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► **To cite this version:**

Sarah Khellaf, Jana Peters. Design and analysis of an unusual curve sketching exercise for first year teacher students. Fourth conference of the International Network for Didactic Research in University Mathematics, Leibnitz Universität (Hanover), Oct 2022, Hannover, Germany. hal-04026618

**HAL Id: hal-04026618**

**<https://hal.science/hal-04026618>**

Submitted on 13 Mar 2023

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# Design and analysis of an unusual curve sketching exercise for first year teacher students

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*Abstract: This paper intends to show how certain theoretical elements and tools of the Anthropological Theory of the Didactic (ATD) can be used in the development of exercises that address specific mathematical difficulties of students at the transition from school to university and in a subsequent analysis of student solutions.*

*Keywords: ATD; REM; Herbartian schema; Teachers' and students' practices at university level; Transition to, across and from university mathematics.*

## INTRODUCTION

The switch from school mathematics to a more reflected use of mathematical practices at university is still considered a challenge for students at the transition from school to university (Liebendörfer, 2018, chap. 2.3). Teacher students, specifically, experience what is known in German mathematics education as “double discontinuity” (Winsløw, 2017). To address such transition challenges in an introductory course of mathematics education for first-year teacher students<sup>1</sup>, we employ exercises and teaching materials which aim to examine specific aspects of the relationship between school mathematics and university mathematics. The materials are created on the basis of design principles which address professionalisation aims of teacher education (Ruge et al., 2019; Ruge et al., 2021, p. 250) and were elaborated for the creation of mathematical exercises using the notion of *praxeology* of the ATD (Khellaf et al., 2021).

This contribution focuses on one particular exercise that deals with the topic of *curve sketching*. Curve sketching is a standard topic in calculus courses in German schools and typically serves as application field for differentiation techniques. The exercise discussed in this contribution intends, on the one hand, to help students overcome certain mathematical difficulties connected with the topic of curve sketching (for an overview of typical difficulties see Roos, 2020, chap. 3). On the other hand, the exercise intends to bring to our students' attention specific limitations of typical school mathematical approaches to the topic. It is designed in such way that the application of standard solution strategies leads to a dead end. A solution can be found by directly applying the definitions of extremum and inflection point to the given graphs (Fig. 1). Our students typically experience difficulties in switching from solution strategies they learned in school to practices such as checking prerequisites of theorems and consulting definitions, which are more closely associated with university mathematics. When the usual criteria do not work, some students engage in a variety of ‘unproductive’ and

<sup>1</sup> Course development happens as part of the BMBF-funded teacher education project *Leibniz-Prinzip* (cf. LSE, 2021)

‘unusual’ practices and many have unexpected difficulty perceiving definitions as a resource. As most students attend several mathematics lectures before encountering our exercise (e. g. Analysis I), this persistence of school approaches surprised us and incited us to investigate this didactical phenomenon.

In this paper, we intend to do two things:

1. We employ the Herbartian schema of the ATD to describe in detail the rationale behind the exercise’s design and successive modifications. This adds to previous work on exercise’s design that drew mainly on Critical Psychology and the ATD’s notion of praxeology (cf. Khellaf et al., 2021; Ruge et al., 2019).
2. We present a *reference epistemological model (REM)* that was created to represent a range of (according to the reference institution) ‘legitimate’, ‘plausible’ and ‘expected’ solutions to the exercise. This REM shall in the future serve as analytical tool for the analysis of student solutions to the exercise.

We finish by discussing potentially generalizable aspects of our investigation and observations that might be of interest to for example DBR or SoTL communities (inside and outside the field of mathematics education).

## **THEORETICAL BACKGROUND: ELEMENTS OF THE ANTHROPOLOGICAL THEORY OF THE DIDACTIC**

The *Anthropological Theory of the Didactic (ATD)* (Chevallard, 1992, 2019) is a research programme for the study of human practices from an institutional perspective<sup>2</sup>. *Institution* in the sense of ATD means any form of legitimised social group. Any form of knowledge, and thus also actions in relation to this knowledge, is located in institutions and subject to institutional conditions and legitimisations. Institutional conditions do not merely represent external societal conditions but are constitutive for knowledge and the actions associated with it. In the ATD, knowledge is understood as human activity – legitimised, justified and explained within the institution – that includes not only practical aspects of “know-how” (i.e. *praxis*, task and technique) but also knowledge in the sense of “know-why” (i.e. *logos*, technology and theory). This is subsumed under the term *praxeology*.

The study of the dissemination of knowledge through institutions and among persons is at the heart of the ATD. One important tool to capture study processes is the Herbartian schema (cf. Bosch, 2019):  $[S(X; Y; Q^\heartsuit) \rightarrow M = \{A_i^\diamond, \dots, Q_i, \dots, W_i, \dots, D_i, \dots\}] \Rightarrow A^\heartsuit$ . It consists of the Didactic System  $S(X; Y; Q^\heartsuit)$ <sup>3</sup> around a question  $Q^\heartsuit$  that is studied by  $X$  (one or more students) with the help of  $Y$  (one or more teachers) to find an answer  $A^\heartsuit$ . The study process takes place in a *didactic milieu*  $M$  consisting of

<sup>2</sup> Due to limited space, we cannot really explain all relevant concepts of the ATD in detail and therefore have to refer the reader to the existing literature. Selecting a few works, we refer to Chevallard (2019) for an introduction to ATD and an overview. Lucas, Fonseca, Gascón and Schneider (2019) focus on the REM and the DEM as important concepts for research in ATD. For a concise introduction to the Herbartian schema we refer to Bosch (2019).

<sup>3</sup> The  $\heartsuit$  is a metaphor used within the ATD to indicate that this question is “at the heart” of the study process.  $A^\heartsuit$  is then the answer to the question  $Q^\heartsuit$ .

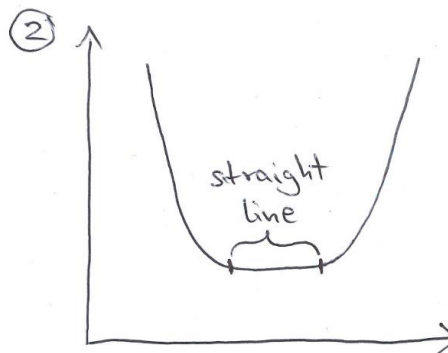
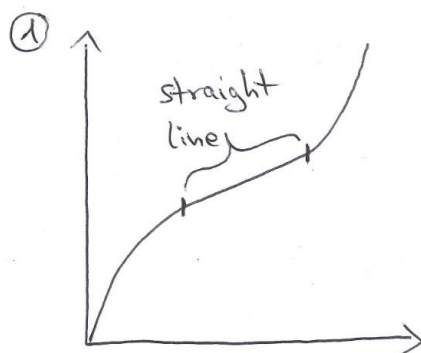
different types of objects: established answers  $A_i^\diamond$  to questions  $Q_i$  that come up in the study process and have to be deconstructed and reconstructed in order to arrive at  $A^\heartsuit$  – this process is known as *question-answer-dialectic*; works  $W_i$  and data  $D_i$ , which help with or provide feedback on the study process. Another category for researching study processes is *media*. Media are all types of systems that issue a message or statements. This can be textbooks, other works, but also fellow students or teachers. The usefulness of a statement to arrive at  $A^\heartsuit$  is evaluated against the milieu and fed back to X and Y. This is called the *media-milieu-dialectic*. Media-milieu- and question-answer-dialectics are analytical tools to describe the dynamics of study processes.

The institutional perspective of the ATD on study processes means that X and Y are institutional positions,  $Q^\heartsuit$  and  $A^\heartsuit$  as well as the milieu are part of institutionally legitimised knowledge. In each institution, e.g. school, there is a predominant way of describing and presenting the knowledge, i.e. the set of relevant questions, what answers, works and data are regarded as legitimate or adequate, in what way the knowledge is used etc. This is called the *dominant epistemological model (DEM)*. To study didactic phenomena of an institution, the ATD proposes to build a *reference epistemological model (REM)* (cf. Lucas et al., 2019). The REM can be seen as a *phénoménotechnique* (in the sense of Bachelard) with which didactic phenomena linked to the DEM can be produced and thus studied in the research process.

As mentioned in the introduction, we suspect that institutional differences between school mathematics and university mathematics might have played a role in the genesis of the phenomenon to be studied. Differences between the knowledge of different educational institutions and their implications for possible study processes are central concerns of the ATD. Therefore, ATD is particularly useful for us, as it makes possible a detailed description of practices typical for the involved institutions.

Give correct mathematical answers to the following questions and provide an acceptable mathematical justification.

- 1) How many inflection points does graph 1 have?
- 2) How many extrema does graph 2 have?



**Fig. 1: Task instruction of the curve sketching exercise**

## DESIGN AND ANALYSIS OF THE EXERCISE

### Task description<sup>4</sup>

Fig. 1 shows *the curve sketching exercise* given to students (cf. Ruge et al., 2021, section 14.2.2). Students are additionally given a chapter on curve sketching from a German mathematics textbook for upper secondary school (Freudigmann et al., 2012, pp. 38-67) as reference material. The chapter contains definitions, theorems, examples and exercises. Figures 2 and 3 show the definitions of *extremum* and *inflection point*.

**Definition:** A function has a

<b>local maximum</b> $f(x_0)$ ,	<b>local minimum</b> $f(x_0)$ ,
---------------------------------	---------------------------------

at  $x_0$ , if there exists an interval  $I$  with  $x_0 \in I$ , so that for all  $x \in I$ :

$f(x) \leq f(x_0)$	$f(x) \geq f(x_0)$ .
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In this case, the point  $(x|f(x_0))$  [sic] is called

<b>“high point”</b> of the graph	<b>“low point”</b> of the graph.
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**Fig. 2: Definition of *extremum* (Freudigmann et al., 2012, p.46, translation by authors)**

**Definition:** Let the function  $f$  be defined on an interval  $I$ , differentiable and let  $x_0$  be an inner point of the interval.

A point  $x_0$  at which the graph of  $f$  changes from being a left-hand<sup>5</sup> curve to being a right-hand<sup>4</sup> curve or the other way around is called **inflection point** of  $f$ .

The respective point  $W(x_0|f(x_0))$  is called **inflection point** of the respective graph.

**Fig. 3: Definition of *inflection point* (Freudigmann et al., 2012, p.56, translation by authors)**

Among the theorems are four which specify criteria for the existence of extremums and inflection points on certain differentiable functions; Fig. 4 shows one such theorem.

**Theorem:** Let the function  $f$  be arbitrarily often differentiable on an interval  $I$  and let  $x_0$  be an inner point of the interval.

1. If  $f''(x_0) = 0$  and  $f''$  has a sign change in the vicinity of  $x_0$ , then  $f$  has an inflection point at  $x_0$ .
2. If  $f''(x_0) = 0$  and  $f'''(x_0) \neq 0$ , then  $f$  has an inflection point at  $x_0$ .

**Fig. 4: Schoolbook theorem: criteria for inflection points (Freudigmann et al., 2012, p. 56, transl. by authors)**

<sup>4</sup> For a more detailed description of the 2020 implementation cf. <https://platinum.uia.no/download/io3-teaching-units-for-student-inquiry/luh-teaching-units/> (last accessed 12.08.2022).

<sup>5</sup> ‘left-hand curve’ = strictly convex curve; ‘right-hand curve’ = strictly concave curve [the authors]

Questions 1) and 2) can be answered with the help of the definitions of extremum and inflection point that can be found in the schoolbook (Fig. 2 & Fig. 3). According to these definitions, there are no inflection points on graph 1 but infinitely many extremums on graph 2 (cf. Fig. 1).

### **Examples of ‘peculiar’ student responses to the task**

In 2019, a student group presented an exercise solution and explained correctly why the schoolbook theorems do not yield useful information, but could still not answer questions 1) and 2) because they had not thought of looking into the definition as a viable course of action. In 2021 a student, who wanted to write an essay about the topic, tried to apply the strategy of approximating the functions pertaining to graphs 1 and 2 with step functions and to transfer knowledge about properties of the step functions onto the limit functions, before we directed him towards the definitions. Overall, we could observe in many students’ approaches a peculiar absence of the strategy of consulting the (formal) definitions of the concepts in question, and a replacement of this strategy by a variety of ‘unproductive’ and ‘unusual’ strategies.

### **Description of the task’s economy and ecology**

In German *school mathematics*, the curve sketching exercise can be considered ‘unusual’ because exercises within the topic of curve sketching typically ask for the application of algorithmic procedures derived from schoolbook theorems which give criteria for the existence of extremums and inflection points. These algorithms constitute the institution’s ‘official answer’  $A_{\text{School}}^{\diamond}$  for virtually all schoolbook exercises which ask to find extremums or inflection points on functions, which are typically given in algebraic form.  $A_{\text{School}}^{\diamond}$  thus constitutes the *dominant epistemological model (DEM)* in the institution of school mathematics regarding the topic of curve sketching. In the curve sketching exercise, however, the standard procedures for determining extrema and inflection points fail: The schoolbook theorems are formulated as unidirectional conditional statements (cf. Fig. 4), but on the straight segments of graph sketches 1 and 2 (Fig. 1) the sufficient conditions do not hold.

In *university mathematics*, a standard technique to approach any mathematical problem is the verification of one’s own knowledge of the involved definitions, which we will call  $A_{\text{University}}^{\diamond}$ . In the case of the curve sketching exercise, making sensible use of the schoolbook definitions will lead to an answer to questions 1) and 2).

The curve sketching exercise was designed in such way, that the *media* provided strongly evoke the DEM of the institution of school mathematics: it is introduced by a fictional conversation between two school students and provides a schoolbook excerpt as material ( $W_{\text{School}}$ ).  $A_{\text{University}}^{\diamond}$ , on the other hand, becomes less associated with the task ( $\rightarrow A_{\text{University}}^{\diamond}$ ). One aim of evoking the DEM was to make our students experience its mathematical limitations and provoke subsequent questioning of the DEM.

The *milieu* actively provided was a page on an online learning platform that contained the media and some instructions. However, as the page is non-responsive and does not

give feedback on student's activities (apart from providing the media), it is not included in the Herbartian schema. The teaching team  $Y$  offered to respond to questions at any time and would add further elements to the milieu upon request (in the form of replies). An 'invisible' but nonetheless crucial part of the milieu are *the results of an application of logical/mathematical reasoning* ( $D_{\text{math}}$ ).  $A^\heartsuit$  was not included in the milieu (e.g. in the form of a sample solution for self-check).

### Question-answer-dialectic, chronogenesis and media-milieu-dialectic

Students are confronted with  $Q^\heartsuit$  (cf. Fig.1) and many first try  $A^\diamond_{\text{School}}$  to solve it. A correct(!) application of *mathematical reasoning* (i. e. interaction with the milieu) should yield  $D_{\text{math}}$ , i.e. the 'feedback' that  $A^\diamond_{\text{School}}$  cannot answer  $Q^\heartsuit$ , which should give rise to new questions  $Q_i$ . This coincides with the broader intention of the curve sketching exercise to initiate a questioning of  $A^\diamond_{\text{School}}$  with regard to its "validity and limitations ..., its adequacy to  $Q^\heartsuit$ , the authors], the adaptations required, etc." (Bosch, 2019, p. 4040).

The original curve sketching exercise can be depicted schematically as:<sup>6</sup>

$$[S(X; Y; Q^\heartsuit) \rightarrow M=\{A^\diamond_{\text{School}}, A^\diamond_{\text{University}}, W_{\text{School}}, D_{\text{math}}, Q_i, \dots\}] \Rightarrow A^\heartsuit$$

However, problems with the chronogenesis of the inquiry can arise and students can get stuck in a particular way: as the only part of the (initially provided) milieu that can give feedback on the correctness of students' answers is  $Y$  or  $D_{\text{math}}$ , which are both spatially and/or temporarily displaced (hence the grey font), it can happen that students end up believing, e.g. due to a logical error in the interpretation of the schoolbook theorems (taking unidirectional conditional statements for equivalences), that they have answered the question  $Q^\heartsuit$  correctly even though they haven't. We suspect, that the DEM is being evoked too strongly and that the initial milieu doesn't provide enough feedback to enable (at least part of) our students to overcome and question the DEM.

As a consequence, in subsequent implementations of the online course, the milieu was enriched by adding new materials  $(\dots)_i$  to the learning platform which consisted in fictional dialogues  $W_{\text{Student}}$  between students who try to solve the curve sketching exercise but who run into the same dead ends as some actual students did and give an incorrect answer. The materials then pose the question  $Q_{\text{DM}}$ , why the displayed reasoning is incorrect, and include an official answer  $A^\diamond_{\text{DM}}$  to  $Q_{\text{DM}}$  by the institution that is our didactics course in the form of a sample explanation.

The modified curve sketching exercise can thus be schematically represented as:

$$[S(X; Y; Q^\heartsuit) \rightarrow M=\{A^\diamond_{\text{School}}, A^\diamond_{\text{University}}, (W_{\text{Student}}, Q_{\text{DM}}, A^\diamond_{\text{DM}})_1, (\dots)_2, \dots, W_{\text{School}}, D_{\text{math}}, Q_i, \dots\}] \Rightarrow A^\heartsuit$$

<sup>6</sup> The symbols stand for:  $X$  = students,  $Y$  = teaching team,  $Q^\heartsuit$  = questions 1) and 2) from Fig.1,  $A^\diamond_{\text{School}}$  = algorithms for checking criteria for the existence of extremums and inflection points,  $A^\diamond_{\text{University}}$  = verification of one's own knowledge of the involved definitions,  $W_{\text{School}}$  = provided schoolbook excerpt,  $D_{\text{math}}$  = results of (correct) mathematical reasoning

## THE REFERENCE EPISTEMOLOGICAL MODEL

In order to better understand the potential of the exercise to induce question-answer- and media-milieu-dialectics and to enable more in-depth analyses of students' solutions we started to work out a REM with reference to the *institution of mathematics education* constituted by our working group and through our courses.<sup>7</sup> The REM contains all praxeological elements which occur in a set of solutions to the curve sketching exercise that we created on the basis of  $W_{\text{School}}$  and our knowledge of mathematics (cf.  $D_{\text{math}}$ ) and which were deemed the most 'obvious', 'plausible' and 'expected' (i.e. 'normal') solutions from the point of view of reference.<sup>8</sup> A visualisation of this set of solutions is given in the form of a flowchart (Fig. 5), which was inspired by the tool of *questions-answers map* (Bosch, 2019, p. 4041). The chart is not meant to be interpreted strictly chronologically: drawing a 'solution path' into the chart is supposed to indicate which praxeological elements occur and don't occur, but not necessarily at which stage of the solution they are employed (they might be employed several times at different stages in the solution). The detailed REM can be accessed by consulting tables 1, 2 and 3 in the supplementary material in combination with the flowchart. The tables specify *three regional praxeologies* that will now be explained.

### **Regional praxeology *Differentiation* (cf. Table 1)**

We regard the differentiation of real functions as a praxeology in its own right, because it is taught in school as separate topic before curve sketching is introduced in later school years. In the context of our course, differentiation is also treated differently from curve sketching as our didactic contract demands that the schoolbook be cited when curve sketching theorems are used, while justifications for the application of differentiation techniques are not necessary. The praxeology *differentiation* includes all praxeological elements that were relevant in our solutions, even if some of them are likely not commonly taught in German schools but in introductory Analysis courses at university (e.g.  $\theta(\text{diff})$ .1 and 2). In this sense, the resulting praxeology is specific to the institution of mathematics education, whose praxeological equipment intersects with that of both school and university mathematics (as well as that of other disciplines).

### **Regional praxeology *Curve Sketching* (cf. Table 2)**

We consider the praxeologies *extrema* ( $E$ ) and *inflection points* ( $IP$ ) to be two subpraxeologies of the regional praxeology *curve sketching*. The two praxeologies possess largely similar logos-blocks (monotony is relevant for both; justifications of theorems are based on the same ideas) and are consolidated on page 67 of Freudigmann et al. (2012) into a sequence of activities commonly called "curve sketching". However, the book's presentation assigns the two praxeologies to two different subchapters with their own specific definitions, theorems, examples and tasks. Table 2

<sup>7</sup> It is prominently characterised by the assessment criteria we apply in our exams and which we communicate to our students, by the knowledge and practices we teach in our courses, etc.

<sup>8</sup> Naturally, included solutions had to be mathematically sound.



represents the praxeology extrema (E); the respective praxeology inflection points (IP) is analogous, but “E” is swapped with “IP” and other schoolbook theorems are relevant.

### **Regional praxeology *Solving Tasks* (cf. Table 3)**

After we had coded all steps of each solution using the above praxeologies as coding manual<sup>9</sup>, there remained passages in our solutions to which no praxeological elements had yet been assigned. For this reason, we defined an additional praxeology *solving tasks* that includes general strategies for solving tasks. These strategies are commonly introduced in school, applied in all subjects (not just mathematics), and are relevant for university mathematics as well, where they undergo a mathematical specialisation as stronger emphasis is placed on mathematical rigour (e.g. in mathematical case differentiations at university, the cases have to cover all logical possibilities and any two cases have to be disjunct from each other).

## **DISCUSSION**

The design of the curve sketching exercise addresses two circumstances: First, it is a reaction to mathematical difficulties students typically experience at the transition from school to university.<sup>10</sup> By posing a mathematical problem which cannot be solved within a well-known DEM from school mathematics, we intend to initiate questioning and further analysis of this DEM. This idea is especially relevant in the education of teacher students, as this group can not only benefit from questioning and increasing their mathematical knowledge, but also from using didactic theory to analyse and reflect upon the teaching of the DEM in school, its goals and effects. Secondly, exercise modifications intend to mitigate teaching difficulties arising from the non-responsiveness of our digital learning environment and the lack of feedback by the milieu. These difficulties are in fact not only a problem of our specific learning platform but also, more generally, a feature of the type of task proposed: the idea for the task is based on the expectation (and experience) that many students are fixated on the DEM so strongly that they overlook solution strategies that are very elementary from the point of view of university mathematics. The potential of the exercise lies in its capacity to incentivise reflection on these issues, i. e. on biases in one’s own mathematical thinking and their possible origins. With regard to the learning platform, the question arises of how to provide feedback that is standardised but will nonetheless relate to individual mistakes to some degree in order to help students who are ‘stuck in the DEM’ and to encourage them to continue their investigation into Q♥. Due to worries that the presence of a sample solution A♥ might discourage further investigation and bar access to certain learning experiences, we did not provide A♥ on the page of the learning platform that contained the exercise. The option of enriching the milieu with discursive material in the way explained, on the other hand, has worked well for us:

<sup>9</sup> Coding is understood here in the sense of qualitative content analysis; the units that were coded were the elements of the 4-T-model, i. e. (the application of) techniques, reasons given for applying a technique, more rarely: intermediate questions/tasks.

<sup>10</sup> In German mathematics education, such exercises are referred to as “Schnittstellenaufgaben”.

some students became aware of mistakes they had made themselves while all of them were provided an opportunity to analyse typical learner mistakes using didactic theory.

In order to create and refine an exercise that targets DEM bias in the described way, it is necessary to gather knowledge about typical student difficulties. Consultation of literature may guide towards promising mathematical topic areas and can generate ideas for potential exercises at the beginning of a design process. Once concrete exercises have been formulated, the potential of an exercise can be illuminated by creating a questions-answers map (our flowchart being a variation thereof). An initial questions-answers map functions as a reflection tool that gives an overview of the knowledge involved in various ways of dealing with the task and may generate (on the basis of knowledge about the DEM) a priori hypotheses about possible student difficulties. After a first trial of the exercise, materials that address mathematical difficulties with the task can be created on the basis of (naïve) observations of typical student mistakes and with the help of the map, which can facilitate a systematic covering of all difficulties of interest.

The questions-answers map can be refined in connection with the creation of a detailed REM, which has happened in our case. The REM does not only help deepen reflection on an exercise's potential and the involved institutional knowledge(s), but can be employed as analytical tool in ATD-based qualitative analyses of (various types of) data documenting student solutions or solution processes. In such analyses, it will serve as a normative reference against which deviations in solutions (or solution processes) can be systematically identified and characterised. This may be useful in cases where student difficulties are yet unknown or not documented sufficiently in existing research. We plan an exploratory investigation into our students' solution strategies on the basis of our REM in order to see if we can learn more about the types of difficulties that occur and how to address them in our teaching. (There is a possibility, too, that unexpected solution strategies which are viable in certain milieus will inform the development of further materials.) Additionally, our REM constitutes a novelty in that it mixes praxeological elements from university and school mathematics to create a representation of a praxeological organisation that stems from the institution of mathematics education. This may open up new possibilities of investigating questions of teacher education more broadly. E.g.: Can the necessity to use praxeological elements from university mathematics in tasks that 'could possibly occur in school' provide a profession-oriented *raison-d'être* for these elements in the study curriculum?

## REFERENCES

- Bosch, M. (2019). Study and Research Paths: A Model for Inquiry. In B. Sirakov, P. Ney de Souza & M. Viana (eds.), *Proceedings of the International Congress of Mathematicians (ICM 2018)* (pp. 4015–4035). World Scientific.
- Chevallard, Y. (1992). Fundamental concepts in didactics: Perspectives provided by an anthropological approach. *Research in Didactique of Mathematics, Selected Papers*. La Pensée Sauvage, 131–167.

- Chevallard, Y. (2019). Introducing the anthropological theory of the didactic: An attempt at a principled approach. *Hiroshima Journal of Mathematics Education*, 12, 71–114.
- Freudigmann, H., Buck, H., Greulich, D., Sandmann, R. & Zinser, M. (2012). *Lambacher Schweizer – Mathematik für Gymnasien. Analysis Leistungskurs*. Ernst Klett Verlag.
- Khellaf, S., Hochmuth, R. & Peters, J. (2021). Aufgaben an der Schnittstelle von Schulmathematik, Hochschulmathematik und Mathematikdidaktik. Theoretische Überlegungen und exemplarische Befunde aus einer einführenden Fachdidaktikveranstaltung. In R. Biehler, A. Eichler, R. Hochmuth, S. Rach, N. Schaper (Eds.), *Lehrinnovationen in der Hochschulmathematik: praxisrelevant - didaktisch fundiert - forschungsbasiert*. Springer Spektrum.
- Leibniz School of Education (LSE) (2021a). *Leibniz-Prinzip*. Retrieved 12.07.2022 from <https://www.lse.uni-hannover.de/de/lse/projekte/qualitaetsoffensive-lehrerbildung/projekt-leibniz-prinzip/>
- Liebendörfer, M. (2018). *Motivationsentwicklung im Mathematikstudium*. Springer Spektrum.
- Lucas, C., Fonseca, C., Gascón, J. & Schneider, M. (2019). The phenomenotechnical potential of reference epistemological models: The case of elementary differential calculus. In M. Bosch, Y. Chevallard, F. J. Garcia & J. Monaghan (Eds.), *Working with the Anthropological Theory of the Didactic in Mathematics Education* (S. 77–98). Routledge.
- Roos, A.-K. (2020). *Mathematisches Begriffsverständnis im Übergang Schule–Universität: Verständnisschwierigkeiten von Mathematik an der Hochschule am Beispiel des Extrempunktbegriffs*. Springer Spektrum.
- Ruge, J.; Khellaf, S.; Hochmuth, R.; Peters, J. (2019). Die Entwicklung reflektierter Handlungsfähigkeit aus subjektwissenschaftlicher Perspektive. In Dannemann, S.; Gillen, J.; Krüger, A.; von Roux, Y. (Hrsg.), *Reflektierte Handlungsfähigkeit in der Lehrer\*innenbildung - Leitbild, Konzepte und Projekte*. Berlin: Logos Verlag.
- Ruge, J., Hochmuth, R., Khellaf, S. & Peters, J. (2021). In Critical Alignment with IBME. In I. M. Gómez-Chacón, R. Hochmuth, B. Jaworski, J. Rebenda, J. Ruge & S. Thomas (eds.), *Inquiry in University Mathematics Teaching and Learning. The Platinum Project*. Brno: Masaryk University Press. <https://munispace.muni.cz/library/catalog/chapter/2132/535>
- Winsløw, C. (2017). The ATD and other approaches to a classical problem posed by F. Klein. In G. Cirade, M. Artaud, M. Bosch, J.-P. Bourgade, Y. Chevallard, C. Ladage & T. Sierra (Eds.), *Évolutions contemporaines du rapport aux mathématiques et aux autres savoirs à l'école et dans la société* (pp. 69–91). <https://citad4.sciencesconf.org/data/pages/ActesCITAD4.pdf>