted research in France in the 1980s, showing that students felt overwhelmed by the many new definitions and theorems, and the students expressed concern with the use of formalism. Dorier et al. (2000) have researched students' difficulties with the generalised part of linear algebra, and they call this *the obstacle of formalism*. According to Dorier and Sierpinska (2001), students also have difficulties with understanding formal concepts in relation to their geometric interpretations. However, it is our perspective that the aspect of formalism needs further conceptualisation. In our study, we chose to define and identify particular elements of mathematical statements as “indicators of (lacking) formalism”, as will be worked out in the next section. To further explore the aspect of formalism, we pose an additional supporting research question:

2. *What indicators of lacking formalism can be found in the students' explanations of the concepts of eigenvector and eigenvalue?*

In this context, we would like to stress that we do not use the term “lacking” in any normative sense here, but only in the function of indicating the absence of something.

## **METHODOLOGY**

### **Setting and participants**

This study took place at the Norwegian University of Science and Technology in Trondheim with first- and second-year students in a basic linear algebra course. The students were majoring in mathematics and mathematics education. The teaching of this course included weekly lectures where the teacher presented key definitions, theorems and relevant examples using the blackboard and/or PowerPoint presentations. In addition, there were optional weekly exercise classes where the students could discuss tasks from the homework with each other and teaching assistants. To gain access to the exam, students had to complete and submit a minimum of eight out of twelve of these

exercise sets and have them graded by a teaching assistant. Out of the 243 students who were enrolled in the course, 52 consented to participate in our study. We admit that our results may not be representative for the student body in the course, yet we argue that it is sufficient to say something about trends *within* the group of participants.

To understand the students' concept images and concept definitions of eigenvectors and eigenvalues, we designed four tasks as part of the students' weekly homework and collected the written works of the students. In addition to explaining the concepts of eigenvector and eigenvalue in their own terms, students were asked to determine whether and why statements about eigenvectors and eigenvalues are true or false, as well as use graphic representations to determine whether a given vector is an eigenvector corresponding to a matrix, and why/why not. These tasks were designed specifically to have students' work with multiple representations of eigenvectors, and consequently modes of description, and to test their abilities to move between them. The students were allowed to work on the exercises for one week and all aids were permitted. In this paper, we will only present our analyses of the first task and our focus is on part a): «*For parts a) and b), explain in your own words. You may also use drawings. a) What is an eigenvector? b) What is an eigenvalue?*». The purpose of this open phrasing was to elicit student thinking and learn about their concept images and concept definitions.

### **Method of analysis**

The students' written works were collected through the digital learning platform Blackboard, that was used for the organisation of the whole course, and analysed qualitatively using a thematic coding approach in two rounds, each having a first and a second level. The coding in the first round was inspired by Wawro et al (2019). In the first level, descriptive codes were constructed inductively from single words or short phrases in the students' written answers. Codes such as «scalar multiple» or «transformation» were assigned to trace the modes of description in the students' answers. In the second level, codes were grouped into themes corresponding to Hillel's (2000) modes of description in an interpretative process.

For the coding in the second round, the students' explanations of the concept were compared to an "ideal" formal concept definition from the textbook used in the course, that is, *Elementary Linear Algebra* (2020, p. 291) by Anton Kaul:

*If  $A$  is an  $n \times n$  matrix, then a nonzero vector  $\mathbf{x}$  in  $\mathbb{R}^n$  is called an **eigenvector** of  $A$  (or of the matrix operator  $T_A$ ) if  $A\mathbf{x}$  is a scalar multiple of  $\mathbf{x}$ , that is, for some scalar  $\lambda$ . The scalar is called an **eigenvalue** of  $A$  (or of  $T_A$ ), and  $\mathbf{x}$  is said to be an **eigenvector corresponding to  $\lambda$** .*

This definition contains all necessary specifications of the used symbols and precise relations between the occurring concepts and was therefore considered as fulfilling the highest relevant standard for formalism in the context of our study. We compared the answers of the students with this definition and identified which if these specifications were missing. These "lacks" were considered as "indicators of lacking formalism", and the categories obtained in this process are listed in the next section.

## RESULTS AND ANALYSIS

### Various modes of description

In this section, examples from students' works will be presented, together with their codes and how they were categorised as relating to the abstract, algebraic or geometric mode of description. For the purpose of this analysis, the tasks and the students' answers were translated from Norwegian to English. As the modes are not specific to eigentheory and the students gave only short explanations, it was necessary to make our own interpretation of this classification and restrict our analysis to single words or short phrases used by students. The codes, their explanation and prevalence obtained in the first round of coding are given in table 1. From a mathematician's point of view, many of these codes are interchangeable. However, we argue that this is not necessarily obvious to students and that realising some of these are interchangeable is related to having an advanced concept image. Some answers are complex, using both symbols and natural language, or connecting the concept of eigenvector to other concepts, while others are more condensed. Consequently, some answers were assigned multiple codes, while others were given only one or two.

***Algebraic modes of description:*** Answers that describe eigenvectors by writing a symbolic definition similar to the one from the textbook, i.e.  $A\mathbf{x} = \lambda\mathbf{x}$ , were considered as using an algebraic mode of description. This was also the case for answers that rephrase this relation in natural language, i.e., a discursive definition of eigenvectors. From table 1, it is evident that most students described eigenvectors using the symbolic definition, a discursive definition or a combination of the two. For example, one student wrote: «*A vector  $\mathbf{x}$  is an eigenvector if you can write  $A\mathbf{x} = \lambda\mathbf{x}$ , where  $A$  is a matrix and  $\lambda$  is a scalar. More thoroughly explained,  $\mathbf{x}$  is an eigenvector if a matrix multiplied by the vector returns the vector scaled by  $\lambda$ .*». In this example, the first sentence defines the concept of eigenvector as vectors fulfilling the eigenequation. In the second sentence, the student tries to elaborate by explaining the equation in natural language. As the answer gives both a symbolic and a discursive definition, it was coded accordingly and categorised as having an algebraic mode of description.

***Abstract modes of description:*** Answers that relate eigenvectors to concepts from the more general part of theory were considered to have an abstract mode of description. Table 1 indicates that fewer answers were assigned these codes, as compared to the codes corresponding to the algebraic mode. Out of the abstract codes, «transformation» is the most recurrent within the data material, with 15 compared to 1–3 occurrences. The following answer was coded as «vector space», «image» and «transformation»: «*An eigenvector  $v \neq 0$  is a vector in the vector space that doesn't change direction when it's imaged by a linear transformation. This means that if a square matrix is multiplied with this vector, the resulting vector will be a scalar multiple of the eigenvector.*». By describing eigenvectors in relation to multiple concepts from the more formal and general part of theory, the answer contains several elements corresponding to an abstract mode of description. As the student described an eigenvector as «not changing direction» (i.e., maintaining direction) and as a «scalar multiple», the answer also has elements from the geometric mode, and was additionally categorised accordingly.

**Table 1: The table explains the 13 codes, how often they occur in the answers of the 52 students and their corresponding modes of description.**

Mode of description	Code	Explanation	Occurrence
Algebraic	Discursive definition	Using natural language to explain the eigenequation, $A\mathbf{x} = \lambda\mathbf{x}$ .	35
	Symbolic definition	Description with the eigenequation, $A\mathbf{x} = \lambda\mathbf{x}$ .	31
	Linear system	Connects eigenvectors to the solution of a linear system.	1
	Linear	Describing $A\mathbf{x}$ and $\mathbf{x}$ as linear.	1
Abstract	Transformation	Description related to the concept of transformation, using the words «transforming», «transformation» etc.	15
	Span	Description related to the concept of span, using the words «spanning», «spans» etc.	3
	Image	Description related to the concept of image, using the words «image», «imaging» etc.	3
	Vector space	Description related to the concept of vector space, using the phrasing «an element of a vector space» or similar.	2
	Transformation definition	Description with the eigenequation in terms of a transformation, e.g. $T(\mathbf{x}) = \lambda\mathbf{x}$ .	1
Geometric	Scalar multiple	Description using the words «scalar multiple», «scaling» etc.	21
	Maintains direction	Describing eigenvectors as vectors that do not change direction.	12
	Dynamic changes in size	Dynamic description, using words like «stretching», «shrinking» etc.	5
	Figure	Included a figure or sketch.	5

**Geometric modes of description:** Students who described eigenvectors by referring to some visual representation in  $\mathbb{R}^2$  or  $\mathbb{R}^3$  were considered as using a geometric mode of description. This includes answers that described eigenvectors as maintaining direction or as being scaled under a transformation (or matrix multiplication), as well as answers where the student made some sketch showing the relation between the matrix, the eigenvector and the eigenvalue. An answer that was coded as both «maintains direction» and «dynamic changes in size» is the following: «An eigenvector is a vector such that when multiplied by a matrix [it] won't change direction, but only length.». In this example, the student correctly described how a matrix may change the length of an eigenvector and how its direction is preserved (however, the option of flipping the vector was not apparent in the students' answer).

## Indicators of lacking formalism

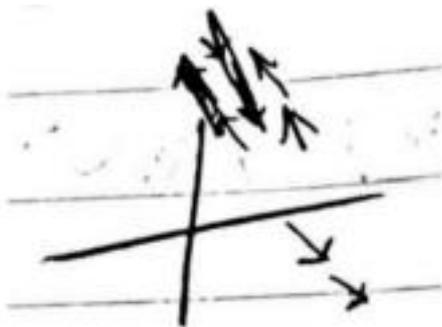
The aspect of formalism was evaluated by comparing what was missing from the students' description of the concepts to the formal concept definition in their textbook. Table 2 gives an indication about which lacks were most prevalent in the dataset. The most common lacks among the students' answers appear to be not to mention the dimensions of the eigenvector  $\mathbf{x}$  (omitted by 42 students), the matrix  $A$  (omitted by 28 students) or specify that  $\mathbf{x}$  may not be equal to the zero vector (omitted by 36 students). From table 2, it is noticeable that the works of most of the students showed several lacks when compared to the formal definition. In the following example, the student correctly explained eigenvector and eigenvalue by referring to the symbolic definition, but the answer has lacks: «*An eigenvector is a vector  $\mathbf{x}$  that can solve  $A\mathbf{x} = \lambda\mathbf{x}$ , where  $A$  is a matrix and  $\lambda$  is called the eigenvalue.*». The student did not specify the dimensions of neither the vector («L/vector dimension») nor the matrix («L/matrix dimension»), did not rule out the eigenvector to be equal to the zero vector («L/nonzero eigenvector»), did not state that  $\lambda$  is a scalar («L/eigenvalue unknown») and did not explicitly state that the eigenvector and eigenvalue correspond to the specific matrix («L/eigenvalue vector» and «L/vector-matrix»). In our analysis, we found several examples where it remains unclear whether the student was aware of the connection between the matrix, the eigenvector and the eigenvalue. For instance, one student wrote «*An eigenvector  $\mathbf{x}$  is a vector that is scaled when multiplied by a matrix.*». This could either indicate that the student thought of an eigenvector as corresponding to a specific matrix, or a misconception that an eigenvector is scaled by every matrix. In another case, the student did not mention the matrix at all: «*An eigenvector is a vector that can be scaled but does not change direction.*». Furthermore, three students gave answers where it is ambiguous if they were aware of all the ways a matrix may act upon its eigenvector(s). For instance, one student wrote: «*An eigenvector is a vector that is stretched either in a positive or negative direction [...].*». This could indicate the idea of an eigenvector as only being stretched (not shrunk etc.) when multiplied by the corresponding matrix.

**Use of visual representations:** In  $\mathbb{R}^2$  and  $\mathbb{R}^3$ , eigenvectors and eigenvalues have visual representations. The task asked the students to explain eigenvectors and eigenvalues, but they were also encouraged to draw a sketch to illustrate the concepts. Out of the 52 students that participated in this study, only five drew a sketch supplementing their verbal description. Figures 1 and 2 give two examples of such sketches. In figure 1, the student drew a coordinate system and multiple arrows pointing in opposite directions to each other. In figure 2, the student drew the eigenvector  $\mathbf{x}$  and the vector  $A\mathbf{x}$  in a coordinate system with scales, as well as the equations  $A\mathbf{x} = \lambda\mathbf{x}$  and  $A\mathbf{x} = -2\mathbf{x}$ , thereby indicating the relationship between the matrix, the eigenvector and the eigenvalue  $\lambda = -2$ . Given this, the sketch in figure 2 is more detailed than the sketch in figure 1. However, as the sketch in figure 1 is not restricted to a particular eigenvector and eigenvalue, it could be interpreted as more general and dynamic. The student wrote that «*An eigenvector says about how much [sic] matrix stretches/shrinks things in a direction. The eigenvalue is how much [sic] eigenvector stretches/shrinks.*». This answer could indicate a developing concept image where the student is able to connect the concept of eigenvector to the geometric idea of scaling. The usage

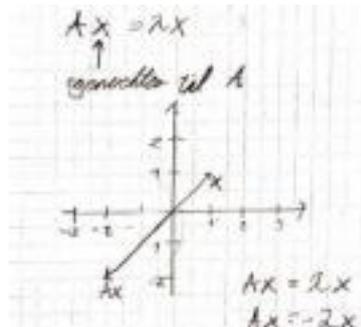
of very informal language here, for instance the usage of the term "things", leaves it unclear what object the student thought is being stretched.

**Table 2: The table explains the seven codes for lacks and how often they occur in the answers of the 52 students.**

Code	Explanation	Occurrence
L/vector dimension	The student does not specify the dimension of the eigenvector.	42
L/nonzero eigenvector	The student does not specify that the eigenvector cannot be equal to the zero vector.	36
L/matrix dimension	The student does not specify that the matrix needs to be square.	28
L/eigenvalue-vector	The student does not specify that the eigenvalue and eigenvector form a corresponding pair.	27
L/vector-matrix	The student does not connect the eigenvector to a specific matrix.	23
L/eigenvalue unknown	The student does not specify that the eigenvalue is a scalar.	7
L/act	The student does not describe the possible ways (i.e. stretching, shrinking, leaving unchanged, rotating by 180 degrees) in which the matrix may act upon its eigenvector(s).	3
L/matrix	The student does not mention matrix or linear transformation at all.	1



**Figure 1: Student sketch apparently depicting the act of scaling.**



**Figure 2: Student sketch showing the correspondence between the matrix, its eigenvector and eigenvalue.**

## DISCUSSION AND CONCLUSION

The purpose of this study was to gain more insight into students' descriptions of eigenvectors. Most of the students used an algebraic mode of description, which is also the mode the book and the lecture set their focus on. However, several students implemented multiple modes in their answers, indicating the development of their concept images. While the usage of only one mode of description in their answer cannot be considered as a proof of a concept image on a low level of development, we do think that an answer including several modes of descriptions and, even more significantly, some meaningful connections between these modes, can be considered as a strong sign of a further developed concept image. Only a handful of students included a sketch in

their answer, despite there being an explicit suggestion to use drawings. This could be because the majority overlooked it, ignored it or perhaps because they did not know how to provide an appropriate sketch. Furthermore, few students connected the concept of eigenvector to the more abstract concepts of image, span or vector space. Of the students who did describe eigenvectors using concepts from the more formal part of theory, the majority used the concept of transformation. However, we wonder if students are aware of the nuances that distinguish a matrix from a transformation.

The works of the students presented a variety of lacks that may or may not result from flawed concept images. However, the results obtained in this study do not allow us to say for certain that these specific students had such misconceptions. It is also difficult to tell to which extent the rather open formulation of our task influenced the formalism of the answers given by the students. Concerning formalism, we got the impression that students are not used to focusing on this aspect in their weekly homework. If a higher level of formalism in the students' works is indeed desired by teachers, it may be constructive to target this shortage by emphasising why formalism is required in mathematical contexts. Furthermore, a discussion (either teacher-student or student-student) or task about «what if» could be productive. For example: «What would happen if we allowed the zero vector to be an eigenvector?».

Our analysis showed that several students gave a discursive rephrasing (as described in table 1) of the eigenequation (i.e.  $A\mathbf{x} = \lambda\mathbf{x}$ ), omitting aspects of the formal concept definition such as the correspondence between the eigenvector, eigenvalue and its matrix. Other students defined eigenvectors without mention of the matrix or transformation at all. As eigenvectors are derived from their corresponding matrix (or transformation), these answers were interpreted as incomplete. However, whether such incomplete definitions were due to a lack of formalism, some losses caused by the translation from a (possibly non-verbal) concept image to a written description or actual misconceptions remains unclear. In future studies, we will work with similar research questions and address the challenges presented in this paper. We acknowledge that it can be problematic to characterise students' concept images from their written answers alone. We believe that by also analysing the students' answers to the other tasks we designed and conducting interviews with students, we can gain deeper insight into their concept images, and consequently, their procedural and conceptual understanding of eigenvectors and eigenvalues. Building on this, we aim to develop tasks that explicitly address changes between different representations and modes of description.

## REFERENCES

- Anton, H., Rorres, C., & Kaul, A. (2020). Elementary linear algebra: Applications version (12th ed.). University of Chicago Press.
- Dorier, J-L. (2017). Duality between formalism and meaning in the learning of linear algebra. In R. Göller, R. Biehler, R. Hochmuth, H.-G. Rück (Eds.). Didactics of Mathematics in Higher Education as a Scientific Discipline – Conference Proceedings (p. 101-105). Kassel, Germany: Universitätsbibliothek Kassel.
- Dorier, J-L., Robert, A., Robinet, J., Rogalski, M. (2000). The Obstacle of Formalism

in Linear Algebra. In Dorier, J.-L. (Eds.) *On the Teaching of Linear Algebra*. Mathematics Education Library, 23, 85-124. Springer, Dordrecht.  
[https://doi.org/10.1007/0-306-47224-4\\_2](https://doi.org/10.1007/0-306-47224-4_2)

- Dorier J.-L. & Sierpinska, A. (2001) Research into the Teaching and Learning of Linear Algebra. In D. Holton (Ed.), *The Teaching and Learning of Mathematics at University Level - an ICMI study* (p. 255-273). Dordrecht: Kluwer.
- Hillel, J. (2000). Modes of description and the problem of representation in linear algebra. In J.-L. Dorier (Ed.), *On the teaching of linear algebra*. Wiley.
- Lapp, D. A., Nyman, M. A., & Berry, J. S. (2010). Student connections of linear algebra concepts: an analysis of concept maps. *International Journal of Mathematical Education in Science and Technology*, 41(1), 1-18.  
<https://doi.org/10.1080/00207390903236665>
- Sierpinska, A. (2000). On some aspects of students' thinking in linear algebra. In J.- L. Dorier (Ed.), *On the Teaching of Linear Algebra* (pp. 209-246). Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Stewart, S. (2018). Moving Between the Embodied, Symbolic and Formal Worlds of Mathematical Thinking with Specific Linear Algebra Tasks. In S. Stewart, C. Andrews-Larson, A. Berman & M. Zandieh. (Eds.) *Challenges and Strategies in Teaching Linear Algebra*, p. 51-67. Springer International Publishing.
- Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *12 (2)*, 151-169.  
<https://doi.org/10.1007/BF00305619>
- Vinner, S. (1983) Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology*, 14(3), 293-305. <https://doi.org/10.1080/0020739830140305>
- Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65-81).
- Vinner, S. (2002). The Role of Definitions in the Teaching and Learning of Mathematics. In Tall, D. (Ed.) *Advanced Mathematical Thinking* (p. 65-81). Springer, Dordrecht. [https://doi.org/10.1007/0-306-47203-1\\_5](https://doi.org/10.1007/0-306-47203-1_5)
- Wawro, M., Zandieh, M., & Watson, K. (2018). Delineating Aspects of Understanding Eigentheory through Assessment. In V. Durand-Guerrier, R. Hochmuth, S. Goodchild, & N.M. Hogstad (Eds.), *Proceedings of the Second Conference of the International Network for Didactic Research in University Mathematics (INDRUM 2018, 5-7 April 2018)* (pp. 275-284), Kristiansand, Norway: University of Agder and INDRUM.
- Wawro, M., Zandieh, M., & Watson, K. (2019). Student understanding of linear combinations of eigenvectors. *ZDM Mathematics Education*, 51, 1111-1123.  
<https://doi.org/https://doi.org/10.1007/s11858-018-01022-8>