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# **Analysing proofs of the Bolzano-Weierstrass theorem**

## **A mean to improve proof skills and understanding of completeness**

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*In this communication, we will analyse several proofs of the Bolzano-Weierstrass theorem. We will first present the method by Bolzano in the memoir of the Intermediate Value Theorem that is known to have inspired the proof by Weierstrass, then we will analyse proofs available for both the set version proved by Weierstrass, and the sequence version. We will emphasize the diversity of modes of reasoning on the one hand, on objects and completeness characterisation on the second hand. We will finally suggest scenarios for undergraduates or for prospective teachers.*

*Keywords: Teaching and learning of logic, reasoning and proof, Epistemological studies of mathematical topics, Bolzano-Weierstrass Theorem, Proofs 'analysis, Teaching and learning of analysis and calculus.*

### **INTRODUCTION**

As accounted by Bergé (2010), undergraduate students having followed four courses on the set of real numbers might face still difficulties with task requiring a sound understanding of completeness. This motivates the search for activities able to contribute to this understanding. This communication falls within a wider research project aiming at identifying didactic means to improve the teaching and learning of the set of real numbers as a complete ordered set, considering the crucial role of epistemology in didactics of mathematics, and focusing on proof and proving. Our main didactic hypothesis is that logical analysis of proof fulfils three main functions: to control validity; to understand the strategy of the author of the proof; to contribute both to the development of proof and proving skills and to the appropriation of the mathematical content at stake. We have already discussed in other papers 1/ the potentiality of fixed-point theorems of increasing functions of a real subset in itself (Durand-Guerrier, 2016); 2/ the relevance of working on proofs of completeness in various settings of real numbers sets (Durand-Guerrier & Tanguay, 2018); 3/ The fecundity of approaching the real exponential function in the more general frame of real function satisfying the algebraic relation  $f(x + y) = f(x)f(y)$  (Durand-Guerrier et al., 2019); 4/ the dialectical relationship between *truth* and *proof* in the case of the emergence of the Intermediate Value Theorem (IVT), while comparing Bolzano and Cauchy approaches (Durand-Guerrier, 2022). In this communication, we are extending our analysis to the *Bolzano-Weierstrass Theorem* (BWT) which plays an important role in Analysis. In his memoir on the IVT published in 1817, Bolzano used a method by *dissection* to prove (in modern term) the existence of a *least upper bound* for a bounded above set. Later, Weierstrass relied on the method described by Bolzano to prove the theorem that is named after the two mathematicians (Oudot, 2017).

Nowadays, we found in the international literature various proofs of this theorem. We will argue that analysing these proofs might contribute to deepen both proofs skills and conceptual understanding of the concept of  $\mathbb{R}$ -completeness, considering both the *version for sets* and the *version for sequences*:

1. BWT for sets: *Every infinite and bounded set of real numbers has at least an accumulation point.*
2. BWT for sequences: *Every bounded sequence has a convergent subsequence.*

We first present and analyse the Bolzano dissection method and four BWT proofs. Then, we propose some guidelines for designing a didactical engineering.

## PROOFS OF THE BOLZANO-WEIERSTRASS THEOREM

Although the name of Bernard Bolzano is associated to the name of Karl Weierstrass, some authors claim that Bolzano neither prove, nor even enounce the theorem (e.g., Oudot, 2017). However, it is agreed that Bolzano used a *dissection method* in a lemma for the proof of the Intermediate Value Theorem (IVT), which inspired Weierstrass to prove that *any infinite bounded subset of the set of real numbers has an accumulation point.* (e.g., Oudot, 2017). We present and briefly analyse 1/ the dissection method by Bolzano; 2/ the main ideas of the proof by Weierstrass of the BWT for sets; 3/ an alternative proof of the BWT for sets in a paper by Mamona-Downs; 3/ two classical proofs of the BWT for sequences in a textbook.

In Durand-Guerrier and Arsac (2009, p. 152), we provided evidence that analysing logically a mathematical proof requires both logical and mathematical competencies. We have identified relevant questions that we will use as a lens for analysing the proofs presented in this section: *what are the data and hypothesis? which objects are introduced along the proof, and with which aim? Which are the modes of reasoning, the explicit and implicit assumptions?* In the case of BWT, we will also focus on the *axioms of completeness*, and on the recourse to *potential infinity versus actual infinity*.<sup>1</sup>

### The dissection method by Bolzano (1817)

In the paragraph 12 of the proof of the IVT, Bolzano enounce and prove the theorem<sup>2</sup>:

Theorem. If a property  $M$  does not belong to all values of a variable  $x$ , but does belong to all values which are less than a certain  $u$ , then there is always a quantity  $U$  which is the greatest of those of which it can be asserted that all smaller  $x$  has property  $M$ . (Russ, 1980, p. 174)

He initiated the proof by introducing a positive quantity  $D$  such that there is at least an element among those less than  $u + D$  that does not hold the property  $M$  (such a quantity exists by hypothesis). Doing this, he implicitly introduces a *real interval*  $[u, u + D[$  in which he will search the quantity  $U$ . Then he considers sequentially the quantities  $u + \frac{D}{2^m}$ , with  $m$  a null or positive integer. In other words, he introduces a *geometrical*

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<sup>1</sup> This paper will not consider the logical issues involved in the use of quantifiers and connectives.

<sup>2</sup> For the theorem, its proof and the comments by Bolzano on this theorem, we use the English Translation by Russ (1980).

sequence with ratio  $\frac{1}{2}$ . It is sure, by the choice of  $D$ , that for  $m = 0$  there is at least an element not satisfying the property  $M$  among the element less than  $u + \frac{D}{2^0}$ . In case where whatever the value of  $m$ , there exists always an element not satisfying the property  $M$  along the element less than  $u + \frac{D}{2^m}$ , then he concludes that  $u$  is the greatest of those quantities of which it can be asserted that all smaller  $x$  has property  $M$ . The proof of this assertion (not provided in this part of the memoire) relies on the fact that given a real number  $v > u$ , there exists an element between  $u$  and  $v$  that does not has the property; indeed there exists a positive integer  $k$  such that  $u < u + \frac{D}{2^k} < v$ . Opposite, if for a certain rank  $m$ , for the first time all elements less than  $u + \frac{D}{2^m}$  holds the property  $M$ , then he reinitiated the process with  $u + \frac{D}{2^m}$  playing the role of  $u$ , and  $u + \frac{D}{2^{m-1}}$  playing the role of  $u + D$ . This is motivated by the fact that at this step, the quantity  $U$  should be searched in the interval  $[u + \frac{D}{2^m}, u + \frac{D}{2^{m-1}}[$ . Bolzano notes that the difference between the two quantities is  $\frac{D}{2^m}$ . The process is then reiterated, considering the sequence<sup>3</sup> $(u + \frac{D}{2^m} + \frac{D}{2^{m+n}})$ . Then either  $U = u + \frac{D}{2^m}$  and the theorem is proved; or there is a value  $r$  such that  $U$  belongs to  $[u + \frac{D}{2^m} + \frac{D}{2^{m+r}}, u + \frac{D}{2^m} + \frac{D}{2^{m+r-1}}[$ , and the process is iterated. There are then two possible issues: either there is a step where the value of  $U$  is reached, and the iterations stop; or the iterations do not stop and hence, it provides an increasing sequence in geometric progression of ratio  $\frac{1}{2}$ , with first term  $u + \frac{D}{2^m}$ , each term being less than  $u + \frac{D}{2^{m-1}}$ , with the difference between two consecutive terms decreasing to 0. This way corresponds to the modern method by dichotomy. Bolzano used then a previous lemma (§9) to conclude of the existence of a quantity  $U$  which is the greatest of those of which it can be asserted that all smaller  $x$  has property  $M$ .

*Lemma.* If, therefore, some given series has the property that each term is finite, but the change which it undergoes on every further continuation is smaller than any given quantity, provided only that the number of terms taken in the first place is large enough, then there is always one and only one constant quantity which comes as close to the value of this series as desired, if it is continued far enough. (Russ, 1980, p. 173).

In this proof, the data are *a property  $M$ , a variable  $x$  and a constant  $u$* ; and the hypothesis:  *$M$  does not belong to all values of  $x$ , but does belong to all values which are less than  $u$* . Then there are several objects that are introduced: a constant  $D$ , a geometric sequence, and along the proof, some specific terms of the sequences identified by indexes of the power of 2; finally, a geometrical progression is introduced. To conclude to the existence, Bolzano refers to what we name nowadays the *Cauchy*

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<sup>3</sup> In this proof, Bolzano does not use the term sequence, and the notion of interval; it remains implicit.

*criteria of convergence for sequence*, one of the possible axioms of completeness. It is noticeable that as Bolzano did not create the set of real numbers (Freudenthal, 1971, p. 387), the proof in the lemma 9 relies on an assumption that he could not prove (an axiom). In this proof, Bolzano recourse to potential infinity through iteration process.

### **The proof by Weierstrass of the BWT for sets**

According to Oudot (2017), while in contemporaneous textbooks the BWT is most often given for sequences, Weierstrass proved the version for *infinite sets*. He first defined the notion of *accumulation point* for an infinite set.

*Definition*: an accumulation point of a given subset of the set of real numbers is a point such that each pointed neighbourhood has a non-empty intersection with the set.

He then proved by dichotomy, a method inspired by Bolzano, the theorem *for sets*: *Every infinite and bounded set of real numbers has at least an accumulation point*.

Here is a summary of the proof (translated from Oudot, 2017).

*Proof* - If a set  $A$  is bounded, then it is included in a closed interval  $[m, M]$ . As  $A$  is infinite, cutting the interval through its midpoint, we get two intervals so that at least one contains an infinite number of points of  $A$ . The process is then iterated on an interval with infinite numbers of points of  $A$  (there exists at least one), leading to the construction of a sequence of nested intervals with length less than  $\frac{1}{2^m}$  at the step  $m$ . The conclusion follows because the set of real numbers being complete, it satisfies the property of *nested intervals with length tending to 0*: there exists a unique element belonging to each interval. This element is an accumulation point for  $A$ .

In this proof, the data is a set, with two hypotheses: the set is *bounded*; the set is *infinite*. Then an interval  $[m, M]$  is introduced, and a process of dichotomy is performed. It is noticeable that the method is simpler than the method by Bolzano, thanks to the explicit introduction of *an interval*, that allows the recourse to *actual infinite*.

### **A proof of the BWT for sets in an educational paper**

In a paper published in 2010, Mamona-Downs suggested that providing students opportunities to contrast the convergence behaviour of a sequence and the accumulation points of the underlying set of the sequence is worthwhile for undergraduates. In the paper, she provides a proof of BWT with recourse to the supremum property (p. 283).

**“Theorem (Bolzano–Weierstrass):** *Let  $S$  be a real set that is both infinite and bounded. Then  $S$  has an accumulation point<sup>4</sup>.*

As a preliminary step to the proof, the author explains that assuming the problem is solved, a candidate appears (the supremum of a well-chosen set, as shown in the proof below).

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<sup>4</sup> The author indicates that there is no loss of generality if we suppose that  $S$  is a subset of a closed interval,  $[a, b]$  say.

Proof: Consider the set  $L: \{x \in [a, b] : [a, x] \text{ contains none or a finite number of elements of } S \text{ less than } x\}$ .  $L$  is non-empty and is bounded by  $b$ . Thus  $\sup(L)$  exists. Denote  $\sup(L)$  by  $r$ .

Take any  $\epsilon > 0$ , and consider the interval  $(r - \epsilon, r + \epsilon)$ . Now  $[a, r - \epsilon]$  contains none or a finite number of elements of  $S$ , and  $[a, r + \epsilon]$  contains an infinite number of elements of  $S$ . Then the interval  $(r - \epsilon, r + \epsilon)$  also contains an infinite number. This implies  $(r - \epsilon, r + \epsilon)$  contains an element of  $S^5$ , so  $r$  indeed is an accumulation point for  $S$ .<sup>5</sup>

In this proof, the data is a set, with two hypotheses: the set is *bounded*; the set is *infinite*. There is a first introduction of an interval, relying on the hypothesis that the set is bounded, followed by the introduction of the set  $L$ , which is the left part of a cut, in Dedekind sense, of the interval  $[a, b]$ . This choice is motivated by the preliminary step because the supremum linked to this cut has been identified as a candidate to be an accumulation point. The axiom for completeness is *the existence of a supremum for any bounded subset of the set of real numbers*. Once done, the author names  $r$  the supremum and introduces a family of intervals centred in  $r$ . She uses implicitly that the complement of a finite subset in an infinite set is infinite, to assert that at each step, there is an interval with infinitely many elements of  $S$ ; hence each interval of the family contains infinitely many elements of  $S$ ; this proves that  $r$  is an accumulation point of  $S$ .

## Two classical proofs of the BWT for sequences

In this section, we present two classical proofs of the BWT out of a textbook for undergraduates (Bartle & Sherbert, 2000). The two proofs of BWT are in the section 3.4 entitled “*Subsequences and the Bolzano-Weierstrass theorem*”. The authors first enounce and prove a theorem guaranteeing the existence of monotone subsequence for any real sequence (p.78). We present the proof of this theorem because it is used in their first proof of BWT.

### “3.4.7 Monotone Subsequence Theorem

If  $X = (x_n)$  is a sequence of real numbers, then there is a subsequence of  $X$  that is monotone.

Proof. For the purpose of this proof, we will say that the  $m$ th term  $x_m$  is a “peak” if  $x_m \geq x_n$  for all  $n$  such that  $n \geq m$  (That is,  $x_m$  is never exceeded by any term that follows it in the sequence.) Note that, in a decreasing sequence, every term is a peak, while in an increasing sequence, no term is a peak.

We will consider two cases, depending on whether  $X$  has infinitely or finitely many peaks.

*Case 1:*  $X$  has infinitely many peaks. In this case, we list the peaks by increasing subscripts:  $x_{m_1}, x_{m_2}, \dots, x_{m_k}, \dots$ . Since each term is a peak, we have

$$x_{m_1} \geq x_{m_2} \geq \dots \geq x_{m_k} \geq \dots$$

Therefore, the subsequence  $(x_{m_n})$  of peaks is a decreasing subsequence of  $X$ .

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<sup>5</sup> Implicitly there is at least an element different from  $r$ .

*Case 2:*  $X$  has a finite number (possibly zero) of peaks. Let these peaks be listed by increasing subscripts  $x_{m_1}, x_{m_2}, \dots, x_{m_r}$ . Let  $s_1 = m_r + 1$  the first index beyond the last peak.

Since  $x_{s_1}$  is not a peak, there exists  $s_2 > s_1$  such that  $x_{s_1} < x_{s_2}$ . Since  $x_{s_2}$  is not a peak, there exists  $s_3 > s_2$  such that  $x_{s_2} < x_{s_3}$ . Continuing in this way, we obtain an increasing subsequence  $(x_{s_k})$  of  $X$ .

It is not difficult to see that a given sequence may have one subsequence that is increasing, and another subsequence that is decreasing.” (op. cit. page 78).

In this proof, the data is a real sequence; there is no additional hypothesis. The notion of “peak” is introduced and depending on whether or not there are infinitely many peaks (actual infinity), one can construct either a decreasing subsequence or a finite sequence of peaks. In the former case, we have a proof of the theorem; in the latter case, an iterative process (potential infinity) provides an increasing sequence, and the theorem is proved. It is noticeable that this proof use only the order properties; the  $\mathbb{R}$ -completeness is not involved, so this theorem holds for example for rational sequences, while BWT does not. Both *actual* infinity and *potential* infinity are involved.

The authors move then to the BWT theorem for sequences for which they provide two proofs, the first one relying on the previous theorem.

“3.4.8 The Bolzano-Weierstrass Theorem. *A bounded sequence of real numbers as a convergent subsequence.*

*First proof* - It follows from the Monotone Sequence Theorem that if  $X = (x_n)$  is a bounded sequence, then it has a subsequence  $X' = (x_{s_k})$  that is monotone. Since this subsequence is also bounded, it follows from the monotone convergence theorem 3.3.2 that the subsequence is convergent.” (op. cit. p. 78-79).

Here the data is a sequence of real numbers; the hypothesis is that it is bounded. Thanks to theorem 3.4.7, the authors introduce a monotone subsequence of the given sequence. They use, without enouncing it, the assertion that *a subsequence of a bounded sequence is bounded*. The conclusion relies on theorem 3.2.2. (*A monotone sequence of real numbers is convergent if and only if it is bounded*). The completeness of the set of real numbers is required for the proof of the reverse implication. This theorem is among those that can be chosen as *axiom of completeness*. It was at the origin of the creation of irrational numbers by Dedekind, who proved it in his ordered complete system of real numbers (Dedekind, 1963). In this textbook, the axiom of completeness is the supremum one: *Every nonempty set of real numbers that has an upper bound has also a supremum (i.e. a least upper bound)* (2.3.6, p.37).

The second proof relies on arguments close to those of the proof by Weierstrass for the version *for sets* by introducing *the set of values of the sequence*.

“*Second proof* – Since the set of values  $\{x_n : n \in N\}$  is bounded, this set is contained in an interval  $I_1 = [a, b]$ . We take  $n_1 := 1$ .

We now bisect  $I_1$  into two equal subintervals  $I'_1$  and  $I''_1$ , and divide the set of indices  $\{n \in \mathbb{N} : n > 1\}$  into two parts:

$$A_1 = \{n \in \mathbb{N} : n > n_1, x_n \in I'_1\} \qquad B_1 = \{n \in \mathbb{N} : n > n_1, x_n \in I''_1\}$$

If  $A_1$  is infinite, we take  $I_2 = I'_1$  and let  $n_2$  be the smallest natural number in  $A_1$ . If  $A_1$  is a finite set, then  $B_1$  must be infinite, and we take  $I_2 = I''_1$  and let  $n_2$  be the smallest natural number in  $B_1$ .

We now bisect  $I_2$  into two equal subintervals  $I'_2$  and  $I''_2$ , and divide the set of indices  $\{n \in \mathbb{N} : n > n_2\}$  into two parts:

$$A_2 = \{n \in \mathbb{N} : n > n_2, x_n \in I'_2\} \qquad B_2 = \{n \in \mathbb{N} : n > n_2, x_n \in I''_2\}$$

If  $A_2$  is infinite, we take  $I_3 = I'_2$  and let  $n_3$  be the smallest natural number in  $A_2$ . If  $A_2$  is a finite set, then  $B_2$  must be infinite, and we take  $I_3 = I''_2$  and let  $n_3$  be the smallest natural number in  $B_2$ .

We continue in this way to obtain a sequence of nested intervals  $I_1 \supseteq I_2 \supseteq \dots \supseteq I_k \supseteq \dots$ , and a subsequence  $(x_{n_k})$  of  $X$  such that  $x_{n_k} \in I_k$  for  $k \in \mathbb{N}$ . Since the length of  $I_k$  is equal to  $\frac{b-a}{2^{k-1}}$ , it follows from Theorem 2.5.3 that there is a (unique) common point  $\xi \in I_k$  for all  $k \in \mathbb{N}$ . Moreover since  $x_{n_k}$  and  $\xi$  both belong to  $I_k$ , we have

$$|x_{n_k} - \xi| \leq \frac{b-a}{2^{k-1}}$$

whence it follows that the subsequence  $(x_{n_k})$  of  $X$  converge to  $\xi$ .” (op. cit. p.79)

In this proof, the data and the hypothesis are the same than for the first proof. The authors introduce the set of values of the sequence; as the sequence is bounded, the set is also; then, an interval is introduced to initiate the dichotomy<sup>6</sup>, that provides both a sequence of nested intervals with length converging to 0, and a subsequence of the initial sequence. By the corresponding axiom of completeness, there exists a unique point in the intersection, that is proved to be the limit of the sequence. Comparing this proof with Weierstrass’s one enlightens the links between the set version and the sequence version of BWT, and the links with axioms of  $\mathbb{R}$ -completeness.

The analysis of these five proofs confirms the potential of the BWT to work with undergraduates or prospective teachers on various characterisations of completeness, and on their links. In the second section, we provide some paths for designing a didactical engineering aiming at developing proof and proving skills together with contribution to a better understanding of the completeness of the set of real numbers.

## **GUIDELINES FOR DESIGNING A DIDACTICAL ENGINEERING**

As Mamona-Downs suggested, we consider that introducing undergraduates or secondary mathematics prospective teachers to both versions of BWT and designing a didactical engineering<sup>7</sup> around the analysis of the various proofs would offer the

<sup>6</sup> The proof is done in case this set of values is infinite

<sup>7</sup> For didactical engineering in university mathematics education, see Gonzales-Martins et al. (2014)



opportunity to contrast the convergence behaviour of a sequence and the accumulation points of the underlying set of the sequence, and to contribute to the cognitive development of proof's skills (Dreyfus et al., 2012).

### **A possible scenario for undergraduates or prospective teachers**

The scenario we now present has not yet been implemented but is based on our experience as a university teacher of undergraduates and as a teacher's trainer for prospective secondary mathematics teachers. The analysis of the five proofs in the first section is part of the a priori analysis; by lack of place, we will not go deeper in it in this section, but only indicates the guidelines of the proposal.

Asynchronous work – Reading the paper by Oudot (2017). This paper is online on the French website “CultureMaths” and provides historical and contextual elements.

First activity – In small groups, reading and analysing the excerpt of the proof by Bolzano presented above, including the lemma of §9. Possible questions are:

1. What are the data, the hypothesis, the objects introduced along the proof?
2. In which respect the method developed in the proof is related to the nowadays *proof by dichotomy*?
3. Considering your own knowledge, indicate in which conditions does the lemma of §9 used in the proof apply.
4. According to you, under which theoretical assumptions the proof is valid?

Second activity<sup>8</sup> – In small subgroups – Half of the subgroups works on the two proofs of the sequence version of the theorem provided by Bartle and Sherter (2000); the others work on two proofs of the set version – the one presented in Oudot (2017), and the one by Mammona-Downs (2010). The students are requested to write down an account of their works. Possible questions are:

1. What are the data, the hypothesis, the objects introduced along the proof?
2. Which are the theorems / axioms that are used explicitly or implicitly?
3. Which mode or reasonings do you identify in the two proofs.
4. According to you, under which theoretical assumptions the proof is valid?

Third activity – Collective work

1. Sharing the analysis by oral presentation with slides.
2. Discussing the following questions: where does the assumptions on completeness of the set of real numbers is called for in the different proofs? Are there some intermediate results that would hold in uncompleted ordered fields?

### **Possible further activities**

1. Design and implementation of an algorithm for methods by dichotomy of the proof by Bolzano of the existence of the monotone subsequence and comparison with algorithmic methods in computer science (Meyer & Modeste, 2018).

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<sup>8</sup> An alternative for prospective secondary mathematics teachers, is to ask them to provide proofs of the BWT.

2. Implementation of the proofs on automated theorem provers (Hanna & Yan, 2022) and combining pencil/paper proofs and formal proofs (Narboux et Durand-Guerrier, 2022).
3. Discussing the relationships between organising and operative dimensions in the various proofs (Battie, 2009).
4. Discussing infinity issues: actual *versus* potential infinity; other definition of the infinite set leading to alternative proofs (e.g. Eidolon & Oman, 2017).

## CONCLUSION

In this paper, we have tried to highlight the opportunities offered by the Bolzano-Weierstrass Theorem, an important theorem in Analysis with a lot of applications, to promote both proof skills and understanding of  $\mathbb{R}$ -completeness. We have presented guidelines for a didactical engineering to be further implemented and refined.

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