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A tale of four cities: reflections of master's students in mathematics on a visual word problem

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A group of master's students in mathematics was asked to reflect on a visual word problem. No one of the students identified the correct solution and defended instead an incorrect one. The results are explained by the difficulty of the problem itself (reflecting into the students' overall difficulty in imagining its solution) as well as by the students' tendency to overgeneralize. Interestingly, some of the students reach a contradictory statement (which they do not dismiss or acknowledge as such) as a consequence of the effort to accommodate their own mathematical reasoning with what they perceive to be a normative characterization of the problem coming from the lecturer. I conclude by discussing psycho-pedagogical considerations on imagination and intuition with related issues of university curriculum reform.

Keywords: Teaching and learning of logic, reasoning and proof, Curricular and institutional issues concerning the teaching of mathematics at university level, Visualization, Imagination, Intuition.

INTRODUCTION

While in general research in tertiary education is extending beyond the level of undergraduate studies (Artigue, 2021, p. 14; cf. also Winsløw et al., 2018), there seems to be very little or no research at all on students of graduate programs or courses in pure and applied mathematics (Winsløw & Rasmussen, 2020, p. 883-884) and specifically on master's students in mathematics. This is possibly a consequence of the two-years master's programs in mathematics being a relatively new phenomenon in Europe. Thus, most research on postsecondary mathematics education appears to concentrate either on undergraduate programs/courses in mathematics and related disciplines or else on graduate programs specifically designed for teachers. The present study contributes to this under-researched field by investigating how master's students in mathematics reflect on an unusual visual problem.

THEORETICAL FRAMEWORK AND RESEARCH QUESTION

Research in visualization within mathematics education originated in the work of Alan Bishop and was later carried out by various authors: see Presmeg (2020) for a compendium. In this paper, I will follow the mainstream lineage of research developed by Abraham Arcavi (2003) and Norma Presmeg (2006) albeit explicitly stressing on some hopefully clarifying preliminary definitions inspired from the work of psychologist Efraim Fischbein (1987) as well as from the writings of mathematicians such as Felix Klein, David Hilbert and Henry Poincaré. These definitions will constitute the framework for carrying out the analysis of the case-study presented below. This framework can be understood as a systematization of the traditional

understanding of cognitive steps happening during a working mathematician's process of proving.

Vision may be defined unambiguously as the faculty by which we directly see things which are there for us to see. On the other hand, *imagination* is the faculty by which we see what is not there to see (in mathematics this usually happens in connection with some properties one wants to prove or show). It may be divided into *passive imagination* (the act of representing to oneself something prompted to us from an outside source) and *active imagination* (the act of representing to oneself something not prompted from the outside). Furthermore, *intuition* is the faculty by which we generalize the properties that we see or imagine.[1]

Notice that the definition of intuition given above (essentially derived from Fischbein, 1987) is somewhat more specific than the usual meaning given to the term "intuition" (mostly found within philosophy of mathematics or mathematicians' introspective accounts) which is generally an umbrella term used by authors to characterize any informal way of grasping mathematical truths outside of formal reasoning. Indeed, for the great majority of authors "intuitive reasoning" is nothing but a synonym of "informal reasoning". Notice also that for simplicity and adherence to tradition, I take in this paper a clear a priori distinction between informal and formal reasoning, albeit agreeing with the philosophical stance taken by Giardino (2010) that the two forms of reasoning are really inextricably intertwined. Notice also that the literature has traditionally distinguished between internal and external acts of visualization. Presmeg (2006) assumes this distinction as unproblematic by adopting the Piagetian view that any act of external visualization depends on internal mental images. I do not want to delve into this issue here, but I would like to remark that the distinction must be made at the level of imagination, i.e., the distinction does not concern vision (always external) and intuition (always internal).

To get a concrete grasp of these definitions and to simultaneously give an example of how these can be applied to analyze mathematical processes, let us look at the usual proof of the following proposition: the opposite sides of a parallelogram are congruent to each other. Provided that we indeed know what a parallelogram is, we can draw it (as an act of passive imagination) by tracing two pairs of parallel lines as in Figure 1.a. At this point we can see the parallelogram $ABCD$ as a direct act of vision. Furthermore, in order to prove the proposition, we may (actively) imagine the segment AC (Figure 1.b) and consider the angles that this new segment forms with the lines. We are then able to conclude that angles DAC and ACB are congruent to each other (Figure 1.c) as well as angles CAB and ACD (Figure 1.d). Thus, triangles DAC and ACB are congruent (by known properties of congruence). Therefore, the opposite sides of the initial parallelogram are congruent to each other. Finally, it is by intuition that we realize that

the property thus proved is not linked to the particular parallelogram considered, but holds in general for all parallelograms.

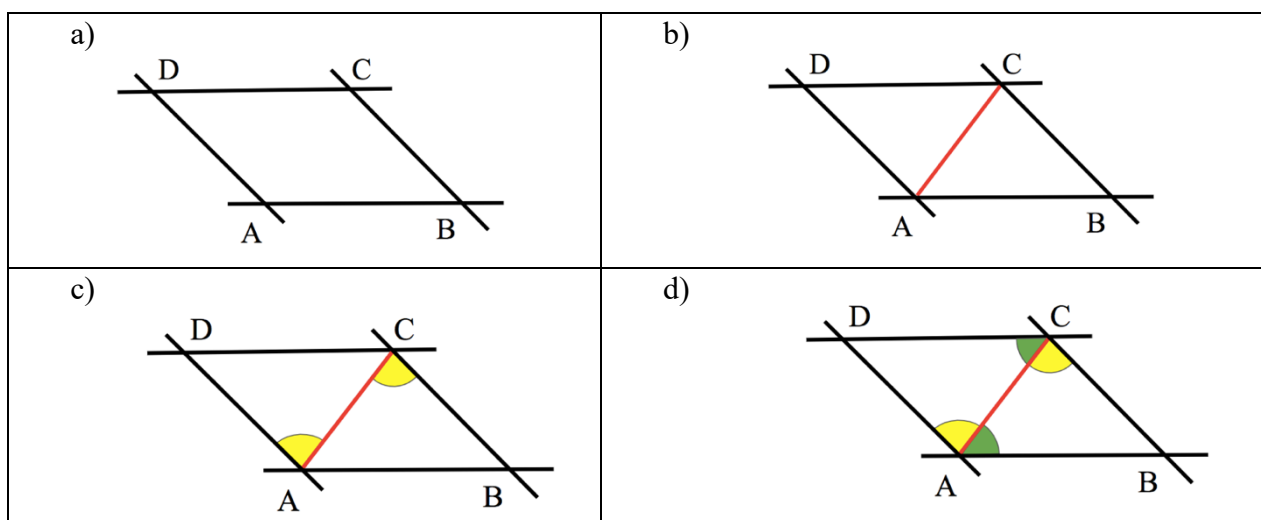


Figure 1: visual steps involved in the proof of the proposition

Now, in the passage above the crucial imaginative step (the proverbial “idea” one must have) is to consider the segment AC and reduce the proof of the proposition to the proof of the congruence of triangles DAC and ACB . In this example the segment or shape that one needs to imagine in order to complete the proof is almost evident to the trained eyes of a mathematician. However, this may not be the case for inexperienced pupils and, similarly, even experienced mathematicians might have trouble when solving a problem involving a difficult imaginative step which is not cued by the figure or diagram naturally representing the problem.

In this paper I will concentrate on a problem of this kind: the four cities problem, which I will describe below. My research question will thus be the following.

How do master’s students in mathematics reason about the four cities problem?

Other than complementing the literature on university mathematics education, answering to this question will also contribute in general to the literature on students engaged in problem solving, which seems to have concentrated primarily on students of compulsory schools (cf. Verschaffel et al., 2020)

THE FOUR CITIES PROBLEM AND RESEARCH CONTEXT

The problem below was given to 28 students enrolled in a master’s program in mathematics at the University of Turin, Italy. This is a competitive program focusing on pure and applied mathematics. The main requirement for entering the program is to have completed a three-year bachelor’s in mathematics with good marks. Such bachelors, in the Italian university system (which does not offer a major-minor arrangement of credits but focuses instead almost entirely on mathematics) usually revolve around learning mathematical content knowledge in the form of theorems and proofs which were customarily tested by means of problem sets usually revolving on the application of these. The problem was given as part of a voluntary assignment

within the students' first course in mathematics education. This is an elective course that students usually take in the first year of the master's program. During the course, other visual mathematical problems were presented, but this problem was the one which caused the most difficulties to the students.

Problem: Four cities are placed at the four corners of a square and an engineer wants to design a road which connects them. What path she has to choose in order to use the least amount of materials?

The problem is equivalent to the problem of finding the minimal path which connects the vertices of a square. The optimal solution to the problem is presented in red in Figure 2.d (modulo a 90-degree rotation), while Figure 2.a, 2.b and 2.c show paths which indeed connect the four cities but are not minimal.[2]

The following hint was given by the lecturer right after the statement of the problem: "the solution is not the path consisting of the square itself" in order to help the students exclude right away the path presented in Figure 2.a. However, this hint may have instead prompted some confusion since a portion of the students interpreted it "normatively", so to speak, as we will see below. After this, no other communication took place between the students and the lecturer.

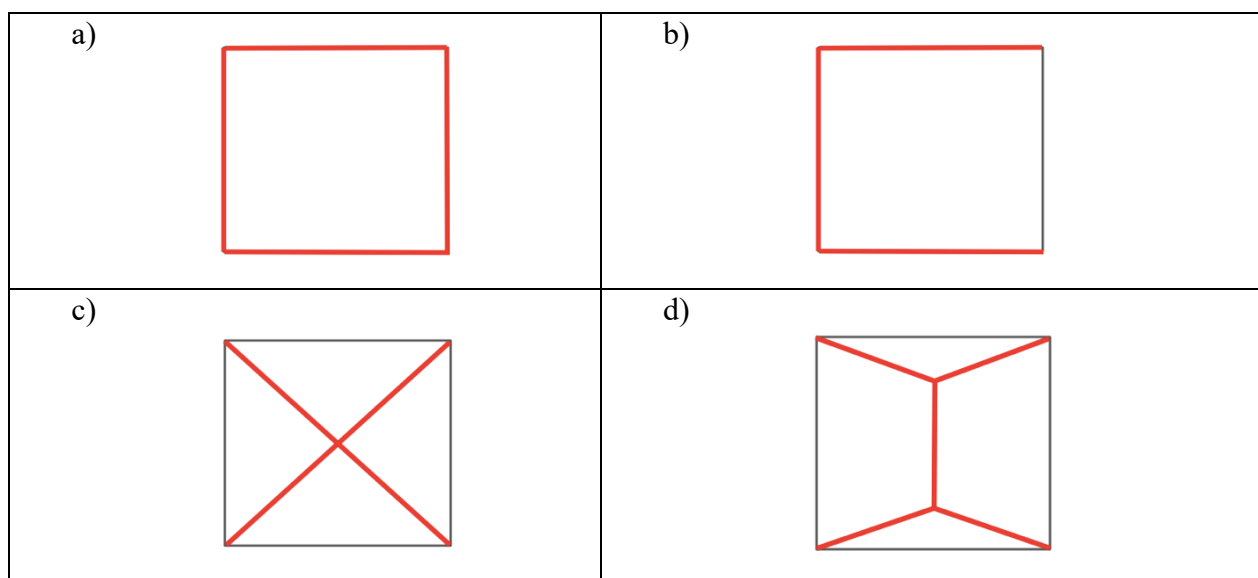


Figure 2: possible paths linking four vertices of a square

The participants were asked to form groups of three or four. In each group one student (the observer) had to write a report observing and elaborating on the way she and her colleagues reasoned about the problem. In the spirit of Arzarello et al. (2002), the students were invited to visualize the problem by means of a dynamic-geometry software (Geogebra) and given about one hour to solve the problem collaboratively.

METHOD

The data for this study consist of the written reports and the Geogebra files produced by the groups. A preliminary categorization of the groups was performed in terms of the final answer they gave to the problem.

Then the data were analysed by means of the framework described above. As seen in the case of the parallelogram theorem, a procedure of proof relying on visual representation can be segmented into steps linked to the given definitions of vision imagination and intuition. In that case, the analysis referred to an unproblematic and correct proof. However, nothing really prevents to apply it to any other correct or incorrect mathematical proof or proof attempt.

Thus, for each group of students, I analysed the respective texts in connection with the Geogebra files they produced in search for instances text and figures signalling acts of vision, imagination and intuition as connected to the logical structure of the argumentation they provided. In particular, the Geogebra files contained traces of both the students' acts of passive imagination (e.g., draw the initial square) as well as to their attempts to concretize products of their active imagination. These in turn were signalled by corresponding textual expressions describing attempts to add to their drawing new lines or figures. Finally instances of acts of intuition were similarly mostly signalled by textual data describing attempts at deduction and generalization.

In presenting the textual data, I translated the relevant passages from Italian as literally as possible.

RESULTS

Only one of the eight groups hinted at the correct solution. However, the students in this group admitted that one of them had already seen the problem before and hence they were excluded from the study. Among the remaining groups, four suggested that the solution was the one depicted in Figure 2.c: “the diagonals”, while three groups suggested that the solution of the problem was the one depicted in Figure 2.a, “the square”, and thus had to conceptually “accommodate” the aforementioned hint, as I will discuss below.

Analysis of two reports concluding that the solution is “the diagonals”

Let us now examine the reports of two representative groups (here called A and B) of the former portion of students. The remaining groups (C and D) had similar reports to Group B. Indeed, Group A drew a square together with its diagonals, and just wrote the following laconic sentence.

The minimal path to unite the 4 cities is through the bisectors of the quadrilateral, given the fact that a straight segment is always the shortest way to unite 2 points.

Here the students enact a false deduction, or an over-generalization as described in (Fischbein, 1987): since the shortest way to unite two points is a straight line, then the shortest way to unite four points is simply two straight lines. This report does not furnish us with any clue as to how they arrived to consider the path consisting of the diagonals, or “the bisectors”, as they say here.

On the other hand, a quotation from Group B's report may let us understand how these other students arrived at the same conclusion: the observer writes that his colleagues

[...] decided to represent the diagonals of the quadrilateral, since they thought that the best idea was that of starting from the properties offered by the quadrilateral [...] they [then] asked themselves if there did not exist a path better than the one just deduced [...] they then decided to construct a second quadrilateral and conjoin the vertices, not by the diagonals, but by segments located in a different way [...] In conclusion both the girls agreed, in light of their reasoning and the tests performed, that the minimal path was the one represented by the diagonals of the quadrilateral.

Thus, the students in Group B chose the diagonals because they were “offered by the quadrilateral” itself. In other words, the imaginative step connected with the decision to conjoin opposite points in the parallelogram example above was, as they seem to mean, suggested or cued by the figure itself.

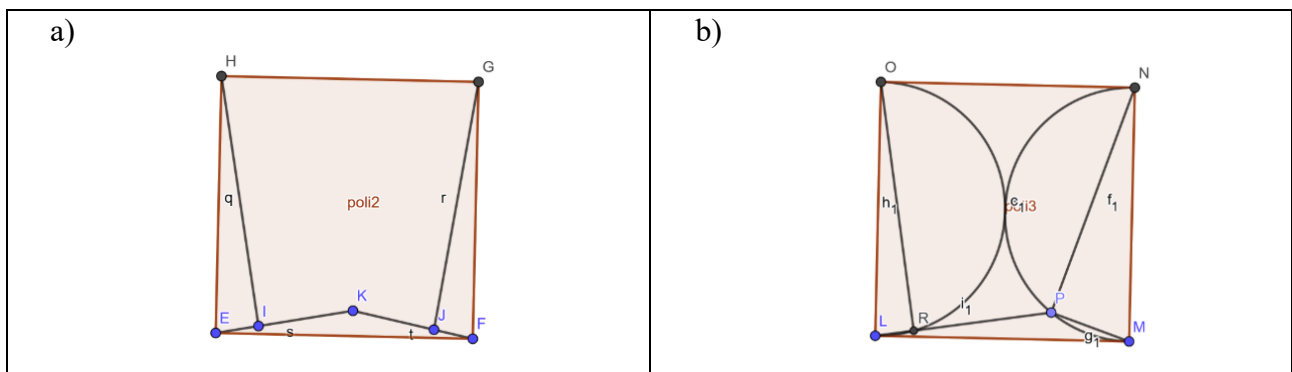


Figure 3: Geogebra protocols from Group B

Notice that these students also in the end overgeneralized as they considered two different configurations (their own drawings in the software are displayed in Figure 3.a and 3.b) and noticed that the path consisting of the diagonals was shorter than those, and then concluded that the former is shorter than any possible path. Interestingly, notice how the path displayed in Figure 3.a is not too distant from a correct solution.

Analysis of two reports concluding that the solution is (approximately) the square

What about the remaining three groups? As said before, the students in these groups stated that the solution the problem was the square itself but were puzzled by the hint which straightforwardly told them that this was not the case. Of course, the hint was specifically given in order to prompt students to think about other non-obvious solutions and help them in identifying the correct one. However, some of the students instead gave an argument for affirming that the solution was the square itself and then, since this had been excluded by the lecturer, proceeded to propose an “approximate solution” to the problem. Let us see how by analysing the reports of two representative groups (here called E and F) of this latter portion of students. The remaining Group G had a similar report as to Group F. Indeed, the observer of Group E wrote that her colleagues considered the diagonals first but then

After some reflection, they discussed on the fact that by considering the diagonals of the square, in order to visit all the cities, they necessarily needed to use one of the sides. Given the impossibility of doing this, they abandoned this idea.

This passage suggests that students in Group E were possibly also imposing to the problem the limitation that in order to visit all the cities a hypothetical traveller must not touch the same city two times (an interpretation which is at odds with the realistic setting within which the problem was presented). In any case, they were convinced that the solution to their interpretation of the problem had to be the square. Since this solution was ruled out by the hint, they then reasoned as follows.

[...] they thought of creating polylines [delle spezzate], not necessarily coinciding with the diagonals, which best approximated the perimeter so that their point of intersection lied on the square's axis. Initially they considered these just on two sides of the square while on the others they considered the diagonals. In order to understand if the minimal path was that formed by the polylines on two sides and the diagonals on the other two or rather was the path consisting of polylines on all four sides they decided [...] to calculate which one was shorter [...] Therefore [...] their final conjecture was that of choosing, as minimal path, the one consisting of polylines which best approximate the square's perimeter.

What happened here? It appears that first the students did not imagine that other paths are possible, and as a consequence this led to a comparison whose result they thus generalized. Indeed, they constructed using Geogebra the two configurations displayed in Figure 4.a and 4.b below, then they calculated their respective perimeters (notice that for Figure 4.a this includes the dotted diagonals) and finally conjectured that the solution should be the latter.

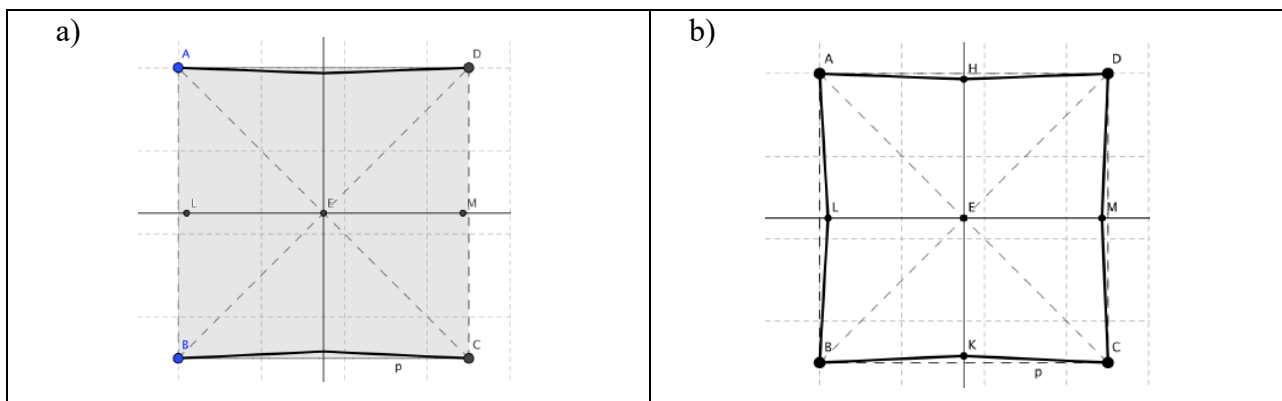


Figure 4: Geogebra protocols from Group E

Furthermore, what is perhaps most striking here is the struggle the students experienced in formulating the consequences of their conjecture. Since the latter points to the fact that the solution should be the square itself, but since also the lecturer had ruled out this possibility, these students are then forced to conclude an impossibility: the minimal path is the one which best approximates the square itself, despite the fact that no such unique path exists!

Similarly, Group F started by considering the diagonals of the square but then

[...] came the idea of creating a square in the centre [of the original square] whose side can vary between 0 and 1 [the side of the square] linking any of its vertex to one and only one city [...]

At this point, they used Geogebra to represent this situation (as displayed in Figure 5 where point R can vary over side QN) and concluded that

[...] point R must be as close as possible to point N for having the minimal path: this means that the two squares must have roughly the same side [...]. The minimal path is given by the approximation of the square having as vertices the four cities.

Here again for these students “the” solution is “the minimal path” consisting of the best approximation to the square itself, despite this being a mathematical impossibility.

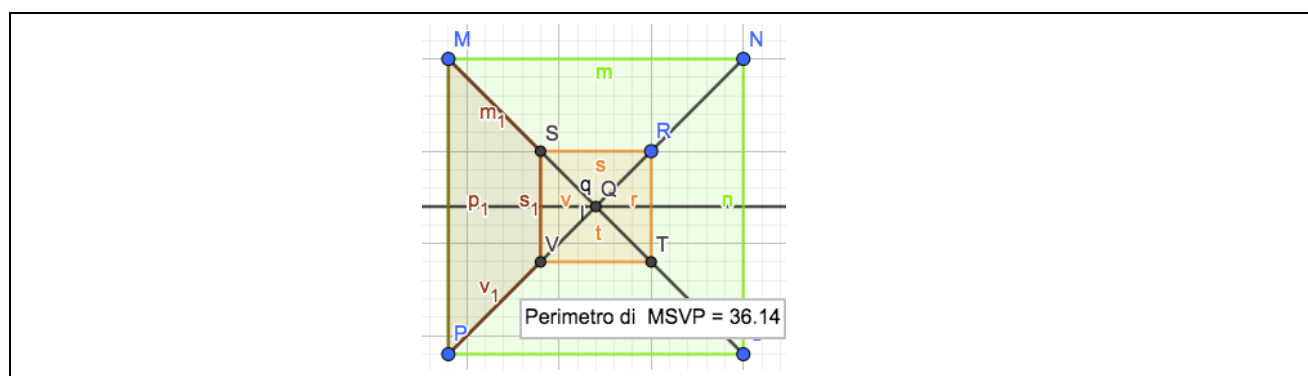


Figure 5: Geogebra protocol from Group F

CONCLUSION

In conclusion, these data may be explained by the difficulty of the problem itself (which in turns reflect into the students’ difficulty to imagine the solution) as well as by the students’ tendency to overgeneralize connected to false deduction. Furthermore, many of the students’ imagination seems to have been crucially impeded by the fact that the square itself cued a false solution: in the words of (some of) the students, the square “offered” the diagonals as the solution for presumably all Groups A, B, C and D. A similar phenomenon is probably safe to assume having played a role also for Groups E, F and G in convincing them that the square itself (or its “approximation”) was the solution. This fact is not surprising perhaps and possibly just tells us something about the difficulty of the problem chosen.

On the other hand, the proneness of master’s students in mathematics to overgeneralize is more interesting given the fact that this phenomenon is present in a form or another in the reports of both the first and the second portions of students. This behaviour could be taken to be desirable and proper when conjecturing. However, the substantial easiness with which these students passed from the particular to the general could be regarded as problematic as pertaining to master’s students of mathematics (i.e., students who supposedly are at the pinnacle of mathematical instruction). As a linguistic observation, notice the awkward usage of the term “deduction” in the quoted passage of Group B, where “deduced” is used as a synonym of “conjectured” or simply “thought of”.

Moreover, with respect to Groups E, F and G, we have seen how the students prefer to conclude an impossibility rather than dismissing their own mathematical reasoning or

rather than dismissing what they perceive to be a normative characterization of the problem coming from the lecturer. This is in itself an interesting phenomenon which could be interpreted as signalling a formal and “ritualistic” way of dealing with mathematical problems. Furthermore, if further replicated with different (but analogous) problems and by means of a larger group of students, similar results may point to the fact that for some university students in mathematics what they perceive to be a correct mathematical argument together with what they perceive to be a normative statement coming from the lecturer has stronger epistemic (or perhaps only cognitive) primacy over acknowledging the contradictory nature of their conclusions.

Finally, I do not mean to understand these preliminary conclusions in purely psychological terms as dependent solely on the students’ internal faculties, them being completely separated from the context in which these students were immersed.[3] On the contrary, the students’ difficulties I have outlined in this paper render possibly evident that the kind of mathematical training to which these students were exposed fails (at least in this case) as a training for problem solving involving a strong conjecturing component. More empirical data agreeing with these results may point to the fact that the type of mathematical rationality into which these students have been steeped in is very different from the kind of rationality which would ideally be that of a working mathematician. Indeed, a deeper and larger study into these phenomena would be required to reach more than tentative conclusions on this matter. Such study could perhaps suggest the need to develop the university mathematics curriculum in favour of a greater exposure of students to problems involving a stronger conjecturing component. It remains however an open question whether such exposure may in general succeed in the training of imagination and intuition and, in general, whether these faculties are susceptible to be trained at all. The latter in turn is a psycho-pedagogical matter over which further investigation would be needed.

NOTES

1. In accordance with the scholarship on visualization within mathematics education (cf. Arcavi, 2003), one can thus generally understand *visualization* in mathematics as all that concerns the faculties/properties/abilities above, i.e., the mode by which we bring mathematical objects at the attention of our senses, we manipulate them and we reflect on them, internally (i.e., in the mind) or externally via some material support, traditionally by hand-drawing on paper or, nowadays, by means of software-generated images.
2. An a priori analysis of the problem would be too long to give here. It is plausible to think that a path very similar to the one which is the solution of the problem must be reached by a unique act of imagination, which arguably seems to not be decomposable into simpler mental actions.
3. For instance, the behavior of Groups E, F and G just summarized may be perhaps explained by a difficulty in reasoning outside of the didactic contract the students assume to be in place (on this concept see Brousseau, Sarrazy and Novotná, 2020), connected to a difficulty in questioning the authority of the teacher.

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