



Transitions in abstract algebra throughout the Bachelor: the concept of ideal in ring theory as a gateway to mathematical structuralism

Julie Jovignot, Thomas Hausberger

► To cite this version:

Julie Jovignot, Thomas Hausberger. Transitions in abstract algebra throughout the Bachelor: the concept of ideal in ring theory as a gateway to mathematical structuralism. Fourth conference of the International Network for Didactic Research in University Mathematics, Leibnitz Universität (Hanover), Oct 2022, Hannover, Germany. hal-04026498

HAL Id: hal-04026498

<https://hal.science/hal-04026498>

Submitted on 13 Mar 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Transitions in abstract algebra throughout the Bachelor: the concept of ideal in ring theory as a gateway to mathematical structuralism

Julie Jovignot¹ and Thomas Hausberger²

¹Haute-Ecole Pédagogique du Valais, Switzerland, julie.jovignot@hepvs.ch;

²Université de Montpellier, France

The concept of ideal, because of its role in the construction of structuralist algebra, is an important entry point for studying the teaching of this field. In this article, we will focus our attention on the management of transitions in abstract algebra (Hausberger, 2018). To do so, we will place ourselves in the framework of the Anthropological Theory of the Didactic (Chevallard, 2000) and provide salient results of praxeological analyses of our corpus. The latter is made up of the teaching material of three French post-secondary teachers. These analyses will allow us to study continuities and ruptures in the praxeologies of the abstract algebra track throughout the Bachelor in France, but also to shed light on the way in which the professors manage the transitions in the development of structuralist praxeologies.

Keywords: teaching and learning of linear and abstract algebra; transition to, across and from university mathematics; structuralist praxeologies; anthropological theory of the didactic; ideal in ring theory.

INTRODUCTION

The issue of transitions has received increasing attention in research on mathematics education, resulting in a recent topical ICME survey (Gueudet, diSessa, Kwon and Verschaffel, 2016). The state of the art of the literature reviewed in the survey underlines different facets (cognitive, epistemological, socio-cultural, institutional,...) of the transitions. The latter are investigated in terms of continuities, discontinuities, ruptures that occur at different transition points (e.g. from school to university) and research-based devices are proposed to accommodate them. Even more recently, Hochmuth, Broley and Nardi (2021) report on the works on this theme carried out within INDRUM.

This paper focuses on transitions across university mathematics courses. Such transitions have been mainly investigated in the context of the analysis path (loc. cit., p. 203-204). In particular, several studies adopt the institutional perspective offered by the Anthropological Theory of the Didactic (ATD; Chevallard and Bosch, 2020) and refer to a model of the calculus to analysis transition introduced by Winsløw (2006). In their concluding section, Hochmuth et al. (2021) raise the question of the situation for the other fields, and, in reference to Winsløw, “what does the observed jump from the first to the third stage of the model mean in a longer-term perspective, e.g. taking into account what students learn in more advanced mathematics studies?” (p. 210).

Our goal is to investigate these questions in the context of the abstract algebra path, with an epistemological and institutional lense. It is based on the perspective opened

up by Hausberger (2018): the concrete to abstract transition identified by Winslow in analysis is generalized in the form of the wider perspective of the teaching and learning of mathematical structuralism at large - in ATD terms, the development of structuralist praxeologies. A new three stage model has been proposed by Hausberger and applied to abstract algebra in a small-scale pioneering study. A larger-scale study, centered on second and third year post-secondary teaching practices in France and Switzerland, is carried out by Candy (2020a) in her PhD project with a focus on the concept of ideal in ring theory. This choice is motivated by the role played by the concept of ideal (Corry, 2004, p. 15): its central importance for the theory of abstract rings and, even more, for the “rise of structures” due to its strong interconnections with other algebraic concepts (fields, modules, groups, etc.). A model for such praxeologies taught at the second year of post-secondary studies in France has been presented at INDRUM2020 (Candy, 2020b).

This paper reports on the results of the PhD project that connect to the issue of transitions, in France. It addresses the following research questions: What continuities and ruptures can be observed in the praxeologies of abstract algebra that involve the concept of ideal, as they are taught throughout the Bachelor in France? How are transitions in the development of structuralist praxeologies handled by abstract algebra teachers? Both questions are related since our methodology is based on the analysis of teaching material provided by selected teachers, under the light of Hausberger’s model. It also refers implicitly to didactic transposition processes (Chevallard, 2020), but the discussion of conditions and constraints that explain the observed states of equilibrium within institutions are out of the scope of the paper.

We begin by presenting our theoretical framework and the model, and then outline the methodology for analyzing the data. We illustrate the methodology through its application to selected excerpts from exercise sheets. Then, we discuss the results obtained in relation to the research questions, before concluding with the highlights of the study and prospects for further development.

THEORETICAL FRAMEWORK

Hochmuth et al (2021) highlight the following main features of ATD that justify its frequent use in research on transitions: the consideration of knowledge as living within *institutions*, the institutionalization of knowledge seen as the result of complex processes of *didactic transposition* subject to a set of conditions and constraints at various levels, and finally the central 4T-model of *praxeologies* (task, technique, technology, theory) that allows researchers to build reference models of the knowledge to be taught for application to teaching-learning phenomena. We direct the reader unfamiliar with ATD to the mathematics education encyclopedia (Chevallard, 2020) for an introduction to these notions and will focus the rest of our discourse on the additional tools, specific to the point of view of mathematical structuralism, that have been developed.

Structuralist praxeologies and their levels. The starting point is the consideration of mathematical structuralism as a methodology, which consists of reasoning in terms of classes of objects, relations between these classes and stability properties for

operations on structures (Hausberger, 2018). The general view of structures thus allows particular properties of objects to be demonstrated by making them appear as consequences of more general facts (theorems about structures). Dually, generalizations are put to the test of objects, hence a *dialectical relationship between objects and structures*. In praxeological terms, the simplification produced lies in the passage from a praxeology $P=[T/?/?/\Theta_{\text{particular}}]$ where it is unclear which technique to apply, to a *structuralist praxeology* $P_s=[T^s/\tau/\theta/\Theta_{\text{structure}}]$ where, modulo generalization of the type of task (T^s), the theory of a given type of structure guides the mathematician in solving the problem. Furthermore, Hausberger (2018) distinguishes several levels of structuralist praxeologies: at *level 1*, structures act as a vocabulary and appear mainly through definitions (e.g., the type of tasks “prove that a ring A is a principal ideal domain (PID)” is solved by showing, by hand, that the definition is satisfied, i.e. that any ideal is monogeneous); at *level 2*, the technique used mobilizes general abstract results about structures (on our example, one shows the existence of an Euclidean algorithm, which invokes in the logos of the praxeology the structuralist theorem that any Euclidean ring is a PID).

Transitions in the development of structuralist praxeologies. Following Winsløw, Hausberger (2018, p. 89) proposes a three-phase model (Figure 1): while the first type of transition amounts to going from P to P_s , the second type leads to praxeologies whose entirety of praxis and logos lies in the abstract. To unfold our example, the student then encounters tasks like “show that a Noetherian integral domain such that any maximal ideal is principal is a PID”.

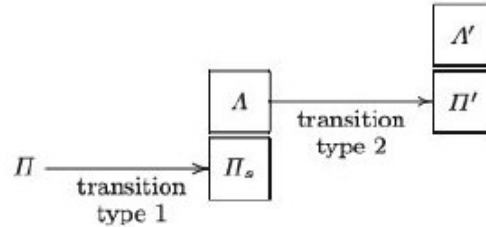


Figure 1: a model for transitions in the development of structuralist praxeologies

Contextualization and decontextualization of structuralist praxeologies in relation to the dialectic of objects and structures. In the example given, the ring A plays the role of a *didactic variable* of the type of tasks: the structuralist praxeology is thus *contextualized to domains of mathematical objects*, whose variation is crucial to lead - in fine - to a *decontextualization* (the ring A is defined *abstractly*). We will be particularly attentive to the choices made by university teachers in relation to these didactic variables which are essential to operate the objects-structures dialectic.

METHODOLOGY

In order to shed light on didactic choices that concern transitions in abstract algebra throughout the Bachelor, we conducted a case study of 9 university teachers considered representative of 5 teaching levels, in France and Switzerland (Candy, 2020a). In this article, we will rely on data from three teachers: MP1, EC2 and EC4. MP1 teaches mathematics in the second year of the Classes Préparatoires aux Grandes Ecoles Mathématiques-Physiques (CPGE-MP; these are classes reserved for

the best students who are destined to enter the French Grandes Ecoles), EC2 teaches in the second year of the Bachelor's degree in a 7.5 ECTS course called “Linear Algebra” and EC4 teaches in the third year of the Bachelor's degree in a 5 ECTS course called “Elements of Ring and Field Theory”. EC4 teaches in the same university as EC2; moreover, some of the students of the third year of the Bachelor's degree come from a CPGE-MP. Thus, in this article, we can study the transitions through two possible curricula experienced by those third year Bachelor students.

In our study, we began by conducting an ecological analysis of the official syllabi in order to bring to light the places where the concept of ideal lives. Then, our praxeological analysis of the course documents (lecture notes and tutorial sheets) consisted in highlighting the praxeologies that mobilize the concept of the ideal within the exercise sheets of the corpus. When the exercises were not corrected, we used the correction of exercises of the same type of tasks present in the institution. We took care to link the exercises to the contents of the lectures, which allow to identify the global organization of the praxeologies (their unification by common technologies or theories, within themes or sectors of study) and to provide certain technological and theoretical elements that remain partially implicit in tutorials.

Finally, the structuralist level of the praxeologies has been carefully noted, as well as the choice of the didactic variables of contextualization of the structuralist praxeologies. The aim is to analyze the continuity and rupture that can be observed in relation to the two types of transitions described in the model, under the hypothesis that the type I (epistemological) transition would be situated at the level of the (institutional) transition between the second and third year of the Bachelor's degree, while the type II transition would be linked to that of the Bachelor's to the Master's degree.

PRAXEOLOGICAL ANALYSES

In this section, we illustrate our analytical tools on salient excerpts from the corpus while reporting on the main findings of our analyses. The discussion of the results in order to answer the research questions will be the subject of the next section.

CPGE-MP: the MP1 corpus

The analysis of the official program of CPGE-MP¹ allows to identify three main habitats of the concept of ideal. The latter is introduced by its general definition in the sector “common algebraic structures”, at the level of the theme “ideals of a commutative ring” where it is linked to the notion of ring homomorphism (as kernel) and to the notion of divisibility (inclusion of ideals), then illustrated in the case of the ring \mathbf{Z} . It is then taken up again at the level of the theme “rings of one-dimensional polynomials” of the same sector, through the item “ideals of $K[X]$ ”. One can note that the program does not mention the principality property (of an ideal, of a ring) and that it does not underline the analogy between the arithmetic of \mathbf{Z} and that of $K[X]$ (which follows from the principality). Nevertheless, the implicit organization of the contents is based on this analogy. Finally, the concept of ideal is mentioned in the

¹ https://cache.media.education.gouv.fr/file/special_1_MEN_ESR/42/4/MP-mathematiques_287424.pdf

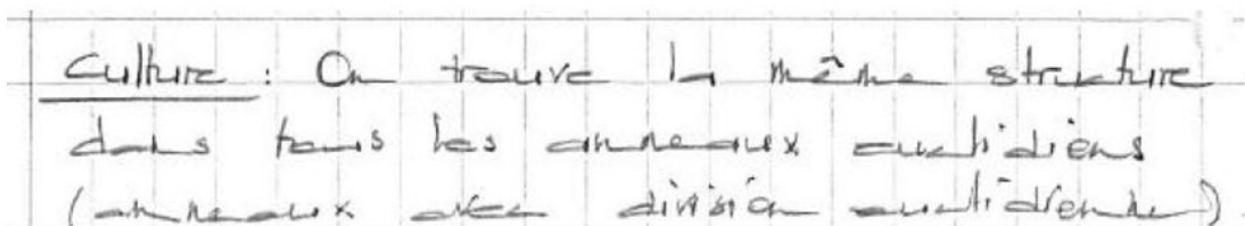
theme “polynomials of an endomorphism, of a square matrix” of the sector “reduction of endomorphisms and square matrices” where the properties of ideals previously studied allow to justify the existence of the minimal polynomial.

The study of the objects at stake in the exercise sheets shows a diversity of contextualizations, although limited to numbers and polynomials. For the principal ideal domains (PID), the classical examples \mathbf{Z} and $K[X]$, quoted in the official program, are mainly worked on through their arithmetic (definition of gcd and lcm in terms of ideals) and the consequences in linear algebra of the principality of $K[X]$. The ring $\mathbf{Z}[X]$ is studied as a non-example of a PID. Finally, MP1 has chosen to introduce $\mathbf{Z}[i]$ (the Gaussian ring of integers, whose historical importance in the development of abstract algebra is well known) and the set \mathbf{D} of decimal numbers to work on the principality of Euclidean rings on less classical examples.

The analysis of the tasks shows that the students' work is mainly situated at level 1 of the structuralist praxeologies. Thus the type of task T_1 (to demonstrate that a subset I of a ring A is an ideal), present in 2 occurrences, gives rise to a praxeology whose technology is based on the definition of an ideal and the type T_2 (to demonstrate that a given Euclidean ring A is a PID, 4 occurrences), in spite of the genericity of the technique, proceeds by hand from the definition of principality. It is at the level of a meta-discourse that the teacher underlines the analogy between the two contexts and the generality of the method, without going so far as to quote a structuralist theorem (figure 2). Indeed, the theorem in question is not on the syllabus; its status is that of a cultural element and the definition of a Euclidean ring is not formalized. Only one occurrence of T_1 leads to a level 2 structuralist praxeology where the technique uses the structuralist theorem characterizing ideals as kernels of ring homomorphisms.

21. Montrer que l'anneau $\mathbf{Z}[i]$ des entiers de Gauss est principal.

22. Montrer que l'ensemble des rationnels du type $10^n x$ avec $n \in \mathbf{Z}$ et $x \in \mathbf{Z}$ est un anneau principal.



Culture : On trouve la même structure dans tous les anneaux euclidiens (anneaux à division euclidienne).

Figure 2: example of a structuralist theorem that remains implicit

The abstract tasks, 3 in total, are situated within the same exercise devoted to the notion of radical \sqrt{I} of an ideal I of a ring A : it is proved to be an ideal (T_1) then appears the type of task T_7 (to prove properties of operations on ideals). This last type, introduced during the lectures on the gcd and lcm (addition and intersection of ideals), is carried out here in an abstract context, about a new operation whose behavior with respect to the two previous ones is studied (e.g. $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$), to finally be contextualized to \mathbf{Z} through the task of determining the radical of an ideal of \mathbf{Z} . These tasks show a dialectic between contextualization and decontextualization, since the general formula may be used to reduce to computing the radical of prime ideals of \mathbf{Z} . It is a local implementation of the dialectic between objects and

structures in the sense of Hausberger (2018), but the notion of radical remains weakly motivated.

Second year of the Bachelor: the EC2 corpus

We notice that the concept of ideal does not appear explicitly in the official syllabus of EC2. However, the same niches as in the case of MPSI-MP are likely to be invested, since the arithmetic of polynomials and the reduction of endomorphisms are part of the study program. Although he starts his course with a chapter “small panorama of algebraic structures” (like MP1), EC2 chooses to introduce the ideal concept at the level of the theme “arithmetic of $K[X]$ ” of the sector “the algebra $K[X]$ ”, which constitutes its main habitat, with as niche the principality of $K[X]$ and the reformulation of the gcd (defined from the divisibility relation) in terms of ideals. Not surprisingly, these results are subsequently applied to the theme “polynomials of endomorphisms” of the sector “reduction of endomorphisms”.

Of the 7 tasks on ideals contained in the tutorial sheets, only two are contextualized (to $K[X]$, one to prove principality and the other to prove the existence of the lcm of two elements a and b of $K[X]$, via the introduction of a generator of the ideal $(a) \cap (b)$). They appear as isolated tasks aimed at proving theoretical elements stated in the course. We identify a single proper praxeology in this corpus, generated by the type of task denoted previously T_1 . Its 4 occurrences are all decontextualized: it consists in proving that the sum and intersection of ideals is still an ideal, starting with the case of principal ideals. This shows a deficit of the objects-structures dialectic. A last abstract task is given in connection with the reduction of endomorphisms: it is to prove that the kernel of a homomorphism of algebras is an ideal. It shall be noted that the definition of a ring homomorphism (and its intimate link with the notion of ideal) are not mentioned in the course.

Third year of the Bachelor: the EC4 corpus

The syllabus of EC4 is quite detailed: it includes both a large panel of concepts (ideal, ideal generated by a subset, quotient rings, prime/maximal ideal, PID/euclidean ring/unique factorization domain) and the study of their properties in a structuralist perspective (behavior of ideals under homomorphisms, isomorphism theorems), from which structuralist theorems result (e.g. characterization of prime and maximal ideals by quotient properties), but it also mentions specific examples that must be treated (\mathbb{Z} , $K[X]$, $\mathbb{Z}[X]$, $\mathbb{Z}[i]$). We thus find the paradigmatic examples used by MP1 and the perspective of unification between the contexts of numbers and polynomials, which gives meaning to the abstract theorems. Unlike MP1, the learning goals are organized around the structures, it is no longer the numbers or the polynomials that are put to the fore.

EC4 reintroduces in his course all the basic notions related to ideals even if EC2 had already introduced some of them. The structure of the course allows to link the 12 exercises dealing with ideals to the 4 following sectors: “ideals and quotients”, “polynomials and ideals”, “prime and maximal ideals”, “operations on ideals”. Considering their complexity (we have identified 41 different tasks), we can only

sketch the corresponding praxeologies and refer the reader to the thesis manuscript (Candy, 2020a, appendix A5, p. 400-410) for a detailed description.

Of these 41 tasks, 18 use a definition and 23 use a theorem about structures. This highlights EC4's didactic intention to reach a structuralist level 2. More precisely, the praxeologies mobilizing prime or maximal ideals (for example, the praxeology generated by T_{18} : show that an ideal I of a ring A is prime, figure 3) mostly use as technology the isomorphism theorems or the Chinese theorem; they thus work at a level 2. On the other hand, those concerning the principality of ideals (e.g. T_{13} : show that an ideal I of a ring A is principal) have as technology the definition of a principal ideal and are thus mostly done "by hand". Indeed, they are mostly situated in the sector "polynomials and ideals" and are contextualized to $K[X]$. We do not note any subsequent development of the praxeology at the structuralist level 2, when the principality of Euclidean rings is known.

7. Idéaux de $\mathbb{Z}[X]$. On travaille dans l'anneau $\mathbb{Z}[X]$.

- 1) Montrer que l'idéal (2) est premier mais pas maximal.
- 2) Montrer que l'idéal (X) est premier mais pas maximal.
- 3) Montrer que l'idéal $(2, X)$ n'est pas principal. Montrer que c'est un idéal maximal.
- 4) Montrer que l'idéal (X^2+1) est premier mais pas maximal. Reconnaissez-vous le quotient $\mathbb{Z}[X]/(X^2+1)$?

Figure 3: illustration of T_{18} in EC4's corpus

Of these 41 tasks, 10 are abstract tasks. Within the corpus, we find occurrences of task types that are first contextualized and then decontextualized in order to demonstrate a general result (but not a generalization of the former contextualized occurrences). This is the case, for example, of T_{18} , contextualized twice in $\mathbb{Z}[X]$ in exercise 7 (figure 3), before being posed to the case of the inverse image of any prime ideal by a ring homomorphism, and then to the case of an ideal P defined abstractly by a condition that may be interpreted as a rewriting, in the set-theoretic terms of ideals, of the definition of a prime ideal (figure 4).

8. Idéaux premiers. Soit A un anneau et P un idéal de A avec $P \neq A$. Montrer que P est premier si et seulement si pour tous I, J idéaux de A on a : $IJ \subset P \Rightarrow I \subset P$ ou $J \subset P$.

Figure 4: one occurrence of T_{18} , decontextualized

The corresponding praxeology is of structuralist level 1, unlike the one applied to $\mathbb{Z}[X]$; we can regret that the usefulness of these results is not highlighted by new contextualizations, in the spirit of the dialectic between objects and structures. The other abstract tasks are related to the discussion of the consequences of the presence of invertible elements in an ideal, to the determination of the ideals of a Cartesian product of rings and its consequences for a quotient of such a product. These are therefore theoretical results which are intended to feed contextualized structuralist praxeologies of level 2. For example, the question of the determination of the ideals

of \mathbf{Z}^2 is posed in application of the principality of \mathbf{Z} and the results on the Cartesian product. A praxeology generated by T_{13} is thus obtained, at structuralist level 2.

DISCUSSION

	Summary of main characteristics	Type transition 1	Type transition 2
MP1	Praxeologies mainly contextualized and limited to the structuralist level 1; a relative diversity of contexts (\mathbf{Z} , $K[X]$, $\mathbf{Z}[i]$, \mathbf{D})	Spotted preparation (via meta discourse)	Absent
EC2	A single type of tasks dealt with at structuralist level 1 and in an abstract context; the objects-structures dialectic is extremely weak	Absent	Absent
EC4	Work at structuralist levels 1 and 2; decontextualized instances of praxeologies are introduced to establish structuralist properties of ideals and punctually serve to develop contextualized praxeologies of structuralist level 2	At the center of the course	Spotted preparation (implicit)

Table 1: summary of the main results

As suggested by the analysis of the syllabi, the type 1 transition does not appear as a learning goal in the CPGE-MP and second year Bachelor institutions under study. Accordingly, when tasks are contextualized to object domains, mainly $K[X]$ and \mathbf{Z} , the technique consists in applying the definitions of concepts without relating the properties at stake to general results. Thus, the structuralist praxeologies involved are all level 1 and the objects-structures dialectics remains largely invisible in these institutions.

Nevertheless, MP1 chose to introduce a relative diversity of examples of PIDs which are all Euclidean, and he uses meta discourse to allude to the underlying general principle (a structuralist theorem). This didactic gesture may be considered an intention to facilitate the type 1 transition. At university, EC2 introduces structures as a “vocabulary” and assigns abstract tasks on basic formal properties of ideals. The type 1 transition is therefore envisaged from a top-down perspective: although the course content is organized around domains of objects (polynomials, matrices, endomorphisms), concepts are introduced beforehand and taken as a given. This strategy may be questioned since it may be argued that the resulting level 1 structuralist praxeologies will tend to be weakly motivated.

It is from the third year of the Bachelor's degree onward that the type 1 transition appears as a real objective. We have seen, in the EC4 course, that the syllabus is organized around structures. Moreover, EC4 deploys praxeologies, in numerous contexts, which mobilize the concept of ideal at a structuralist level 2, notably around prime/maximal ideals and the isomorphism theorems or the Chinese theorem. The objects-structures dialectic is at play and assigned abstract tasks punctually serve to subsequently develop contextualized level 2 structuralist praxeologies.

Type 2 transition does not appear as a learning objective in the institutions under study. Abstract tasks are rare in CPGE-MP and restricted to the first basic properties

of ideals in the second year of Bachelor. At the third year, EC4 assigns tasks to explore structuralist properties of ideals (e.g. behavior under inverse image, Cartesian product). Nevertheless, the main application is the enrichment of contextualized level 2 structuralist praxeologies. A body of abstract knowledge in which ideals play a major role (e.g. Elimination Theory, Algebraic Geometry) is not in the horizon. However, the scope of exercise 8 (figure 4) may be connected to Algebraic Geometry. The teacher probably had this connection in mind, but it remains invisible to the students. Thus, the type 2 transition is left to the Master's degree.

Our analyses are limited to the case of three professors chosen as representatives of their respective institutions. However, Candy (2020a) provides results on a larger corpus that support these analyses. In the context of this article, we can provide some initial answers to our research questions.

To answer the question of continuities and ruptures which can be observed in the abstract algebra track around the concept of ideal through the Bachelor in France, it seems important to us to recall that the students who take the course of EC4 could have first followed the course of EC2 or that of MP1. However, the treatment between EC2 and MP1 appears different. In both cases, the praxeologies involved are of structuralist level 1. But, if EC2 chooses to work on praxeologies in a decontextualized way, MP1 chooses to work on contextualized praxeologies and to accompany the type 1 transition by meta-discourse. Thus, students from the MP1 course could be better prepared for the upcoming transition since the structuralist perspective is pointed out as an horizon.

Structuralist praxeologies at the third year of Bachelor are either of level 1 or 2. However, discontinuities in the type 1 transition can be noted since structuralist praxeologies are most of the time elaborated from theoretical elements provided a priori, in a top-down perspective. The analogy between the arithmetic of numbers and polynomials, carefully developed by MP1 in a bottom-up perspective, remains a missed opportunity to develop a structuralist praxeology in the EC4 course, since all students do not share such a background. Finally, the transition of type 2 is not worked out; it would be necessary to analyze a corpus of teaching material at the Master's degree to measure the epistemological gap in praxeological terms.

As for the management of transitions by teachers, our study tends to show, for both type 1 and type 2 transition, that it is the objects-structures dialectics which is central in the management of transitions by teachers. For the type 1 transition, this can be done through meta-discourse which deals with the identification of a technique present in the structuralist level 1 praxeologies which could be generalized to a structure class in order to create a technological element of level 2 (figure 2, MP1). The type 1 transition can also be managed through the choice of variables contextualizing the tasks which favour the use of level 2 praxeologies and make level 1 praxeologies costly. This is the case, for example, of EC4 in the framework of prime and maximal ideals (figure 4). Finally, for the type 2 transition, we have seen that it can be punctually favored by a set of contextualization then contextualization as in figures 3 and 4 of the EC4 corpus.

GENERAL CONCLUSION AND PERSPECTIVES

This analysis allows us to highlight the objects-structures dialectic as a nodal point of the management of transitions in abstract algebra. The interplay of contextualizations and decontextualizations allows the passage from structuralist praxeologies of level 1 to 2 by recognition of a unifying technology on contextualized praxeologies (type 1 transition); then, decontextualization offers an opportunity to engage in abstract tasks with familiarity gained from contextualized praxeologies (type 2 transition).

The phenomena observed in this reduced corpus would benefit from being tested in the context of a larger corpus, as is the case with Candy (2020a), so as to be able to argue for the presence of dominant praxeological models within institutions. They are also to be related to the ecological study of the conditions and constraints that are exercised at the level of the different institutions, in order to shed light on the states of equilibrium reflected in these dominant models. Moreover, our results suggest that this equilibrium is likely to be unstable with respect to the management of transitions in structuralist algebra, and this instability should also be investigated.

REFERENCES

- Candy, J. (2020a). *Etude de la transposition didactique du concept d'idéal : écologie des savoirs et problématique de l'entrée dans la pensée structuraliste en France et en Suisse romande* [doctoral thesis]. University of Montpellier.
- Candy, J. (2020b). Etude de l'enseignement du concept d'idéal dans les premières années post-secondaires : élaboration de modèles praxéologiques de référence. In T. Hausberger et al. (Eds.), *Proceedings of INDRUM2020* (pp. 308-317).
- Chevallard, Y., & Bosch, M. (2020). Anthropological Theory of the Didactic (ATD). In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 53-61). Springer.
- Corry, L. (2004). *Modern Algebra and the Rise of Mathematical Structures*. Birkhäuser.
- Gueudet, G., Bosch, M., diSessa, A. A., Kwon, O. N., & Verschaffel, L. (2016). *Transitions in Mathematics Education*. Springer.
- Hausberger, T. (2018). Structuralist praxeologies as a research program on the teaching and learning of abstract algebra. *International Journal of Research in Undergraduate Mathematics Education*, 4(1), 74–93.
<https://doi.org/10.1007/s40753-017-0063-4>
- Hochmuth, R., L. Broley, & E. Nardi (2021). Transitions to, across and beyond university. In V. Durand-Guerrier et al. (Eds.), *Research and Development in University Mathematics Education: overview produced by INDRUM* (pp.193-215). Routledge.
- Winsløw, C. (2008). Transformer la théorie en tâches : la transition du concret à l'abstrait en analyse réelle. In A. Rouchier & I. Bloch (Eds.), *Actes de la XIIIème école d'été en didactique des mathématiques* (cd-rom). La Pensée Sauvage.