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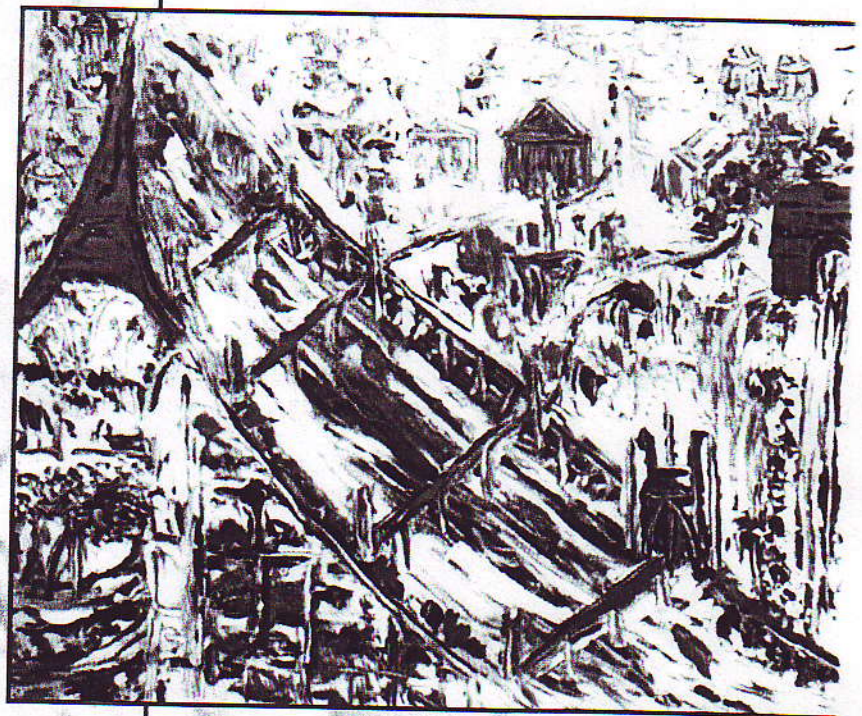
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Possibilistic likelihood relations

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Abstract

This paper proposes two refinements of the orderings induced by possibility and necessity measures in finite settings. The first one is a strict partial ordering that possesses all properties of a comparative probability (cancellation, autoduality, especially). It is closely related to nonmonotonic reasoning and naturally comes up as a model of subjective uncertainty in a Savage-like framework for decision, where indifference between acts is not transitive, and the preference ordering is recovered from two weak orderings, on states and consequences respectively. The second possibilistic likelihood relation is based on a lexicographic refinement of the maximum operation. In some sense these relations on events recover the additivity property of probabilities, without using numbers.

1 Introduction

In fuzzy set theory, and in possibility theory, it is very common to compute the maximum or the minimum of a function over a set, and then to compare sets on such a basis. This method of ordering for multiatribute items is coarse and has been criticized on such grounds. For instance, in fuzzy optimization after Bellman and Zadeh [1], not all optimal solutions to a maximum problem are interesting because some of them fail to be Pareto-efficient. In order to cope with this situation while keeping the aggregation by the minimum, refinement of the minimum-based (and maximum-based) orderings have been introduced (Fargier et al. [4]). Dubois et al. [7]), that preserve Pareto-efficiency. Namely letting V and W be vectors of evaluations with components v_i and w_i the following relations, that refine the Pareto ordering can be considered:

— *discrimin relation* between vectors V and W , defined as follows:

$$W >_{disc} V \text{ iff } \min_{j \in \mathcal{D}(V, W)} w_j > \min_{j \in \mathcal{D}(V, W)} v_j$$

where $\mathcal{D}(V, W)$ is the set $\{j, w_j \neq v_j\}$ of discriminating components. The relation $>_{disc}$ is a partial ordering;

— the so-called *leximin ordering* between vectors, defined as follows:

$$W >_{lm} V \text{ iff } \exists k, w_{[k]} = v_{[k]} \text{ for } 1 \leq k < k \text{ and } w_{[k]} > v_{[k]}$$

where $w_{[1]} \leq w_{[2]} \leq \dots \leq w_{[n]}$ and $v_{[1]} \leq v_{[2]} \leq \dots \leq v_{[n]}$. $W =_{lm} V$ iff neither $W >_{lm} V$ nor $V >_{lm} W$ holds, which means $w_{[i]} = v_{[i]}$ for all i . It corresponds to a lexicographic ordering of increasingly rearranged vectors. The relation \geq_{lm} is a complete preordering that refines the discrimin relation. The leximin ordering is well-known in the decision-theoretic literature [21]. It has been found as the limit of other types of multiple criteria rankings based on triangular norms and ordered weighted averages (Dubois et al. [6]). Similarly, changing min into max (resp. reversing the ordering of the $w_{[i]}$'s) yields the *disctmax* (resp. *lexmax*) orderings.

This paper exploits these kinds of relations, originally introduced in multicriteria evaluation, within possibility theory itself, for achieving a finer-grained representation of qualitative uncertainty. It enables possibility and necessity orderings of events to be refined on the basis of the only knowledge of the underlying possibility distribution, and without necessarily resorting to the use of numbers. These orderings are purely ordinal, yet being as much discriminant as comparative probability orderings (Fishburn

[15]) with which they share many properties. These extended possibility orderings naturally arise from considerations in qualitative decision theory and in problems of inconsistency management in rank-ordered knowledge bases.

2 Qualitative Possibility Theory

Let S be a finite set of elements called "states". States encode possible situations, states of affairs, etc. Let (\mathcal{S}, \geq_L) be a confidence relation on events A, B, C, \dots . We assume that \geq_L is a complete preordering: \geq_L is transitive and for any $A, B, A \geq_L B$ or $B \geq_L A$; this is also called a weak order [22].

ii) \geq_L is non-trivial ($\mathcal{S} \geq_L \emptyset$, but not $\emptyset \geq_L \mathcal{S}$),
iii) \geq_L is faithful to deductive inference: $A \subseteq B \Rightarrow A \leq_L B$ (inclusion-monotony)

$A \leq_L B$ means that an agent's confidence in event B is at least as high as in A and the above property states that if A implies B , the agent cannot trust B less than A . As usual, let us define the indifference \sim_L and the strict ordering $>_L$ induced from \geq_L by:

$$A \sim_L B \text{ iff } A \leq_L B \text{ and } B \leq_L A \\ A >_L B \text{ iff } A \leq_L B \text{ and not } (B \leq_L A).$$

The most common choice for \geq_L is a comparative probability [15]:

Definition 1: \geq_L is a comparative probability iff

- \geq_L is complete and transitive
- $\mathcal{S} >_L \emptyset$ (non-triviality),
- $\forall A, A \leq_L \emptyset$ (consistency)
- if $A \cap (B \cup C) = \emptyset$ then:
 $B \leq_L C \Leftrightarrow A \cup B \leq_L A \cup C$ (additivity)

Any probability measure induces a comparative probability ordering, but not conversely (see [15]). Among the simplest types of other comparative uncertainty relations are those induced by possibility and necessity measures [5]:

Definition 2: \geq_Π is a possibility ordering iff

- \geq_Π is complete and transitive
- $\mathcal{S} >_\Pi \emptyset$,
- $\forall A, A \geq_\Pi \emptyset$
- $\forall A, B \geq_\Pi C \Rightarrow A \cup B \geq_\Pi A \cup C$
 $B \geq_\Pi C$ reads B is at least as possible (= plausible) as C .

Definition 3: \geq_N is a necessity ordering iff

- \geq_N is complete and transitive
- $\mathcal{S} >_N \emptyset$,
- $\forall A, S \geq_N A$
- $\forall A, B \geq_N C \Rightarrow A \cap B \geq_N A \cap C$
 $B \geq_N C$ reads B is at least as necessary (= certain) as C .

Necessity orderings are also epistemic entrenchments in the sense of Gärdenfors [17] (see Dubois and Prade [12]), and possibility orderings have been introduced by Lewis [20], then rediscovered by Dubois [5]. Possibility and necessity orderings are dual in the sense that for a given possibility ordering \geq_Π , the relation defined by $A \geq_N B$ iff $B \geq_\Pi A$ is a necessity ordering, and conversely. A pair (\geq_Π, \geq_N) such that $A \geq_N B$ iff $B \geq_\Pi A$ is called a pair of dual uncertainty relations (while comparative probability in the sense of Definition 1 is autodual). $B \geq_N C$ is intuitively understood as " B is at least as certain as C ", in the sense that arguments supporting B are at least as compelling as the arguments supporting C . The dual ordering $B \geq_\Pi C$ means that B is at least as plausible as C where plausibility refers to consistency with an agent's knowledge: B is plausible when it does not contradict the agent's beliefs. Any necessity ordering on a finite set can be represented by a necessity function on a linearly ordered scale $(L, >)$:

Proposition 1 [5]: There is a finite totally ordered set L and a set-function $N: 2^S \rightarrow L$ such that: $B \geq_N C \Leftrightarrow N(B) \geq N(C)$, where $N(F) \in L$ and $N(F \cap G) = \min(N(F), N(G)) \forall F$ and G .

L is isomorphic to the quotient set $2^S / \sim_N$ when \sim_N is the indifference part of \geq_N . $0 \in L$ denote the bottom element of L (equivalence class of \emptyset) and $1 \in L$ denotes the top element of L (equivalence class of S), and $1 > 0$.

The same kind of property holds for possibility orderings with respect to possibility functions:

Proposition 2 [5]: There is a finite totally ordered set L and a set-function $\Pi: 2^S \rightarrow L$ such that: $B \geq_\Pi C \Leftrightarrow \Pi(B) \geq \Pi(C)$ where $\Pi(B) \in L$ and $\Pi(B \cup C) = \max(\Pi(B), \Pi(C)) \forall B$ and C .

A pair of dual necessity and possibility orderings is simple since that they both can be derived from a complete preordering on states only. This complete preordering, denoted \geq_π can be

encoded as a possibility distribution π , which is a mapping $S \rightarrow L$. Namely $\Pi(A) = \max_{s \in A} \pi(s)$ and $N(A) = n(\Pi(A)) = \min_{s \in A} n(\pi(s))$ where n is the order reversing function $L \rightarrow L$. In terms of possibility measures (Zadeh [26]; Dubois and Prade [11]), $\Pi(A) = n(N(\bar{A})) = (1 - N(\bar{A}))$ if $L = [0, 1]$.

3 - Discrimimax likelihood

Two events A and B cannot be discriminated using a dual pair (N, Π) as soon as $\Pi(A) = \Pi(B)$ and $N(A) = N(B)$, which occurs quite often, namely, as soon as $\max_{s \in A} \pi(s) = \max_{s \in B} \pi(s) < 1$. $\Pi(A) = \Pi(B)$ may be due to a high value of $\Pi(B \cap A)$. However in the spirit of the discrimin relation, this weight should not affect the comparison between A and B , since it is common to both. The same reasoning applies to $\Pi(B \cap \bar{A})$. Only $\bar{B} \cap A$ and $B \cap \bar{A}$ should matter in telling A from B . Let us introduce the relation \triangleright_{TL} on 2^S , different from although closely related to the original necessity and possibility orderings modelled by N and its dual Π :

Definition 4:

$A \triangleright_{TL} B$ iff $\Pi(\bar{B} \cap A) > \Pi(B \cap \bar{A})$.
 $A \not\triangleright_{TL} B$ iff not $(B \triangleright_{TL} A)$.

The relation \triangleright_{TL} is a refinement of both possibility and necessity orderings:

Proposition 3:

• $N(A) > N(B) \Rightarrow A \triangleright_{TL} B$;
 • $\Pi(A) > \Pi(B) \Rightarrow A \triangleright_{TL} B$.

Proof: Let us prove that: $B \not\triangleright_{TL} A \Rightarrow N(B) \geq N(A)$.

First notice that: $B \not\triangleright_{TL} A \Leftrightarrow N(\bar{A} \cup B) \geq N(A \cup \bar{B})$ composing Def 4. Thus

$$\min(N(\bar{A} \cup B), N(A \cup \bar{B})) \geq \min(N(A \cup B), N(\bar{A} \cup \bar{B})).$$

Noticing that:

$$\begin{aligned} N(A) &= N((A \cup B) \cap (A \cup \bar{B})) \\ &= \min(N(A \cup B), N(A \cup \bar{B})). \\ N(B) &= N((\bar{A} \cup B) \cap (A \cup B)) \\ &= \min(N(\bar{A} \cup B), N(A \cup B)). \end{aligned}$$

we get $N(B) \geq N(A)$.

Let us prove that: $B \not\triangleright_{TL} A \Rightarrow \Pi(B) \geq \Pi(A)$. Notice that $B \not\triangleright_{TL} A \Leftrightarrow \Pi(\bar{A} \cap B) \geq \Pi(A \cap \bar{B})$. Thus $\max(\Pi(\bar{A} \cap B), \Pi(A \cap B)) \geq \max(\Pi(A \cap \bar{B}), \Pi(A \cap B))$. Noticing that:

$$\Pi(A) = \Pi((A \cap B) \cup (A \cap \bar{B}))$$

$$\begin{aligned} &= \max(\Pi(A \cap B), \Pi(A \cap \bar{B})), \\ \Pi(B) &= \Pi((\bar{A} \cap B) \cup (A \cap B)) \\ &= \max(\Pi(\bar{A} \cap B), \Pi(A \cap B)), \end{aligned}$$

we get: $\Pi(B) \geq \Pi(A)$. ■

N, B . It may happen that both $N(A) = N(B)$ and $\Pi(A) = \Pi(B)$ hold and that $A \triangleright_{TL} B$. Note that \geq_{Π} and \geq_{TL} coincide when restricting to mutually exclusive events A and B since then $\Pi(A \cap B) = 0$. More generally \geq_{Π} and \geq_{TL} coincide whenever $\Pi(A \cap B) < \min(\Pi(A \cap \bar{B}), \Pi(\bar{A} \cap B))$ which is to say when the property $\Pi(A \cap B) = \min(\Pi(A), \Pi(B))$ does not hold, a possibilistic counterpart to independence (e.g., Naimias [23]; Dubois et al. [10]). More precisely

Proposition 4: The equivalence $\Pi(A) \geq \Pi(B) \Leftrightarrow A \geq_{TL} B$ holds if and only if

$$\Pi(A \cap B) < \Pi(A \cap \bar{B}).$$

Proof:

The two orderings differ only when $\Pi(A) = \Pi(B)$ while $\Pi(A \cap B) > \Pi(A \cap \bar{B})$. This case is when $\Pi(A \cap B) \geq \Pi(A \cap \bar{B})$. ■

The partial ordering \triangleright_{TL} on events is a special case of the so-called "discrimimax" relation between vectors:

Definition 5: $V \geq_{discrimimax} W \Leftrightarrow \max_{i \in \mathcal{I}} (v_i - w_i) \geq \max_{i \in \mathcal{J}} (v_i - w_i)$ where $\mathcal{I}(V, W) = \{i, v_i \neq w_i\}$ is the set of non-equal components of the vectors V and W .

The discrimimax relation is a refinement of both Pareto partial ordering between vectors ($V \geq_{Pareto} W$ iff $\forall i, v_i \geq w_i$) and the maximum ordering ($\max_i v_i \geq \max_i w_i$) that gets rid of non-discriminating components.

Now if π denotes the possibility distribution associated to Π (i.e., such that $\Pi(A) = \max_{s \in A} \pi(s)$), and let a be the vector (a_1, \dots, a_n) such that, denoting $S = \{s_1, s_2, \dots, s_n\}$:

$$\begin{aligned} a_i &= \pi(s_i) \text{ if } s_i \in A \\ a_i &= 0 \text{ otherwise} \end{aligned}$$

then $A \triangleright_{TL} B \Leftrightarrow a \triangleright_{discrimimax} b$.

Notice that the relation $A \sim_{TL} B$ iff neither $A \triangleright_{TL} B$ nor $B \triangleright_{TL} A$ hold on 2^S is not transitive since:

$$\Pi(A \cap \bar{B}) = \Pi(\bar{A} \cap B) \text{ and } \Pi(B \cap \bar{C}) = \Pi(\bar{B} \cap C)$$

$\Pi(A \cap \bar{C})$ do not imply $\Pi(A \cap \bar{C}) = \Pi(\bar{A} \cap C)$.

This is because $\max(\pi_5, \pi_3) = \max(\pi_4, \pi_6)$ and $\max(\pi_2, \pi_6) = \max(\pi_3, \pi_7)$ do not imply

$$\begin{aligned} \max(\pi_4, \pi_7) &= \max(\pi_2, \pi_5) \text{ denoting } \pi_i = \pi(s_i), \\ A &= \{s_2, s_3, s_5\}, B = \{s_3, s_4, s_6\}, C = \{s_3, s_4, s_7\} \end{aligned}$$

(even if π takes on two values, 0 and 1 only).

Interestingly, the refined possibility and necessity relation satisfy not only monotony, but also the additivity condition (iv) of comparative probabilities, and the relation \geq_{TL} turns out to be self dual:

Proposition 5: $A \subseteq B \Rightarrow A \leq_{TL} B$

Moreover if A is a proper subset of B then $A <_{TL} B$ always hold for positive possibility distributions.

Proof: If $A \subseteq B$ then $A \cap \bar{B} = \emptyset$. Hence $\Pi(\bar{A} \cap B) \geq \Pi(A \cap \bar{B}) = 0$. Moreover if A is a proper subset of B then $\bar{A} \cap B \neq \emptyset$ and if π is positive $\Pi(\bar{A} \cap B) > \Pi(A \cap \bar{B}) = 0$. ■

Proposition 6:

$$\begin{aligned} \bullet A \cap (B \cup C) = \emptyset &\Rightarrow \\ (B \triangleright_{TL} C \Leftrightarrow A \cup B \triangleright_{TL} A \cup C) &\text{ (additivity).} \end{aligned}$$

$$\bullet A \geq_{TL} B \Leftrightarrow \bar{B} \geq_{TL} \bar{A} \text{ (self duality)}$$

Proof:

$$\begin{aligned} \bullet A \cup B \geq_{TL} A \cup C &\Leftrightarrow \Pi(\bar{A} \cap B) \geq \Pi(\bar{A} \cap C) \\ &\Leftrightarrow \Pi(B \cap \bar{A}) \geq \Pi(C \cap \bar{A}) \\ &\Leftrightarrow \Pi(B \cap \bar{A} \cap \bar{C}) \geq \Pi(C \cap \bar{A} \cap \bar{C}). \end{aligned}$$

Since $A \cap (B \cup C) = \emptyset$, $B \cap \bar{A} \cap \bar{C} = B \cap \bar{C}$ and $C \cap \bar{A} \cap \bar{C} = C \cap \bar{C} = \emptyset$.

$$\text{Thus: } A \cup B \geq_{TL} A \cup C \Leftrightarrow \Pi(B \cap \bar{C}) \geq \Pi(C \cap \bar{B})$$

$$\Leftrightarrow B \geq_{TL} C.$$

$$\bullet A \geq_{TL} B \Leftrightarrow$$

$$(A \cap B) \cup (A \cap \bar{B}) \geq_{TL} (A \cap B) \cup (A \cap \bar{B})$$

$$\Leftrightarrow (A \cap B) \geq_{TL} (A \cap B) \text{ (additivity) } \Leftrightarrow$$

$$(\bar{A} \cap B) \cup (A \cap \bar{B}) \geq_{TL} (\bar{A} \cap B) \cup (A \cap \bar{B})$$

$$\text{ (additivity again) } \Leftrightarrow$$

$$\bar{B} \geq_{TL} \bar{A}. \quad \blacksquare$$

So, the relation \geq_{TL} possesses all properties of a comparative probability, but for the transitivity of indifference. In the following, the possibilistic relation \geq_{TL} will be called *discrimimax likelihood* and $A \geq_{TL} B$ reads "A is at least as likely as B", so as to emphasize the self-transitivity of the indifference relation, we cannot generally represent such relations by

means of probability measures,¹ nor can we assume properties that are usually derived from comparative probabilities. For instance, the following property of comparative probabilities is NOT satisfied in general by discrimimax likelihood:

$$\begin{aligned} \text{If } A \cap B = \emptyset \text{ and } C \cap D = \emptyset, \\ \text{then } A \sim_{TL} B \text{ and } C \sim_{TL} D \\ \Leftrightarrow A \cup B \sim_{TL} C \cup D \end{aligned}$$

Counter-example:

$$\begin{aligned} S &= \{a, b, c, d, e, f, g, h\} \\ A &= \{a, b, c\} \quad B = \{e, f, g\} \\ C &= \{b, g, h\} \quad D = \{c, d, e\} \\ \pi(a) &= \pi(f) = \pi(c) = \pi(g) = 1 \\ \pi(b) &= \pi(d) = \pi(e) = \pi(h) = 0. \\ \Pi(A \cap C) &= \Pi(\{a, c\}) = 1 \text{ and } \Pi(C \cap \bar{A}) = \\ \Pi(\{g, h\}) &= 1: A \sim_{TL} C \\ \Pi(B \cap D) &= \Pi(\{f, g\}) = 1 \text{ and } \Pi(D \cap \bar{B}) = \\ \Pi(\{c, d\}) &= 1: B \sim_{TL} D \\ \Pi((A \cup B) \cap (C \cup D)) &= \Pi(\{a, f\}) = 1 \\ &> \Pi((C \cup D) \cap (A \cap B)) = \Pi(\{d, h\}) = 0. \end{aligned}$$

Thus: $A \cup B \triangleright_{TL} C \cup D$. ■

However, the following very strong, non probabilistic property holds for the discrimimax likelihood:

Proposition 7: For A, B, C disjoint, $A \cup C \triangleright_{TL} B$ and $B \cup C \triangleright_{TL} A$ imply $C \triangleright_{TL} A \cup B$

Proof: Obvious due to the disjointness assumption. The property holds for possibility orderings, which then coincides with the discrimimax likelihood. ■

Hence although similar, the discrimimax likelihood noticeably differs from a comparative probability.

Any qualitative possibility distribution \geq_{π} can be viewed as a well-ordered partition² $\{E_0, \dots, E_n\}$ of S such that:

$$\begin{aligned} \omega \geq_{\pi} \omega' \text{ iff} \\ \omega \in E_i, \omega' \in E_j \text{ and } i < j \text{ (for } 0 \leq i, j \leq n), \\ \omega \geq_{\pi} \omega' \text{ iff} \\ \omega \in E_i, \omega' \in E_j \text{ and } i \leq j \text{ (for } 0 \leq i, j \leq n). \end{aligned}$$

¹ This representation is already not always possible with standard comparative probability on finite sets (Fishburn, 1986).

² i.e. $\Omega = E_0 \cup \dots \cup E_n$. For $i \neq j$ there holds $E_i \cap E_j = \emptyset$, and $\forall k, E_k \neq \emptyset$.

By convention E_0 gathers the worlds which are totally possible (i.e., $\forall \omega \in E_0, \pi(\omega) = 1$). Qualitative possibility distributions allow us to represent a possibility distribution in terms of classes of equally possible worlds. Well-ordered partitions are used by Spohn [25]. In terms of well-ordered partitions, discrimmax likelihood can be described as follows: $A \succ_{\text{TLex}} B$ if there is an E_k such that $A \cap B \cap E_k \neq \emptyset$ and $B \cap A \cap E_k = \emptyset$, while $A \cap B \cap E_i = \emptyset$ and $B \cap A \cap E_i = \emptyset$ for all $i < k$. In particular if π is uniform (total ignorance), $A \succ_{\text{TLex}} B$ if and only if B is a proper subset of A . Hence the discrimmax likelihood coincides with the inclusion relation. At the opposite, if π is linear, that is for a certain ranking of $S, \pi_1 > \pi_2 > \dots > \pi_n$, for any $A \neq B$, either $A \succ_{\text{TLex}} B$ or $B \succ_{\text{TLex}} A$. Then the discrimmax likelihood can be represented by a very special kind of probability functions, called big-stepped probabilities (Dubois et al. [9]), such that $\forall A \subseteq S, \exists s \in A, P(s) > P(A \setminus \{s\})$ so that $P(A) \geq P(B)$ if and only if $\max_{s \in A} P(s) \geq \max_{s \in B} P(s)$: the magnitude of $P(s)$ can never be attained by summing the probabilities of states that are individually less probable than s . Moreover $P(s_1) > P(s_2) > \dots > P(s_{n-1}) > P(s_n)$ for states s_1, \dots, s_n of nonzero probability. Clearly a lot of probability measures are ruled out by these conditions.

More generally the relation $A \succ_{\text{TLex}} B$ can be generated from the restriction of the possibilistic relation to disjoint events (this is a partial ordering) completed by the additivity property for non disjoint events.

4. Leximax likelihood

Similarly, one can further refine the possibility relations between events A and B by considering a lexicographic ranking of reordered elements in A and B :

Definition 5: $V \succ_{\text{Leximax}} W \Leftrightarrow \exists k, w[k] = v[i]$ for $i < k$ and $w[k] > v[k]$ where $w[1] \geq w[2] \geq \dots \geq w[m]$ and $v[1] \geq v[2] \geq \dots \geq v[m]$.

Now if \vec{a} is again the vector (a_1, \dots, a_n) such that, denoting $S = \{s_1, s_2, \dots, s_n\}$:

$a_i = \pi(s_i)$ if $s_i \in A$
 $a_i = 0$ otherwise,
then $A \succ_{\text{TLex}} B \Leftrightarrow \vec{a} \succ_{\text{Leximax}} \vec{b}$.

This relation will be called "possibilistic lexicographic likelihood" ("leximax" likelihood for short). It is a complete preordering of events.

Proposition 8: The leximax likelihood relation refines the discrimmax likelihood one.

Proof: obvious.

In fact, the leximax likelihood relation coincides with the discrimmax one for linear possibility distributions. In terms of well-ordered partitions, leximax likelihood can be described as follows: $A \succ_{\text{TLex}} B$ if there is an E_k such that $|A \cap E_k| > |B \cap E_k|$ while $|A \cap E_i| = |B \cap E_i|$ for all $i < k$. The leximax relation for a uniform possibility distribution coincides with the comparative probability relation that is induced by a uniform probability. This is not surprising in view of the following results:

Proposition 9: The leximax likelihood relation is a comparative probability.

Proof: The leximax relation is a complete preordering. $S \succ_{\text{TLex}} \emptyset$ and $A \not\geq_{\text{TLex}} \emptyset$ obviously hold. The additivity property stems from the following remark: if $W > V$ then turn the n -sized vectors W and V into V' and W' by augmenting the number of components from n to $n+m$ where $V'_i = V_i$ (resp. $W'_i = W_i$) for $i = 1, n, \dots, n+m$ and $V'_i = 1$ (resp. $W'_i = 1$) for $i = n+1, \dots, n+m$. Clearly $W' > V'$. Indeed if j is such that $w[j] = v[j]$ for $i > j$ and $w[j] > v[j]$ then if all k_i are greater than $w[j]$ or smaller than $v[j]$ in the wide sense, a simple shift of index j to $j' \geq j$ may only occur and $w' > v'$ results. If $w[j] > k_i > v[j]$, then $w'[j'] > v'[j']$ for $j' \geq j$. $w'[j'] = w[j] > v[j] = k_i$ and $w' > v'$ results again. Conversely going from W' and V' to W and V by removing components of equal value will not alter the leximax ordering. The additivity axiom is a special case of such a stability property. ■

The leximax likelihood relation can be generated from the well-ordered partition (E_0, \dots, E_p) of S as follows

- (i) $\{s\} \succ_{\text{TLex}} \emptyset$
- (ii) if $s, s' \in E_i$, then $\{s\} \sim_{\text{TLex}} \{s'\}$
- (iii) if $s \in E_p$ and $A \subseteq E_{i+1} \cup \dots \cup E_p$ then $\{s\} \succ_{\text{TLex}} A$.
- (iii) apply additivity: $B \succ_{\text{TLex}} C \Leftrightarrow A \cup B \succ_{\text{TLex}} A \cup C$

Proposition 10: Any leximax likelihood relation can be represented by a probability measure.

Proof: Consider the well-ordered partition (E_0, \dots, E_p) of S . Let $f_p = |E_p|$ and $f_i = |E_i|(f_i + 1)$, for $i < p$ decreasing to 0. Let K be the sum of the f_i s. Define a probability distribution P such that $P(s) = f_i/(K - |E_i|)$ if $s \in E_i$. Then $B \not\geq_{\text{TLex}} C \Leftrightarrow P(B) \geq P(C)$ ■

The family of probability measures in the proof include big-stepped probabilities. Indeed the discrimmax likelihood and the leximax one coincide for linear possibility distributions. In fact the discrimmax likelihood relations marginally intersect comparative probabilities (and leximax likelihoods), while leximax likelihoods form a subclass of comparative probabilities.

5. Applications

5.1 Decision theory

If S is a set of states, an act is viewed as a mapping $f: S \rightarrow X$, where X is another finite set that represents consequences of acts. It is assumed that X has at least three elements x and y . In each state s , act f produces a result $f(s) \in X$. In this approach, due to Savage, the set of possible acts is represented by the set X^S of mappings from states to consequences.

Let X^S be equipped with a strict preference relation \succ and let \succcurlyeq denote the associated non strict relation. A set of axioms, that slightly differs from the one of Savage, is as follows:
• $S^1: (X^S, \succ)$ is a transitive, irreflexive, partially ordered set.
Let $A \subseteq S$ be an event, f and g two acts, and denote by $f \#_A g$ the act such that

$$f \#_A g(s) = \begin{cases} f(s) & \text{if } s \in A \\ g(s) & \text{if } s \notin A. \end{cases}$$

- $S_2: (X^S, \succ)$ satisfies the sure-thing principle: $f \#_A h \succcurlyeq g \#_A h$ if and only if $f \#_A h \succcurlyeq g \#_A h$. The sure-thing principle enables two notions to be simply defined, namely conditional preference and null events. Act f is said to weakly preferred to g , *conditioned* on A if and only if $\forall h, f \#_A h \succcurlyeq g \#_A h$. This is denoted by $f \succcurlyeq_A g$. An event A is said to be *null* if and only if $\forall f, \forall g, f \succcurlyeq_A g$ holds.
- $U: (f \succcurlyeq_A g \text{ and } (f \succcurlyeq_B g) \Rightarrow f \succcurlyeq g)$.

3 Axioms of the form S_i are originally from Savage. Axioms of the form S_i are variants of Savage axioms. Other ones are new.

Among acts in X^S are *constant acts* such that: $\exists x \in X, \forall s \in S, f(s) = x$. It is denoted f_x . It seems reasonable to identify the set of constant acts $\{f_x, x \in X\}$ and X . The preference on X can be induced from (X^S, \succ) as follows. Given (X^S, \succ) , the preference relation \succ_P on X is of the form $\forall x, y \in X, x \succ_P y$ if and only if $f_x \succcurlyeq f_y$.

- $S_3: \forall A \subseteq S, A$ not null, $(f_x \succcurlyeq f_y)_A$ if and only if $x \succ_P y$.
- Projecting X^S on 2^S yields a likelihood relation, that is, a binary relation \succ_L among events defined by $A \succ_L B$ iff $x \#_A y \succcurlyeq x \#_B y$ for some $x \succ_P y$.
- $S_4: \forall x, y, x', y' \in X \text{ s.t. } x \succ_P y, x' \succ_P y', x \#_A y \succcurlyeq x \#_A y' \Rightarrow x \#_B y \succcurlyeq x \#_B y'$.
- $S_5: \exists x, y, z$ three constant acts such that $f_x \succcurlyeq f_y \succcurlyeq f_z$.

Property S_4 enables events A and B to be consistently compared by fixing $x \succ_P y$ arbitrarily when selecting binary events $x \#_A y$ and $x \#_B y$. The last axiom enables the partial ordering on acts to be reconstructed from a likelihood relation on events and a preference relation on consequences.

1. $f \succcurlyeq g \Leftrightarrow [f \#_P g] \succ_L [g \#_P f]$ where \succ_P is the projection of \succ on X (due to S_3) and \succ_L is the projection of \succ on events (due to S_4). It can be proved that if a discrimmax ordering is chosen for a likelihood relation, then the partial ordering on acts does satisfy $S^1, S_2, S_3, U, S_4, S^5$ and the lifting axiom L . Starting with a regular possibilistic ordering \succ_{Π} on events, lifting it to acts using (L) and reprojecting the relation on acts back to events, yields the discrimmax relation refining \succ_{Π} . Moreover: for A, B, C disjoint, $A \cup C \succ_L B$ and $B \cup C \succ_L A$ imply $C \succ_L A \cup B$.

$A \succ_L A$ and $A \subseteq B$ imply $B \succ_L A$. These properties are satisfied by the discrimmax likelihood relation (Dubois et al. [8,9]) and are characteristic of non-monotonic reasoning in the sense of Kraus, Lehman and Magidor (Friedman and Halpern [16]).

5.2 Nonmonotonic reasoning

Consider nonmonotonic inferences of the form $A \sim B$ which means that if all is known by an agent is that event A obtains, then the agent will reason as if B obtained as well. In other words, in the context where the agent knows A only, he accepts B . Basic properties of nonmonotonic inference have been advocated by Kraus, Lehmann and Magidor [18]:

Right weakening:

$A \vdash B$ and $B \subseteq C$ imply $A \vdash C$
 AN : $A \vdash B$ and $A \vdash C$ imply $A \vdash B \cap C$
 OR : $A \vdash C$ and $B \vdash C$ imply $A \cup B \vdash C$
Cautious monotony (CM): $A \vdash B$ and $A \vdash C$ imply $A \cap B \vdash C$
Cut: $A \vdash B$ and $A \cap B \vdash C$ imply $A \vdash C$.

The above rules of inference embody the notion of plausible inference in the presence of incomplete information. Namely, they describe the properties of deduction under the assumption that the state of the world is as normal as can be. Given a subjective likelihood ordering \succsim , define now $A \vdash B$ iff $A \cap B \succsim A \cap B$. $A \vdash B$ intuitively means that B is more likely than its negation when A is true. Possibility theory is closely related to preferential inference in nonmonotonic reasoning, as defined by Kraus, Lehmann and Magidor [18]. Assume no null events. Then, an inference relation \vdash is preferential if and only if there exists a set of possibility orderings \mathcal{P} such that if $A \neq \emptyset$: $A \vdash B$ iff $\forall \pi \in \mathcal{P}, A \cap B \succsim A \cap B$.

(Dubois and Prade [13]).

Note that only disjoint events are compared, so that again the discrimmax likelihood (that then coincides with the possibilistic one) yields a nonmonotonic inference that obeys the above properties.

5.3 Inconsistency management

Consider layered inconsistent knowledge bases of the form $K = L_1 \cup \dots \cup L_n$, such that formulas ϕ_k in L_i have the same level of priority or certainty and are more reliable than ϕ_j in L_j where $j > i$. Formulas of L_1 are the most important beliefs and those in L_n are the least important ones. There exist two criteria for defining such preferred subbases of K : set-inclusion (Brewka, [4]) or cardinality (Benferhat et al. [2]).

Definition 7: A consistent subbase $M = M_1 \cup \dots \cup M_n$ is an *inclusion-preferred* subbase of K iff there does not exist a consistent subbase $N = N_1 \cup \dots \cup N_n$ of K such that: $\exists i \leq n$, where $M_i \subset N_i$ and for $j > i$, $N_j = M_j$.

Definition 8: A consistent subbase $M = M_1 \cup \dots \cup M_n$ is called a *lex-preferred* subbase of K iff there does not exist a consistent subbase $N = N_1 \cup \dots \cup N_n$ such that: $\exists i \leq n$,

where $|N_i| > |M_i|$ and for $j > i$, $|N_j| = |M_j|$ where $|M|$ is the cardinality of M .

Define a complete preordering on K that accounts for the layers, namely $\phi_i \succsim \pi \phi_k$ iff $i < k$. Then it is easy to check that

Proposition 11 $M = M_1 \cup \dots \cup M_n$ is an inclusion-preferred subbase (resp. *lex-preferred*) of K iff M is a maximal element of $\{N \mid N \text{ consistent, } N \subseteq K\}$ for the discrimmax (resp. leximax) relation induced by \succsim .

6. Conclusion

This paper has proposed two natural extensions of possibility and necessity orderings that exploit the ordinal information contained in a possibility distribution. These relations are self-dual and come close to comparative probability structures. The discrimmax likelihood relation, being not complete is a new kind of uncertainty relation, that naturally arises in decision theory under uncertainty. The problem of axiomatically characterising these uncertainty relations remain open. However they belong to a more general family of what can be called partial additive belief structures on events, that are characterized by the following axioms:

- A0. \geq is reflexive and transitive
 - A0'. $A \geq \emptyset$, (and not $\emptyset \geq A \neq \emptyset$ if there are no null events)
 - A2. $A \subseteq B$ implies $B \geq A$, and
 - A3. If A, B, C are disjoint subsets then: $A \cup B \geq A \cup C \Leftrightarrow B \geq C$
- A large class of such relations can be defined by means of unknown numerical weights $p = (p_1, \dots, p_n)$ assigned to elements of S , and a set of constraints on the p_i 's defining a domain Δ of feasible-tuples. Then define (Lang [19]): $B \geq A$ iff $\sum_{\phi \in B(p)} \phi \geq \sum_{\phi \in A(p)} \phi \forall p \in \Delta$. By suitably restricting Δ , the discrimmax relation and the leximax relation can be recovered (Benferhat et al. [3]).

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