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# Logical Analysis of Fuzzy Constraint Satisfaction Problems

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## Abstract

*Fuzzy sets naturally model elastic constraints. Fuzzy constraints satisfaction problem (FCSP) enable the introduction of different kinds of flexibility. Levels of priority can be attached to constraints, and satisfaction levels can be fuzzily thresholded. Fuzzy constraints are aggregated by min operation into a fuzzy set membership function to be maximized (discrimin and leximin refinements of the min ordering can be also used in this maximization). This representation framework, originally expressed in terms of membership functions, is equivalently translated into a set of prioritized crisp constraints, in this paper. We take advantage of this representation for modelling aggregations different from min, expressing either reinforcement and compensation. This offers a logical understanding of fuzzy constraints. In relation to this new representation scheme, computational aspects are briefly exemplified and discussed.*

## 1. Introduction

Fuzzy constraint satisfaction problems (FCSP) are an extension of constraint satisfaction problems (CSP), where elastic constraints can replace crisp ones [1]. In a FCSP a constraint is satisfied to a degree (rather than satisfied or not satisfied), and the acceptability of a potential solution to an aggregated set of fuzzy constraints becomes a gradual notion. Another way of providing flexibility, is by attaching priority levels to constraints, or by introducing fuzzy thresholds. Constraints may be also conditioned. These different ways of relaxing crisp CSPs have been proposed and investigated by different authors [23][4][7][8]. FCSPs are naturally encountered in different areas such as structural design [18], or scheduling [16] for instance.

Solving a FCSP amounts to find an instantiation for its variables that maximizes the satisfaction level of the least satisfied constraint, since the constraints are aggregated by the min operator. Due to the idempotence of the min operation, most CSP computation techniques easily extend to the fuzzy setting [16][8]. Since min-optimal solutions to a FCSP may be numerous, two refinements of the ranking of solutions have been proposed, namely the *discrimin* and *leximin orderings* [9][10]. These refinements do not only take into account the least satisfied constraint but also compare the other satisfaction levels.

It is well known that a fuzzy set can be viewed as a weighted collection of crisp sets corresponding to its  $\alpha$ -level cut. This has been extensively used in fuzzy

numbers computation and in fuzzy programming. When a finite set of satisfaction degrees is used, which is often sufficient in FCSPs, this provides a convenient, finite representation of a fuzzy set as a nested collection of crisp sets. One step further is to turn the  $\alpha$ -level cuts into weighted logical proposition. Namely, in the framework of possibilistic logic [13], it has been shown that a fuzzy set over a set of interpretations can faithfully represent a set of classical propositions, weighted in terms of necessity degrees. This idea applies both to fuzzy sets which represent uncertain pieces of information (as in reasoning under uncertainty) and to fuzzy sets encoding preference profiles (as in fuzzy optimization). Following this view, this paper systematically investigates the expression of FCSPs in terms of collections of crisp constraints weighted in terms of priority.

This logical interpretation of FCSPs has several benefits. Apart from the fact it should contribute to a better understanding of FCSP issues, it corresponds to an expression mode (classical constraints + priority levels) which is also natural for the user. The logical setting also enable us to represent fuzzy constraints aggregated by operations other than min. Thus logical handling of constraints can be also useful for communicating with the user and for providing him with explanations about of the quality of the solution found. This point will not be further discussed in this paper. Lastly, the logical view may help to design algorithms for solving FCSPs, especially when looking for discrimin or leximin optimal solutions.

The rest of the paper is organized into three parts. Section 2 gives the background on FCSPs. Section 3 provides the logical counterpart of the FCSP model, and Section 4 briefly discusses computational issues.

## 2. Background on FCSPs

A FCSP is defined by a set of decision variables  $X = \{X_1, \dots, X_N\}$ , a set of domains  $D = \{D_1, \dots, D_N\}$  where  $D_i$  is the finite domain of  $X_i$ , and a set of fuzzy constraints  $C = \{C_1, \dots, C_K\}$  where  $C_k$  is a fuzzy set defined over the Cartesian product of the domains of the variables related by  $C_k$ . Therefore, a membership value is associated to each tuple of values of the variables related by  $C_k$ . The membership degrees express preferences among solutions by ranking the instantiations which are more or less acceptable for the satisfaction of the flexible constraint  $C_k$ .

In the following, we consider fuzzy sets defined by a finite linearly ordered valuation set. For instance, the

following sets  $\{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$ , or  $\{0, 0.2, 0.5, 0.8, 1\}$  can be used as a qualitative valuation set. If only the ordering between grades is meaningful, fuzzy set intersection, union and complementation can be only defined by min, max and the order-reversing operation of the valuation set. The valuation set is denoted  $L = \{\alpha_0 = 0, \alpha_1, \dots, \alpha_L = 1\}$ , with  $L + 1$  elements, where 0 and 1 denote the bottom and top elements respectively; the order-reversing is still denoted by  $1 - (\cdot)$ . Finite scales go well with finite domains; however a finite scale still provides a reasonable approximation for continuous domains, when dealing with finite sets of constraints.

A solution to a FCSP is denoted by  $\mathbf{x}$  and is a vector of values in  $D_1 \times \dots \times D_N$ . The level of satisfaction of the constraints by  $\mathbf{x}$  is denoted by  $\text{sat}(\mathbf{x})$  and is defined, as proposed by Bellman and Zadeh [1]:

$$\text{sat}(\mathbf{x}) = \min_k \mu_{C_k}(\mathbf{x}),$$

where  $\mu_{C_k}(\mathbf{x})$  is short for  $\mu_{C_k}(\mathbf{x}^{\downarrow V(C_k)})$  where  $\mathbf{x}^{\downarrow V(C_k)}$  is the restriction of  $\mathbf{x}$  to the values of the variables of the constraint  $C_k$ . The satisfaction degree of a solution is therefore the satisfaction of the least satisfied constraint. An optimal solution, so called "min optimal", is  $\mathbf{x}^*$  such that  $\text{sat}(\mathbf{x}^*) = \max_{\mathbf{x}} \min_k \mu_{C_k}(\mathbf{x})$ .

The satisfaction level of a min optimal solution of a FCSP  $P$  is said to be the consistency degree of  $P$  and is denoted by  $\text{Cons}(P)$ . It is important to remark that the aggregation of the satisfaction levels of the constraints by min presupposes that the degrees of satisfaction of the different constraints commensurate.

## 2.1. Representation enhancements

In this framework prioritized constraints can be also modelled. Solutions that satisfy all constraints are preferred if any. In case of a completely inconsistent problem  $P$  ( $\text{Cons}(P) = 0$ ), less priority constraints have to be dropped, and the chosen solution satisfies only constraints with higher priorities.

We use a finite linearly ordered scale for assessing priorities, where a priority equal to 1 means that the constraint is imperative, while a priority equal to 0 means that it is completely allowed to violate the constraint, and intermediate levels estimate to what extent the constraint is imperative. A pair  $(C, \rho)$  where  $C$  is a crisp constraint and  $\rho$  a priority level can be represented by a fuzzy constraint  $C'$  [8] such that:

$$\begin{aligned} \mu_{C'}(\mathbf{x}) &= 1 \text{ if } \mathbf{x} \text{ satisfies } C \\ &= 1 - \rho \text{ if } \mathbf{x} \text{ violates } C. \end{aligned}$$

Thus, constraint  $C$  is considered as satisfied at least to degree  $1 - \rho$  whatever the considered solution, whether it satisfies  $C$  or not. This way of modelling priority presupposes a connection between levels of constraint satisfaction and levels of constraint priority, which is established by the order-reversing operation  $1 - (\cdot)$ . From a possibility theory point of view,  $\rho$  is a necessity degree and  $1 - \rho$  a possibility degree [15].

A priority  $\rho$  can also be attached to a fuzzy constraint  $C$ ; it is modelled by the fuzzy set  $C'$

$$\mu_{C'}(\mathbf{x}) = \max(1 - \rho, \mu_C(\mathbf{x})). \quad (1)$$

In practice, it may be useful to add a safeguard constraint  $C^*$  (with priority 1) in case some solutions among the ones which violates  $C$  are really unacceptable, thus changing (1) into

$$\mu_{C'}(\mathbf{x}) = \min(\mu_{C^*}(\mathbf{x}), \max(1 - \rho, \mu_C(\mathbf{x}))). \quad (2)$$

Another type of relaxation of a fuzzy constraint is based on the use of a threshold [10]. The idea is that a constraint  $C$  will be regarded, after relaxation, as fully satisfied by  $\mathbf{x}$  if its level of satisfaction by  $\mathbf{x}$  reaches at least a threshold  $\theta$ . It express a discounting. It amounts to replace  $C$  by the fuzzy set  $C'$  defined by:

$$\mu_{C'}(\mathbf{x}) = \theta \rightarrow \mu_C(\mathbf{x}), \quad (3)$$

where  $a \rightarrow b$  is Gödel implication ( $a \rightarrow b = 1$  if  $a \leq b$ ,  $a \rightarrow b = b$  if  $a > b$ ). Note that if  $\theta$  is not reached by  $\mathbf{x}$ , the actual level which is reached is kept,  $\mu_{C'}(\mathbf{x}) = \mu_C(\mathbf{x})$ .

Obviously, it is possible to have a fuzzy constraint  $C$  with priority  $\rho$  and threshold  $\theta$ , together with a safeguard constraint  $C^*$ . It is represented by (see Fig. 1)

$$\mu_{C'}(\mathbf{x}) = \min(\mu_{C^*}(\mathbf{x}), \max(1 - \rho, \theta \rightarrow \mu_C(\mathbf{x}))). \quad (4)$$

Interestingly enough, the min-aggregation of prioritized and thresholded constraints is also encountered in the expression of extended division operation in fuzzy relational databases [14].

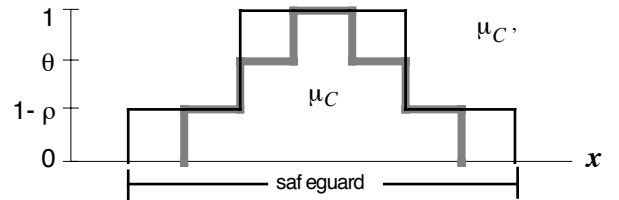


Figure 1. Thresholded prioritized constraint with safeguard

Conditional constraints can also be modelled in this framework. Such a constraint applies only if another one is satisfied. Let  $C_i / C_k$  denote the constraint  $C_i$  conditioned by a constraint  $C_k$ . This should be interpreted as follows.  $C_i$  is imperative if  $C_k$  is satisfied and  $C_i$  can be discarded otherwise. Formally, we have:

$$\mu_{C_i / C_k}(\mathbf{x}) = \max(\mu_{C_i}(\mathbf{x}), 1 - \mu_{C_k}(\mathbf{x})). \quad (5)$$

When  $C_k$  is crisp, it expresses that  $C_i$  has priority 1 (in the sense of (2)) when  $C_k$  is satisfied, and 0 otherwise. Indeed if  $C_k$  is not satisfied,  $\mu_{C_i / C_k}(\mathbf{x}) = 0$ , which means that constraint  $C_i$  is ignored. In general, when  $C_k$  is a fuzzy constraint,  $\mu_{C_k}(\mathbf{x})$  can be considered as the priority level of  $C_i$ , which thus depends on the considered instantiation  $\mathbf{x}$ .

## 2.2 Refinements of min ordering

To solve a FCSP, we search for an instantiation that maximizes the satisfaction level of the least satisfied constraint. This approach does not compare the satisfaction levels of the other constraints, and therefore, min optimal solutions cannot be discriminated.

There exist two main refinements for the min ordering: the discrimin, and the leximin orderings [9][10]. With the discrimin ordering we look for the lowest satisfied constraint among the constraints that are not equally satisfied. Namely, let  $\Delta(\mathbf{x}, \mathbf{y})$  be the subset of constraints that are not equally satisfied by  $\mathbf{x}$  and  $\mathbf{y}$ , i.e., the  $C_k$ 's such that  $\mu_{C_k}(\mathbf{x}) \neq \mu_{C_k}(\mathbf{y})$ . Formally the discrimin is a partial ordering defined by:

$$\mathbf{x} >_{\text{disc}} \mathbf{y} \Leftrightarrow \min_{C_k \in \Delta(\mathbf{x}, \mathbf{y})} \mu_{C_k}(\mathbf{x}) > \min_{C_k \in \Delta(\mathbf{x}, \mathbf{y})} \mu_{C_k}(\mathbf{y})$$

Consider  $\mathcal{S}_x$  the fuzzy set of constraints satisfied by  $\mathbf{x}$  which associates to each constraint  $C_k$  its satisfaction level  $\mu_{C_k}(\mathbf{x})$ . The  $\alpha$ -cut of  $\mathcal{S}_x$  is the set of constraints that are satisfied at least at level  $\alpha$ , and is denoted by  $(\mathcal{S}_x)_\alpha = \{C_k \in \mathcal{C} \mid \mu_{C_k}(\mathbf{x}) \geq \alpha\}$ . The discrimin ordering can be interpreted in terms of inclusion of  $\alpha$ -cuts. It consists in comparing the  $\alpha$ -cuts of  $\mathcal{S}_x$  and of  $\mathcal{S}_y$ , from the lowest level to the greatest one, until reaching a level  $\alpha$  such that  $(\mathcal{S}_x)_\alpha \neq (\mathcal{S}_y)_\alpha$ . At this level if a strict inclusion  $(\mathcal{S}_x)_\alpha \supset (\mathcal{S}_y)_\alpha$  holds, then  $\mathbf{x} >_{\text{disc}} \mathbf{y}$ . Otherwise, the solutions are incomparable.

The leximin ordering (e.g., [9]) is a refinement of min and discrimin orders. Consider the  $s_x$  as an *increasingly arranged* vector of the satisfaction degrees in  $\mathcal{S}_x$ ,  $s_x = (s_{[1]}, \dots, s_{[k]})$ , i.e., is such that  $s_{[i]} \leq s_{[i+1]}$  where  $s_{[i]} = \mu_{C_{[i]}}(\mathbf{x})$  (where the constraints have been renumbered). Then, the leximin ordering is defined by:

$$\mathbf{x} >_{\text{leximin}} \mathbf{y} \Leftrightarrow \exists i \forall j < i \ s_{x[j]} = s_{y[j]} \text{ and } s_{x[i]} > s_{y[i]}.$$

This corresponds to a lexicographic comparison of the vectors  $s_x$  and  $s_y$  [9].

A solution  $\mathbf{x}$  is preferred to a solution  $\mathbf{y}$ , in a leximin order, if there is a threshold  $\alpha$  such that for all  $\beta < \alpha$ , the number of constraints satisfied by  $\mathbf{x}$  at level at least  $\beta$  is equal to the number of constraints satisfied by  $\mathbf{y}$ , and  $\mathbf{x}$  satisfies more constraints than  $\mathbf{y}$  at level  $\alpha$ .

### 3. Logical representation of FCSPs

#### 3.1 Logical expression of fuzzy constraints

We are now going to turn the definitions of the preceding section into logical terms, by replacing the fuzzy sets by sets of prioritized crisp constraints. Using the scale  $L = \{\alpha_0 = 0, \alpha_1, \dots, \alpha_L = 1\}$ , let  $C_j = (C)_{\alpha_{j+1}}$  denote the  $\alpha$ -cut of a constraint  $C$  at level  $\alpha_{j+1}$  for  $j = 0$  to  $L - 1$ . The priority attached to  $C_j$  is  $1 - \alpha_j$ .  $C_0$ , the support of  $C$ , is the crisp constraint with the highest level of priority, say  $1 = 1 - \alpha_0$ ;  $C_{L-1}$ , the core of  $C$ , is the crisp constraint with the lowest level of strictly positive priority,  $1 - \alpha_{L-1}$ . Figure 2 shows an example of a fuzzy constraint and its representation in terms of crisp constraints with attached priorities. We can write:

$$\mu_{C \cdot}(\mathbf{x}) = \min_j \max(\alpha_j, \mu_{C_j}(\mathbf{x})), \quad (6)$$

where  $\mu_{C_j}(\mathbf{x}) = 1$  if  $\mathbf{x} \in C_j$ , and 0 otherwise. In other words, the fuzzy constraint  $C$  is translated into the set of pairs  $(C_j, 1 - \alpha_j)$  which can be viewed as a set of possibilistic logic formulas [13], where we use the same notation  $(C_j)$  for a proposition and its set of models.

If a priority level  $\rho$  is attached to the fuzzy constraint  $C$  the corresponding fuzzy constraint  $C'$  is defined by:

$$\begin{aligned} \mu_{C'}(\mathbf{x}) &= \max(1 - \rho, \min_j \max(\alpha_j, \mu_{C_j}(\mathbf{x}))) \\ &= \min_j \max(1 - \min(\rho, 1 - \alpha_j), \mu_{C_j}(\mathbf{x})). \end{aligned}$$

Let  $j^\rho$  such that  $1 - \rho = \alpha_{j^\rho}$ , and remind that  $\mu_{C_j}(\mathbf{x}) \geq \mu_{C_{j+1}}(\mathbf{x})$  and that  $\alpha_j < \alpha_{j+1}$ . Then the previous equation can be rewritten as:

$$\mu_{C'}(\mathbf{x}) = \min_{j=j^\rho, L-1} \max(\alpha_j, \mu_{C_j}(\mathbf{x})). \quad (7)$$

Therefore, attaching the priority  $\rho$  to  $C$ , means that, in terms of prioritized constraints  $C_j$ , we should consider only the constraints  $C_j$  with priorities lower or equal to  $\rho$ , i.e., such that  $1 - \alpha_j \leq \rho$ , or equivalently  $j \geq j^\rho$ . This can be observed on the example of Figure 2, for  $\rho = 0.6$  ( $j^\rho = 2$ ), where the constraints eliminated by this relaxation ( $C_0$  and  $C_1$ ), are in a dotted area.

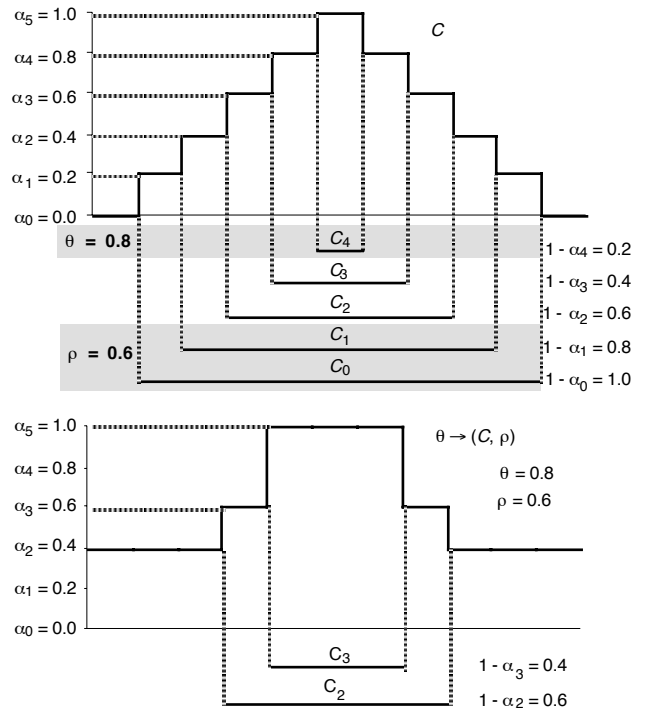


Figure 2. Fuzzy constraint as a set of constraints with priority

In case of a threshold  $\theta$ , the fuzzy constraint  $C$  is relaxed into the fuzzy constraint  $C' = \theta \rightarrow C$  by:

$$\mu_{C'}(\mathbf{x}) = \theta \rightarrow \min_j \max(\alpha_j, \mu_{C_j}(\mathbf{x})),$$

which can be rewritten as:

$$\mu_{C'}(\mathbf{x}) = \min_{j=0, j^\theta-1} \max(\alpha_j, \mu_{C_j}(\mathbf{x})), \quad (8)$$

where  $j^\theta$  is such that  $\alpha_{j^\theta} = \theta$ . This means that we should retain only the crisp constraints  $C_i$ , with priorities higher than  $1 - \theta$ . For  $\theta = 0.8$  ( $j^\theta = 4$ ) in Figure 2, the discarded constraint ( $C_4$ ) is in the other dotted area. Thus, applying threshold  $\theta$  to the formula  $C$  amounts to replacing  $C$  by the set of prioritized logical formulas ( $C_j, \alpha_j$ ) for  $j = 0$  to  $j^\theta - 1$ .

Introducing a safeguard constraint  $C^*$  is simply adding  $C^*$  with a priority 1, i.e., the logical formula ( $C^*, 1$ ).

Conditioning a fuzzy constraint  $C_i$  by a crisp constraint  $C_k$  is modelled by a fuzzy set, as defined by equation (5), which can be written in terms of a set of prioritized constraints:

$$\mu_{C_i/C_k}(x) = \min_j \max(\mu_{C_{ij}}(x), 1 - \mu_{C_k}(x), \alpha_j). \quad (9)$$

This corresponds to the possibilistic logic formulas ( $C_{ij}, \min(1 - \alpha_j, \mu_{C_k}(x))$ ), which means that  $C_{ij}$  has priority  $1 - \alpha_j$  only for instantiations satisfying  $C_k$ . Thus, the operation is achieved by conditioning each constraint  $C_{ij}$  by  $C_k$ . Thus, conditioning  $C_i$  by  $C_k$  amounts to replace  $C_i$  by the set of prioritized logic formulas ( $\neg C_k \vee C_{ij}, 1 - \alpha_j$ ); see [13]. When  $C_k$  is itself a fuzzy constraint, which is equivalent to the set of prioritized formulas ( $C_{kl}, 1 - \alpha_{kl}$ ), we obtain a set of formulas of the form:

$$(C_{ij}, \min(1 - \alpha_{ij}, \min_l \max(\alpha_{kl}, \mu_{C_{kl}}(x)))).$$

This can be shown to be equivalent to the set of formulas: ( $\neg C_{kl} \vee C_{ij}, \min(1 - \alpha_{ij}, \alpha_{kl})$ ).

We have shown the equivalence between a fuzzy constraint, which may be prioritized, thresholded or conditioned, with a knowledge base made of possibilistic logic formulas. In particular,  $\rho$ - and  $\theta$ -relaxation's amount to discard the highest(s) or the lowest(s) priority constraints respectively.

### 3.2 Symbolic aggregation of fuzzy constraints

The aggregation of two fuzzy constraints  $C_1$  and  $C_2$  defined by  $\mu_c(x) = \min(\mu_{C_1}(x), \mu_{C_2}(x))$ , can be easily interpreted in the prioritized constraints framework. Applying (6) to  $\mu_{C_1}(x)$  and  $\mu_{C_2}(x)$ , the min aggregation can be written as follows:

$$\mu_c(x) = \min_{j=0, L-1} \max(\alpha_j, \min(\mu_{C_{1j}}(x), \mu_{C_{2j}}(x))).$$

This expresses that the min aggregation corresponds to the union of the two sets of possibilistic logic formulas ( $C_{1j}, 1 - \alpha_j$ ) and ( $C_{2j}, 1 - \alpha_j$ ). This is a particular case of the syntactic fusion of possibilistic pieces of information [3].

The min aggregation of prioritized constraints can model hierarchically organized sets of constraints, that have the form:  $C_1$  should be satisfied (its priority is 1), and among the solutions of  $C_1$  (if any) the ones satisfying  $C_2$  are preferred (the priority of  $C_2$  is  $1 - \alpha_2$ ), and among those satisfying  $C_1$  and  $C_2$ , those satisfying  $C_3$  are preferred (the priority of  $C_3$  is  $1 - \alpha_3$  with

$1 - \alpha_2 > 1 - \alpha_3$ ), where  $C_1, C_2$  and  $C_3$  are crisp constraints. It should be understood that satisfying  $C_2$  if  $C_1$  is not satisfied, is of no interest. Therefore if all the three constraints are satisfied then the satisfaction level is 1; if only  $C_1$  and  $C_2$  are satisfied then the satisfaction level is  $\alpha_3$ ; if  $C_1$  is satisfied but not  $C_2$  (whatever  $C_3$ ) then the satisfaction level is  $\alpha_2$ ; otherwise the satisfaction level is 0. This can be easily expressed by the three prioritized crisp constraints ( $C_1, 1$ ), ( $\neg C_1 \vee C_2, 1 - \alpha_2$ ) and ( $\neg C_1 \vee \neg C_2 \vee C_3, 1 - \alpha_3$ ).

Aggregation operations other than min can be also accommodated. Indeed reinforcement and compensation operators, such as the product and the average respectively, can also be interpreted in terms of operations on prioritized constraints. Let ( $A, \alpha$ ) and ( $B, \beta$ ) be two crisp constraints with priorities  $\alpha$  and  $\beta$  and  $\square$  an increasing aggregation operator. The aggregation ( $A, \alpha$ )  $\square$  ( $B, \beta$ ) is the fuzzy constraint defined by

$$\mu_{(A, \alpha) \square (B, \beta)}(x) = \mu_{(A, \alpha)}(x) \square \mu_{(B, \beta)}(x), \quad (10)$$

which can be easily interpreted in terms of prioritized constraints.

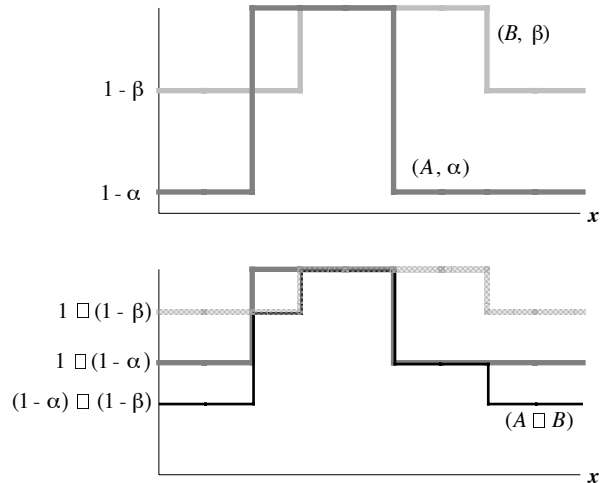


Figure 3. Aggregation ( $A, \alpha$ )  $\square$  ( $B, \beta$ )

As it can be checked in Figure 3, the aggregation ( $A, \alpha$ )  $\square$  ( $B, \beta$ ) is equivalent to the (min) conjunction of the prioritized constraints ( $A, 1 - (1 \square (1 - \alpha))$ ), ( $B, 1 - (1 \square (1 - \beta))$ ), ( $A \cup B, 1 - (1 - \alpha) \square (1 - \beta)$ ). This symbolic combination of prioritized constraints, ( $A, \alpha$ ) and ( $B, \beta$ ), comes from the method for fusing pieces of uncertain evidence recently proposed in [3]. Note that the combination amounts to adding the constraint  $A \cup B$  to a level of priority higher than the ones of  $A$  and  $B$ . Indeed, provided that  $\square$  is an increasing operation, we have,  $1 - ((1 - \alpha) \square (1 - \beta))$  greater or equal to  $1 - (1 \square (1 - \alpha))$  and  $1 - (1 \square (1 - \beta))$ .

This can be generalized to fuzzy constraints  $A$  and  $B$ . Using equation (6), we rewrite the fuzzy constraints  $A \equiv \bigcap_i (A_i, 1 - \alpha_i)$  and  $B \equiv \bigcap_k (B_k, 1 - \beta_k)$ , where the highest priorities are  $1 - \alpha_0$  and  $1 - \beta_0$  respectively. Then the repeated use of (10) shows that  $A \square B$  is equivalent to the conjunction of the following sets of prioritized constraints:  $\{(A_i \cup B_k, 1 - (\alpha_i \square \beta_k))$  for all  $(i, k)\}$ ,

$\{(A_i, 1 - (1 \square \alpha_i))$  for all  $i\}$  and  $\{(B_k, 1 - (1 \square \beta_k))$  for all  $k\}$ . It should be emphasize that the translation of aggregation  $\square$  into a possibilistic prepositional logic base is done at the expense of introduction of new levels in the scale. Indeed  $\square$  is not closed on  $\mathbf{L}$  generally.

Consider an example with three constraints  $A$ ,  $B$  and  $C$  where  $A$  and  $B$  are fuzzy constraints and  $(C, \rho)$  is a weighted crisp constraint. Constraint  $A$  is supposed to be thresholded. Moreover  $(C, \rho)$  and  $B$  are supposed to be aggregated by a compensatory operation, here the arithmetic mean ( $s \square t = (s + t)/2$ ). This can be formally written as the problem  $P$ :  $(\theta \rightarrow A) \cap ((C, \rho) \square B)$  where  $\cap$  stands for the min aggregation. In the example we use the satisfaction scale  $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\} = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$  for  $A$  and  $B$  and we take  $\theta = 0.8, \rho = 0.6$ . The problem  $P$  is translated into  $P'$  which is presented in a stratified way in the next table. Remember  $A_j = A_{\alpha_{j+1}}$ , e.g.,  $A_1 = A_{\alpha_2}$ , i.e., the 0.4-cut.

Priority.	Constraints
1.0	$A_0$
0.8	$A_1$ <span style="float: right;"><math>B_0 \cup C</math></span>
0.7	<span style="float: right;"><math>B_1 \cup C</math></span>
0.6	$A_2$ <span style="float: right;"><math>B_2 \cup C</math></span>
0.5	$B_0$ <span style="float: right;"><math>B_3 \cup C</math></span>
0.4	$A_3$ $B_1$ <span style="float: right;"><math>B_4 \cup C</math></span>
0.3	$B_2$ $C$
0.2	$B_3$
0.1	$B_4$

More generally, let  $P = \{C_1, \dots, C_K\}$  be a set of fuzzy constraints (with priorities, thresholds, and specified aggregations). Thus it translates into a (larger) set of crisp constraints  $P' = \{C'_1, \dots, C'_M\}$  with new computed priorities. However as it can be seen in the above example,  $P'$  may include redundancies which can be easily detected from the symbolic writing. In our example  $\{A_3, B_1, B_4 \cup C\}$  is simplified into  $\{A_3, B_4 \cup C\}$  ( $B_1 \supseteq B_4$ ). Indeed  $(C', \lambda')$  is redundant w. r. t.  $(C, \lambda)$  iff  $C' \supseteq C$  and  $\lambda' \leq \lambda$ , and we have the  $\alpha$ -cut property  $(C)_\alpha \supseteq (C')_{\alpha'}$  and  $\alpha' \geq \alpha$ . Thus problem  $P'$  in the above example reduces to the table below which exhibits the problem relaxation space [17].

Satisfaction	Priority	Problem relaxation
0.2	1	$\{A_0\}$
0.3	0.8	$\{A_1, B_0\}$
0.4	0.7	$\{A_1, B_1\}$ <span style="float: right;"><math>\{A_1, C\}</math></span>
0.5	0.6	$\{A_2, B_2\}$ <span style="float: right;"><math>\{A_2, C\}</math></span>
0.6	0.5	$\{A_2, B_3\}$ <span style="float: right;"><math>\{A_2, B_0, C\}</math></span>
0.7	0.4	$\{A_3, B_4\}$ <span style="float: right;"><math>\{A_1, B_1, C\}</math></span>
0.8	0.3	$\{A_3, B_2, C\}$
0.9	0.2	$\{A_3, B_3, C\}$
1	0.1	$\{A_3, B_4, C\}$

It can be checked that the discrimin ordering between solutions of  $P$  (maximization of the aggregated fuzzy sets  $C_i$ ) is equivalent to the discrimin ordering between solutions of the translated problem  $P'$ , in terms of weighted crisp constraints. The same holds for the leximin. When  $P'$  is such that all its crisp constraints have different priorities, then the discrimin and the

leximin orderings became the same. In this situation a discrimin optimal solution is also a leximin optimal.

## 4. Computational issues

It is well known that CSP problems are NP-complete [21]. The main tools for solving CSPs are basically backtracking-based search algorithms and constraint propagation techniques [21]. Finding a min-optimal solution is NP-hard, and can be done using a branch and bound-based algorithm [17][19]. The min produces a good upper bound estimation of the global satisfaction level which enables an efficient pruning of useless branches.

The use of the min operator allows for a direct generalization of the constraint propagation algorithms used in the classical CSP framework (for example the AC3 by Mackworth [21]). The levels of satisfaction are propagated by extensions of existing algorithms (see for example [8][16]). This is essentially due to the idempotence of min operator. Min-optimal solutions for FCSP can be obtained by solving a logarithmic number of CSP's [6].

Finding a discrimin or leximin optimal solution for a FCSP is much more complex than finding a min-optimal solution. Dubois and Fortemps [11] discuss how to compute discrimin-optimal and leximin-optimal solutions for special classes of FCSPs and present algorithms for some specific problems such as, for instance, shortest path problems with flexible constraints, and flexible assignment problems. The underlying idea is to find a min-optimal solution and then by saturating the least satisfied constraint(s) then to generate a new sub-problem, whose min-optimal solutions are better (in the sense of discrimin or leximin) than the first solution found. This process is repeated until no more sub-problems can be generated and a leximin or discrimin optimal solution is reached. However, for a general FCSP the generation of the sub-problems involves a NP-complete step.

It is also possible to encode a leximin FCSP as an additive CSP, turning leximin into an ordered weighted average aggregation as shown in [10] but, unfortunately, it becomes a harder optimization problem.

The strong connection between FCSPs, prioritized CSPs and logical inference in possibilistic knowledge bases, suggest that we can translate, as explained in this paper, a FCSP  $P$  into a stratified knowledge base  $K$  and then find the set of discrimin optimal solutions using a procedure proposed by Brewka [5]; see also [2] for a presentation in a possibilistic logic framework. Let  $K_0$  be the subset of  $K$  containing the formulas of higher priority,  $K_1$  be the subset of  $K$  containing the formulas of second priority, say at level  $1 - \alpha_1$ , and so on. Let  $K_0^\diamond$  be a maximal consistent subset of  $K_0$ . Then let  $K_1^\diamond$  be the maximal consistent subset of formulas formed by adding to  $K_0^\diamond$  as many formulas from  $K_1$  as possible. Continue down to the lowest level of priority. A consistent subset so obtained,  $K_{L-1}^\diamond$ , is called a preferred sub-base by Brewka, and all its solutions are discrimin optimal, and conversely. Problem relaxation

(see Section 3) provides us with a partial ordering for organizing the consistency checks. Brewka's procedure can also be used to check if a min-optimal solution (eventually founded by a standard method) is a discrimin-optimal solution.

Besides, one of the authors [22] has suggested the use of non-enumerative optimization algorithms for obtaining leximin optimal solutions or near-optimal solutions. The use of hill-climbing based algorithms on a min-FCSP faces the difficulty that the solutions are distributed through only at most  $L$  satisfaction levels, which can lead to many plateau's for the function to be optimized. In leximin-FCSPs, since the number of leximin-levels is much higher than  $L$ , the corresponding plateau's will be in a much lower number. See [20] for the use of such algorithms in the CSP framework.

## 5. Concluding remarks

We have shown in this paper the strong connection between FCSPs and CSPs with priorities and therefore with stratified knowledge bases (in the sense of possibilistic logic). All the relaxations (prioritized, thresholded constraints, and conditional constraints, etc) on FCSPs have a clear counterpart in CSPs with priorities and thus in stratified knowledge bases. The same occurs for the refinements of min-optimization, namely discrimin and leximin optimal solutions can be redefined in this logical framework. Computational issues are in a preliminary phase, and the research in this area will deal, particularly, with the computation of discrimin or leximin optimal solutions, or the complete set of all these solutions.

Lastly, it would be of interest to consider FCSP problems where explicit track is kept of the sources which assign the constraints, and to introduce priorities between the sources, taking advantage of the possibilistic logic counterpart of this problem which has been already studied [12].

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