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Issam Chekakta, Dalel Jabri, Koffi Motchon, Kevin Guelton, Djamel E C Belkhiat. Design of asynchronous switched Takagi-Sugeno model-based  $H_\infty$  filters with nonlinear consequent parts for switched nonlinear systems. *International Journal of Adaptive Control and Signal Processing*, 2023, 37 (6), pp.1511-1535. 10.1002/acs.3588 . hal-04026284

HAL Id: hal-04026284

<https://hal.science/hal-04026284v1>

Submitted on 12 Jul 2024

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# Design of asynchronous switched Takagi–Sugeno model-based $H_\infty$ filters with nonlinear consequent parts for switched nonlinear systems

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## Summary

This paper investigates the design of asynchronous switched nonlinear  $H_\infty$  filters for a class of continuous-time nonlinear switched systems with mismatching switching laws and  $L_2$  norm-bounded disturbances. In this context, the switched nonlinear system is modelled as a switched Takagi–Sugeno (T-S) model with nonlinear consequent parts, that is, where unmeasured nonlinear terms are kept in the nonlinear consequent parts in order to circumvent the occurrence of unmeasured premise variables, which is usually faced in conventional T-S modeling without nonlinear consequent parts. In this framework, asynchronous switched T-S filters with unmeasured nonlinear consequent parts are proposed to estimate unmeasured and/or disturbed system's outputs, even when the filter's switching law mismatches the switched nonlinear system's one, which can be in practice unknown or imprecisely measured. Based on a candidate multiple Lyapunov function, combined with a  $H_\infty$  criterion, conditions are proposed in terms of Linear Matrix Inequalities for the design of the considered asynchronous switched T-S filters with unmeasured nonlinear consequent parts. Compared with previous related works, these conditions have the advantage of being dwell-time-independent and less conservative, thanks to the incremental quadratic constraints employed to deal with the unmeasured nonlinear consequent parts. Furthermore, acknowledging that T-S models are only representing nonlinear ones on subsets of their state space, an optimization procedure to estimate the filtering error's domain of attraction is developed. Two illustrative examples are considered to validate the proposed results. An academic one is presented to illustrate the improvements brought in terms of conservatism by the proposed switched T-S filter design methodology with regards to previous related studies. Then, a case study illustrates the effectiveness of this proposal from a switched nonlinear mass-spring system inspired from related literature.

## KEYWORDS

asynchronous switching laws,  $H_\infty$  filters' design, LMI-based conditions, nonlinear consequent parts, switched Takagi–Sugeno models

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## 1 | INTRODUCTION

Over the last decades, switched systems have attracted widespread attention since they provide a convenient description for various systems that exhibit switching phenomena, where these systems can be seen naturally as a combination of continuous dynamics and discrete events. In this context, we can quote as examples, power electronics, air traffic control, robotics, chemical processes, as well as other systems in various fields. A look at the literature shows that a large number of studies related to many issues have been performed, such as modeling, stability analysis, state estimation, filtering, and so on (see e.g., References 1–7). However, many physical switched systems are not linear by nature, that is, involving switched nonlinear behaviors. Fortunately, an attractive framework to deal with nonlinear systems, namely Takagi–Sugeno (T-S) models,<sup>8</sup> allows to describe nonlinear systems,<sup>9</sup> and by extension switched nonlinear systems.<sup>10–16</sup> Indeed, each nonlinear switched mode can be represented by a T-S model remaining on smoothly weighted combinations of vertices, valid in a compact set of the whole state space. Thanks to their polytopic convex structure, the T-S framework allows extending some of the linear control concepts to the nonlinear cases (see e.g.<sup>9,17–19</sup>).

Among the issues under consideration in different fields of application, there is a growing interest in  $H_\infty$  filtering techniques since such filters are particularly useful to estimate or filter some of the system's outputs, which cannot be measured or affected by noise disturbances.<sup>20–24</sup> Compared with other filtering approaches such as Kalman filters, the main interest of  $H_\infty$  filters are their robustness against unmodeled dynamics as well as they don't require any knowledge on the noise disturbance signals, excepted some bounded-energy condition.<sup>20</sup> In this context, several studies have been carried out to design  $H_\infty$  filters. For example, the authors in<sup>25</sup> have dealt with the problem of robust  $H_\infty$  filtering for a class of T-S fuzzy neutral systems with time-varying delays and parameter uncertainties. A delay-dependent sufficient condition for the design of the filter matrices has been formulated in terms of LMIs. The premise variables have been considered measurable in this study. In the same way, a robust  $H_\infty$  filtering for T-S sampled-data fuzzy systems with uncertain parameters has been designed in Reference 26. Sufficient conditions ensuring the convergence of the filtering error with  $H_\infty$  disturbance attenuation have been given in terms of LMIs. Two cases have been considered in this study depending on whether the premise variables are measurable or unmeasurable. Moreover, the design of a robust  $H_\infty$  filtering for a class of T-S fuzzy systems with time-varying delay have been investigated in Reference 27. Based on a Lyapunov–Krasovskii functional in the delta domain, sufficient conditions ensuring the convergence of the filtering error were established in terms of LMIs in the case when the premise variables are known. Recently, the authors in<sup>28</sup> investigated the design of an event-triggered fuzzy filter for vehicle sideslip angle estimation. The nonlinear dynamics were modeled as an uncertain T-S fuzzy model. By establishing an augmented Lyapunov–Krasovskii functional candidate, the convergence conditions of the robust asynchronous fuzzy filter have been formulated in terms of LMIs. In addition, the authors in Reference 24 investigated the design of a delay-dependent  $H_\infty$  filter for nonlinear systems with a time varying delay described by T-S fuzzy models. Based on a nonquadratic Lyapunov–Krasovskii functional, sufficient conditions ensuring the convergence of the robust  $H_\infty$  filter with measurable premise variables have been formulated in terms of LMIs. Moreover, the design of a distributed fuzzy  $H_\infty$  filter for a class of nonlinear systems interconnected over an undirected graph has been investigated in Reference 29. T-S fuzzy model of each subsystem was developed. By applying membership-dependent multi-Lyapunov functions, a sufficient condition was given to obtain the desired fuzzy filter parameters by solving a set of LMIs.

Furthermore,  $H_\infty$  filter design for switched systems has been the subject of recent investigations. The design of an event-driven asynchronous  $H_\infty$  filter for a class of networked switched linear systems has been studied in Reference 30 under constrained switching. Some sufficient conditions were established in terms of LMIs to obtain the desired filter parameters by using a Lyapunov–Krasovskii functional and the average dwell time approach. In the same way, the design of an event-triggered  $H_\infty$  filter for networked switched linear systems subject to stochastic cyber attacks has been studied in Reference 31. Based on Lyapunov functions and average dwell time approach, a set of LMI conditions have been given to guarantee the realization of  $H_\infty$  performance of the filtering error system. In Reference 11, the  $H_\infty$  filtering problem for a class of switched T-S fuzzy systems under constrained and asynchronous switching has been investigated. Using multiple Lyapunov functions approach and mode-dependent average dwell time technique, some sufficient conditions ensuring the convergence of the filtering error with a weighted  $H_\infty$  performance index have been formulated in terms of LMIs. Moreover, the premise variables are considered measurable. The design of a synchronous  $H_\infty$  filter for a class of switched T-S fuzzy systems with measurable premise variables and under constrained switching has been studied in Reference 13. Using Lyapunov theory with persistent dwell time condition, a set of LMI conditions have been developed to ensure the convergence of the filtering error with a prescribed nonweighted  $H_\infty$  noise attenuation performance. In the

same context, the authors in Reference 15 investigated the design of a nonweighted asynchronous  $H_\infty$  filter for a class of continuous time switched T-S fuzzy systems. Using fuzzy multiple Lyapunov function, sufficient conditions have been formulated in terms of LMIs to guarantee the filtering error system is globally asymptotically stable with a nonweighted  $H_\infty$  performance. The filter was designed under minimum dwell time constraint and under the condition that the premise variables are measurable. In our recent work,<sup>16</sup> dwell-time free LMI-based conditions for the design of an asynchronous  $H_\infty$  filters for switched T-S systems under arbitrary switching have been investigated. To reduce the conservatism of the design conditions, LMI-based relaxation techniques<sup>32,33</sup> were considered together with a descriptor redundancy approach to ensure the convergence of the filtering error with  $H_\infty$  disturbance attenuation performances. However, the main weakness of this study is that the premise variables in the considered T-S models are considered measurable, which is often unrealistic in practice.

From the above review of the literature, it seems that the design of  $H_\infty$  filters for switched T-S systems with Unmeasured Premise Variables (UPVs) has been rarely investigated. Hence, dealing with UPVs is the major issue that should be taken into account to design this kind of filters. Close to  $H_\infty$  filter context, significant works have been carried out in the last decades to deal with UPVs in the case of observer design. Broadly speaking, these works can be summarized as follows:

- The most commonly used technique to deal with UPVs problem is the Lipschitz condition, which is very easy to implement. This latter is often considered to avoid the additive nonlinear term that occurs in the dynamics of the state estimation error. Hence, several studies using the Lipschitz condition have been published. For example, we can quote the research works carried out by References 34–38. Then, quite a few improvements have been introduced by using Lipschitz condition. To this end, relaxed results are proposed in Reference 39 using the Lipschitz conditions and quasi input-to-state stability to ensure bounded state estimation errors. Another Lipschitz condition-based approach has been proposed in References 35 and 36 to avoid UPVs by introducing the nonlinear terms in the consequent parts of the T-S models.
- Further improvements in terms of conservatism are proposed in Reference 40 to substitute the Lipschitz condition by applying the differential mean value theorem in order to deal with the additive term in the dynamic of the state error.
- Another approach to deal with UPVs is proposed in Reference 41, which consists in considering the error between the measured and unmeasured premise variables as model uncertainties, then applying robust control approaches.
- More recently, in Reference 42, auxiliary dynamics and immersion techniques are considered to rewrite and augment the T-S model with UPVs as a new T-S model with weighting functions depending only on measured variables. However, this approach cannot be easily generalized and may fail to provide the required transformations for the initial model due to the possible infinite number of iterations and the nature of the nonlinear entries of the system.

It is worth highlighting that the above-mentioned results are based on T-S observers or filters with an interconnection fuzzy structure involving UPVs, that is, where both the measured and unmeasured nonlinearities of the original system are considered together to provide the whole T-S fuzzy modeling interconnection structure (see e.g., References 18,34,35,38, and 41). Hence, this leads to introduce, in the LMI-based design conditions, the bounds of the mismatches occurring between the plant's T-S model membership functions, which actually depend on the real systems' states or inputs, and the observers' or filters' ones, which bring conservatism. However, an elegant alternative way can also be drawn from recent literature to circumvent the occurrence of UPVs in T-S modeling. It consists in separating the measured and unmeasured nonlinearities of the original nonlinear system, then applying T-S modeling transformations only on the measured ones, leading to T-S models with nonlinear consequent parts N-TS, see for example, References 21,43–49. In this case, the membership functions of the N-TS model only depend on the measured premises, while the unmeasured terms of the original nonlinear system are reported in the nonlinear consequent parts. In the previous literature, the nonlinear consequent parts have been firstly treated as bounded uncertainties in the design conditions<sup>21,43</sup> (applying Young's inequalities), then by considering sector bounded conditions,<sup>44</sup> or more frequently by assuming Lipschitz constraints.<sup>46–49</sup> Another appealing approach can be drawn from Reference 50, where incremental quadratic constraints are considered for nonlinear observers design to cope with some of the systems nonlinearities, then seldom extended to the context of N-TS fuzzy models.<sup>45</sup> We believe that such approach may bring a nice advantage since it allows to characterize the unknown nonlinearities satisfying incremental quadratic constraints as a set of multiplier matrices and includes as a special case Lipschitz conditions.<sup>50</sup> Moreover, note that the N-TS approach provides another interesting advantage since it allows to reduce the number of vertices (fuzzy rules) compared to the classical T-S representations. Even more

so for their extension to switched T-S models, where the number of vertices may increase significantly according to the number of switching modes. Consequently, the switched N-TS approach appears to be the most convenient to deal with filters or observers' design for switched nonlinear systems, where only the measured nonlinearities are kept in the convex fuzzy interconnection structure and the remaining unmeasured terms are reported as nonlinear consequent parts.

Another concern regarding the above literature review is related to the nature of the switching phenomena. There exists a vast literature dealing with such phenomena in the switched linear system's framework; see, for example, References 1-7,30,51-53. The most easiest case remain when the switched control law, observer or filter, share the same switching law as the switched linear plant, that is the so-called switched synchronous case.<sup>1,2,4,23,31,54</sup> However, it is often unrealistic in practice to consider the synchronous case because it requires the availability of the plant's switching signals, which may be difficult to measure or estimate. As a result, several studies have been carried out in the asynchronous case, either in the switched linear case, see for example,<sup>3,5,7,30,52,53,55</sup> or extended to the switched nonlinear system framework.<sup>11,13,14,16,38,56-58</sup> Hence, the asynchronous case refers to a scenario in which the switching rules of the observer or filter do not match those of the plant. In this more realistic scenario, the mismatch can be a delay (for instance, the amount of time it takes to estimate the plant's switching rule),<sup>3,5,7,53,59</sup> or it can be due to modeling uncertainties in the context of state-dependant switching laws remaining on mismatching switching hyper-planes.<sup>16,38,55,60</sup> So far, design conditions for switched systems can be classified into two classes:

- Dwell-time dependent conditions (see e.g., References 3,5,7 and 51), which are useful to relax the conditions but require some knowledge on the switching rules. Especially, in this case, the considered switched systems are required to stay in each mode during at least a minimum positive dwell-time. However, such approach may be unsuitable for some switched systems, for instance when the switched systems involve state-dependent switching laws that are evolving regarding to uncertain switching hyper-planes.
- Dwell-time free conditions (see e.g., References 16,38,55,60), which are useful for arbitrary or state-dependent switching laws evolving regarding to uncertain switching hyper-planes. Despite these conditions cannot benefit from the conservatism reduction brought by dwell-time dependent conditions, they allows to cope with a larger class of switched systems since they require less restrictive knowledge about the switching phenomena.

Let us highlight that the vast majority of previous works dealing with the design of  $H_\infty$  filters for switched T-S systems focused on switching under dwell-time conditions. Hence, to the best of the authors' knowledge,  $H_\infty$  filter design for switched T-S systems under arbitrary switching has been rarely studied so far.

Given the earlier issues, the present proposal focuses on the asynchronous switched N-TS  $H_\infty$  filters design for a class of switched nonlinear systems represented by N-TS (to circumvent the occurrence of UPVs), subject to external disturbances, and where the switching sequence can be uncontrolled, arbitrary or unknown. In this respect, the contributions of the present study can be summarized by the following points:

- Extending our preliminary study presented in Reference 16 to a wider class of switched N-TS fuzzy systems. To this end, the first contribution of the present proposal consists in the design asynchronous switched N-TS  $H_\infty$  filters by considering incremental quadratic constraints,<sup>50</sup> enlarging the Lipschitz conditions (see the above-mentioned references), to cope with the additive nonlinear term arising from the error between the measured and UPVs.
- The second contribution of the present proposal is the ability to deal with asynchronous switching modes between the considered switched nonlinear systems and their designed switched  $H_\infty$  filters. In this context, we assume uncontrolled, arbitrary, or unknown switching sequences, which outperform previous studies where dwell-time considerations are made (see the references mentioned earlier in this introduction). Likewise, the proposed asynchronous initialization of the switching modes between the systems and the filters is an interesting feature, particularly since all the above-mentioned studies assumed synchronous initialization.
- The third contribution of the present study consists of proposing an optimization procedure for the estimation of the attraction domain of the filtering error based on the enlargement of Lyapunov level sets. Indeed, recall that N-TS fuzzy models being only valid on a subset of the state space of a nonlinear system, and also due to the system's and the filter's membership functions mismatches, the proposed results can only be valid locally. While, to the best of the authors' knowledge, no previous N-TS model-based filters design studies from the literature took into account such an important feature.
- The last contribution concerns the common objective of most of the works in the field of T-S fuzzy model-based design, that is, the reduction of the conservatism of the proposed conditions (see, e.g., Reference 61). To this end, the use of

quadratic constraints approaches, in addition to the usual techniques,<sup>32,33</sup> provide a significant improvement compared to previous related studies<sup>11,13,15,16</sup> in terms of feasibility fields, but also for the minimization of the obtained  $H_\infty$  attenuation level.

The remainder of this paper is organized as follows. Section 2 presents the preliminaries and problem statement. In Section 3, LMI conditions for the design of asynchronous switched N-TS  $H_\infty$  filters with unmeasured premise variables are presented. Finally, simulation results were given to compare the conservatism and to illustrate the efficiency of the designed filters with respect to previous results in recent literature.

## 2 | PRELIMINARIES AND PROBLEM STATEMENT

Let us consider a class of nonlinear switched systems with  $m$  switching modes given by:

$$\begin{cases} \dot{x}(t) = f_{x,\sigma(t)}(x(t)) + f_{x,\sigma(t)}^w(x(t))w(t) \\ z(t) = f_{z,\sigma(t)}(x(t)) \\ y(t) = Cx(t) + Dw(t), \end{cases} \quad (1)$$

where  $x(t) \in \mathbb{R}^{n_x}$  is the state vector;  $y(t) \in \mathbb{R}^{n_y}$  is the measured output vector and  $z(t) \in \mathbb{R}^{n_z}$  is the unmeasured output to be estimated;  $w(t) \in \mathbb{R}^{n_w}$  is a time-varying  $L_2$  norm bounded exogenous disturbance input vector;  $\sigma(t) \in \{1, \dots, m\}$  is the switching law;  $f_{x,\sigma(t)}(x(t)) \in \mathbb{R}^{n_x}$ ,  $f_{z,\sigma(t)}(x(t)) \in \mathbb{R}^{n_z}$  and  $f_{x,\sigma(t)}^w(x(t)) \in \mathbb{R}^{n_x \times n_w}$  are nonlinear vector or matrix-valued functions that describe the dynamics of the considered system,  $C \in \mathbb{R}^{n_y \times n_x}$  and  $D \in \mathbb{R}^{n_y \times n_w}$  are, respectively, the state selection and direct transfer matrices of the measured output equation.

**Assumption 1.** In this paper, we assume that there is no state jumps when switching occur in the switched nonlinear system (1), that is, all the switched modes share the same state vector  $x(t)$ . Hence, without loss of generality, we can assume that the output equation is common and linear for all the switched modes since it is usually dedicated to select the state variables (or their linear combinations) which are seen by the output  $y(t)$ , as well as to model the direct transfer from the exogenous disturbance input  $w(t)$  to the output  $y(t)$ .

Note that, in (1), from the time-driven switching signal  $\sigma(t) \in \{1, \dots, m\}$ , we can define, for  $j \in \{1, \dots, m\}$ :

$$\sigma_j(t) = \begin{cases} 1 & \text{if } \sigma(t) = j, \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

where obviously  $\sum_{j=1}^m \sigma_j(t) = 1$ .

Hence, the switched nonlinear system (1) can be equivalently rewritten as:

$$\begin{cases} \dot{x}(t) = \sum_{j=1}^m \sigma_j(t) \left( f_{x,j}(x(t)) + f_{x,j}^w(x(t))w(t) \right) \\ z(t) = \sum_{j=1}^m \sigma_j(t) f_{z,j}(x(t)) \\ y(t) = Cx(t) + Dw(t). \end{cases} \quad (3)$$

**Assumption 2.**  $\forall j \in \{1, \dots, m\}$ , the nonlinear vector-valued functions  $f_{x,j}(x(t)) \in \mathbb{R}^{n_x}$  and  $f_{z,j}(x(t)) \in \mathbb{R}^{n_z}$  are smooth sector bounded nonlinear functions (where  $f_{x,j}(0) = 0$  and  $f_{z,j}(0) = 0$ ). Also, together with the matrix-valued function  $f_{x,j}^w(x(t)) \in \mathbb{R}^{n_x \times n_w}$ , we assume that their entries contain  $p$  scalar bounded state-dependant nonlinear functions  $\eta_\rho(x(t)) \in [\underline{\eta}_\rho, \bar{\eta}_\rho]$ ,  $\rho \in \{1, \dots, p\}$ .

From Assumption 2, exactly matching T-S models<sup>8</sup> for each switched mode  $j$  of the switched nonlinear system (3) can be obtained by applying the well-known sector nonlinearity approach<sup>9</sup> on each nonlinearity  $\eta_\rho(x(t)) \in [\underline{\eta}_\rho, \bar{\eta}_\rho]$ . However, if some state variables cannot be measured, this way of doing would leads to T-S models with UPVs (see e.g., References 34-38), which is, for the reasons explained in the introduction of the present paper, what we want to overcome in the sequel. Therefore, to circumvent the occurrence of UPVs, let us rewrite the sector bounded nonlinear vector-valued functions in (3), under Assumption 2, as:

$$f_{x,j}(x(t)) = \bar{f}_{x,j}(Mx(t))x(t) + g_{x,j}(Mx(t))\bar{\phi}(Nx(t)), \quad (4)$$

and

$$f_{z,j}(x(t)) = \bar{f}_{z,j}(Mx(t))x(t) + g_{z,j}(Mx(t))\bar{\phi}(Nx(t)), \quad (5)$$

where  $M \in \mathbb{R}^{n_m \times n_x}$  and  $N \in \mathbb{R}^{(n_x - n_m) \times n_x}$  are known matrices selecting respectively the measured and unmeasured state variables, so the nonlinear functions  $\bar{f}_{x,j}(Mx(t)) \in \mathbb{R}^{n_x \times n_x}$ ,  $\bar{f}_{z,j}(Mx(t)) \in \mathbb{R}^{n_z \times n_x}$ ,  $g_{x,j}(Mx(t)) \in \mathbb{R}^{n_x \times n_s}$  and  $g_{z,j}(Mx(t)) \in \mathbb{R}^{n_z \times n_s}$  contain only measurable nonlinear terms while the unmeasured terms are reported in the vector-valued sector-bounded nonlinear functions  $\bar{\phi}(Nx(t)) \in \text{co}\{\underline{U}x(t), \bar{U}x(t)\} \subseteq \mathbb{R}^{n_s}$ , where  $\underline{U} \in \mathbb{R}^{n_s \times n_x}$  and  $\bar{U} \in \mathbb{R}^{n_s \times n_x}$ .

*Remark 1.* Let us notice that, for the sake of convenience and to lighten further mathematical developments, we consider in the present study that  $f_{x,j}^w(x(t))$  do not depends on unmeasured state variable, that is,  $f_{x,j}^w(x(t)) \equiv f_{x,j}^w(Mx(t))$ . Obviously, this choice can be considered as slightly restrictive w.r.t. this class of systems but it does not affect the proof of concept in our proposal since this term only relates to the external disturbances  $\omega(t)$ , which in practice usually affect the system's dynamics without any nonlinear transformation (i.e.,  $f_{x,j}^w(x(t)) = B$  constant).

Now, applying the well-known sector nonlinearity approach,<sup>9</sup> only on the nonlinear terms involving measured state variables, we can write:

$$\begin{aligned} \bar{f}_{x,j}(Mx(t)) &= \sum_{i=1}^{r_j} h_i(Mx(t))\mathcal{A}_{ij}, & \bar{f}_{z,j}(Mx(t)) &= \sum_{i=1}^{r_j} h_i(Mx(t))\mathcal{F}_{ij}, & f_{x,j}^w(Mx(t)) &= \sum_{i=1}^{r_j} h_i(Mx(t))B_{ij} \\ g_{x,j}(Mx(t)) &= \sum_{i=1}^{r_j} h_i(Mx(t))H_{ij}^x, & g_{z,j}(Mx(t)) &= \sum_{i=1}^{r_j} h_i(Mx(t))H_{ij}^z, \end{aligned} \quad (6)$$

for any  $x(t)$  belonging to the validity domain  $D_x$  defined as:

$$D_x = \{x \in \mathbb{R}^{n_x} : \mathfrak{L}x(t) \leq Q\}, \quad (7)$$

where  $\mathfrak{L} \in \mathbb{R}^{v \times n_x}$  is a given matrix and  $Q \in \mathbb{R}^v$  ( $v$  is the number of state variable bounds to be satisfied);  $r_j$  is the number of vertices (or fuzzy rules) in the  $j$ th mode;  $\forall i_j \in \{1, \dots, r_j\}$ ,  $h_{i_j}(Mx(t)) \geq 0$  are normalized membership functions, which depend only on measured state variables and satisfy the convex sum property  $\sum_{i=1}^{r_j} h_{i_j}(Mx(t)) = 1$ ; the matrices  $\mathcal{A}_{ij} \in \mathbb{R}^{n_x \times n_x}$ ,  $\mathcal{F}_{ij} \in \mathbb{R}^{n_z \times n_x}$ ,  $B_{ij} \in \mathbb{R}^{n_x \times n_w}$ ,  $H_{ij}^x$  and  $H_{ij}^z$  define the vertices in each mode  $j$ .

*Remark 2.* Let us denote  $j$  and  $j^+$ , respectively, the switched system's active mode at time  $t$  and its successor. Without loss of generality, we consider in this paper that the switches in (1) and (3) occur within switching sets defined by linear hyper-planes defined by:

$$\mathcal{S}_{jj^+} = \{x \in \mathbb{R}^{n_x} : S_{jj^+}x = 0\}, \quad (8)$$

where  $S_{jj^+}$  are real matrices with appropriate dimensions.

From the above developments, the switched nonlinear system (1) can be exactly represented on the compact subset  $D_x$  of its state space by a switched T-S model with nonlinear consequent parts (switched N-TS) as follows:

$$\begin{cases} \dot{x}(t) = \sum_{j=1}^m \sum_{i=1}^{r_j} \sigma_j(t) h_{i_j}(Mx(t)) \left( \mathcal{A}_{ij}x(t) + H_{ij}^x \bar{\phi}(Nx(t)) + B_{ij}w(t) \right) \\ z(t) = \sum_{j=1}^m \sum_{i=1}^{r_j} \sigma_j(t) h_{i_j}(Mx(t)) (\mathcal{F}_{ij}x(t) + H_{ij}^z \bar{\phi}(Nx(t))) \\ y(t) = Cx(t) + Dw(t) \end{cases} \quad (9)$$

Furthermore, a change of origin for the nonlinear consequent term is performed such that  $\bar{\phi}(Nx(t)) = \bar{\phi}(x(t)) - \underline{U}x(t)$ . Thus we have  $\bar{\phi}(Nx(t)) \in \text{co}\{0, Ux(t)\} \subseteq \mathbb{R}^{n_s}$ , with  $U = \bar{U} - \underline{U}$ , and the N-TS (9) can be rewritten as:

$$\begin{cases} \dot{x}(t) = \sum_{j=1}^m \sum_{i=1}^{r_j} \sigma_j(t) h_{i_j}(Mx(t)) \left( A_{ij}x(t) + H_{ij}^x \bar{\phi}(Nx(t)) + B_{ij}w(t) \right) \\ z(t) = \sum_{j=1}^m \sum_{i=1}^{r_j} \sigma_j(t) h_{i_j}(Mx(t)) (F_{ij}x(t) + H_{ij}^z \bar{\phi}(Nx(t))) \\ y(t) = Cx(t) + Dw(t), \end{cases} \quad (10)$$

with  $A_{ij} = \mathcal{A}_{ij} + H_{ij}^x \underline{U}$  and  $F_{ij} = \mathcal{F}_{ij} + H_{ij}^z \underline{U}$ .

Such a change of variable is useful for design purpose so that the following property holds.

**Property 1.** (Reference 44) The vector of nonlinearities  $\phi(Nx(t)) \in \text{co}\{0, Ux(t)\}$  satisfies the following sector-boundedness condition:

$$\phi(Nx(t))^T \Upsilon (\phi(Nx(t)) - Ux(t)) \leq 0, \quad (11)$$

where  $\Upsilon \in \mathbb{R}^{n_s \times n_s}$  is any positive-definite diagonal matrix.

*Remark 3.* Recall that an interesting feature of switched N-TS systems is that they can be used to overcome the occurrence of UPVs efficiently, see for example, References 21,43-48. Indeed, as highlighted from the above developments, a switched N-TS model consists in transforming the original switched nonlinear system (1) into a nominal switched T-S model with additive nonlinear consequent parts. In this case, the premise variables of the N-TS will only depend on the measured states, while the nonlinear terms depending on unmeasured variables are reported into the nonlinear consequent part. Moreover, another interesting feature of N-TS approaches is that the resulting number of vertices involved in the design conditions can be significantly reduced to help relaxing the conservatism or the computational complexity versus classical T-S modeling approaches.

In this paper, given switched N-TS systems (10), we propose the design of asynchronous switched N-TS filters given by:

$$\begin{cases} \hat{x}(t) = \sum_{j=1}^m \sum_{i_j=1}^{r_j} \hat{\sigma}_j(t) h_{i_j}(M\hat{x}(t)) \left( \hat{A}_{i_j} \hat{x}(t) + H_{i_j}^x \phi(N\hat{x}(t)) + \hat{B}_{i_j} y(t) \right) \\ \hat{z}(t) = \sum_{j=1}^m \sum_{i_j=1}^{r_j} \hat{\sigma}_j(t) h_{i_j}(M\hat{x}(t)) \left( \hat{F}_{i_j} \hat{x}(t) + H_{i_j}^z \phi(N\hat{x}(t)) \right), \end{cases} \quad (12)$$

where  $\hat{x}(t)$  is the filter's state vector,  $\hat{A}_{i_j}$ ,  $\hat{B}_{i_j}$ ,  $\hat{F}_{i_j}$  are gain matrices to be synthesized;  $\forall j \in \{1, \dots, m\}$ ,  $\hat{\sigma}_j(t) \in \{0, 1\}$ , with  $\sum_{i_j}^m \hat{\sigma}_j(t) = 1$ , define the switching law of the filter, which is assumed to evolve according to the filter's own switching sets  $\hat{\mathcal{S}}_{\hat{j}\hat{j}^+}$  defined by the hyper-planes:

$$\hat{\mathcal{S}}_{\hat{j}\hat{j}^+} = \{\hat{x} \in \mathbb{R}^{n_x} : \hat{S}_{\hat{j}\hat{j}^+} \hat{x} = 0\}, \quad (13)$$

where  $\hat{S}_{\hat{j}\hat{j}^+}$  are real matrices with appropriate dimensions,  $\hat{j}$  and  $\hat{j}^+$  denote, respectively, the switched N-TS filter's active mode at time  $t$  and its successor.

*Remark 4.* Let us recall that considering the filter's switching signals  $\hat{\sigma}_j(t)$  asynchronous with regards to the switched system's ones  $\sigma_j(t)$  is relevant when the latter are unavailable or imprecisely measured, as mentioned in the introduction of this paper.

## 2.1 | Notation

In the sequel, we will denote switched convex combinations of matrices  $\mathcal{M}_{(\cdot)}$  with appropriate dimensions by:

$$\begin{aligned} \mathcal{M}_{h_\sigma} &= \sum_{j=1}^m \sum_{i_j=1}^{r_j} \sigma_j(t) h_{i_j}(Mx(t)) \mathcal{M}_{i_j}, \quad \mathcal{M}_{h_{\hat{\sigma}}} = \sum_{j=1}^m \sum_{i_j=1}^{r_j} \hat{\sigma}_j(t) h_{i_j}(M\hat{x}(t)) \mathcal{M}_{i_j}, \\ \mathcal{M}_{h_\sigma h_{\hat{\sigma}}} &= \sum_{j=1}^m \sum_{i_j=1}^{r_j} \sum_{i_j=1}^{r_j} \sigma_j(t) \hat{\sigma}_j(t) h_{i_j}(Mx(t)) h_{i_j}(M\hat{x}(t)) \mathcal{M}_{i_j i_j}, \quad \text{and} \quad \mathcal{M}_{\hat{\sigma}} = \sum_{j=1}^m \hat{\sigma}_j(t) \mathcal{M}_j. \end{aligned}$$

Moreover, the time  $t$  as functions argument will be omitted when no ambiguity arises.  $\mathcal{M}^\dagger$  stands for the pseudo-inverse of non square matrices  $\mathcal{M}$ . For real square matrices  $\mathcal{M}$ , we denote  $\mathcal{H}e(\mathcal{M}) = \mathcal{M} + \mathcal{M}^T$ . Finally,  $\mathcal{M}_{(\nu)}$  denotes the  $\nu^{\text{th}}$  row of a matrix  $\mathcal{M}$ .

Let us define the filtering errors  $e_x(t) = x(t) - \hat{x}(t)$  and  $e_z(t) = z(t) - \hat{z}(t)$ , we can write:

$$\begin{aligned} \dot{e}_x(t) &= \left( A_{h_\sigma} x(t) + H_{h_\sigma}^x \phi(Nx(t)) + B_{h_\sigma} w(t) \right) - \left( \hat{A}_{h_{\hat{\sigma}}} \hat{x}(t) + H_{h_{\hat{\sigma}}}^x \phi(N\hat{x}(t)) + \hat{B}_{h_{\hat{\sigma}}} y(t) \right) \\ &= A_{h_\sigma} e_x(t) + (A_{h_\sigma} - \hat{A}_{h_{\hat{\sigma}}}) \hat{x}(t) + H_{h_\sigma}^x \phi_e(t) + (H_{h_\sigma}^x - H_{h_{\hat{\sigma}}}^x) \phi(N\hat{x}(t)) + B_{h_\sigma} w(t) - \hat{B}_{h_{\hat{\sigma}}} (Cx(t) + Dw(t)), \end{aligned} \quad (14)$$



and:

$$\begin{aligned} e_z(t) &= \left( F_{h_\sigma} x(t) + H_{h_\sigma}^z \phi(Nx(t)) \right) - \left( \hat{F}_{h_\sigma} \hat{x}(t) + H_{h_\sigma}^z \phi(N\hat{x}(t)) \right) \\ &= F_{h_\sigma} e_x(t) + (F_{h_\sigma} - \hat{F}_{h_\sigma}) \hat{x}(t) + H_{h_\sigma}^z \phi_e(t) + (H_{h_\sigma}^z - \hat{H}_{h_\sigma}^z) \phi(N\hat{x}(t)), \end{aligned} \quad (15)$$

where  $\phi_e(t) = \phi(Nx(t)) - \phi(N\hat{x}(t))$ . Then, considering the augmented state error vector  $e(t) = [e_x(t)^T e_z(t)^T]^T$  and introducing  $\phi_a(t) = [\phi^T(N\hat{x}(t)) \ \phi_e^T(t)]^T$  and  $\tilde{w}(t) = [x^T(t) \ \hat{x}^T(t) \ w^T(t)]^T$ , a compact form of the filtering-error dynamics can be written as the following descriptor:

$$E\dot{e}(t) = \tilde{A}_{h_\sigma} e(t) + \Gamma_{h_\sigma h_\delta} \phi_a(t) + \tilde{B}_{h_\sigma h_\delta} \tilde{w}(t), \quad (16)$$

with

$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \tilde{A}_{h_\sigma} = \begin{bmatrix} A_{h_\sigma} & 0 \\ F_{h_\sigma} & -I \end{bmatrix}, \Gamma_{h_\sigma h_\delta} = \begin{bmatrix} H_{h_\sigma}^x - H_{h_\delta}^x & H_{h_\sigma}^x \\ H_{h_\sigma}^z - H_{h_\delta}^z & H_{h_\sigma}^z \end{bmatrix}, \tilde{B}_{h_\sigma h_\delta} = \begin{bmatrix} -\hat{B}_{h_\sigma} C & A_{h_\sigma} - \hat{A}_{h_\sigma} & B_{h_\sigma} - \hat{B}_{h_\sigma} D \\ 0 & F_{h_\sigma} - \hat{F}_{h_\sigma} & 0 \end{bmatrix}. \quad (17)$$

**Assumption 3.** Let us assume that the characterization of the nonlinear term  $\phi(Nx(t))$  can be made based on a set  $\mathcal{W}$  of symmetric matrices  $W_{h_\sigma} = \text{diag}(W_{h_\sigma}^{11}, W_{h_\sigma}^{22})$ , with  $W_{h_\sigma}^{11T} > 0$  and  $W_{h_\sigma}^{22} = W_{h_\sigma}^{22T} < 0$ . Hence, Following the work of Açıkmeye and Corless,<sup>50</sup> all  $W_{h_\sigma} \in \mathcal{W}$  satisfies the Incremental Quadratic Constraint ( $\delta QC$ ) given by:

$$\phi(t, q_1, q_2) = \begin{bmatrix} q_1 - q_2 \\ \phi(q_1, t) - \phi(q_2, t) \end{bmatrix}^T W_{h_\sigma} \begin{bmatrix} q_1 - q_2 \\ \phi(q_1, t) - \phi(q_2, t) \end{bmatrix} \geq 0, \quad (18)$$

where  $q_1 = Nx(t)$  and  $q_2 = N\hat{x}(t)$ .

*Remark 5.* In order to cope with the term  $\phi_e(t)$  in the filtering error dynamics (16), an alternative is to consider Lipschitz conditions, however this may lead to conservatism. Instead, the  $\delta QC$  (18) is more general since it includes as a special case the Lipschitz condition. Indeed, note that the Lipschitz condition  $\|\phi(q_1, t) - \phi(q_2, t)\| \leq \mu \|q_1 - q_2\|$ , with  $\mu \geq 0$ , can be rewritten as:

$$\begin{bmatrix} q_1 - q_2 \\ \phi(q_1, t) - \phi(q_2, t) \end{bmatrix}^T W_{h_\sigma} \begin{bmatrix} q_1 - q_2 \\ \phi(q_1, t) - \phi(q_2, t) \end{bmatrix} \geq 0,$$

with the incremental multiplier matrix  $W_{h_\sigma} = \begin{bmatrix} \mu^2 I & 0 \\ 0 & -I \end{bmatrix}$  satisfying the  $\delta QC$  (18).

Now, let us recall that the switched N-TS model (10) is valid and guarantee an exact representation of the switched nonlinear system (1) inside a domain of validity  $\mathcal{D}_x$  defined in (7). Hence, to design a switched N-TS filter that has the same structure as the considered switched N-TS system, it is necessary to consider a domain of validity  $\mathcal{D}_{\hat{x}}$ , similarly to the one of the switched N-TS system (10) (i.e., using the same nonlinear sectors). These allow to define the domain of validity of the estimation error  $\mathcal{D}_{e_x}$  as follows:

$$\mathcal{D}_{e_x} = \{e_x \in \mathbb{R}^{n_x} : \mathfrak{L}e_x(t) \leq 2\mathcal{Q}\}. \quad (19)$$

*Remark 6.* Note that the initialization of the filter's state is arbitrary by nature. Indeed, for practical reasons, it is well-known that the whole system's state is not supposed to be available (else, filtering or any other estimation techniques would be irrelevant). Therefore, it is not realistic to assume that the system's initial state  $x(0)$  is known and the filter can be initialized in any other initial state  $\hat{x}(0) \in \mathcal{D}_{\hat{x}}$ . For the sake of convenience, in this study, we choose to set the filter's initial state as  $\hat{x}(0) = 0$  so that the estimation of the domain of attraction will be easier since, in this case we have  $e_x(0) = x(0)$ .

From now, the following problem statement for the design of the asynchronous switched N-TS filter (12) can be made.

## 2.2 | Problem statement

The objective of this work is to synthesize the gain matrices of the asynchronous switched N-TS filters (12) such that the following requirements are satisfied, together with maximizing the estimate  $D_a \subseteq D_{e_x}$  of the domain of attraction, that is, the guaranteed domain of convergence of the filtering errors.

- i.  $\forall e_x \in D_a \subseteq D_{e_x}$ , the unmeasured output filtering error  $e_z(t)$  is converging to the origin, that is,  $\lim_{t \rightarrow +\infty} e_z(t) = 0$ , when  $w(t) = 0$ .
- ii. For all nonzero  $\tilde{w}(t) \in L_2[0, \infty)$ , the filtering error dynamics (16) has a prescribed disturbance attenuation level  $\gamma$ , such that:

$$\frac{\|e_z(t)\|_S^2}{\|\tilde{w}(t)\|_R^2} \leq \gamma^2, \quad (20)$$

that is to say, if the following  $H_\infty$  criterion is satisfied:

$$\int_0^\infty e_z^T(t) S e_z(t) dt \leq \gamma^2 \int_0^\infty \tilde{w}^T(t) R \tilde{w}(t) dt, \quad (21)$$

where  $R$  and  $S$  are known weighting positive diagonal matrices with appropriate dimension.

Furthermore, the following lemmas will be used to relax the LMI conditions derived as the main results in the next section.

**Lemma 1.** (Reference 32) For any matrices  $Z, R, L, P$ , and  $Q$  with appropriate dimensions, one has:

$$Z^T P + P^T Z + Q < 0 \iff \exists R, L : \begin{bmatrix} Z^T L^T + LZ + Q & (*) \\ P - L^T + R^T Z & -R^T - R \end{bmatrix} < 0. \quad (22)$$

**Lemma 2.** (Reference 33) Let  $i = 1, \dots, r$  and  $\delta_i(\cdot) > 0$  being scalar functions satisfying the convex sum property  $\sum_{i=1}^r \delta_i(\cdot) = 1$ . For same sized symmetric matrices  $\Gamma_{ij}$ , the inequality  $\sum_{i=1}^r \sum_{k=1}^r \delta_i(\cdot) \delta_j(\cdot) \Gamma_{ij} < 0$  is verified if,  $\forall(i, k) = \{1, \dots, r\}^2$ :

$$\frac{1}{r-1} \Gamma_{ii} + \frac{1}{2} (\Gamma_{ki} + \Gamma_{ik}) < 0. \quad (23)$$

This concludes the preliminaries. The following section will be devoted to provide LMI conditions for the design of asynchronous switched N-TS  $H_\infty$  filters with nonlinear consequent parts and incremental quadratic constraints, such that the above problem statement is fulfilled.

## 3 | MAIN RESULTS

In this section, the problem of  $H_\infty$  filtering for the switched N-TS systems (10) and the estimation of filtered unmeasured output vector  $z(t)$  is considered. To this end, sufficient LMI-based conditions for the design of asynchronous switched N-TS filters (12) with nonlinear consequent parts, incremental quadratic constraints and arbitrary switching (without dwell-time conditions) are provided. These are summarized in the following theorem.

**Theorem 1.** Consider the switched N-TS system (10), and the switched N-TS  $H_\infty$  filters (12).  $\forall(j, \hat{j}) \in \{1, \dots, m\}^2, \forall(i_j, k_j) \in \{1, \dots, r_j\}^2$  and  $\forall q_j \in \{1, \dots, r_j\}$ , both requirements of the above problem statement are satisfied if there exist a scalar  $\gamma > 0$  and real matrices  $\hat{A}_{i_j}, \hat{B}_{i_j}, \hat{F}_{i_j}, G_j, L_{k_j}, R_{k_j}, X_j^1 = X_j^{1T}, X_j^2, X_j^3$  and diagonal matrices  $Y_{i_j} \geq 0$ , which verify the following optimization problem:

$$\begin{cases} \min \gamma^2, \max \text{trace}(X_j^1) \\ \text{s.t. (25), (26), (27) and } X_j^1 > 0 \end{cases} \quad (24)$$

where

$$\frac{1}{r_j - 1} \Xi_{i_j q_j} + \frac{1}{2} \left( \Xi_{i_j k_j} + \Xi_{k_j i_j} \right) < 0, \forall (i_j, k_j) \in \{1, \dots, r_j\}^2, \quad (25)$$

$$X_j^{1+} = X_j^1 + G_j^T C + C^T G_j, \forall (\hat{j}, \hat{j}^+) \in \{1, \dots, m\}^2 \quad (26)$$

$$\begin{bmatrix} X_j^1 & (*) \\ \mathfrak{Q}_{(\nu)} X_j^1 & 4Q_{(\nu)}^2 \end{bmatrix} \geq 0, \forall \nu \in \{1, \dots, \nu\}, \forall \hat{j} \in \{1, \dots, m\}, \quad (27)$$

with

$$\Xi_{i_j k_j, q_j} = \begin{bmatrix} He(L_{k_j} \tilde{A}_{i_j}^T) & (*) & (*) & (*) & (*) & (*) & (*) \\ X_j - L_{k_j}^T + R_{k_j}^T \tilde{A}_{i_j}^T & -R_{k_j} - R_{k_j}^T & (*) & (*) & (*) & (*) & (*) \\ B_{i_j q_j}^T & 0 & -\mathcal{R} \gamma^2 & (*) & (*) & (*) & (*) \\ \Gamma_{i_j q_j}^{1T} & 0 & \Upsilon_{i_j} U \bar{\Psi} & -2\Upsilon_{i_j} & (*) & (*) & (*) \\ \Gamma_{i_j k_j}^{2T} & 0 & 0 & 0 & W_{i_j}^{22} & (*) & (*) \\ \psi X_j & 0 & 0 & 0 & 0 & -I & (*) \\ \tilde{N} X_j & 0 & 0 & 0 & 0 & 0 & -\bar{W}_{i_j}^{11} \end{bmatrix}, \quad (28)$$

where:

$$\tilde{A}_{i_j} = \begin{bmatrix} A_{i_j} & 0 \\ F_{i_j} & -I \end{bmatrix}, \tilde{B}_{i_j q_j} = \begin{bmatrix} -\hat{B}_{q_j} C & A_{i_j} - \hat{A}_{q_j} & B_{i_j} - \hat{B}_{q_j} D \\ 0 & F_{i_j} - \hat{F}_{q_j} & 0 \end{bmatrix}, \Gamma_{i_j q_j}^1 = \begin{bmatrix} H_{i_j}^x - H_{q_j}^x \\ H_{i_j}^z - H_{q_j}^z \end{bmatrix}, \Gamma_{i_j q_j}^2 = \begin{bmatrix} H_{i_j}^x \\ H_{i_j}^z \end{bmatrix}, \quad (29)$$

and

$$\psi = \text{diag}(0, S^{1/2}), \tilde{N} = \begin{bmatrix} N & 0 \end{bmatrix}, \bar{\Psi} = \begin{bmatrix} 0 & I & 0 \end{bmatrix}, \bar{W}_{i_j}^{11} = \left( W_{i_j}^{11} \right)^{-1}. \quad (30)$$

In this case, the filter's state vector must be updated at the switching times according to:

$$\hat{x}^+(t) = \left( I - \mathcal{L}_j^{-1} \left( C \mathcal{L}_j^{-1} \right)^\dagger C \right) \hat{x}(t) + \mathcal{L}_j^{-1} \left( C \mathcal{L}_j^{-1} \right)^\dagger y(t), \quad (31)$$

with  $\mathcal{L}_j = \mathbf{V}_j \sqrt{\Lambda_j} \mathbf{V}_j^T$ , where  $\mathbf{V}_j \in \mathbb{R}^{n \times n}$  are composed of the orthonormal eigenvectors of  $P_j^1 = \left( X_j^1 \right)^{-1}$ , and where  $\Lambda_j \in \mathbb{R}^{n \times n}$  are diagonal matrices, which entries are the eigenvalues of  $P_j^1$ .

Furthermore, the intersection of Lyapunov level sets  $\mathcal{L}(1)$ , defined by (32), provides an estimate of the domain of attraction  $\mathcal{D}_a$  of  $e_x$ .

$$\mathcal{L}(1) = \bigcap_{j \in I_m} \left\{ e_x(0) \in \mathbb{R}^{n_x} : e_x^T(0) P_j^1 e_x(0) \leq 1 \right\}, \quad (32)$$

*Proof.* Let us consider a multiple quadratic Lyapunov candidate function given by:

$$V(t, e(t)) = e(t)^T E P_{\hat{\sigma}} e(t), \quad (33)$$

where  $EP_{\hat{\sigma}} = P_{\hat{\sigma}}^T E > 0$ , with  $P_{\hat{\sigma}} = \begin{bmatrix} P_{\hat{\sigma}}^1 & 0 \\ P_{\hat{\sigma}}^2 & P_{\hat{\sigma}}^3 \end{bmatrix}$  and  $P_{\hat{\sigma}}^1 = P_{\hat{\sigma}}^{1T} > 0$ .

Taking the time derivative of (33), yields:

$$\dot{V}(t, e(t)) = 2e^T(t) P_{\hat{\sigma}}^T E \dot{e}(t) = 2e^T(t) P_{\hat{\sigma}}^T \left( \tilde{A}_{h_{\hat{\sigma}}} e(t) + \Gamma_{h_{\hat{\sigma}}} \phi_a(t) + \tilde{B}_{h_{\hat{\sigma}}} \tilde{w}(t) \right). \quad (34)$$

Hence, the items i. and ii. of the problem statement defined in Section 2 are satisfied if there exists  $\gamma$  such that the following inequality holds:

$$\dot{V}(t, e(t)) + e_z(t)^T S e_z(t) - \gamma^2 \tilde{w}(t)^T \mathcal{R} \tilde{w}(t) = \begin{bmatrix} e(t) \\ \tilde{w}(t) \\ \phi(N\hat{x}(t)) \\ \phi_e(t) \end{bmatrix}^T \begin{bmatrix} He(\tilde{A}_{h_\sigma}^T P_{\hat{\sigma}}) + \psi^T \psi & (*) & (*) & (*) \\ \tilde{B}_{h_\sigma h_\sigma}^T P_{\hat{\sigma}} & -\mathcal{R} \gamma^2 & (*) & (*) \\ \Gamma_{h_\sigma h_\sigma}^{1T} P_{\hat{\sigma}} & 0 & 0 & (*) \\ \Gamma_{h_\sigma h_\sigma}^{2T} P_{\hat{\sigma}} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ \tilde{w}(t) \\ \phi(N\hat{x}(t)) \\ \phi_e(t) \end{bmatrix} < 0. \quad (35)$$

On the other hand, the  $\delta QC$  (18) is equivalent to:

$$\begin{bmatrix} e(t) \\ \phi_e(t) \end{bmatrix}^T \begin{bmatrix} \tilde{N}^T W_{h_\sigma}^{11} \tilde{N} & 0 \\ 0 & W_{h_\sigma}^{22} \end{bmatrix} \begin{bmatrix} e(t) \\ \phi_e(t) \end{bmatrix} \geq 0. \quad (36)$$

From (11) and (36), the inequality (35) holds,  $\forall [e^T(t) \ \tilde{w}^T(t) \ \phi^T(N\hat{x}(t)) \ \phi_e^T(t)] \neq 0$ , if:

$$\begin{bmatrix} He(\tilde{A}_{h_\sigma}^T P_{\hat{\sigma}}) + \tilde{N}^T W_{h_\sigma}^{11} \tilde{N} + \psi^T \psi & (*) & (*) & (*) \\ B_{h_\sigma h_\sigma}^T P_{\hat{\sigma}} & -\mathcal{R} \gamma^2 & (*) & (*) \\ \Gamma_{h_\sigma h_\sigma}^{1T} P_{\hat{\sigma}} & \Upsilon_{h_\sigma} U \bar{\Psi} & -2\Upsilon_{h_\sigma} & (*) \\ \Gamma_{h_\sigma h_\sigma}^{2T} P_{\hat{\sigma}} & 0 & 0 & W_{h_\sigma}^{22} \end{bmatrix} < 0. \quad (37)$$

Performing a congruence transformation to (37) with the diagonal matrix  $\text{diag}(X_{\hat{\sigma}}, I, I, I)$  where  $X_{\hat{\sigma}} = P_{\hat{\sigma}}^{-1}$ , the following inequality is obtained:

$$\begin{bmatrix} He(X_{\hat{\sigma}}^T \tilde{A}_{h_\sigma}^T) + X_{\hat{\sigma}}^T \tilde{N}^T W_{h_\sigma}^{11} \tilde{N} X_{\hat{\sigma}} + X_{\hat{\sigma}}^T \psi^T \psi X_{\hat{\sigma}} & (*) & (*) & (*) \\ B_{h_\sigma h_\sigma}^T & -\mathcal{R} \gamma^2 & (*) & (*) \\ \Gamma_{h_\sigma h_\sigma}^{1T} & \Upsilon_{h_\sigma} U \bar{\Psi} & -2\Upsilon_{h_\sigma} & (*) \\ \Gamma_{h_\sigma h_\sigma}^{2T} & 0 & 0 & W_{h_\sigma}^{22} \end{bmatrix} < 0. \quad (38)$$

By applying the Schur complement, the inequality (38) is equivalent to:

$$\begin{bmatrix} He(X_{\hat{\sigma}}^T \tilde{A}_{h_\sigma}^T) & (*) & (*) & (*) & (*) & (*) \\ B_{h_\sigma h_\sigma}^T & -\mathcal{R} \gamma^2 & (*) & (*) & (*) & (*) \\ \Gamma_{h_\sigma h_\sigma}^{1T} & \Upsilon_{h_\sigma} U \bar{\Psi} & -2\Upsilon_{h_\sigma} & (*) & (*) & (*) \\ \Gamma_{h_\sigma h_\sigma}^{2T} & 0 & 0 & W_{h_\sigma}^{22} & (*) & (*) \\ \psi X_{\hat{\sigma}} & 0 & 0 & 0 & -I & (*) \\ \tilde{N} X_{\hat{\sigma}} & 0 & 0 & 0 & 0 & -\bar{W}_{h_\sigma}^{11} \end{bmatrix} < 0. \quad (39)$$

Then, by applying Lemma 1, the inequality (39) holds if  $\exists(L_{h_\sigma}, R_{h_\sigma})$  such that:

$$\begin{bmatrix} He(L_{h_\sigma} \tilde{A}_{h_\sigma}^T) & (*) & (*) & (*) & (*) & (*) & (*) \\ X_{\hat{\sigma}} - L_{h_\sigma}^T + R_{h_\sigma}^T \tilde{A}_{h_\sigma}^T & -R_{h_\sigma} - R_{h_\sigma}^T & (*) & (*) & (*) & (*) & (*) \\ B_{h_\sigma h_\sigma}^T & 0 & -\mathcal{R} \gamma^2 & (*) & (*) & (*) & (*) \\ \Gamma_{h_\sigma h_\sigma}^{1T} & 0 & \Upsilon_{h_\sigma} U \bar{\Psi} & -2\Upsilon_{h_\sigma} & (*) & (*) & (*) \\ \Gamma_{h_\sigma h_\sigma}^{2T} & 0 & 0 & 0 & W_{h_\sigma}^{22} & (*) & (*) \\ \psi X_{\hat{\sigma}} & 0 & 0 & 0 & 0 & -I & (*) \\ \tilde{N} X_{\hat{\sigma}} & 0 & 0 & 0 & 0 & 0 & -\bar{W}_{h_\sigma}^{11} \end{bmatrix} < 0, \quad (40)$$

which lead, after the application of Lemma 2, to the conditions expressed as LMIs in (25).

Moreover, in order to ensure the decreasing of the Lyapunov function at the switching instants, the following inequality must be verified:

$$V(t^+, e^+(t)) \leq V(t, e(t)), \quad (41)$$

where  $e^+(t)$  is the updated error of the filter in the forthcoming mode  $\hat{j}^+$ .

Following the pioneer work of Pettersson,<sup>55</sup> it can be deduced from (33) that (41) is satisfied if:

$$(x - \hat{x}^+)^T P_{\hat{j}^+}^1 (x - \hat{x}^+) \leq (x - \hat{x})^T P_{\hat{j}}^1 (x - \hat{x}). \quad (42)$$

Moreover, recall that we have  $y = Cx + Dw$ , then assume that when the switches occurs at switching instants,  $\hat{x}^+$  verifies  $y = C\hat{x}^+ + Dw$ . In this case, we can write  $C(x - \hat{x}^+) = 0$  and for any matrix  $G$  of appropriate dimension we have<sup>55</sup>:

$$(x - \hat{x}^+)^T (G^T C + C^T G) (x - \hat{x}^+) = 0. \quad (43)$$

Therefore, from (43), if there exist  $G_j$  such that the equality (26) is satisfied, then the inequality (42) yields to:

$$(x - \hat{x}^+)^T P_{\hat{j}}^1 (x - \hat{x}^+) \leq (x - \hat{x})^T P_{\hat{j}}^1 (x - \hat{x}). \quad (44)$$

Furthermore, the updated value of the filter's state  $\hat{x}^+(t) \in \bar{S}_{\hat{j}^+}$  have now to be determined such that the previous inequality is satisfied. To this end, consider the spectral decomposition  $P_{\hat{j}}^1 = \mathcal{L}_{\hat{j}}^T \mathcal{L}_{\hat{j}}$ , with  $\mathcal{L}_{\hat{j}} = V_{\hat{j}} \sqrt{\Lambda_{\hat{j}}} V_{\hat{j}}^T \in \mathfrak{R}^{n \times n}$ ,<sup>62</sup> (44) is satisfied if:

$$\|\mathcal{L}_{\hat{j}}(x - \hat{x}^+)\| \leq \|\mathcal{L}_{\hat{j}}(x - \hat{x})\|. \quad (45)$$

To find the updated value  $\hat{x}^+$ , lying on the hyper-plane  $y(t) = C\hat{x}^+(t)$ , such that the distance  $\|\mathcal{L}_{\hat{j}}(x - \hat{x}^+)\|$  is minimized, the optimization problem is defined as follows:

$$\begin{aligned} & \min_{\hat{x}^+} \|\mathcal{L}_{\hat{j}}(\hat{x}^+ - \hat{x})\| \\ & \text{subject to} \quad : C\hat{x}^+ = y. \end{aligned} \quad (46)$$

By introducing a scalar  $\alpha_j = \mathcal{L}_{\hat{j}}(\hat{x}^+ - \hat{x})$ , we have  $\mathcal{L}_{\hat{j}}\hat{x}^+ = \alpha_j + \mathcal{L}_{\hat{j}}\hat{x}$ , then the above stated optimization problem can be reformulated as:

$$\begin{aligned} & \min_{\hat{x}^+} \|\alpha_j\| \\ & \text{subject to} \quad : C\mathcal{L}_{\hat{j}}^{-1}\alpha_j = y - C\hat{x}, \end{aligned} \quad (47)$$

which admits for solution the minimum least square length to  $y(t) - C\hat{x}$ , that is:

$$\alpha_j = \left( F\mathcal{L}_{\hat{j}}^{-1} \right)^\dagger (y(t) - C\hat{x}), \quad (48)$$

and so:

$$\mathcal{L}_{\hat{j}}\hat{x}^+ = \mathcal{L}_{\hat{j}}\hat{x} + \left( C\mathcal{L}_{\hat{j}}^{-1} \right)^\dagger (y - C\hat{x}). \quad (49)$$

Finally, left multiplying (49) by  $\mathcal{L}_{\hat{j}}^{-1}$ , the updated value  $\hat{x}^+$  can be computed as (31).

To conclude this proof, it remains to provide an estimate of the domain of attraction  $D_a \subseteq D_{e_x}$  (see requirement i. in the problem statement defined in the previous section). Notice that the Lyapunov function (33) can be rewritten as  $V(t, e(t)) = e_x^T(0)P_{\hat{j}}^1 e_x(0)$ . Hence, let us consider the Lyapunov level set  $\mathcal{L}(1)$  defined, at  $t = 0$ , by (32). Performing a congruence transformation on (27) by  $\text{diag} \left( P_{\hat{j}}^1, 1 \right)$ , then applying the Schur Complement, we get:

$$P_{\hat{j}}^1 - \frac{\mathfrak{Q}_{(v)}^T \mathfrak{Q}_{(v)}}{4\mathcal{Q}_{(v)}^2} \geq 0, \forall \hat{j} \in \{1, \dots, m\}. \quad (50)$$

Pre- and post-multiplying (50) by  $e_x^T(0)$  and its transpose provides:

$$e_x^T(0)P_j^1 e_x^T(0) - \frac{e_x^T(0)\mathfrak{Q}_{(v)}^T \mathfrak{Q}_{(v)} e_x^T(0)}{4Q_{(v)}^2} \geq 0, \forall j \in \{1, \dots, m\}. \quad (51)$$

Consequently, for any  $e_x(0) \in \mathcal{L}(1)$ , the inequality  $e_x^T(0)\mathfrak{Q}_{(v)}^T \mathfrak{Q}_{(v)} e_x^T(0) \leq 4Q_{(v)}^2$  holds and, from the definition (19) of the error domain  $D_{e_x}$ , one gets that  $\mathcal{L}(1) \subseteq D_{e_x}$ . Finally, a simple procedure to enlarge  $\mathcal{L}(1)$  is to maximize the trace of  $X_j^1$  with  $\hat{j} \in \{1, \dots, m\}$ , as proposed in (24). ■

*Remark 7.* The asynchronous switched N-TS  $H_\infty$  filters design conditions includes as special case the design of synchronous switched Filter by replacing the index  $\hat{j}$  with  $j$  in the LMI-based conditions of Theorem 1. Moreover, another special case includes our preliminary results for the design of asynchronous switched T-S filter without UPVs presented in,<sup>16</sup> which can be recovered by eliminating the fourth, fifth, and seventh rows, and by setting  $\mathcal{R} = I$ ,  $\mathcal{S} = I$ . Finally, the special case which considers Lipchitz conditions to deal with the nonlinear consequent part (see Remark 5) can be retrieved by setting  $W_{i_j}^{11} = \mu^2 I$  and  $W_{i_j}^{22} = -I$ .

This concludes the main results. The following section will be dedicated to illustrate its effectiveness over numerical simulations, compared with related recent results from the literature.

## 4 | SIMULATION RESULTS AND DISCUSSION

In order to illustrate the effectiveness of the proposed asynchronous switched N-TS  $H_\infty$  filters design methodology, two simulation examples are presented in this section. The first one is an academic example devoted to compare the conservatism of the LMI conditions provided in Theorem 1 with regards to related recent results proposed in References 11,13,15, and 16. The second example illustrates the effectiveness of the present proposal in simulation by considering a switched model of a mass-spring system drawn from References 30 and 54, extended to the switched nonlinear framework by considering nonlinear stiffness, as proposed in Reference 63, subject to external disturbances and considering arbitrary mismatching switching laws.

### 4.1 | Academic example

Let us consider a switched nonlinear system with three modes ( $m = 3$ , i.e.,  $\sigma(t) \in \{1, 2, 3\}$ ) defined by:

$$\left\{ \begin{array}{l} \begin{array}{l} \dot{x}_1(t) \\ \dot{x}_2(t) \end{array} = \underbrace{\begin{bmatrix} -1 & \Delta_{\sigma(t)}\eta_1(x_1(t)) + \theta_{\sigma(t)} \\ \beta_{\sigma(t)}(3 + \eta_1(x_1(t))) & -2\sigma(t) \end{bmatrix}}_{\bar{f}_{x,\sigma(t)}(x_1(t))} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{g_{x,\sigma(t)}=H^x} \underbrace{\sin(x_2(t))}_{\bar{\phi}(x_2(t))} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{5} + \sigma(t)e^{-2} \end{bmatrix}}_{f_{x,\sigma(t)}^w=B_{\sigma(t)}} w(t) \\ z(t) = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{f_{z,\sigma(t)}} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \frac{1}{2}w(t) \end{array} \right. \quad (52)$$

where  $\eta_1(x_1(t)) = x_1(t)$  and with the switching parameters  $\Delta_1 = \frac{3}{2}$ ,  $\Delta_2 = 2 + \frac{1+2a-b}{6}$ ,  $\Delta_3 = 3 + \frac{3+ab}{6}$ ,  $\theta_1 = 2$ ,  $\theta_2 = 1 + a - \frac{b}{2}$ ,  $\theta_3 = 2 + \frac{ab}{2}$ ,  $\beta_1 = \frac{b}{6}$  and  $\beta_2 = \beta_3 = 0$ . Moreover,  $a$  and  $b$  are two scalar parameters dedicated to further check the feasibility fields of the design conditions proposed in Theorem 1, then to compare their conservatism with the following previous related results from the literature:

- Theorem 1 in Reference 11, which proposes the design of  $H_\infty$  filters for switched T-S systems with asynchronous switching and measured premises,

- Theorem 3 in Reference 13, which considers synchronous filtering for switched T-S systems with persistent dwell-time with measured premises.
- Theorem 1 in Reference 15, which considers non-weighted asynchronous  $H_\infty$  filtering design for switched T-S systems with minimum dwell time switching,
- Theorem 1 in Reference 16, which constitutes a special case (published preliminary results) of this paper, where the premise variables were assumed measurable.

To obtain a classical switched T-S representation of the above nonlinear switched system, we should consider applying the sector nonlinearity approach<sup>9</sup> on both the terms  $\eta_1(x_1(t)) = x_1(t)$  and  $\eta_2(x_2(t)) = \text{sinc}(x_2(t))$ , leading to four vertices in each switched mode for standard T-S modeling without nonlinear consequent parts. However, assuming that the second state variable  $x_2(t)$  is unmeasured, this would make  $\eta_2(x_2(t))$  unavailable. Hence, to circumvent the problem of UPVs in conventional T-S modeling, let us consider the N-TS modeling approach, which allows to reduce the number of vertices to two in each modes. To do so, let us assume that  $x_2(t) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$  is unmeasured. Then, let  $\bar{\phi}(x_2(t)) = \sin(x_2(t)) \in \text{co} \left\{ \frac{2}{\pi}x_2(t), x_2(t) \right\}$  and apply a change of origin such that  $\phi(x_2(t)) = \bar{\phi}(x_2(t)) - \frac{2}{\pi}x_2(t) \in \text{co} \left\{ 0, \frac{\pi-2}{\pi}x_2(t) \right\}$ , leading to  $U = \left[ 0 \quad (\pi-2)/\pi \right]$  (see Property 1). So, applying the sector nonlinearity approach<sup>9</sup> for  $x_1(t) \in [-3, 3]$ , we can write for each switched mode  $j = \{1, 2, 3\}$ :

$$\eta_1(x_1(t)) = h_{1j}(x_1(t)) \times (-3) + h_{2j}(x_1(t)) \times 3, \quad (53)$$

with the membership functions:

$$h_{1j}(x_1(t)) = \frac{3-x_1(t)}{6}, \quad h_{2j}(x_1(t)) = \frac{x_1(t)+3}{6} \quad \text{and} \quad h_{1j}(x_1(t)) + h_{2j}(x_1(t)) = 1, \quad (54)$$

Hence, the switched nonlinear system (52) can be rewritten as a N-TS system (10) with  $r_j = 2$  vertices in each of its  $m = 3$  modes ( $j = \{1, 2, 3\}$ ) with the matrices:

$$\begin{aligned} A_{1_1} &= \begin{bmatrix} -1 & -3\Delta_1 + \theta_1 \\ 0 & -2 + \frac{2}{\pi} \end{bmatrix}, A_{2_1} = \begin{bmatrix} -1 & 3\Delta_1 + \theta_1 \\ b & -2 + \frac{2}{\pi} \end{bmatrix}, A_{1_2} = \begin{bmatrix} -1 & -3\Delta_2 + \theta_2 \\ 0 & -4 + \frac{2}{\pi} \end{bmatrix}, A_{2_2} = \begin{bmatrix} -1 & 3\Delta_2 + \theta_2 \\ 0 & -4 + \frac{2}{\pi} \end{bmatrix}, \\ A_{1_3} &= \begin{bmatrix} -1 & -3\Delta_3 + \theta_3 \\ 0 & -6 + \frac{2}{\pi} \end{bmatrix}, A_{2_3} = \begin{bmatrix} -1 & 3\Delta_3 + \theta_3 \\ 0 & -6 + \frac{2}{\pi} \end{bmatrix}, B_{1_1} = B_{2_1} = \begin{bmatrix} 0 \\ \frac{1}{5} + 1e^{-2} \end{bmatrix}, B_{1_2} = B_{2_2} = \begin{bmatrix} 0 \\ \frac{1}{5} + 2e^{-2} \end{bmatrix}, \\ B_{1_3} &= B_{2_3} = \begin{bmatrix} 0 \\ \frac{1}{5} + 3e^{-2} \end{bmatrix}, H_{ij}^x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, F_{ij} = \begin{bmatrix} 0 & 1 \end{bmatrix}, H_{ij}^z = 0 \quad (\forall j \in \{1, 2, 3\} \text{ and } \forall i_j \in \{1, 2\}), \\ M &= \begin{bmatrix} 1 & 0 \end{bmatrix}, N = \begin{bmatrix} 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \text{and} \quad D = \frac{1}{2}. \end{aligned} \quad (55)$$

Note that, since we consider here that  $x_1(t) \in [-3, 3]$  and  $x_2(t) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ , the obtained N-TS represents exactly the switched nonlinear system (52) on a validity domain  $D_x$  defined in (7) with:

$$\mathfrak{Q} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \mathcal{Q} = \begin{bmatrix} 3 \\ 3 \\ \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix}. \quad (56)$$

For different values of  $a \in [-1, 3]$  and  $b \in [-2, 0.5]$  with a step of 0.1, the feasibility regions obtained from Theorem 1 using YALMIP and SeduMi in MATLAB<sup>64</sup> is compared with the ones obtained from the previous related studies.<sup>11,13,15,16</sup> These results are shown in Figure 1. For these tests, each considered LMI-based conditions are checked over 1066 points  $(a, b)$ . The feasibility field provided by theorem 1 in Reference 11 (65 solutions, 6.1%) contains the one provided by theorem 3 in Reference 13 (91 solutions, 8.54%), which in turn includes the one obtained with theorem 1 in Reference 15 (179

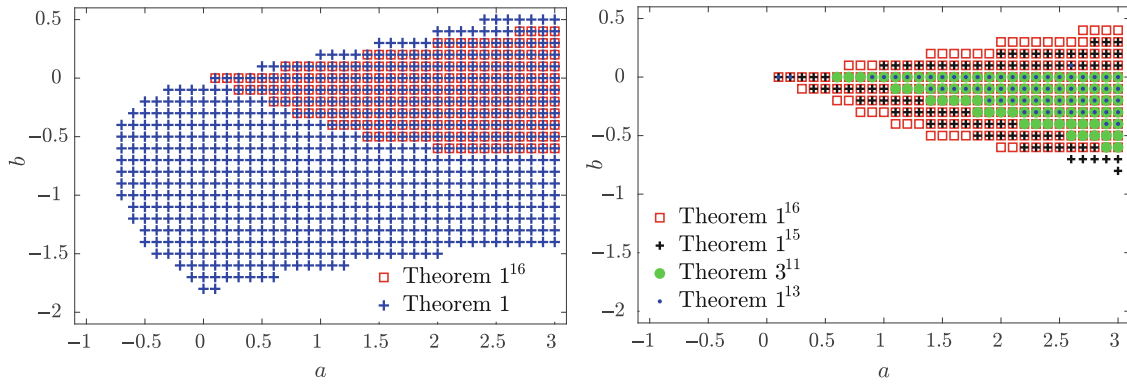


FIGURE 1 Feasibility regions provided by Theorem 1 and the considered studies.

TABLE 1 Computational complexity of the considered studies.

Method	Feasibility	$S_{lr}$	$\mathcal{N}_c$	$\mathcal{N}_d$	$\gamma$	Solver time (s)
Theorem 1 in Reference 11	6.1%	$5n_x + n_z + n_w$	136	181	0.1936	4.77
Theorem 3 in Reference 13	8.54%	$3n_x + n_w + n_y$	52	148	0.4725	1.21
Theorem 1 in Reference 15	16.8%	$12n_x + n_w + n_z$	652	1097	0.1530	24.79
Theorem 1 in Reference 16	19.7%	$4n_x + 3n_z + n_w$	580	2693	1.0	18.80
Theorem 1 in this paper	63.79%	$5n_x + 3n_z + 2n_\phi + n_w$	90	218	0.0245	0.84

solutions, (16.8%). Theorem 1 in Reference 16, which is a particular result of Theorem 1 of the present paper, provides 210 feasible solutions (19.7%) and it is logically included in the feasibility field we have obtained with the proposed dwell-time free LMI conditions. This clearly confirms that our proposal, using N-TS modeling to circumvent the occurrence of UPVs, provides better results and thus a significant conservatism improvement.

Moreover, to emphasize the computational complexity of the LMI-based conditions proposed in Theorem 1, compared with the ones proposed in References 11,13,15, and 16, Table 1 provides the size  $S_{lr}$  of the LMI rows, the number  $\mathcal{N}_c$  of conditions, and the number  $\mathcal{N}_d$  of decision variables. This shows that the conditions presented in Theorem 1 require fewer decision variables and LMI conditions than the ones proposed in References 16 and 15. Then, compared to References 11 and 13, the difference of computational complexity remains reasonable and above all, the dwell-time free LMI conditions proposed in this paper achieve the lowest  $H_\infty$  performance  $\gamma$ , listed in Table 1 as the minimal values obtained from each results over all the test points  $(a, b)$ . Furthermore, for the particular point  $(a, b) = (2, 0)$ , the computational time achieved to solve the different conditions are also given in Table 1 (we use for this test a 2013 HP notebook-2000 laptop having a 2.0GHz Intel Core I7 2nd generation processor and 8GB of memory). This emphasizes that the conditions of Theorem 1, which always provide the best results, are less computationally costly than the other tested results from the literature.

Now, for simulation purposes, let us consider this first example at the particular point  $(a = 1, b = -1)$ , where no solution can be found from previous results.<sup>11,13,15,16</sup> Let the weighting parameters of the  $H_\infty$  criterion (21) be set as  $\mathcal{R} = \text{diag}(150I_2, 150I_2, 1)$ ,  $\mathcal{S} = 1$  and the  $\delta QC$  constraints (see Assumption 3) be defined by  $W_{i_j}^{11} = 0.2I$ ,  $W_{i_j}^{22} = -1$ . Using YALMIP and SeDuMi in Matlab<sup>64,65</sup> to solve the conditions of Theorem 1, we obtain a minimized  $H_\infty$  performance index of  $\gamma = 0.0735$  and the switched N-TS filter gain matrices given by:

$$\hat{A}_{1_1} = \begin{bmatrix} -1.0395 & -5.3603 \\ 0.0023 & -2.7921 \end{bmatrix}, \hat{A}_{2_1} = \begin{bmatrix} -0.4579 & 8.0586 \\ -0.5248 & -2.1121 \end{bmatrix}, \hat{A}_{1_2} = \begin{bmatrix} -1.0393 & -5.4133 \\ 0.0028 & -2.7823 \end{bmatrix}, \hat{A}_{2_2} = \begin{bmatrix} -0.4271 & 8.1495 \\ -0.5397 & -2.1560 \end{bmatrix},$$

$$\hat{A}_{1_3} = \begin{bmatrix} -1.0392 & -5.4596 \\ 0.0037 & -2.7776 \end{bmatrix}, \hat{A}_{2_3} = \begin{bmatrix} -0.4209 & 8.2376 \\ -0.5528 & -2.1814 \end{bmatrix}, \hat{B}_{1_1} = \begin{bmatrix} -0.0077 \\ 0.4232 \end{bmatrix}, \hat{B}_{2_1} = \begin{bmatrix} 0.0123 \\ 0.4206 \end{bmatrix}, \hat{B}_{1_2} = \begin{bmatrix} -0.0074 \\ 0.4236 \end{bmatrix},$$



$$\hat{B}_{2_2} = \begin{bmatrix} 0.0115 \\ 0.420 \end{bmatrix}, \hat{B}_{1_3} = \begin{bmatrix} -0.0076 \\ 0.4240 \end{bmatrix}, \hat{B}_{2_3} = \begin{bmatrix} 0.0104 \\ 0.4188 \end{bmatrix}, \hat{F}_{1_1}^T = \begin{bmatrix} -0.0001 \\ 1.0092 \end{bmatrix}, \hat{F}_{2_1}^T = \begin{bmatrix} -0.0073 \\ 1.0136 \end{bmatrix},$$

$$\hat{F}_{1_2}^T = \begin{bmatrix} 0.0003 \\ 1.00092 \end{bmatrix}, \hat{F}_{2_2}^T = \begin{bmatrix} -0.0087 \\ 1.0146 \end{bmatrix}, \hat{F}_{1_3}^T = \begin{bmatrix} 0.0005 \\ 1.0094 \end{bmatrix}, \hat{F}_{2_3}^T = \begin{bmatrix} -0.0107 \\ 1.0148 \end{bmatrix},$$

as well as the Lyapunov matrices:

$$P_1^1 = \begin{bmatrix} 35.8408 & -0.3675 \\ -0.3675 & 3.4015 \end{bmatrix}^{-1}, P_2^1 = \begin{bmatrix} 34.6381 & -0.4356 \\ -0.4356 & 3.4015 \end{bmatrix}^{-1}, P_3^1 = \begin{bmatrix} 33.6029 & -0.5111 \\ -0.5111 & 3.4015 \end{bmatrix}^{-1}.$$

We assume that the system is switching according to hyper-planes  $S_{j^+}$  which depend on the system's states, while the filter is switching according to the estimated hyper-planes  $\hat{S}_{j^+}$  depending on the estimated states. This makes the system's and the filter's switching laws asynchronous, despite the fact that they are evolving according to the same hyper-planes. For simulation purpose, these linear hyper-planes (8) are defined by  $S_{12} = [3 \ -1]$ ,  $S_{13} = [4 \ -5]$ ,  $S_{21} = [2 \ -9]$ ,  $S_{23} = [1 \ 2]$ ,  $S_{31} = [3 \ 2]$  and  $S_{32} = [-1 \ 0.2]$ . Moreover, to better highlight such an asynchronous switching behavior, the system and the filter have been initialized in different switching modes ( $j_{t=0} = 3, \hat{j}_{t=0} = 1$ ) with their respective initial conditions.  $x(0) = [0 \ 0.9]^T$  and  $\hat{x}(0) = [0 \ 0]^T$  (see Remark 6). Finally, to perform the simulation, we also assume a disturbance input set as  $w(t) = 20 \cos(3.2\pi t)e^{-0.04t} + v(t)$  where  $v(t)$  is a normally distributed random signal (with a sampling period of 0.00025 s, a zero mean value, a maximal amplitude of 4 and a SD equal to 1). Figure 2 shows the evolution of the switching signals, where the asynchronous switching behavior of the designed switched N-TS filter with regard to the switched nonlinear system can be clearly observed. Figure 3 shows the measured output  $y(t)$  and the input disturbance signal  $w(t)$ . Then, the unmeasured output  $z(t)$  and its estimate  $\hat{z}(t)$ , as well as the unmeasured output filtering error  $e_z(t)$ , are plotted in Figure 4.

Moreover, let us notice that from these simulations, for  $t \in [0, t_f]$  with  $t_f = 9$  s, we can provide an estimate of the achieved disturbance attenuation level as:

$$\sqrt{\frac{\int_0^{t_f} e_z^T(t)e_z(t)dt}{\int_0^{t_f} \tilde{w}^T(t)R\tilde{w}(t)dt}} = 0.0079. \quad (57)$$

It shows that the  $\gamma$ -level disturbance attenuation criterion (21) is satisfied since it is lower than  $\gamma = 0.0735$  obtained from the application of Theorem 1. Furthermore, the evolution of the  $\delta QC$  function  $\varphi(t, q_1, q_2) = [(q_1 - q_2)^T \ (\phi(q_1, t) - \phi(q_2, t))^T]^T W_{h_e} [(q_1 - q_2)^T \ (\phi(q_1, t) - \phi(q_2, t))^T]^T$  is depicted in Figure 5, which is always positive. Hence, it verifies the Incremental Quadratic Constraint (18).

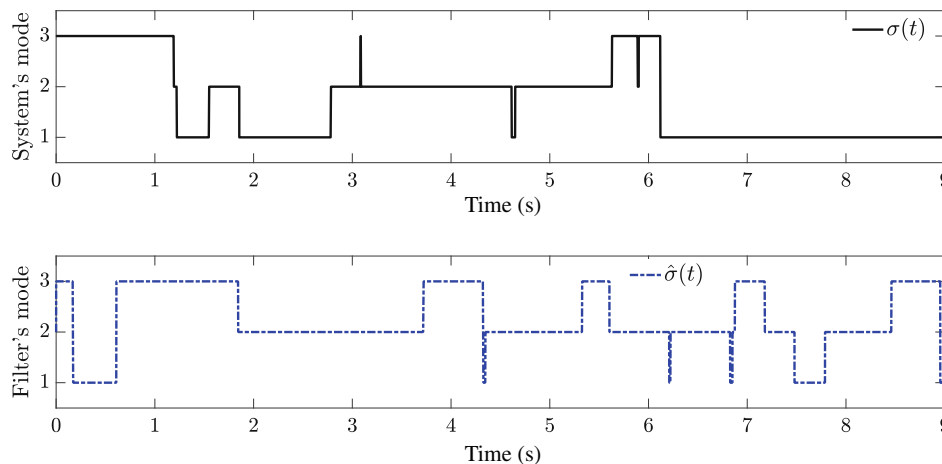


FIGURE 2 Switched modes evolution of the switched system (52) and the designed switched filter (12).

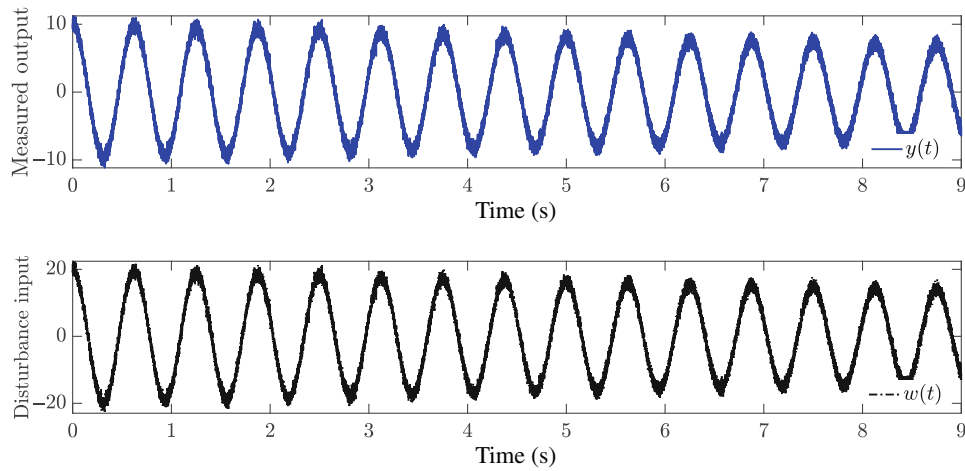


FIGURE 3 Measured output  $y(t)$  and disturbance input  $w(t)$ .

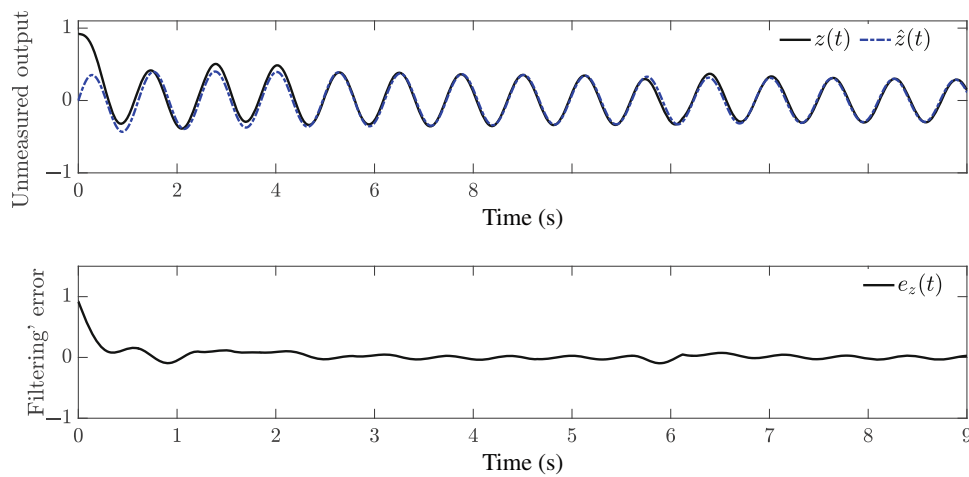


FIGURE 4 Unmeasured output  $z(t)$ , its estimate  $\hat{z}(t)$  and filtering error  $e_z(t)$ .

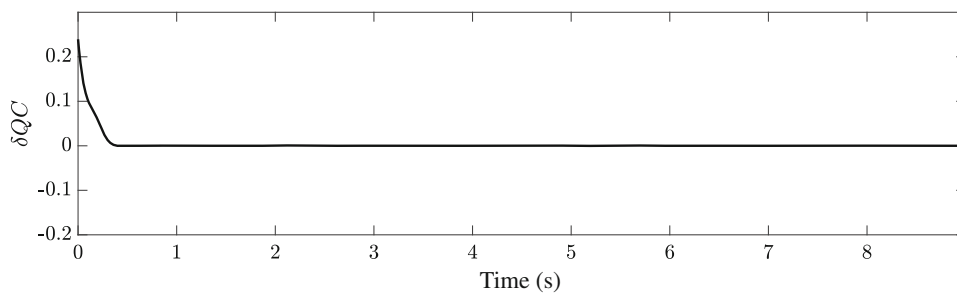
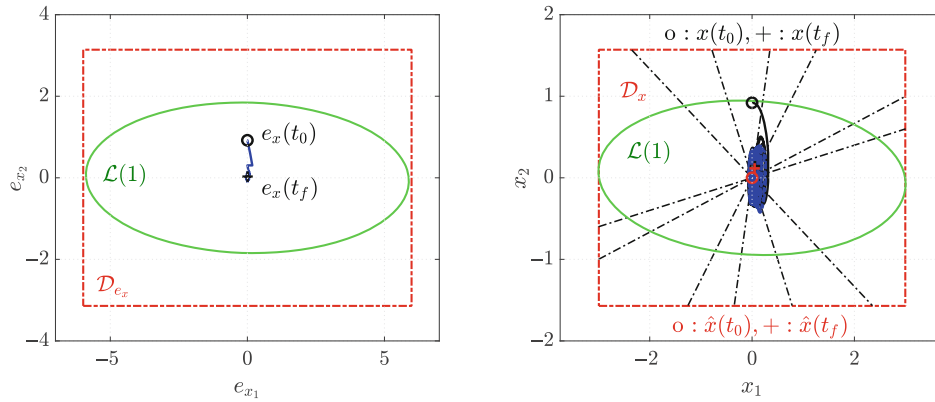


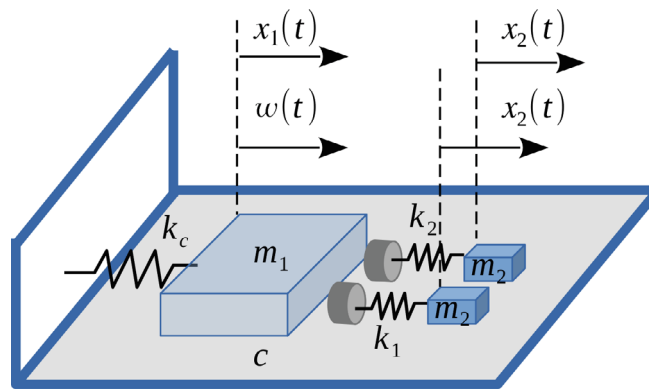
FIGURE 5 Satisfaction of the incremental quadratic constraint (18).

Finally, Figure 6 shows the estimate of the filtering error domain of attraction  $D_a$ , projected on the planes  $(e_{x_1}, e_{x_2})$  and  $(x_1, x_2)$ . We can observe that the state and error trajectories remain in  $D_a$  for initial conditions taken on its edge.

From this first numerical example, we can conclude that the presented simulations show the efficiency of the proposed switched N-TS  $H_\infty$  filter design methodology in providing good estimation  $\hat{z}(t)$  of the unmeasured output  $z(t)$ , with significant conservatism improvements compared to previous related results,<sup>11,13,15,16</sup> despite the occurrence of asynchronous switching and the application of a noisy input disturbance signal.



**FIGURE 6** Projection of  $D_a = \mathcal{L}(1)$  (green lines),  $D_{e_x}$ ,  $D_x$  (red dashed-lines) on planes of interest, states and output error trajectories (blue lines for the filter, black lines for the system), switching hyper-plane frontiers (dashed-dotted black lines).



**FIGURE 7** Switched mass-spring system.

## 4.2 | Switched nonlinear mass-spring system

Let us consider the switched mass-spring system depicted in Figure 7, drawn from References 30 and 54, and governed by the following switched state space model with  $\sigma(t) \in \{1, 2\}$ :

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_c+k_{\sigma(t)}}{m_1} & \frac{k_{\sigma(t)}}{m_1} & -\frac{c}{m_1} & 0 \\ \frac{k_{\sigma(t)}}{m_2} & -\frac{k_{\sigma(t)}}{m_2} & 0 & -\frac{c}{m_2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix} w(t) \\ z(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} x(t) \\ y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x(t) + 0.8w(t), \end{cases} \quad (58)$$

where  $x_1(t)$  and  $x_2(t)$  denotes the displacements of respectively the masses  $m_1 = 6 \text{ kg}$  and  $m_2 = 1 \text{ kg}$ ;  $k_c$  is the stiffness of the left spring while  $k_1$  and  $k_2$  are the stiffness of both the right springs which are assumed to switch according to further defined switching hyper-planes.

Now, to extend the mass-spring system (58) in the nonlinear framework, let us assume hardening springs, as modeled in Reference 63, where:

$$k_c = \kappa_c (1 + a_c^2 x_1^2(t)) \quad \text{and} \quad k_{\sigma(t)} = \kappa_{\sigma(t)} (1 + a_{\sigma(t)}^2 x_2^2(t)), \quad \text{for } \sigma(t) \in \{1, 2\}. \quad (59)$$

where  $\kappa_c = 10 \text{ N/m}$ ,  $\kappa_1 = 10 \text{ N/m}$  and  $\kappa_2 = 20 \text{ N/m}$  are the nominal springs' stiffness;  $a_c = 0.4$ ,  $a_1 = 0.1$  and  $a_2 = 0.2$  are the spring's hardening coefficients.

Hence, by substituting (59) in (58), and assuming that  $x_2$  is unmeasured, it yields the following nonlinear switched model of the considered nonlinear mass-spring system:

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\kappa_c}{m_1} (1 + a_c^2 x_1^2(t)) - \frac{\kappa_{\sigma(t)}}{m_1} & \frac{\kappa_{\sigma(t)}}{m_1} & -\frac{c}{m_1} & 0 \\ \frac{\kappa_{\sigma(t)}}{m_2} & -\frac{\kappa_{\sigma(t)}}{m_2} & 0 & -\frac{c}{m_2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{\kappa_{\sigma(t)}}{m_1} a_{\sigma(t)}^2 \\ -\frac{\kappa_{\sigma(t)}}{m_2} a_{\sigma(t)}^2 \end{bmatrix} (x_2^3(t) - x_2^2(t)x_1(t)) + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix} w(t) \\ z(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} x(t) \\ y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x(t) + 0.8w(t) \end{cases} \quad (60)$$

Let us assume that  $x_1(t) \in [-2, 2]$  and  $x_2(t) \in [-1, 1]$ . Thus, we can define the validity domain  $D_x$  in (7) with:

$$\mathfrak{g} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \text{ and } \mathcal{Q} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}. \quad (61)$$

Also, we can write:

$$\bar{\phi}(x(t)) = x_2^3(t) - x_2^2(t)x_1(t) = \begin{bmatrix} -x_2^2(t) & x_2^2(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in \text{co} \{-Ux(t), Ux(t)\}, \quad (62)$$

with  $U = [-1 \ 1 \ 0 \ 0]$ .

Hence, we can apply the change of variable  $\phi(x(t)) = \bar{\phi}(x(t)) + Ux(t) \in \text{co} \{0, 2Ux(t)\}$ , so that the switched nonlinear system (60) can be rewritten as:

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\kappa_c}{m_1} (1 + a_c^2 \eta(x_1(t))) + \frac{\kappa_{\sigma(t)}}{m_1} (a_{\sigma(t)}^2 - 1) & \frac{\kappa_{\sigma(t)}}{m_1} (1 - a_{\sigma(t)}^2) & -\frac{c}{m_1} & 0 \\ \frac{\kappa_{\sigma(t)}}{m_2} (1 - a_{\sigma(t)}^2) & \frac{\kappa_{\sigma(t)}}{m_2} (a_{\sigma(t)}^2 - 1) & 0 & -\frac{c}{m_2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{\kappa_{\sigma(t)}}{m_1} a_{\sigma(t)}^2 \\ -\frac{\kappa_{\sigma(t)}}{m_2} a_{\sigma(t)}^2 \end{bmatrix} \phi(x(t)) + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix} w(t) \\ z(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} x(t) \\ y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x(t) + 0.8w(t), \end{cases} \quad (63)$$

with  $\eta(x_1(t)) = x_1^2(t)$ , which can be decomposed, by apply the sector nonlinearity approach,<sup>9</sup> as:

$$\eta(x_1(t)) = \underbrace{\frac{x_1^2(t)}{4}}_{h_1(x_1(t)) \geq 0} \times 4 + \underbrace{\frac{4 - x_1^2(t)}{4}}_{h_2(x_1(t)) \geq 0} \times 0, \text{ where } h_1(x_1(t)) + h_2(x_1(t)) = 1, \forall j \in \{1, 2\}, \quad (64)$$

leading to a N-TS model (10) with two switched modes ( $j \in \{1, 2\}$ ) and two T-S vertices ( $i_j \in \{1, 2\}$ ) specified by the matrices:

$$\begin{aligned} A_{1_j} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\kappa_c}{m_1} (1 + 4a_c^2) + \frac{\kappa_j}{m_1} (a_j^2 - 1) & \frac{\kappa_j}{m_1} (1 - a_j^2) & -\frac{c}{m_1} & 0 \\ \frac{\kappa_j}{m_2} (1 - a_j^2) & \frac{\kappa_j}{m_2} (a_j^2 - 1) & 0 & -\frac{c}{m_2} \end{bmatrix}, A_{2_j} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\kappa_c}{m_1} + \frac{\kappa_j}{m_1} (a_j^2 - 1) & \frac{\kappa_j}{m_1} (1 - a_j^2) & -\frac{c}{m_1} & 0 \\ \frac{\kappa_j}{m_2} (1 - a_j^2) & \frac{\kappa_j}{m_2} (a_j^2 - 1) & 0 & -\frac{c}{m_2} \end{bmatrix}, \\ B_{1_j} &= B_{2_j} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix}, H_{1_j}^x = H_{2_j}^x = \begin{bmatrix} 0 \\ 0 \\ \frac{\kappa_j}{m_1} a_j^2 \\ -\frac{\kappa_j}{m_2} a_j^2 \end{bmatrix}, F_{1_j} = F_{2_j} = N = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, H_{1_j}^z = H_{2_j}^z = 0, C = M = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, D = \frac{4}{5}. \end{aligned} \quad (65)$$

To illustrate the effectiveness of the proposed switched nonlinear design procedure, let the weighting parameters of the  $H_\infty$  criterion (21) be set as  $\mathcal{R} = \text{diag}(250I_4, 250I_4, 1)$  and  $S = 1$  and the  $\delta QC$  constraints (see Assumption 3) be defined by  $W_{ij}^{11} = 0.26I$ ,  $W_{ij}^{22} = -0.1$ . Solving the LMI conditions of Theorem 1 in Matlab with the use of YALMIP and SeDuMi,<sup>64</sup> one obtains a minimized  $H_\infty$  performance index  $\gamma = 0.0775$  and the switched N-TS filter (12) gain matrices given by:

$$\begin{aligned} \hat{A}_{1_1} &= \begin{bmatrix} 0.07530.1450 & 1.00000.0000 \\ 0.04440.0673 & -0.00001.0000 \\ -4.12672.1224 & -0.8333 - 0.0000 \\ 10.4952 - 10.2954 & 0.0000 - 5.0000 \end{bmatrix}, \hat{A}_{2_1} = \begin{bmatrix} 0.0833 & 0.1442 & 1.0000 & -0.0000 \\ 0.0619 & 0.0922 & -0.0000 & 1.0000 \\ -3.4674 & 1.3555 & -0.8333 & 0.0000 \\ 9.8202 & -9.4648 & -0.0000 & -5.0000 \end{bmatrix}, \\ \hat{B}_{1_1} &= \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.2071 \\ -0.0003 \end{bmatrix}, \hat{B}_{2_1} = \begin{bmatrix} 0.0000 \\ 0.0001 \\ 0.2070 \\ -0.0000 \end{bmatrix}, \\ \hat{A}_{1_2} &= \begin{bmatrix} 0.1238 & 0.0954 & 1.0000 & -0.0000 \\ 0.0836 & 0.0693 & -0.0000 & 1.0000 \\ -5.7447 & 3.7354 & -0.8333 & 0.0000 \\ 19.7566 & -19.9103 & 0.0000 & -5.0000 \end{bmatrix}, \hat{A}_{2_2} = \begin{bmatrix} 0.1015 & 0.1080 & 1.0000 & -0.0000 \\ 0.0971 & 0.0910 & 0.0000 & 1.0000 \\ -5.0383 & 2.9238 & -0.8333 & -0.0000 \\ 19.0457 & -18.9358 & -0.0000 & -5.0000 \end{bmatrix}, \\ \hat{B}_{1_2} &= \begin{bmatrix} 0.0000 \\ 0.0001 \\ 0.2071 \\ 0.0000 \end{bmatrix}, \hat{B}_{2_2} = \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.2071 \\ 0.0001 \end{bmatrix}, \\ \hat{F}_{1_1}^T &= \begin{bmatrix} 0.0029 \\ 1.0014 \\ 0.0000 \\ 0.0000 \end{bmatrix}, \hat{F}_{2_1}^T = \begin{bmatrix} 0.0015 \\ 1.0029 \\ -0.0000 \\ -0.0000 \end{bmatrix}, \hat{F}_{1_2}^T = \begin{bmatrix} 0.0044 \\ 1.0046 \\ 0.0000 \\ -0.0000 \end{bmatrix}, \hat{F}_{2_2}^T = \begin{bmatrix} 0.0039 \\ 1.0037 \\ -0.0000 \\ 0.0000 \end{bmatrix}, \end{aligned} \quad (66)$$

and the Lyapunov matrices:

$$P_1^1 = \begin{bmatrix} 1.9411 & 1.7887 & -1.1547 & 0.1795 \\ 1.7887 & 3.9090 & -3.0718 & -4.2428 \\ -1.1547 & -3.0718 & 5.2046 & 0.8520 \\ 0.1795 & -4.2428 & 0.8520 & 26.4345 \end{bmatrix}^{-1}, P_2^1 = \begin{bmatrix} 1.8585 & 1.6720 & -1.1351 & 0.2570 \\ 1.6720 & 3.9090 & -3.0718 & -4.2428 \\ -1.1351 & -3.0718 & 5.2046 & 0.8520 \\ 0.2570 & -4.2428 & 0.8520 & 26.4345 \end{bmatrix}^{-1}. \quad (67)$$

For simulation purposes, we assume that the switched nonlinear system (60) switches according to the hyper-planes (8) with  $S_{12} = [1 \ 1 \ 0 \ 0]$  and  $S_{21} = [1 \ -2 \ 0 \ 0]$ . Moreover, to better highlight the asynchronous behavior of the switched mass-spring system and the designed switched N-TS filter, it is assumed that the latter switches according to different hyper-planes (13) with  $\hat{S}_{12} = [2 \ -1 \ 0 \ 0]$  and  $\hat{S}_{21} = [1 \ -4 \ 0 \ 0]$ . Furthermore we set the initial conditions to  $x(0) = [0.5 \ 0.4 \ 0.01 \ 0.2]^T$  and  $\hat{x}(0) = [0 \ 0 \ 0 \ 0]^T$ , so that the system and the filter are respectively initialized in different modes, i.e.  $j_{t=0} = 2$  and  $\hat{j}_{t=0} = 1$ . We set the noise disturbance input as  $w(t) = 8 \sin(0.8\pi t)e^{-0.01t} + v(t)$ , where  $v(t)$  is a normally distributed random signal (with a sampling period of 0.001 s, a zero mean value, a maximal amplitude of 2 and a standard deviation equal to 1).

Figure 8 shows the evolution of the measured output  $y(t)$  and the noisy disturbance  $w(t)$ . Figure 9 shows the evolution of the asynchronous switching signals. The unmeasured output  $z(t)$  and its estimate  $\hat{z}(t)$  are depicted in Figure 10, together with the filtering error  $e_z(t)$ . Figure 11 confirms that the  $\delta QC$  constraints (18) is always verified since it remains positive. Finally, the estimation of the domain of attraction  $\mathcal{D}_a$  are plotted in Figure 12, projected on the planes of

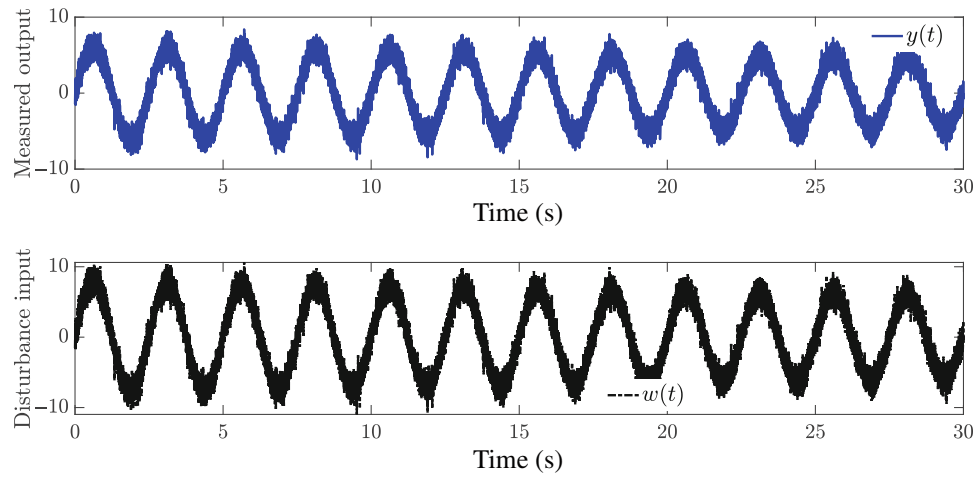


FIGURE 8 Measured output  $y(t)$  and disturbance input  $w(t)$ .

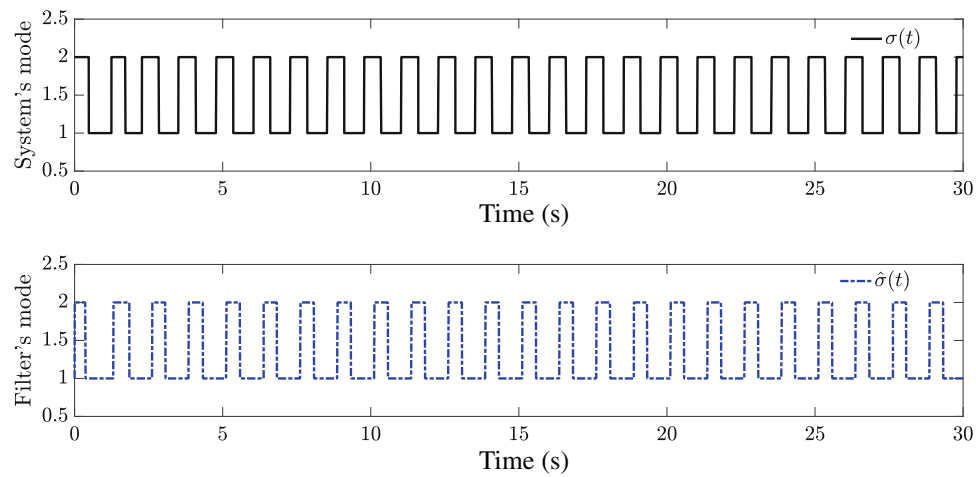


FIGURE 9 Switched modes evolution of the switched system (60) and the designed switched filter (12).

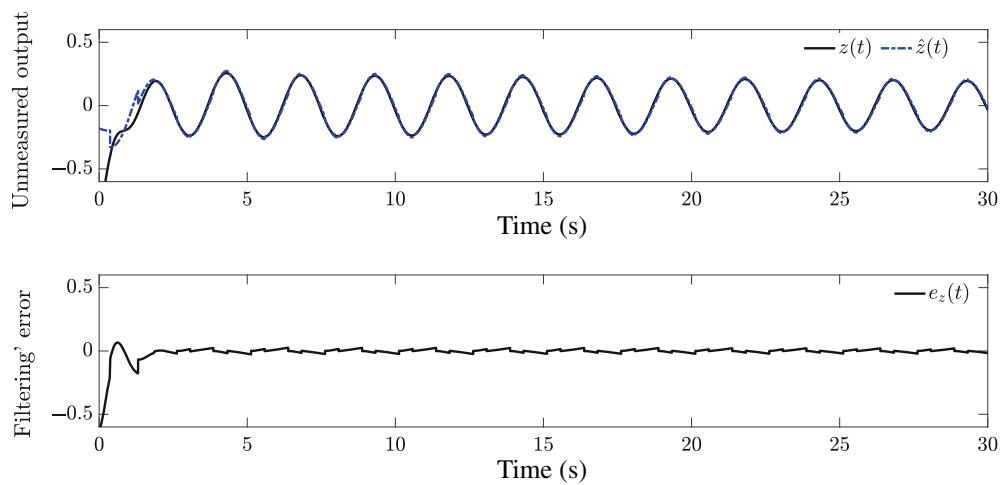


FIGURE 10 Unmeasured output  $z(t)$ , its estimate  $\hat{z}(t)$  and filtering error  $e_z(t)$ .

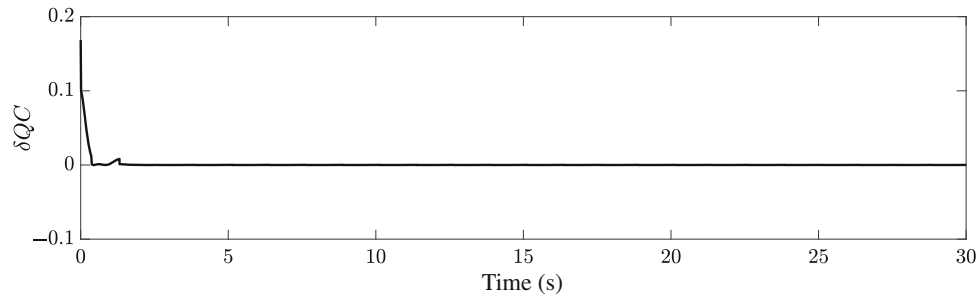


FIGURE 11 Satisfaction of the incremental quadratic constraint (18).

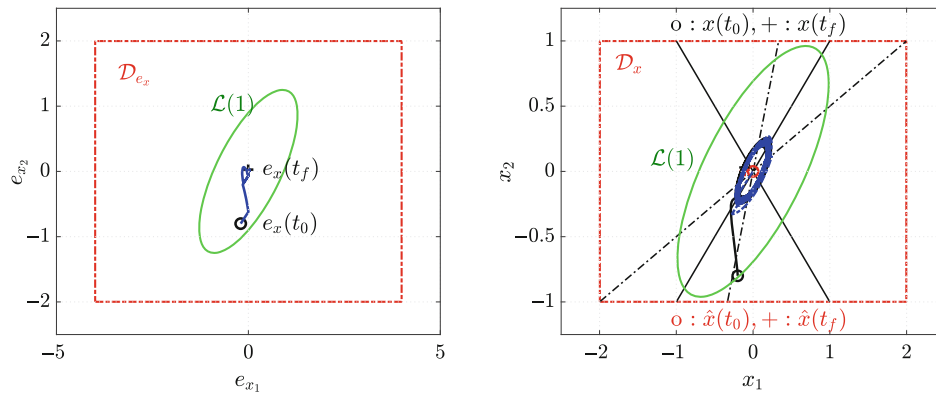


FIGURE 12 Projection of  $\mathcal{D}_a = \mathcal{L}(1)$  (green lines),  $\mathcal{D}_{e_x}$ ,  $\mathcal{D}_x$  (red dashed-lines) on planes of interest, states and output error trajectories (blue lines for the filter, black lines for the system), system's switching hyper-plane frontiers (black lines), filter's switching hyper-plane frontiers (dashed-dotted black lines)

interest  $(e_{x_1}, e_{x_2})$  and  $(x_1, x_2)$ , together with the system's and filter's state trajectories. As expected, for initial conditions taken inside  $\mathcal{D}_a$ , the state estimation error  $e_x(t)$ , the state trajectory  $x(t)$  and the filter's state  $\hat{x}(t)$  remains in  $\mathcal{D}_a$ . Furthermore, from the simulation results depicted in Figures 8 and 9, for  $t \in [0, t_f]$  with  $t_f = 30s$ , we obtain an estimate of the achieved disturbance attenuation level as:

$$\sqrt{\frac{\int_0^{t_f} e_z^T(t) e_z(t) dt}{\int_0^{t_f} \tilde{w}^T(t) R \tilde{w}(t) dt}} = 0.0039, \quad (68)$$

which satisfies the  $\gamma$ -level disturbance attenuation criterion (21) since it is lower than the obtained  $\gamma = 0.0775$  from the solution of Theorem 1.

From these simulations, we can conclude that the designed switched N-TS filter successfully estimates the unmeasured displacement  $z(t) = x_2(t)$  despite the switched asynchronous phenomena as well as the noisy input disturbances affecting the dynamic behavior of the considered switched nonlinear mass-spring system.

## 5 | CONCLUSION

This paper has investigated the design of asynchronous switched T-S  $H_\infty$  filters with nonlinear consequent parts (switched N-TS filters) for a class of continuous-time switched nonlinear systems involving sector bounded nonlinearities, subject to bounded exogenous disturbances and mismatching switching laws. Such asynchronous switched N-TS filters are proposed to estimate unmeasured output under arbitrary switching, without any consideration of dwell-time (dwell-time independent), and to circumvent the occurrence of unmeasured premise variables brought by standard T-S modeling.

The design conditions, expressed in terms of Linear Matrix Inequalities, have been derived based on a multiple Lyapunov functional and an  $H_\infty$  criterion to attenuate the input disturbances. Incremental Quadratic Constraints have been assumed to cope with the system's unmeasured nonlinear terms, rewritten as nonlinear consequent parts with the considered N-TS modeling approach. Such obtained N-TS model being valid on a compact subset of the system's state space, an optimization procedure has also been proposed to estimate the closed-loop domains of attraction of the filtering-error. Two simulation examples have been provided to illustrate the effectiveness of this proposal. First, an academic example, used to compare the conservatism of the proposed conditions with the recent related literature, shown the superiority of the proposed design conditions in terms of relaxation as well as in terms of lower computational complexity. The second example consists in the simulation of a switched mass-spring system with input disturbances, drawn from the literature then extended to the nonlinear framework by considering hardening springs. Again, with this second example, the effectiveness of the proposed design conditions have been confirmed, highlighting the ability of the designed switched N-TS filter to provide good estimation of the unmeasured output despite noisy input disturbances and mismatching switching sets (i.e., switched asynchronous behavior). Starting from this proposal, many extensions are possible. Among them, we intend, for our further works, to focus on the adaptation of such filtering techniques in the context of non-fragile design or fault-tolerant control.


## DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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**How to cite this article:** Chekakta I, Jabri D, Motchon KMD, Guelton K, Belkhiat DEC. Design of asynchronous switched Takagi–Sugeno model-based  $H_\infty$  filters with nonlinear consequent parts for switched nonlinear systems. *Int J Adapt Control Signal Process.* 2023;37(6):1511-1535. doi: 10.1002/acs.3588