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Modelling periodic phenomena through trigonometric polynomials using digital tools

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The teaching of trigonometrical functions usually is carried out from relationships with measurement of arcs and the unit circle. Within the properties of trigonometric functions, periodicity is one of the most important. We present the first results of a sequence of tasks whose objective was to model sounds using trigonometric polynomials and Fourier's theory, through experimentation in 1st university year with Audacity and Geogebra software. The circulations and activation of planes in the sense of the Mathematical Working Space of the students were analysed while the students' transits for the different stages of the modeling cycle. Among the main results, we recognize that students build the concept of trigonometric function from time-amplitude variables, avoiding the concept of angle or radian.

Keywords: digital technologies, teaching analysis, periodical functions, trigonometric polynomial, modelling,

INTRODUCTION

The study of periodic phenomena has been developed for centuries cultures such as the Babylonian and the Egyptian used the periodic to make predictions (Montiel, 2005). One of the first steps that transitioned trigonometric functions away from geometry was the recognition of their periodicity. In 1670, in *Wallis's mechanica* as sine curve with two complete periods may be found. The conversion and full admittance of trigonometric quantities to the family of functions was accomplished by Euler (Van Brummelen, 2021).

The first periodic functions taught in secondary education are the trigonometric functions sine and cosine, this is due to the importance they have within mathematics and science (Fourier series, electricity, waves, sound, harmonic oscillator, diffusion of heat, etc.). The teaching of these functions in secondary education may vary depending on the country or the educational institution, however, they are often taught based on the teaching of trigonometry.

In relation to the trigonometric function, research has recognized that for the student there is no distinction between ratios and functions, or at least that there is a mixture of concepts to solve problems related to functions (Tanguay, 2010; Winsløw, 2016; Loeng, 2019).

Sound is one of the periodic phenomena that is modeled by trigonometric functions. From a physical point of view, sound is a mechanical wave produced by a vibrating body that propagates in an elastic medium. Any sound, such as a musical note, can be

described in physical terms by specifying the frequency, amplitude or air pressure, and timbre or waveform.

In the other hand, the integration of technologies in education has had a sustained boom during the last decades, however, the results have not been as expected (OECD, 2015). Due to its importance, over time the interest in carrying out studies on the uses of technologies to improve calculus concepts at the university level has increased. In addition, one main difference between secondary school and university work is the use of artifacts, mainly digital tools, to make calculations and develop multiple representations of mathematical objects. Artifacts and tools are widely used at secondary school whereas practices at university level tends to banish technologies from students' activities.

Based on the above, we have set ourselves the objective: study the mathematical work of first-year university students in a task of modelling periodic phenomena related to sound, which integrates technological tools. This task also allows students to conceptualize trigonometric functions from variables related to time and amplitude, rather than angles.

FRAMEWORK

Based on the proposed objective and the design of the task, we used the mathematical modeling cycle (Borromeo-Ferri, 2010) and the Mathematical Working Space [MWS] (Kuzniak, Montoya-Delgadillo & Richards, 2022) articulately.

The MWS allows describing and characterizing the mathematical work of an individual from a specific task. With this approach it is possible to describe how epistemological and cognitive elements are articulated in the solution of a task, showing the circulations between the activated MWS components (Kuzniak et al., 2022).

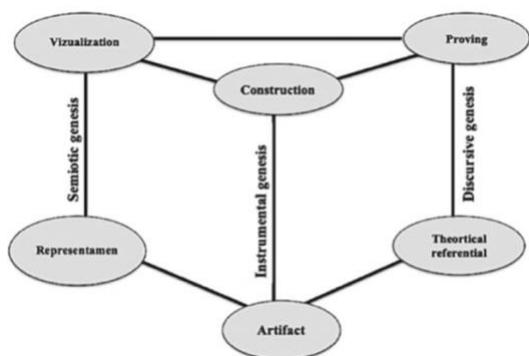


Fig 1a. Mathematical Working Spaces diagram (Kuzniak et al., 2022)

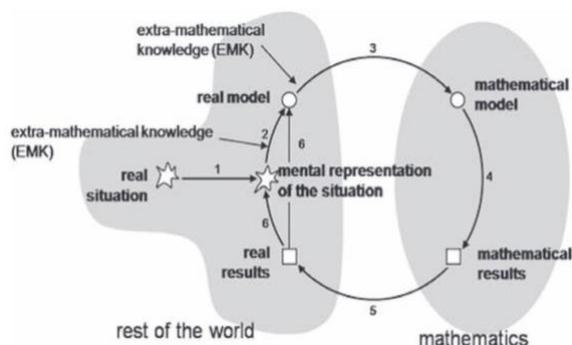


Fig 1b. Modeling cycle (Borromeo-Ferri, 2010)

The MWS consists of the articulation of two planes: Epistemological and Cognitive, which interact with each other, through a genesis process (Kuzniak et al., 2016; Kuzniak et al., 2022). It is worth mentioning that there is also an articulation between the different genesis, generating three vertical planes: [Sem-Ins], [Ins-Dis], [Sem-Dis].

Mathematical model is defined as a set of symbols and mathematical relations that represents, in some way (graphic, numerical or algebraic), the phenomenon in question. We incorporate certain concepts related to mathematical modeling from the cognitive point of view, for this, we will use the Blum-Borromeo modeling cycle (Borromeo-Ferri, 2010). The starting point of the modeling process is a real world situation, which the students read and understands, then idealized (2), simplified or structured to obtain a real model. The real model is mathematized (3), i.e., translated into mathematical language to obtain a mathematical model of the initial situation. A mathematical treatment (4) leads to mathematical results, interpreted as "real results" (5) which will be validated or not in relation to the model situation (6). Then, if the results obtained seem to be coherent, they will be consistent, they will be presented as predictions about the real situation (7).

CONTEXT AND METOD

The methodology that guides this research is Didactic Engineering (Artigue, 1995). To respond to the proposed research objective, the methodology used is qualitative (Bikner-Ashbahr, Knipping & Presmeg, 2015), where the Case Study (Yin, 2009) was used to know and understand the context of the students. The implementation is made up of 5 second semester undergraduate students in mathematical (2021-2022, period), who are taking the subject of *Mini-Projet de Mathématique* who formed 3 work groups. Each group makes up a case. The first group (G1) was made up of one student, the second and third groups were made up of two students each (G2, G3). The proposal was developed during 4 classes of 2 hours each. The first task was developed during classes 1 - 2, the second task was developed in class 3, and the third task in the last session. As part of the course, students were given a document with a brief definition of Fourier Series and its application, from which the definition of trigonometric polynomial is presented. This last concept should be used by the students in this proposal.

The design of the proposal had the intention of generating a modeling process, which would encourage the incorporation of technological tools such as Geogebra or Audacity by students. Audacity is free software that allows you to record and edit audio. Audacity's interface provides a graphical representation of the sound using the variables time and amplitude. Both software's allow you to export the data in Excel files.

The participant observation technique was used by the researcher, since he was the group's teacher during the class sessions. The data has been extracted from the written productions of the students, from the geogebra and Excel files constructed by the students. Mathematical work in the sense of the MWS will be analyzed, as the students go through the different phases of the modeling cycle. We have considered the methodology carried out by Reyes-Avenidaño (2020), in which analyzes the activation of the genesis, plans [Sem-Ins], [Ins-Dis], [Sem -Dis] and the circulations between them, in each one of the phases of the modeling cycle, indicating with arrows within the MWS diagram the activation chronology at each phase.

PRESENTATION OF THE TASKS

The objective of the situation is for students to model periodic phenomena through trigonometric polynomials presented as a Project that included three tasks. The first two tasks given to the students and the objective of these tasks will be described below, in the next section the results of these tasks will be discussed. The third task only its objective will be presented.

First task the objective was to recognize that pure sounds are those that are modeled by trigonometric functions. To accomplish this objective, two subtasks were designed. The subtask 1.1 given was: *Characterize the sounds present in the folder*. Students are presented with a folder with 10 different sounds, and the software Audacity to play these sounds. Students are expected to build the real model (RM) of the situation using their extra mathematical knowledge (EMK) in relation to music or physics. The classification is expected to be made in terms of the nature of the sound, the shape of the wave through what is observed in the Audacity interface or from the vibrating body that produces it. The transition to Mathematical Model (MM) is guided by subtask 1.2: *calculate the function associated with each of these sounds*. To do this, students export data from Audacity to Excel spreadsheet and using Geogebra's Two Variable Regression Analysis tool obtain a graphical representation of the sound. Mathematical Results (MR) obtained are given in terms of the analytical expression of each sound, thanks to the adjustment made by Geogebra. In the transition to the real results (RR) it is expected that students generate a second classification, since pure sounds can be adjusted by a sinusoidal function and complex sounds cannot.

In terms of MWS, Audacity is an artifact that allows students to activate semiotic genesis to construct RM, because they visualize the behavior of the sound graph in the software interface. This artifact allows students to obtain approximations of the periodicity of the wave to calculate the frequency. In the construction of RM and the transition to the mathematical world, it is the instrumental genesis that directs the activation processes of the semiotic genesis through the visualization of the wave and its representation on the Cartesian plane. It is the semiotic genesis that leads the testing processes when adjusting process is developed in Geogebra. Finally, the proof processes in the transition to RR can be based on the activation of the [Ins-Dis] and then [Sem-Dis] plane since the new classification is made in relation to the adjust made it and the analytical expression provided by Geogebra and what is visualized in it.

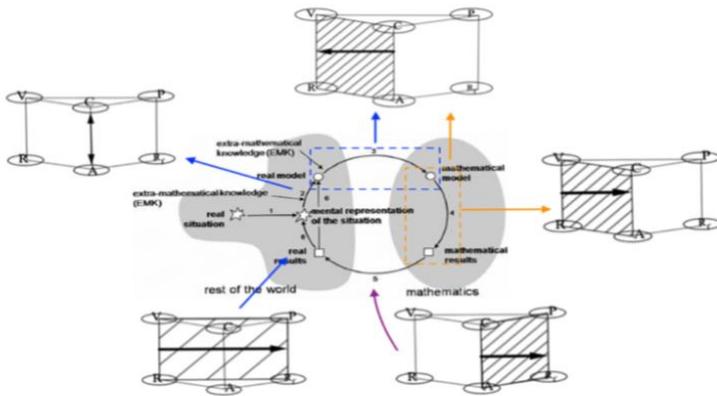


Fig 2. Genesis and plans activated during the modeling cycle of first task.

Second task. The objective was to determine the trigonometric polynomial associated with the superposition of two pure sounds. To accomplish this objective, were designed two subtasks. Subtask 2.1 given to students is: *Using the Audacity mix tool, mix the sounds LA 440hz and DO 264hz present in the given folder and determine the function that models this sound.* The RM is obtained from the sound mixed in Audacity, which provides a graphic representation in its interface that allows it to be classified as a complex sound, following the classification made it in the first task. In the transition to MM, students are expected to relate the superposition of sounds to the sum of the functions that model each sound separately, obtaining that the function is $S(x) = \sin(2\pi \cdot 440 x) + \sin(2\pi \cdot 264 x)$, which can be written as $S(x) = \sin(2\pi \cdot 3 \cdot 88 x) + \sin(2\pi \cdot 5 \cdot 88 x)$ which allows transit from MM to MR, concluding that the function S is obtained as a superposition of two sound with a frequency of 88hz, whose period is approximately $T = \frac{1}{88} = 0,01136$. On the way to the RR, students can export the mixed sound data and graph it in geogebra, to graphically contrast it with the function S obtained earlier.

Subsequently, subtask 2.2 given to students was: *Determine the first coefficients of trigonometric polynomial from the data set provided by Audacity.* This subtask will allow the students to resignify the previously described model, since it will be obtained from Fourier Theory. In terms of modeling cycle, the resignification process is oriented from RR to MM. For the construction of the new MM, students must determine the period of the graphed wave, obtaining the interval $[t_0, t_0 + T]$, whose values they will use from Excel cells. After that, students must calculate the Fourier coefficients a_n and b_n as: $a_0 = \frac{1}{T} \int_0^T f(t) dt$, $a_n = \frac{2}{T} \int_0^T f(t) \cdot \cos(n\omega t) dt$ and $b_n = \frac{2}{T} \int_0^T f(t) \cdot \sin(n\omega t) dt$, with $\omega = \frac{2\pi}{T}$.

However, since the coefficients will be calculated from a data set, in the transition to MR, students must use the Riemann sum to approximate the value of the integral, therefore, they must calculate the coefficients as: $a_n = \sum_{t_0}^{t_0+T} f(t_n) \cdot \cos(\frac{2\pi}{T} t_n)$ and $b_n = \sum_{t_0}^{t_0+T} f(t_n) \cdot \sin(\frac{2\pi}{T} t_n)$. As MR students are expected to determine that $b_3 = 1$ and $b_5 = 1$. In the validation process towards the RR, students can use Geogebra's

PlaySound tool and analyze the difference between the sound created in Audacity and the one delivered by Geogebra, or graphically determine the error between the calculated trigonometric polynomial and the set of data delivered by Audacity.

Figure 3 shows the mathematical work developed during subtasks 2.1 and 2.2. Audacity is an artifact that allows groups to recognize the wave graph to construct the RM as a complex sound related with a superposition of sounds. Semiotic genesis is activated from the software interface in the transition to MM. Students can relate the wave modeled by Audacity with the function obtained through the visualization in Geogebra. They can use the graphical view as a pragmatic proof to determine that this function is the one that models the mix of sounds. In relation to subtask 2.2, when calculating the trigonometric polynomial with the data provided by Audacity, Excel spreadsheet is a first available resource. The organization of mathematical work in the spreadsheet is based on the students' theoretical referential of the concept of integral and Riemann sum. Students must organize a set of interrelated operations in the software to develop the calculation of the coefficients. It observes semiotic components associated with reasoning in relation to definitions and mathematical properties that are articulated, thanks to the intentionality given to the use of this digital artifacts, therefore, it is from the discursive genesis that the plane [Sem-Ins] is activated.

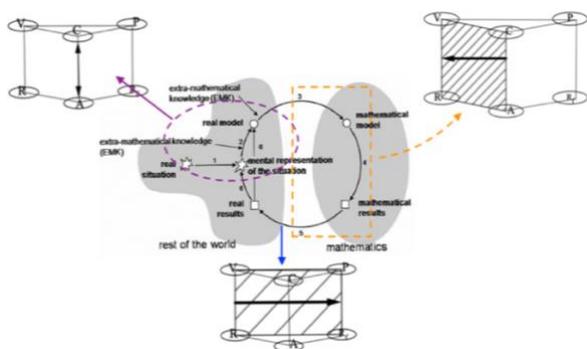


Fig 3a. Mathematical work developed in subtask 2.1

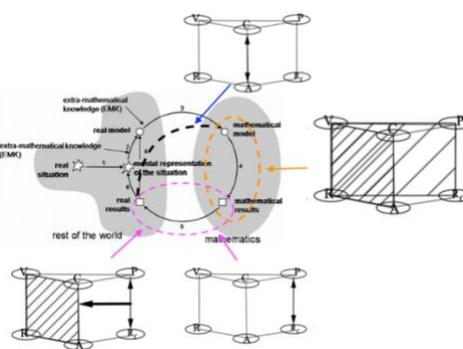


Fig 3b. Mathematical work developed in subtask 2.2

Third task. The objective of the third task was to model a complex sound through a trigonometric polynomial. The students were given only one task: *Determine the function that models this sound.* The students received an audio file with the sound of a flute playing the note LA. Students are expected to carry out a modeling process like the one developed in the second task.

DATA ANALYSIS AND RESULTS

The results obtained in the implementation of the elaborated proposal are presented below.

First Task

In relation to subtasks 1.1 and 1.2, the three groups achieved the proposed objective: to characterize pure and complex sounds and calculate the function associated to each sound. Students made a transit through the different phases of the modeling cycle when

developing both tasks. They play the sounds in Audacity and that allows him to create a classification between pure and complex sounds due graph provides them (Fig 5a) and their EMK. Particularly, G1 associates the pure sounds to sinusoidal functions whose algebraic expression is: $f(t) = A\sin(tx + \varphi)$. Likewise, it describes complex sounds as those that have a fundamental frequency and other frequencies called harmonics (Fig 5b).

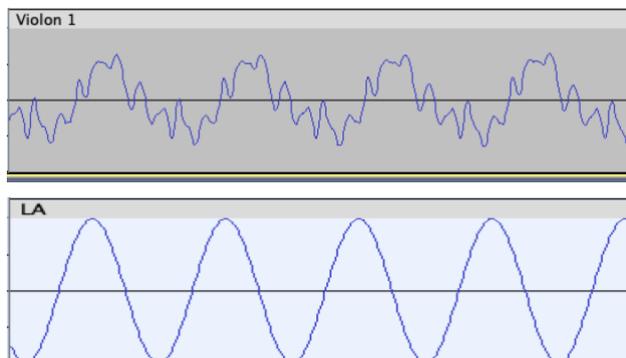


Fig 5a. view of violin and LA sounds in Audacity

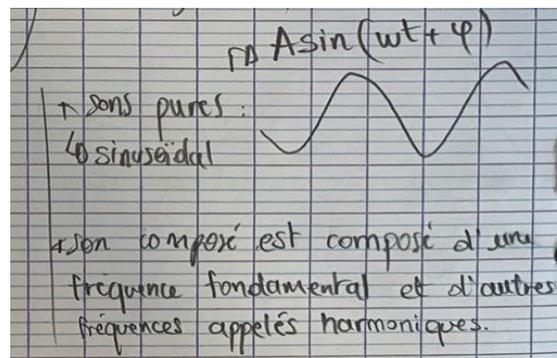


Fig 5b. G1 analysis, evidencing knowledge of music

Students used the Audacity data export tool and thanks to the Geogebra's tool *Two Variable Regression Analysis* get the graphical approximation. Fig 6 shows the function obtained (red one) for the adjustment of the LA 440hz sound. This value is sound closely approximates the theoretical result, which is $f(x) = \sin(2764.5992x)$.

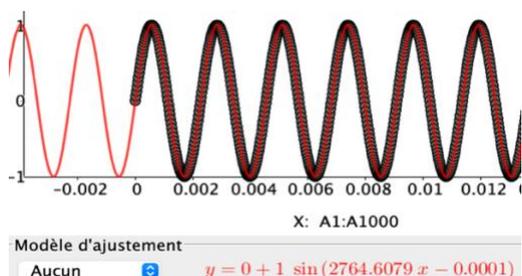


Fig 6. Adjustment obtained LA-tone 440hz made by G2

In relation to *a priori* analysis carried out, Audacity is an artifact that allowed students to activate semiotic genesis thanks to its interface. This artifact allowed the students to obtain approximations of the periodicity of the wave to calculate the frequency. In the construction of the RM and the transition to the mathematical world, it is the instrumental genesis that directs the visualization processes. In turn, in the adjustment processes, it is from the semiotic genesis that the mathematical work is oriented. Finally, in the transition to RR, the students classify that the pure sounds are those that are approximated by trigonometric functions as expected.

Second task

Subtask 2.1 was to determine the function associated with two-tone superposition created in Audacity. To perform this subtask, Audacity “mix” tool is used. The three groups were able to perform this task and determine the function associated with the

superposition of sounds. In relation to the a priori analysis, students export the data from Audacity to Excel and then graph it in Geogebra. G1 and G3 after graphing the data set, immediately input a function that is the sum of $LA(x)$ and $DO(x)$. G2 uses the Geogebra adjustment (Fig 7a), without obtaining results, later it uses the strategy of the sum of the known functions (Fig 7b). Three groups determine that the function $S(x) = \sin(880\pi x) + \sin(528\pi x)$, is the one that approximates the behavior of the data.

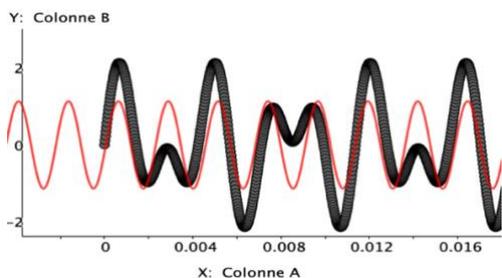


Fig 7a. Adjustment developed by G2

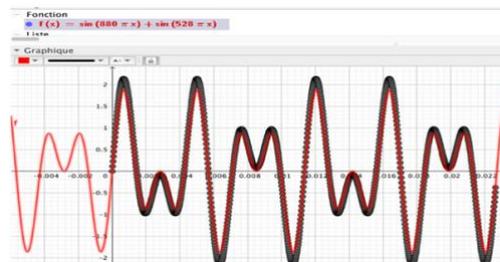


Fig 7b. Graph of the mix obtained by G3

In the validation process, groups mobilize the concept of approximation error through the graph of the function. In it they recognize that the distance between the values of the data set and those of the function is good except in the relative maximums and minimums of the function.

In relation to subtask 2.2, students were asked if they could determine the trigonometric polynomial associated with the first coefficients of Fourier Series using the data's provided by Audacity. In Fig 8 is presented how G1 construct their MM, previously be entered in the spreadsheet to calculate the coefficients using Riemann Sum knowledge, identifying the interval of integration as $T = [0, 0.0113]$ and the fundamental frequency $\omega = \frac{2\pi}{0,0113}$ of the periodic wave.

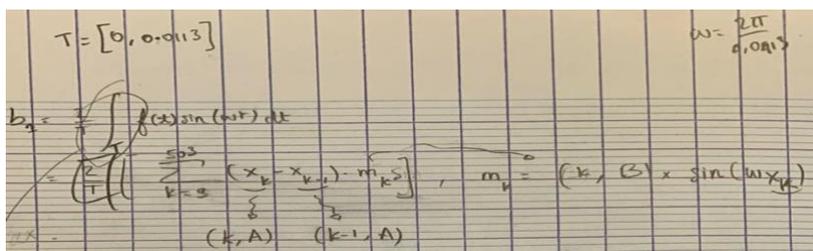


Fig 8. Expression to calculate the coefficients of trigonometric polynomial in Excel, G1

The spreadsheet developed by G3 is shown in Fig 9, unlike the above, builds the MM directly on the spreadsheet of Excel using Riemann Sum knowledge. In the developed programming, it is observed that G3 determines that the period is $T = 0.011$.

| F3 | | | | | | | | |
|----------------------------|---------|----------|---|-------------|---|------------|-----------------|-------------|
| =SENO(((2*PI())/0,011)*A3) | | | | | | | | |
| | A | B | C | D | E | F | G | H |
| 1 | Tiempo | Amplitud | | intervalles | | sinus | sinus*amplitude | aire |
| 2 | 0 | 0 | | | | | | |
| 3 | 0,00002 | 0,11503 | | 0,00002 | | 0,01142372 | 0,001314071 | 2,62814E-08 |
| 4 | 0,00005 | 0,22972 | | 0,00003 | | 0,02855605 | 0,006559896 | 1,96797E-07 |
| 5 | 0,00007 | 0,34373 | | 0,00002 | | 0,03997325 | 0,013740006 | 2,748E-07 |

Fig 9. Excel spreadsheet built by G3

Columns A and B are the ones that Audacity delivers when exporting the data. G3 creates in column D the distance between two consecutive cells of time variable, e.g., $D_3 = A_3 - A_2$, which represents the base of the rectangles in the Riemann method. In column F, G3 calculates the sine value for a given time, and then multiplies this sine value by its amplitude (column G). Finally, in column H, determine the area of each rectangle by multiplying the values in the columns *D* and *G*. The value of the sum, Excel delivers it by selecting all the values of column H, finally the value of the sum is multiplied by $\frac{2}{T}$.

As expected in the a priori analysis, the students mobilized knowledge from their theoretical referential related to the definite integral. The validation process towards the RR occurs in terms of the graphical behavior of the trigonometric polynomial calculated in relation to the set of values delivered by Audacity, however, they do not develop a testing process from an extra-mathematical point of view.

CONCLUSION

In terms of the **research objective**, we can conclude that students build the concept of trigonometric function as one that models pure sounds. Likewise, the trigonometric functions are defined in relation to the time and amplitude variables, proposing an approximation to these functions different that using the concepts of angles and radians. From this conceptualization of trigonometric functions, integrating knowledge of physics and music, students make sense of the phenomenon in question by relating elements of trigonometric functions such as amplitude and periodicity to the sounds reproduced. In relation to complex sounds, these encourage the student to mobilize the knowledge related to Fourier Theory to be able to model them mathematically.

In terms of digital artifact, audacity made a link between the two disciplines, music and mathematics. This allowed understanding that pure sounds were associated with trigonometric functions from the software interface. Audacity allows it to export data in Excel to be graphed in Geogebra, thus providing graphical, numerical and acoustic representations of sounds. Geogebra also allows to develop pragmatic testing processes based on the graphic representation obtained and the adjustment function delivered.

The design of a modeling proposal allows the connection of different mathematical fields through the integration of technologies. The proposal allows connecting knowledge of physics, music and mathematics, often disconnected, to conceptualize trigonometric functions as those that model pure sounds. This conceptualization promotes the understanding and modeling of periodic phenomena, such as sound, through the calculation of trigonometric polynomials associated with Fourier's Theory. We conclude that the design of mathematical modeling tasks integrating technological tools seems to be a good way to transition from high school to university.

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