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Linearity and Nonlinearity of Groups of Polynomial Automorphisms of the Plane

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Abstract

Given a field K , we investigate which subgroups of the group $\text{Aut } \mathbb{A}_K^2$ of polynomial automorphisms of the plane are linear or not.

The results are contrasted. The group $\text{Aut } \mathbb{A}_K^2$ itself is nonlinear, except if K is finite, but it contains some large “finite-codimensional” subgroups which are linear. This phenomenon is specific to dimension two: it is easy to prove that any “finite-codimensional” subgroup of $\text{Aut } \mathbb{A}_K^3$ is nonlinear, even for a finite field K .

When $\text{ch } K = 0$, we also look at a similar questions for f.g. subgroups, and the results are also contrasted. Some “finite-codimensional” subgroups are locally linear but not linear.

This paper is respectfully dedicated to the memory of Jacques Tits.

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Summary

Introduction

1. Main Definitions and Conventions
2. Mixed Products
3. Linearity over Rings vs. over Fields
4. A Nonlinear f.g. Subgroup of $\text{Aut}_0 \mathbb{A}_{\mathbb{Q}}^2$
5. The Linear Representation of $\text{Aut}_1 \mathbb{A}_K^2$
6. The Linear Representation of $\text{SAut}_0^{<n} \mathbb{A}_K^2$
7. Semi-algebraic Characters
8. A Nonlinearity Criterion for $\text{Aut}_S \mathbb{A}_K^2$
9. Two Linearity Criteria for $\text{Aut}_S \mathbb{A}_K^2$
10. Examples of Linear or Nonlinear $\text{Aut}_S \mathbb{A}_K^2$
11. Nonlinearity of Finite-Codimensional Subgroups of $\text{Aut} \mathbb{A}_K^3$

Introduction

Let K be a field given once and for all and let $\text{Aut} \mathbb{A}_K^2$ be the group of polynomial automorphisms of the affine plane \mathbb{A}_K^2 over K .

0.1 General Introduction

A group Γ is called *linear over a ring*, respectively *linear over a field*, if there is an embedding $\Gamma \subset \text{GL}(n, R)$, resp. $\Gamma \subset \text{GL}(n, L)$, for some positive integer n and some commutative ring R , resp. some field L .

Various authors have shown that the automorphism groups of algebraic varieties share many properties with linear groups, see e.g. [2] [21][4], e.g. the Tits alternative holds in $\text{Aut} \mathbb{A}_K^2$ [16]. However, these groups are not always linear [13][19] In this paper, we will investigate the following related

Question: which subgroups of $\text{Aut} \mathbb{A}_K^2$ are indeed linear or not?

Roughly speaking, the answer is that $\text{Aut} \mathbb{A}_K^2$ contains large linear subgroups and small ones which are not.

In order to be more specific, let us consider the following subgroups

$$\begin{aligned}\text{Aut}_0 \mathbb{A}_K^2 &= \{\phi \in \text{Aut} \mathbb{A}_K^2 \mid \phi(\mathbf{0}) = \mathbf{0}\}, \\ \text{Aut}_1 \mathbb{A}_K^2 &= \{\phi \in \text{Aut}_0 \mathbb{A}_K^2 \mid d\phi|_{\mathbf{0}} = \text{id}\}, \text{ and} \\ \text{SAut}_0 \mathbb{A}_K^2 &= \{\phi \in \text{Aut}_0 \mathbb{A}_K^2 \mid \text{Jac}(\phi) = 1\},\end{aligned}$$

where $\text{Jac}(\phi) := \det d\phi$ is the jacobian of ϕ . Since $\text{Aut} \mathbb{A}_K^2 / \text{Aut}_0 \mathbb{A}_K^2$ is naturally isomorphic to \mathbb{A}_K^2 , *informally speaking* $\text{Aut}_0 \mathbb{A}_K^2$ is a subgroup of codimension 2. Similarly, since $\text{Jac}(\phi)$ is a constant polynomial, $\text{SAut}_0 \mathbb{A}_K^2$

has codimension 3 and since $\text{SAut}_0 \mathbb{A}_K^2 = \text{SL}(2, K) \times \text{Aut}_1 \mathbb{A}_K^2$, the group $\text{Aut}_1 \mathbb{A}_K^2$ has codimension 6. Anyhow they are viewed as large subgroups.

It was known that the Cremona group $\text{Cr}_2(\mathbb{C})$ and $\text{Aut} \mathbb{A}_{\mathbb{C}}^2$ are not linear over a field, see [8][9]. For the large subgroups of $\text{Aut}_0 \mathbb{A}_K^2$, the linearity results are much more contrasted, as it is shown by the following

Theorem A. (A.1) *Whenever K is infinite, the group $\text{SAut}_0 \mathbb{A}_K^2$ is not linear, even over a ring.*

(A.2) *However, there is an embedding $\text{Aut}_1 \mathbb{A}_K^2 \subset \text{SL}(2, K(t))$.*

For a finite field K , the index $[\text{Aut} \mathbb{A}_K^2 : \text{Aut}_1 \mathbb{A}_K^2]$ is finite. Therefore Theorem A.1 admits the following converse

Corollary. *If K is finite, the group $\text{Aut} \mathbb{A}_K^2$ is linear over the field $K(t)$.*

The existence of large linear subgroups in $\text{Aut} \mathbb{A}_K^n$ is specific to the dimension 2. The case $n = 3$ is enough to show this. For a finite-codimensional ideal \mathfrak{m} of $K[x, y, z]$, let $\text{Aut}_{\mathfrak{m}} \mathbb{A}_K^3$ be the group of all automorphisms

$$(x, y, z) \mapsto (x + f, y + g, z + h),$$

where f, g and h belongs to \mathfrak{m} . Equivalently, ϕ fixes some infinitesimal neighborhood of a finite subset in \mathbb{A}_K^3 . E.g. for $\mathfrak{n} = (x, y, z)^2$, we have

$$\text{Aut}_{\mathfrak{n}} \mathbb{A}_K^3 = \{\phi \in \text{Aut} \mathbb{A}_K^3 \mid \phi(\mathbf{0}) = \mathbf{0} \text{ and } d\phi|_{\mathbf{0}} = \text{id}\}.$$

However the groups $\text{Aut}_{\mathfrak{m}} \mathbb{A}_K^3$ are not linear, even if K is finite, as shown by

Theorem B. *The group $\text{Aut}_{\mathfrak{m}} \mathbb{A}_K^3$ is not linear, even over a ring.*

We will now turn our attention to the small subgroups, namely the finitely generated (f.g. in the sequel) subgroups of $\text{Aut} \mathbb{A}_K^2$. Let $\Gamma \subset \text{Aut} \mathbb{A}_{\mathbb{Q}}^2$ be

$$\Gamma = \langle S, T \rangle, \text{ where } S(x, y) = (y, 2x) \text{ and } T(x, y) = (x, y + x^2).$$

The question of the existence of nonlinear f.g. subgroups in $\text{Aut} \mathbb{A}_{\mathbb{C}}^2$, which is explicitly raised by Y. Cornulier in [9], is answered by the following

Theorem C. *Let K be a field of characteristic zero.*

(C.1) *The subgroup $\Gamma \subset \text{Aut}_0 \mathbb{A}_K^2$ is not linear, even over a ring.*

(C.2) *Any f.g. subgroup of $\text{SAut}_0 \mathbb{A}_K^2$ is linear over $K(t)$.*

It turns out that the group Γ , which is presented by

$$\langle \sigma, \tau \mid \sigma^2 \tau \sigma^{-2} = \tau^2 \rangle,$$

appears in [11] as the first example of a 1-related group which is residually finite but not linear over a field.

By Theorem C.1, Γ is also the first example of a 1-related group which is not linear (even over a ring) but which is embeddable in the automorphism group of an algebraic variety. Residual finiteness of Γ follows from general principles [2], but the observation that Γ acts on the finite sets \mathbb{F}_p^2 , for any odd p , and faithfully on their product, provides a concrete proof.

For an infinite field K , Theorem A.2 suggests to ask which groups $G \supset \text{Aut}_1 \mathbb{A}_K^2$ are linear or not. Indeed, either this group contains $\text{SAut} \mathbb{A}_K^2 := \{\phi \in \text{Aut}_0 \mathbb{A}_K^2 \mid \text{Jac}(\phi) = 1\}$, which is not linear, or it is isomorphic to

$$\text{Aut}_S \mathbb{A}_K^2 := \{\phi \in \text{Aut}_0 \mathbb{A}_K^2 \mid d\phi|_{\mathbf{0}} \in S\},$$

for some subgroup S of $\text{GL}(2, K)$. Therefore we ask

For a given subgroup $S \subset \text{GL}(2, K)$, is the group $\text{Aut}_S \mathbb{A}_K^2$ linear?

For the subgroups S of $\text{SL}(2, K)$, three criteria provide an almost complete answer, see Theorem D. Some examples of application of the criteria are

Example A. *Let q be a quadratic form on K^2 and $S = \text{SO}(q)$.*

If q is anisotropic, $\text{Aut}_S \mathbb{A}_K^2$ is linear over a field extension of K .

Otherwise $\text{Aut}_S \mathbb{A}_K^2$ is not linear, even over a ring.

Example B. *For some lattices $S \subset \text{SL}(2, \mathbb{R})$, $\text{Aut}_S \mathbb{A}_{\mathbb{C}}^2$ is linear over \mathbb{C} .*

For any lattice $S \subset \text{SL}(2, \mathbb{C})$, $\text{Aut}_S \mathbb{A}_{\mathbb{C}}^2$ is not linear, even over a ring.

Example C. *Let d be a squarefree integer, let \mathcal{O} be the ring of integers of $k := \mathbb{Q}(\sqrt{d})$. Set $K = k(x)((t))$ and $S = \text{SL}(2, \mathcal{O}[x, x^{-1}, t])$.*

If $d > 0$, the group $\text{Aut}_S \mathbb{A}_K^2$ is not linear, even over a ring.

Otherwise, $\text{Aut}_S \mathbb{A}_K^2$ is linear over some field of characteristic zero.

0.2 About the main points of the paper

Since the topics are not ordered as in the general introduction, a summary has been provided. Also note that the statements in the introduction are often weaker than those in the main text.

For a group S , we define in Section 2 the notion of a mixed product of S as a semi-direct product $S \ltimes *_{p \in P} E_p$, where $(E_p)_{p \in P}$ is a family of groups and S acts by permuting the factors of the free product. Hence P is a S -set such that $E_p^s = E_{s \cdot p}$ for any $s \in S$ and $p \in P$.

In section 3, it is shown that, under a mild assumption, a mixed product (or an amalgamated product) which is linear over a ring is automatically linear over a field. This explains the dichotomy *linear over a field/not linear, even over a ring* in our statements. Then, we can use the theory of algebraic groups to show that some mixed products are not linear, even over a ring.

Let S be a subgroup of $\mathrm{SL}(2, K)$. The main question of the paper is to decide if $\mathrm{Aut}_S \mathbb{A}_K^2$ is linear or not. It turns out that this group is mixed product $S \times *_{\delta \in \mathbb{P}_K^1} E_\delta(K)$. Hence, there are two obstructions for the linearity.

First, the groups $S_\delta \times E_\delta(K)$ have to be linear with a uniform bound on the degree. This problem is solved by using the notion of *semi-algebraic characters* for subgroups $\Lambda \subset K^*$. It was inspired by the famous paper of Borel and Tits [5], proving that the abstract isomorphisms of simple algebraic groups are semi-algebraic. Strictly speaking, our paper only provides a partial answer, because otherwise it would had been too long.

The second obstruction is the possibility, or not, to glue together some representations of the groups S and $S_\delta \times E_\delta(K)$ to get a representation of $\mathrm{Aut}_S \mathbb{A}_K^2$. Our linearity criterion is stronger in characteristic zero than in finite characteristics. In characteristic zero, we can use for the gluing process the following stronger version of

Theorem C.2. *Assume K of characteristic 0. There is an embedding*

$$\mathrm{SAut}_0^{<n} \mathbb{A}_K^2 \subset \mathrm{SL}(1 + \mathrm{lcm}(1, 2, \dots, n), K(t)).$$

Here $\mathrm{SAut}_0^{<3} \mathbb{A}_K^2$ denotes the subgroup of $\mathrm{SAut}_0 \mathbb{A}_K^2$ generated by the automorphisms of degree $< n$, for some $n \geq 3$.

The proof of the strong version of Theorem C.2 uses one idea, the ping-pong lemma and one trick, based on the Lie superalgebra $\mathfrak{osp}(1, 2)$. The ping-pong lemma was originally invented by Fricke and Klein for the dynamic of groups with respect to the metric topologies, see [12]. Later, Tits used it in the context of the ultrametric topologies [23], and we follow the Tits idea. Here, the ping-pong setting requires a representation of large dimension.

As a brief conclusion

- if $\mathrm{ch} K = 0$, then $\mathrm{Aut}_0 \mathbb{A}_K^2$ is not even locally linear, $\mathrm{SAut}_0 \mathbb{A}_K^2$ is locally linear but not linear, and $\mathrm{Aut}_1 \mathbb{A}_K^2$ is linear. Therefore, the main questions arise for the groups G with $\mathrm{Aut}_1 \mathbb{A}_K^2 \subset G \subset \mathrm{SAut}_0 \mathbb{A}_K^2$.
- if $\mathrm{ch} K = p$, $\mathrm{SAut}_0 \mathbb{A}_K^2$ is linear iff K is finite, while $\mathrm{Aut}_1 \mathbb{A}_K^2$ is always linear. In particular, $\mathrm{Aut} \mathbb{A}_{\mathbb{F}_p}^2$ is locally linear but not linear. The existence of nonlinear f.g. subgroups is an open question.
- These phenomena are specific to dimension two.

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1 Main Definitions and Conventions

Throughout the whole paper, K will denote a given field. Its *ground field* is $\mathbb{F} = \mathbb{F}_p$ if $\text{ch } K = p$ or $\mathbb{F} = \mathbb{Q}$ otherwise.

1.1 Group theoretical notations

Let S be a group and let $x, y \in S$. The symbols y^x and (x, y) are defined by

$$y^x := xyx^{-1} \text{ and } (x, y) := xyx^{-1}y^{-1}.$$

By definition, a S -set is a set P endowed with an action of S . The *stabilizer* of a point $p \in P$ is the subgroup $S_p := \{s \in S \mid s.p = p\}$. The *core* $\text{Core}_S(A)$ of a subgroup A of S is the kernel of the action of S on S/A .

Similarly, a S -group is a group E endowed with a homomorphism $S \rightarrow \text{Aut}(E)$. The corresponding semi-direct product of S by E is denoted by $S \ltimes E$. Given another S -groups E' , a homomorphism, respectively an isomorphism, $\phi : E \rightarrow E'$ is called an S -homomorphism, resp. an S -isomorphism if it commutes with the S -action.

1.2 Commutative rings and group functors

Throughout the whole paper, a *commutative ring* means an associative commutative unital ring.

A *group functor* is a functor $G : R \mapsto G(R)$ from the category of commutative rings R to the category of groups (see e.g. [10] and [26] for the functorial approach to group theory). The standard example of a group functor is $R \mapsto \text{GL}(n, R)$, where n is a given positive integer.

Given an ideal I of a commutative ring R , we denote by $G(I)$ the kernel of the homomorphism $G(R) \rightarrow G(R/I)$. It is called the *congruence subgroup* associated to the ideal I . For example, $\text{GL}(n, I)$ is the subgroup of $\text{GL}(n, R)$ of all matrices of the form $\text{id} + A$, where all entries of A are in I .

In most cases, *we will only define the group $G(K)$ when K is a field* and the reader should understand that the definition over a ring is similar.

For the group $\text{Aut } \mathbb{A}_K^2$ of polynomial automorphisms of \mathbb{A}_K^2 and its con-sorts, our notation is not consistent, namely K is an index.

1.3 The group functors $\text{Elem}_*(K)$, $\text{Aff}(2, K)$, and their subgroups

By definition, an *elementary automorphism* of \mathbb{A}_K^2 is an automorphism

$$\phi : (x, y) \mapsto (z_1x + t, z_2y + f(x))$$

for some $z_1, z_2 \in K^*$, some $t \in K$ and some $f \in K[x]$. The group of elementary automorphisms of \mathbb{A}_K^2 is denoted $\text{Elem}(K)$. Set

$$\begin{aligned} \text{Elem}_0(K) &= \text{Elem}(K) \cap \text{Aut}_0 \mathbb{A}_K^2, \\ \text{SElem}_0(K) &= \text{Elem}(K) \cap \text{SAut}_0 \mathbb{A}_K^2, \text{ and} \\ \text{Elem}_1(K) &= \text{Elem}(K) \cap \text{Aut}_1 \mathbb{A}_K^2. \end{aligned}$$

However, *we will use the simplified notation $E(K)$ for $\text{Elem}_1(K)$.*

Let $\text{Aff}(2, K)$ be the subgroup of affine automorphisms of \mathbb{A}_K^2 . Set

$$\begin{aligned} B_{\text{Aff}}(K) &:= \text{Aff}(2, K) \cap \text{Elem}(K), \\ B_{\text{GL}}(K) &:= B_{\text{Aff}}(K) \cap \text{GL}(2, K), \text{ and} \\ B(K) &:= B_{\text{Aff}}(K) \cap \text{SL}(2, K). \end{aligned}$$

Indeed $B_{\text{Aff}}(K)$, $B_{\text{GL}}(K)$ and $B(K)$ are the standard Borel subgroups of $\text{Aff}(2, K)$, $\text{GL}(2, K)$ and $\text{SL}(2, K)$. We have

$$\begin{aligned} \text{Aff}(2, K)/B_{\text{Aff}}(K) &= \text{GL}(2, K)/B_{\text{GL}}(K) = \text{SL}(2, K)/B(K) \simeq \mathbb{P}_K^1, \text{ and} \\ \text{Elem}_0(K) &= B_{\text{GL}}(K) \rtimes E(K) \text{ and } \text{SElem}_0(K) = B(K) \rtimes E(K). \end{aligned}$$

1.4 An informal definition of finite-codimensional group subfunctor

Informally speaking, a group subfunctor $H \subset G$ has *finite codimension* if the functor $R \mapsto G(R)/H(R)$ is “represented” by a scheme X of finite type. In particular, it means that there is a natural transform $G/R \rightarrow X$ which induces a bijection $G(K)/H(K) \rightarrow X(K)$ whenever K is an algebraically closed field. Since this notion is used only for presentation purpose, we will not provide a formal definition.

For example, the subgroup $\text{Aut}_0 \mathbb{A}_K^2$ has codimension 2 in $\text{Aut} \mathbb{A}_K^2$, since the quotient $\text{Aut} \mathbb{A}_K^2/\text{Aut}_0 \mathbb{A}_K^2$ is naturally \mathbb{A}_K^2 . This fact does not require to involve the elusive theory of ind-algebraic groups.

2 Mixed Products

Let S be a group and let $(E_p)_{p \in P}$ be a collection of groups indexed by some S -set P . Since we did not find a name in the literature, we will call *mixed product of S* any semi-direct product $S \times *_P E_p$ where S acts on the free product $*_P E_p$ by permuting its factors, i.e. we have $E_p^s = E_{s \cdot p}$ for any $s \in S$ and $p \in P$ (see [17] ch.4 for the definition of free products).

The connections between mixed products, amalgamated products and free products are investigated. As a consequence of the van der Kulk Theorem, we show that the groups $\text{Aut}_S \mathbb{A}_K^2$ are mixed products and that the groups $\text{Aut}_1 \mathbb{A}_K^2$ and $\text{Aut}_{U(K)} \mathbb{A}_K^2$ are free products.

2.1 Amalgamated products

Let A , G_1 and G_2 be groups and let $f_1 : A \rightarrow G_1$ and $f_2 : A \rightarrow G_2$ be two group homomorphisms. Let $G_1 *_A G_2$ be the *amalgamated product* of G_1 and G_2 over A , see e.g. [20], ch. I. Since the amalgamated products satisfy a universal property, they are often *free amalgamated products*, see [17] ch.8.

In what follows, we will *always assume that f_1 and f_2 are injective*. Hence the homomorphisms $G_1 \rightarrow G_1 *_A G_2$ and $G_2 \rightarrow G_1 *_A G_2$ are injective, by the Theorem 1 of ch. 1 in [20]. Therefore, we will use a less formal terminology. The groups G_1 and G_2 are viewed as subgroups of $G_1 *_A G_2$ and we will say that G_1 and G_2 *share A as a common subgroup*.

2.2 Reduced words

The usual definition [20] of reduced words is based on the right A -cosets. In order to avoid a confusion between the set difference notation $A \setminus X$ and the A -orbits notation $A \backslash X$, we will use a definition based on the left A -cosets.

Let G_1, G_2 be two groups sharing a common subgroup A , and let $\Gamma = G_1 *_A G_2$. Set $G_1^* = G_1 \setminus A$, $G_2^* = G_2 \setminus A$ and let $T_1^* \subset G_1^*$ (respectively $T_2^* \subset G_2^*$) be a set of representatives of G_1^*/A (resp. of G_2^*/A).

Let Σ be the set of all finite alternating sequences $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ of ones and twos. A *reduced word* of *type* ϵ is a word (x_1, \dots, x_n, x_0) where x_0 is in A , and $x_i \in T_{\epsilon_i}^*$ for any $1 \leq i \leq n$. Let \mathcal{R} be the set of all reduced words. The next Lemma is well-known, see e.g. [20], Theorem 1.

Lemma 1. *The map*

$$(x_1, \dots, x_n, x_0) \in \mathcal{R} \mapsto x_1 \dots x_n x_0 \in G_1 *_A G_2$$

is bijective.

Set $\Gamma = G_1 *_A G_2$. For $\gamma \in \Gamma \setminus A$ there is some integer $n \geq 1$, some $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \Sigma$ and some $g_i \in G_{\epsilon_i}^*$ such that $\gamma = g_1 \dots g_n$. It follows from loc. cit. that $\gamma = x_1 \dots x_n x_0$ for some reduced word $(x_1 \dots x_n x_0)$ of type ϵ . Since it is determined by γ , the sequence ϵ is called *the type* of γ .

2.3 Amalgamated product of subgroups

Let G_1, G_2 be two groups sharing a common subgroup A and set $\Gamma = G_1 *_A G_2$. Let $G'_1 \subset G_1$, $G'_2 \subset G_2$ and $A' \subset A$ be subgroups such that

$$G'_1 \cap A = G'_2 \cap A = A'.$$

Lemma 2. (i) *The natural map $G'_1 *_{A'} G'_2 \rightarrow \Gamma$ is injective.*

(ii) *Let $\Gamma' \subset \Gamma$ be a subgroup such that $\Gamma'.A = \Gamma$. Then we have*

$$\Gamma' = G'_1 *_{A'} G'_2,$$

where $G'_1 = G_1 \cap \Gamma'$, $G'_2 = G_2 \cap \Gamma'$ and $A' = A \cap \Gamma'$.

Proof. Proof of Assertion (i). For $i = 1, 2$ set $G_i^* = G_i \setminus A$, $G_i'^* = G_i' \setminus A'$. Let $T_i^* \subset G_i^*$ and $T_i'^* \subset G_i'^*$ be a set of representatives of G_i^*/A and $G_i'^*/A'$.

Since the maps $G_i'/A' \rightarrow G_i/A$ are injective, it can be assumed that $T_i'^* \subset T_i^*$. Let \mathcal{R} and \mathcal{R}' be the set of reduced words of $G_1 *_{A'} G_2$, and respectively of $G'_1 *_{A'} G'_2$. By definition, we have $\mathcal{R}' \subset \mathcal{R}$, thus by Lemma 1 the map $G'_1 *_{A'} G'_2 \mapsto G_1 *_{A'} G_2$ is injective.

Proof of Assertion (ii). We will use the notations of the previous proof. Since $\Gamma'.A = \Gamma$, it follows that the maps $G'_1/A' \rightarrow G/A$ and $G'_2/A' \rightarrow G_2/A$ are bijective. Therefore \mathcal{R}' is the set of all reduced words $(x_1, \dots, x_n, x_0) \in \mathcal{R}$ such that $x_0 \in A'$. It follows easily that

$$G_1 *_{A'} G_2 / G'_1 *_{A'} G'_2 \simeq A/A' = \Gamma/\Gamma',$$

and therefore we have $\Gamma' = G'_1 *_{A'} G'_2$. □

2.4 The group $\text{Aut } \mathbb{A}_K^2$ is an amalgamated product

Indeed, it is the classical

van der Kulk Theorem. [27] *We have*

$$\text{Aut } \mathbb{A}_K^2 \simeq \text{Aff}(2, K) *_{B_{\text{Aff}}(K)} \text{Elem}(K).$$

2.5 Mixed products

Let S be a group, let P be a S -set and let $Q \subset P$ be a set of representatives of P/S . A mixed product $S \times *_{p \in P} E_p$ satisfies the following universal property

Lemma 3. *Let $\Gamma \supset S$ be a group. Assume given, for any $q \in Q$, a S_q -homomorphism $\phi_q : E_q \rightarrow \Gamma$. Then there is a unique group homomorphism*

$$\phi : S \times *_{p \in P} E_p \rightarrow \Gamma$$

such that $\phi|_S = \text{id}$ and $\phi|_{E_q} = \phi_q$ for any $q \in Q$.

Proof. Let us define, for any $p \in P$, a S_p -homomorphism $\phi_p : E_p \rightarrow \Gamma$ as follows. Let $s \in S$ such that $q := s.p$ belongs to Q . Set

$$\phi_p(u) = s^{-1} \phi_q(sus^{-1})s,$$

for any $u \in E_p$. Since ϕ_q is a S_q -homomorphism, the defined homomorphism ϕ_p only depends on s modulo S_p . Moreover the collection of homomorphisms $(\phi_p)_{p \in P}$ induces a S -homomorphism from $*_{p \in P} E_p$ to Γ , which extends to the required homomorphism $\phi : S \times *_{p \in P} E_p \rightarrow \Gamma$. \square

It follows that a mixed product $S \times *_{p \in P} E_p$ is entirely determined by S and the S_q -groups E_q for $q \in Q$. For the record, let's state

Lemma 4. *Let $\Gamma = S \times *_{p \in P} E_p$ and $\Gamma' = S \times *_{p \in P} E'_p$ be two mixed groups. If for any $q \in Q$, the groups E_q and E'_q are S_q -isomorphic, then the groups Γ and Γ' are isomorphic.*

2.6 Mixed products with a transitive action on P

In this subsection, we show that the mixed products with a transitive action of S on P are the amalgamated products $S *_A G$ where A is a retract in G .

First, let S, G be two groups sharing a common subgroup A with the additional assumption that A is a retract in G . Therefore, we have $G = A \times E$, for some normal subgroup E of G . Set $\Gamma = S *_A G$ and let Γ_1 be the kernel of the map $\Gamma \rightarrow S$ induced by the retraction $G \rightarrow A \simeq G/E$. It is clear that

$$\Gamma = S \times \Gamma_1.$$

Lemma 5. *Let P be a set of representatives of S/A . We have*

$$\Gamma_1 \simeq *_{\gamma \in P} E^\gamma.$$

*In particular $S *_A G$ is isomorphic to the mixed product $S \times *_{\gamma \in P} E^\gamma$.*

Proof. We can assume that $1 \in P$. By Lemma 3, there is a unique homomorphism $\phi : S \times *_{\gamma \in P} E^\gamma \rightarrow \Gamma$ such that its restriction to $E^1 = E$ and to S is the identity. Conversely, the group $S \times *_{\gamma \in P} E^\gamma$ contains the subgroups S and $G \simeq A \times E^1$ whose intersection is A . Hence, the universal property of amalgamated products provides a natural homomorphism $\psi : \Gamma \rightarrow *_{\gamma \in P} E^\gamma$. Clearly, ϕ and ψ are inverses of each other, what shows the lemma. \square

Conversely, let $\Gamma = S \times *_{p \in P} E_p$ be a mixed product.

Lemma 6. *Assume that S acts transitively on P . Then we have*

$$S \times (*_{p \in P} E_p) \simeq S *_S (S_q \times E_q),$$

where q is any chosen point in P .

The proof of the Lemma 6, based on the same use of universal properties as the previous proof, will be skipped.

2.7 The group $\text{Aut}_S \mathbb{A}_K^2$ is a mixed product

For a subgroup S of $\text{GL}(2, K)$, recall that

$$\text{Aut}_S \mathbb{A}_K^2 := \{\phi \in \text{Aut}_0 \mathbb{A}_K^2 \mid d\phi_{\mathbf{0}} \in S\}.$$

As usual, a line $\delta \in \mathbb{P}_K^1$ has *projective coordinates* $(a; b)$ if $\delta = K.(a, b)$. For such a δ , let $E_\delta(K) \subset \text{Aut} \mathbb{A}_K^2$ be the subgroup

$$E_\delta(K) := \{(x, y) \mapsto (x, y) + f(bx - ay)(a, b) \mid f \in t^2 K[t]\}.$$

Let $\gamma \in \text{GL}(2, K)$ such that $\gamma.\delta_0 = \delta$ where $\delta_0 \in \mathbb{P}_K^1$ has coordinates $(0; 1)$. Then we have $E_{\delta_0}(K) = E(K)$ and $E_\delta(K) = E(K)^\gamma$.

Lemma 7. *We have*

$$\text{Aut}_S \mathbb{A}_K^2 \simeq S \times *_{\delta \in \mathbb{P}_K^1} E_\delta(K).$$

Proof. Clearly it is enough to prove the statement for $S = \text{GL}(2, K)$. Since $B_{\text{Aff}}(K)$ contains the translations, we have $B_{\text{Aff}}(K).\text{Aut}_0 \mathbb{A}_K^2 = \text{Aut} \mathbb{A}_K^2$. Therefore by van der Kulk Theorem and Lemma 2, we have

$$\text{Aut}_0 \mathbb{A}_K^2 \simeq \text{GL}(2, K) *_{B_{\text{GL}}(K)} \text{Elem}_0(K).$$

Since $\text{Elem}_0(K) = B_{\text{GL}}(K) \times E(K)$, it follows from Lemma 5 that

$$\text{Aut}_0 \mathbb{A}_K^2 \simeq \text{GL}(2, K) \times *_{\delta \in \mathbb{P}_K^1} E_\delta(K). \quad \square$$

2.8 Mixed product with an almost free transitive action on P

The action of S on a set P is called *almost free transitive* if P consists of a fixed point and a free orbit under S . (It will be tacitly assumed that $S \neq 1$, so the fixed point and the free orbit are well defined.) In this subsection we show that mixed product with an almost free transitive action of S on P are the free products $G * G'$, where S is a retract in G .

First let $\Gamma = S \times *_{p \in P} E_p$ be a mixed product.

Lemma 8. *Assume that the action of S on P is almost free transitive. Then Γ is isomorphic to the free product*

$$(S \times E_{p_0}) * E_{p_\infty},$$

where $p_0 \in P$ is the fixed point and $p_\infty \in P$ is any point of the free orbit.

Conversely let $\Gamma = (S \times E) * F$ be a free product, where E is a S -group and F is another group.

Lemma 9. *The group Γ is isomorphic to the mixed product*

$$S \times (E * (*_{s \in S} F_s)),$$

where F_s denotes a copy of F , for any $s \in S$.

The easy proofs of the previous two lemmas, which follows the same pattern as Lemma 4, will be skipped.

2.9 The group $\text{Aut}_{U(K)} \mathbb{A}_K^2$ is a free product

Let $U(K)$ be the group of linear transforms $(x, y) \mapsto (x, y + ax)$ for some $a \in K$. Let $\delta_0, \delta_\infty \in \mathbb{P}_K^1$ be the points with projective coordinates $(0; 1)$ and $(1; 0)$. The group $E_{\delta_0}(K) = E(K)$ commutes with $U(K)$.

Lemma 10. *We have*

$$\text{Aut}_{U(K)} \mathbb{A}_K^2 \simeq (U(K) \times E_{\delta_0}(K)) * E_{\delta_\infty}(K).$$

Proof. By Lemma 7, we have $\text{Aut}_{U(K)} \mathbb{A}_K^2 \simeq U(K) \times *_{\delta \in \mathbb{P}_K^1} E_\delta(K)$. Since the action of $U(K)$ on \mathbb{P}_K^1 is almost free transitive, the lemma follows from Lemma 8. \square

2.10 A corollary

Corollary 1. *Let K, L be fields such that $\text{Card } K = \text{Card } L$ and $\text{ch } K = \text{ch } L$. We have*

$$\text{Aut}_1 \mathbb{A}_K^2 \simeq \text{Aut}_1 \mathbb{A}_L^2 \text{ and } \text{Aut}_{U(K)} \mathbb{A}_K^2 \simeq \text{Aut}_{U(L)} \mathbb{A}_L^2.$$

Proof. It can be assumed that K is infinite. Let \mathbb{F} be its prime subfield let E be a \mathbb{F} -vector space with $\dim_{\mathbb{F}} E = \aleph_0 [K : \mathbb{F}] = \text{Card } K$.

By Lemma 7, $\text{Aut}_1 \mathbb{A}_K^2$ is a free product of $\text{Card } K$ copies of E , from which it follows that $\text{Aut}_1 \mathbb{A}_K^2$ only depends on the cardinality and the characteristic of the field K , hence we have $\text{Aut}_1 \mathbb{A}_K^2 \simeq \text{Aut}_1 \mathbb{A}_L^2$.

The proof that $\text{Aut}_{U(K)} \mathbb{A}_K^2 \simeq \text{Aut}_{U(L)} \mathbb{A}_L^2$ is identical. \square

3 Linearity over Rings vs. over Fields

In this section, we show that, under a mild assumption, a mixed product $S \times *_{p \in P} E_p$ or an amalgamated product $G_1 *_A G_2$ which is linear over a ring is automatically linear over a field, see Corollaries 2 and 3.

3.1 Linearity Properties

For a group Γ , the strongest form of linearity is the linearity over a field. Besides the case of *prime* rings, i.e. the subrings of a field, a group which is linear over a ring R is not necessarily linear over a field. Two relevant examples are provided in the subsection 8.6.

On the opposite, there are also groups Γ which contain a f.g. subgroup which is not linear, even over a ring. These groups are nonlinear in the strongest sense.

3.2 Minimal embeddings

Let R be a commutative ring and let Γ be a subgroup of $\mathrm{GL}(n, R)$ for some $n \geq 1$. The embedding $\Gamma \subset \mathrm{GL}(n, R)$ is called *minimal* if for any ideal $J \neq \{0\}$ we have $\Gamma \cap \mathrm{GL}(n, J) \neq \{1\}$.

Lemma 11. *Let Γ be a subgroup of $\mathrm{GL}(n, R)$. For some ideal J , the induced homomorphism $\Gamma \rightarrow \mathrm{GL}(n, R/J)$ is a minimal embedding.*

Proof. Since R is not necessarily noetherian, the proof requires Zorn Lemma.

Let \mathcal{S} be the set of all ideals J of R such that $\Gamma \cap \mathrm{GL}(n, J) = \{1\}$. With respect to the inclusion, $\mathcal{S} \ni \{0\}$ is a nonempty poset. For any chain $\mathcal{C} \subset \mathcal{S}$, the ideal $\cup_{I \in \mathcal{C}} I$ belongs to \mathcal{S} . Therefore Zorn Lemma implies that \mathcal{S} contains a maximal element J . It follows that the induced homomorphism $\Gamma \rightarrow \mathrm{GL}(n, R/J)$ is a minimal embedding. \square

3.3 Groups with trivial normal centralizers

By definition, a group Γ has the *trivial normal centralizers property* if, for any subset $S \not\subset \{1\}$, its centralizer $C_\Gamma(S)$ is not normal, except if $C_\Gamma(S)$ is the trivial group. Equivalently, if H_1 and H_2 are commuting normal subgroups of Γ , then one of them is trivial.

Lemma 12. *Let Γ be a group with the trivial normal centralizers property.*

If Γ is linear over a ring, then Γ is also linear over a field.

Proof. By hypothesis and Lemma 11, there exists a minimal embedding $\rho : \Gamma \subset \mathrm{GL}(n, R)$ for some commutative ring R . The case $\Gamma = \{1\}$ can be excluded, so we will assume that $R \neq \{0\}$.

Let I_1, I_2 be ideals of R with $I_1 I_2 = 0$. Since $H_1 := \Gamma \cap \mathrm{GL}(n, I_1)$ and $H_2 := \Gamma \cap \mathrm{GL}(n, I_2)$ are commuting normal subgroups of Γ , one of them is trivial. Since ρ is minimal, I_1 or I_2 is the zero ideal. Thus R is prime.

It follows that $\Gamma \subset \mathrm{GL}(n, K)$, where K is the fraction field of R . \square

3.4 Amalgamated products with a trivial core

Let G_1, G_2 be two groups sharing a common subgroup A and set $\Gamma = G_1 *_A G_2$.

Let Σ be the set of all finite alternating sequences $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ of ones and twos. For $i, j \in \{1, 2\}$, let $\Sigma_{i,j}$ be the subset of all $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \Sigma$ starting with i and ending with j and let $\Gamma_{i,j}$ be the set of all $\gamma \in \Gamma$ of type ϵ for some $\epsilon \in \Sigma_{i,j}$. Therefore we have

$$\Gamma = A \sqcup \Gamma_{1,1} \sqcup \Gamma_{2,2} \sqcup \Gamma_{1,2} \sqcup \Gamma_{2,1}.$$

By definition, the amalgamated product $G_1 *_A G_2$ is called *nondegenerate* if $G_1 \neq A$ and $G_2 \neq A$. It is called *dihedral* if $G_1 = G_2 = \mathbb{Z}/2\mathbb{Z}$, and $A = \{1\}$, and *nondihedral* otherwise.

Lemma 13. *Let $\Gamma = G_1 *_A G_2$ be a nondegenerate and nondihedral amalgamated product such that $\text{Core}_\Gamma(A)$ is trivial.*

For any element $g \neq 1$ of Γ , there are $\gamma_1, \gamma_2 \in \Gamma$ such that

$$g^{\gamma_1} \in \Gamma_{1,1} \text{ and } g^{\gamma_2} \in \Gamma_{2,2}.$$

In particular Γ has the trivial normal centralizers property.

Proof. First it should be noted that A cannot be simultaneously a subgroup of index 2 in G_1 and in G_2 . Otherwise the core hypothesis implies that $A = \{1\}$ and Γ would be the dihedral group. Hence we can assume that G_2/A contains at least 3 elements.

Next it is clear that $G_i^* \cdot \Gamma_{j,k} \subset \Gamma_{i,k}$ and $\Gamma_{k,j} \cdot G_i^* \subset \Gamma_{k,i}$ whenever $i \neq j$.

Proof that the conjugacy class of any $g \neq 1$ intersects both $\Gamma_{1,1}$ and $\Gamma_{2,2}$. Let $\gamma_1 \in G_1^*$ and $\gamma_2 \in G_2^*$. We have $\Gamma_{2,2}^{\gamma_1} \subset \Gamma_{1,1}$ and $\Gamma_{1,1}^{\gamma_2} \subset \Gamma_{2,2}$. Therefore the claim is proved for any $g \in \Gamma_{1,1} \cup \Gamma_{2,2}$. Moreover it is now enough to prove that the conjugacy class of any $g \neq 1$ intersects $\Gamma_{2,2}$.

Assume now $g \in \Gamma_{2,1}$. We have $g = u.v$ for some $u \in G_2^*$ and $v \in \Gamma_{1,1}$. Since $[G_2 : A] \geq 3$, there is $\gamma \in G_2^*$ such that $\gamma.u \notin A$. It follows that $\gamma.g$ belongs to $\Gamma_{2,1}$, and therefore g^γ belongs to $\Gamma_{2,2}$.

For $g \in \Gamma_{1,2}$, the claim follows from the fact that g^{-1} belongs to $\Gamma_{2,1}$.

Last, let $g \in A \setminus \{1\}$. Since $\text{Core}_A(\Gamma)$ is trivial, there is $\gamma \in \Gamma$ such that g^γ is not in A . Thus g^γ belongs to $\Gamma_{i,j}$ for some i, j . So g is conjugate to some element in $\Gamma_{2,2}$ by the previous considerations.

Proof that Γ has the trivial normal centralizers property. Let H_1, H_2 be nontrivial normal subgroups. By the previous point, there are elements g_1, g_2 with

$$g_1 \in H_1 \cap \Gamma_{1,1} \text{ and } g_2 \in H_2 \cap \Gamma_{2,2}.$$

Since we have $g_1 g_2 \in \Gamma_{1,2}$ and $g_2 g_1 \in \Gamma_{2,1}$, it follows that $g_1 g_2 \neq g_2 g_1$. Therefore H_1 and H_2 do not commute. \square

Corollary 2. *Let $\Gamma = G_1 *_A G_2$ be a nondegenerate amalgamated product such that $\text{Core}_\Gamma(A)$ is trivial¹.*

If Γ is linear over a ring, then Γ is linear over a field.

Proof. Since the infinite dihedral group is linear over a field, we will assume that the amalgamated product $\Gamma = G_1 *_A G_2$ is also nondihedral. Thus the result is an obvious corollary of Lemmas 12 and 13. \square

3.5 Mixed products with trivial core

Let S be a group, and let $S \rtimes_{*_{p \in P} E_p}$ be a mixed product of S .

Let Σ be the set of all finite sequences $\boldsymbol{\pi} = (p_1, \dots, p_m)$ of elements of P with $p_i \neq p_{i+1}$, for any $i < m$. Set $\Gamma_1 = *_{p \in P} E_p$ and $E_p^* = E_p \setminus \{1\}$. Any element $u \in \Gamma_1 \setminus \{1\}$ is uniquely written as $u = u_1 \dots u_m$, where $u_i \in E_{p_i}^*$ for some $m \geq 1$ and some sequence $\boldsymbol{\pi} = (p_1, \dots, p_m) \in \Sigma$. The decomposition $u = u_1 \dots u_m$ is called the *reduced decomposition* of u , $\boldsymbol{\pi}$ is called its *type* and m is called its *length*. For $p, p' \in P$, let $E_{p,p'}$ be the set of all elements $u \in \Gamma_1 \setminus \{1\}$ whose type is a sequence $\boldsymbol{\pi}$ starting with p and ending with p' .

By definition, the free product $*_{p \in P} E_p$, or, by extension, the mixed product $S \rtimes_{*_{p \in P} E_p}$, is called *nondegenerate* if $\text{Card } P \geq 2$ and $E_p \neq \{1\}$ for any $p \in P$. For a nondegenerate mixed product $S \rtimes_{*_{p \in P} E_p}$, we have

$$\text{Core}_\Gamma(S) = \text{Core}_\Gamma(\cap_P S_p).$$

The mixed product $S \rtimes_{*_{p \in P} E_p}$ is called *dihedral* if $\text{Card } P = 2$, if $E_p = \mathbb{Z}/2\mathbb{Z}$ for any $p \in P$ and if

$$S = \{1\} \text{ or } S \simeq \mathbb{Z}/2\mathbb{Z} \text{ permutes the two factors.}$$

It is called *nondihedral* otherwise.

Lemma 14. *Let $\Gamma = S \rtimes_{*_{p \in P} E_p}$ be a nondegenerate and nondihedral mixed product such that $\text{Core}_\Gamma(\cap_P S_p) = \{1\}$. Let $p \in P$.*

(i) *For any element $\gamma \in \Gamma_1 \setminus \{1\}$, there is $v \in \Gamma_1$ such that γ^v belongs to $E_{p,p}$.*

(ii) *For any element $\gamma \in \Gamma \setminus \Gamma_1$, there is $v \in \Gamma_1$ such that $(\gamma, v) \neq 1$.*

In particular Γ has the trivial normal centralizers property.

Proof. *Proof of Assertion(i).* By hypothesis, Γ_1 is not the infinite dihedral group $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. So the assertion follows from Lemma 13 when $\text{Card } P = 2$. Therefore, we will assume that $\text{Card } P \geq 3$.

¹As it has been noticed by the referee, this is equivalent to the faithfulness of the action of Γ on the associated Bass-Serre tree.

The element γ belongs to E_{p_1, p_2} for some $p_1, p_2 \in P$. Let $p_3 \in P \setminus \{p_1, p_2\}$ and let $v \in E_{p, p_3}$. Thus the element γ^v belongs to $E_{p, p}$.

Proof of Assertion (ii). Let $\gamma = su$, where $s \in S \setminus \{1\}$ and $u \in \Gamma_1$.

Obviously Γ/S and Γ_1 are isomorphic S -sets. Since $\text{Core}_\Gamma(S)$ is trivial, there is $v \in \Gamma_1$ such that $(s, v) \neq 1$. Thus, we will now assume that $u \neq 1$.

Let $u = u_1 \dots u_m$ be its reduced decomposition, let (p_1, \dots, p_m) be its type and let $v \in E_{p'}^*$, where $p' \neq p_m$. By definition, $u_1 \dots u_m \cdot v \cdot u_m^{-1} \dots u_1^{-1}$ is a reduced decomposition of the element $w = uvu^{-1}$. Hence w and w^s have length $2m + 1 \geq 2$. Thus we have $w^s \neq v$, or, equivalently $v^\gamma \neq v$. Hence we have $(\gamma, v) \neq 1$.

Proof that Γ has the trivial normal centralizers property. Let H, H' be two nontrivial normal subgroups of Γ . Let $p \neq p'$ be elements of P . By the Assertions (i) and (ii), there are elements g, g' with

$$g \in H \cap E_{p, p} \text{ and } g' \in H' \cap E_{p', p'}.$$

Since we have $gg' \in E_{p, p'}$ and $g'g \in E_{p', p}$, it follows that $gg' \neq g'g$. Therefore H and H' do not commute. \square

Corollary 3. *Let $\Gamma = S \times *_{p \in P} E_p$ be a nondegenerate mixed product such that $\text{Core}_\Gamma(\cap_P S_p) = \{1\}$.*

If Γ is linear over a ring, then Γ is linear over a field.

Proof. Since the infinite dihedral group is linear over a field, we will assume that the mixed product $\Gamma = S \times *_{p \in P} E_p$ is also nondihedral. Then the result is an obvious corollary of Lemmas 12 and 14. \square

4 A Nonlinear f.g. Subgroup of $\text{Aut}_0 \mathbb{A}_{\mathbb{Q}}^2$

Let Γ be the group with presentation

$$\langle \sigma, \tau \mid \sigma^2 \tau \sigma^{-2} = \tau^2 \rangle.$$

In [11], C. Drutu and M. Sapir showed that Γ is not linear over a field². We show that Γ is neither linear over a ring, and that Γ is isomorphic to an explicit subgroup of $\text{Aut}_0 \mathbb{A}_{\mathbb{Q}}^2$, which proves Theorem A.2.

4.1 The amalgamated decomposition $\Gamma = G_1 *_A G_2$

Let us consider the following subgroups of Γ

²In the first version of this paper that appeared in the arXiv, I was unaware of [11]. I'm grateful to T. Delzant for providing this reference.

$$G_1 = \langle \sigma \rangle, G_2 = \langle \sigma^2, \tau \rangle \text{ and } A = \langle \sigma^2 \rangle.$$

The groups G_2 is isomorphic to $\mathbb{Z} \ltimes \mathbb{Z}[1/2]$ where any $n \in \mathbb{Z}$ acts over $\mathbb{Z}[1/2]$ by multiplication by 2^n . The group Γ is the amalgamated product

$$\Gamma \simeq G_1 *_A G_2.$$

Lemma 15. *The group Γ has the trivial normal centralizers property.*

Proof. Set $H = \mathbb{Z}[1/2].\tau$. The A -sets G_2/A and H are isomorphic, hence A acts faithfully on G_2/A . Therefore $\text{Core}_\Gamma(A) \subset \text{Core}_{G_2}(A)$ is trivial, and the assertion follows from Lemma 13. \square

4.2 Quasi-unipotent endomorphisms

Let V be a finite-dimensional vector space over an algebraically closed field K . An element $u \in \text{GL}(V)$ is called *quasi-unipotent* if all its eigenvalues are roots of unity. The *quasi-order* of a quasi-unipotent endomorphism u is the smallest positive integer m such that u^m is unipotent.

If u is unipotent and $\text{ch } K = 0$, set

$$\log u = \log(1 - (1 - u)) := \sum_{k \geq 1} (1 - u)^k / k,$$

which is well-defined since $1 - u$ is nilpotent.

Lemma 16. *Let $h, u \in \text{GL}(V)$. Assume that u has infinite order and $huh^{-1} = u^2$. Then u is quasi-unipotent of quasi-order m for some odd integer m . Moreover K has characteristic zero, and*

$$heh^{-1} = 2e,$$

where $e := \log u^m$.

Proof. Let $\text{Spec } u$ be the spectrum of u . By hypothesis the map $\lambda \in \text{Spec } u \mapsto \lambda^2 \in \text{Spec } u$ is bijective. Hence all eigenvalues of u are odd roots of unity, what proves that u is quasi-unipotent of odd quasi-order m .

Over any field of finite characteristic, the unipotent endomorphisms have finite order. Hence we have $\text{ch } K = 0$. Moreover, we have

$$heh^{-1} = h(\log u^m)h^{-1} = \log(hu^mh^{-1}) = \log(u^{2m}) = 2e. \quad \square$$

4.3 Nonlinearity of Γ

Drutu-Sapir Lemma. *The group Γ is not linear over a field.*

The result is a particular case of Corollary 4 in [11]. Since their proof is based on an earlier result of [28], we shall provide a direct proof.

Proof. Assume otherwise and let $\rho' : \Gamma \rightarrow \mathrm{GL}(V)$ be an embedding, where V is a finite-dimensional vector space over an algebraically closed field K . Since τ has infinite order and $\sigma^2\tau\sigma^{-2} = \tau^2$, it follows from Lemma 16 that K has characteristic zero, $\rho'(\tau)$ is quasi-unipotent of odd quasi-order m .

Step 1: there is another embedding $\rho : \Gamma \rightarrow \mathrm{GL}(V)$ such that $\rho(\tau)$ is unipotent. Let $\psi : \Gamma \rightarrow \Gamma$ be the group homomorphism defined by $\psi(\sigma) = \sigma$, and $\psi(\tau) = \tau^m$. Since $\psi(G_1) \cap \psi(G_2) = A$, it follows from Lemma 2 that the natural homomorphism $\psi(G_1) *_A \psi(G_2) \rightarrow \Gamma$ is injective. Hence ψ is injective and $\rho := \rho' \circ \psi$ is an embedding such that $\rho(\tau) = \rho'(\tau)^m$ is unipotent.

Step 2: the unipotent subgroup $U \subset \mathrm{GL}(V)$. Set $h = \rho(\sigma^2)$, set $\Pi = \mathrm{Spec} h$ be its spectrum, and for each $\lambda \in \Pi$, let $V_{(\lambda)}$ be the corresponding generalized eigenspace. An eigenvalue $\lambda \in \Pi$ is called *minimal* if $2^{-n}\lambda \notin \Pi$ for any integer $n > 0$. Let Π_{\min} be the set of minimal eigenvalues and set

$$\Pi_{\geq k} = \{\lambda \in \Pi \mid \lambda \in 2^l \Pi_{\min} \text{ for some } l \geq k\},$$

for $k \geq 0$. The filtration $\Pi = \Pi_{\geq 0} \supset \Pi_{\geq 1} \supset \dots$ of the set Π induces the following filtration of V

$$V = V_{\geq 0} \supset V_{\geq 1} \supset \dots,$$

where $V_{\geq k} = \bigoplus_{\lambda \in \Pi_{\geq k}} V_{(\lambda)}$. Let U be the group of all $g \in \mathrm{GL}(V)$ such that $(g - \mathrm{id})V_{\geq k} \subset V_{\geq k+1}$ for all $k \geq 0$. For some suitable basis, U is a group of upper triangular matrices. Therefore U is nilpotent.

Step 3: $\rho(\Gamma)$ is nilpotent by commutative. Since $\rho(G_1)$ commutes with h , we have $\rho(G_1).V_{\geq k} = V_{\geq k}$ for any integer k . Therefore $\rho(G_1)$ normalizes U .

Set $u = \rho(\tau)$ and $e = \log u$. By Lemma 16, we have $heh^{-1} = 2e$ and therefore we have $e.V_{\geq k} \subset V_{\geq k+1}$. It follows that $u = \exp e$ belongs to U . Since $\rho(\Gamma) = \langle \rho(G_1), u \rangle$ we have

$$\rho(\Gamma) \subset \rho(G_1) \ltimes U,$$

and therefore $\rho(\Gamma)$ is nilpotent by commutative. Hence $\rho(\Gamma)$ contains a non-trivial normal abelian subgroup. This contradicts Lemma 15, which states that Γ has the trivial normal centralizers property. \square

Lemma 17. *The group Γ is not linear, even over a ring.*

Proof. By Lemma 15 the group Γ has the trivial normal centralizers property. It follows from Lemmas 12 and 4 that Γ is not linear, even over a ring. \square

4.4 Proof of Theorem C.1

Theorem C.1. *The subgroup $\langle S, T \rangle$ of $\mathrm{Aut}_0 \mathbb{A}_{\mathbb{Q}}^2$ is not linear, even over a ring, where*

$$S(x, y) = (y, 2x) \text{ and } T(x, y) = (x, y + x^2).$$

Proof. Set $H_1 = \langle S \rangle$, $H_2 = \langle S^2, T \rangle$, $C = \langle S^2 \rangle$.

We have $S^2 = 2.\text{id}$, therefore we have $H_1 \cap B_{\text{Aff}}(K) = C$. Moreover H_2 is the group of automorphisms of the form

$$(x, y) \mapsto (2^k x, 2^k y + r x^2),$$

for $k \in \mathbb{Z}$ and $r \in \mathbb{Z}[1/2]$, therefore $H_2 \cap B_{\text{Aff}}(K) = C$. It follows from Lemma 2 and van der Kulk Theorem that the natural homomorphism $H_1 *_C H_2 \rightarrow \text{Aut}_0 \mathbb{A}_{\mathbb{Q}}^2$ is injective.

There is a group isomorphism $\Gamma \rightarrow H_1 *_C H_2$ sending σ to S^{-1} and τ to T . Thus, by Lemma 17, the subgroup of $\text{Aut}_0 \mathbb{A}_K^2$ generated by S and T is not linear, even over a ring. \square

5 The Linear Representation of $\text{Aut}_1 \mathbb{A}_K^2$

In this section and the next one, we will use a ping-pong lemma. It will be convenient to define the *highest component* of any nonzero vector valued polynomial $f(t) = \sum_k f_k t^k$ by $\text{hc}(f) := f_m$, where m is the degree of f .

5.1 A ping-pong lemma

Let S be a group, let P be a S -set and let $\Gamma = S \rtimes *_{p \in P} F_p$ be a mixed product of S .

Lemma 18. *Let Ω be a Γ -set, and let $(\Omega_p)_{p \in P}$ be a collection of subsets in Ω . Assume*

(i) *the free product $*_{p \in P} F_p$ is nondegenerate and nondihedral,*

(ii) *$\text{Core}_{\Gamma}(\cap_P S_p)$ is trivial,*

(iii) *the subsets Ω_p are nonempty and disjoint, and*

(iv) *we have $F_p^* \cdot \Omega_q \subset \Omega_p$ whenever $p \neq q$,*

where F_p^* denotes $F_p \setminus \{1\}$.

Then the action of Γ on Ω is faithful.

Proof. Let $p \neq p'$ be two elements in P . By Lemma 14, Assertion (i) any nontrivial normal subgroup $H \subset \Gamma$ contains to some $\gamma \in E_{p,p}$. We have $\gamma \cdot \Omega_{p'} \subset \Omega_p$, therefore γ acts nontrivially on Ω . Since any nontrivial normal subgroup H acts nontrivially, the action of Γ on Ω is faithful. \square

5.2 The group $\text{SL}(2, tK[t])$ is generated by its transvections

It is well known that $\mathrm{SL}(2, K[t])$ is generated by its transvections, but this property does not extend to most congruence subgroups. It is nevertheless the case for the congruence subgroup associated with the ideal $tK[t]$.

More precisely, for any $\delta \in \mathbb{P}_K^1$, let $e_\delta \in \mathrm{End}(K^2)$ be a nilpotent element with $\mathrm{Im} e_\delta = \delta$. Set

$$U_\delta(tK[t]) := \{\mathrm{id} + f(t)e_\delta \mid f \in tK[t]\},$$

which is a subgroup of $\mathrm{SL}(2, tK[t])$. Let $\gamma \in \mathrm{GL}(2, K)$ such that $\gamma.\delta_0 = \delta$ where $\delta_0 \in \mathbb{P}_K^1$ has coordinates $(0; 1)$. Then we have

$$U_{\delta_0}(tK[t]) = U(tK[t]) \text{ and } U_\delta(tK[t]) = U(tK[t])^\gamma.$$

The elements of $\cup_{\delta \in \mathbb{P}_K^1} U_\delta(tK[t])$ are called the *transvections* of $\mathrm{SL}(2, tK[t])$.

Lemma 19. *The group $\mathrm{SL}(2, tK[t])$ is generated by its transvections.*

Proof. Any element $G(t) \in \mathrm{SL}(2, tK[t])$ is a polynomial $G(t) = \sum_{n \geq 0} A_n t^n$ where A_n belong to $\mathrm{End}(K^2)$ and $A_0 = \mathrm{id}$. The proof that $G(t)$ is a product of transvections runs by induction on the degree of $G(t)$.

We can assume that $M := \deg G(t)$ is positive. Since $\det G(t)$ is constant, we have $\det A_M = 0$, i.e. A_M has rank one. Set $\delta = \mathrm{Im} A_M$, $\delta' = \mathrm{Ker} A_M$ and let (u, v) be a basis of K^2 with $v \in \delta'$. The equation $\det G(t) = 1$ means

$$\sum_{n, m \leq N} A_n \cdot u \wedge A_m \cdot v t^{n+m} = u \wedge v.$$

There exists an integer N such that

$$\mathrm{Im} A_N \not\subset \delta, \text{ but } \mathrm{Im} A_m \subset \delta, \text{ for any } m > N.$$

The previous identity implies that

$$\sum_{n+m=M+N} A_n \cdot u \wedge A_m \cdot v = 0.$$

We have $A_n \cdot u \wedge A_m \cdot v = 0$, whenever n, m are bigger than N and $A_M \cdot v = 0$. Hence we get $A_M \cdot u \wedge A_N \cdot v = 0$, or, equivalently $A_N \cdot \delta' \subset \delta$. Therefore we have $A_M = ce_\delta \cdot A_N$ for some $c \in K$. Set

$$H(t) = (1 - ce_\delta t^{M-N}) \cdot G(t).$$

Since $e_\delta \cdot A_m = 0$ for any $m > N$, it follows that $H(t)$ has degree $\leq M$. Moreover, its degree M component is $A_M - ce_\delta \cdot A_N = 0$. Therefore $H(t)$ has degree $< N$, what proves the lemma by induction. \square

5.3 The groups $\mathrm{SL}(2, tK[t])$ and $\mathrm{SL}_{U(K)}(2, K[t])$ are free products

Let δ_0, δ_∞ be the points in \mathbb{P}_K^1 with coordinates $(0; 1)$ and $(1; 0)$. Set

$$\mathrm{SL}_{U(K)}(2, K[t]) = \{G(t) \in \mathrm{SL}(2, K[t]) \mid G(0) \in U(K)\}.$$

Lemma 20. *We have*

$$\mathrm{SL}(2, tK[t]) = *_{\delta \in \mathbb{P}_K^1} U_\delta(tK[t]) \text{ and } \mathrm{SL}_{U(K)}(2, K[t]) = U(K[t]) * U_{\delta_\infty}(tK[t]).$$

Proof. The elements of $\Omega := K[t]^2 \setminus \{0\}$ are nonzero K^2 -valued polynomials. For $\delta \in \mathbb{P}_K^1$, set

$$\Omega_\delta = \{v \in \Omega \mid \text{hc}(v) \in \delta\}.$$

For any $\delta \neq \delta'$ in \mathbb{P}_K^1 , the restriction of e_δ to δ' is injective, hence

$$\text{hc}(H(t).v) = \text{hc}(H(t).\text{hc}(v)),$$

for any $H(t) \in U_\delta(tK[t]) \setminus \{1\}$ and any $v \in \Omega_{\delta'}$. It follows that

$$(U_\delta(tK[t]) \setminus \{1\}).\Omega_{\delta'} \subset \Omega_\delta.$$

Hence by Lemma 18, the free product $*_{\delta \in \mathbb{P}_K^1} U_\delta(tK[t])$ embeds in $\text{SL}(2, tK[t])$, and by Lemma 19, we have

$$*_{\delta \in \mathbb{P}_K^1} U_\delta(tK[t]) = \text{SL}(2, tK[t]),$$

what proves the first assertion. Moreover, we have

$$\text{SL}_{U(K)}(2, K[t]) \simeq U(K) \times *_{\delta \in \mathbb{P}_K^1} U_\delta(tK[t]).$$

Since $U(K)$ acts freely and transitively on $\mathbb{P}_K^1 \setminus \{\delta_0\}$ and since $U(K[t]) = U(K) \times U_{\delta_0}(tK[t])$, it follows from Lemma 8 that

$$\text{SL}_{U(K)}(2, K[t]) = U([K[t]) * U_{\delta_\infty}(tK[t]). \quad \square$$

Remark. The group $\text{SL}_{U(K)}(2, K[t])$ is the “lower nilradical” of the affine Kac-Moody group $\text{SL}(2, K[t, t^{-1}])$. In [24], Tits defined the “lower nilradical” of any Kac-Moody group in term of an inductive limit, which is essentially equivalent to the previous lemma for $\text{SL}_{U(K)}(2, K[t])$.

Since the notes [24] are not widely distributed, let us mention that an equivalent result is stated in [26], Section 3.2 and 3.2, see also [25].

5.4 Proof of Theorem A.2

Lemma 21. *There are isomorphisms*

$$\text{Aut}_1 \mathbb{A}_K^2 \simeq \text{SL}(2, tK[t]) \text{ and } \text{Aut}_{U(K)} \mathbb{A}_K^2 \simeq \text{SL}_{U(K)}(2, K[t]).$$

Proof. By Lemmas 7 and 20, $\text{Aut}_1 \mathbb{A}_K^2$ and $\text{SL}(2, tK[t])$ are free products of $\text{Card } \mathbb{P}_K^1$ copies of a K -vector space of dimension \aleph_0 . Therefore these two groups are isomorphic.

The proof of the isomorphism $\text{Aut}_{U(K)} \mathbb{A}_K^2 \simeq \text{SL}_{U(K)}(2, K[t])$ follows similarly from Lemmas 10 and 20. \square

Theorem A.2. *The groups $\text{Aut}_1 \mathbb{A}_K^2$ and $\text{Aut}_{U(K)} \mathbb{A}_K^2$ are linear over $K(t)$.*

Moreover if $K \supset k(t)$ for some infinite field k , then there exists an embedding $\text{Aut}_1 \mathbb{A}_K^2 \subset \text{Aut}_{U(K)} \mathbb{A}_K^2 \subset \text{SL}(2, K)$.

Proof. It follows from Lemma 21 that $\text{Aut}_1 \mathbb{A}_K^2$ and $\text{Aut}_{U(K)} \mathbb{A}_K^2$ are subgroups of $\text{SL}(2, K(t))$, and therefore they are linear over $K(t)$.

Assume now that $K \supset k(t)$ for some infinite field k . We claim that there exists a field L with $L(t) \subset K$ and $\text{Card } L = \text{Card } K$. If $\text{Card } K = \aleph_0$, then the subfield k satisfies the claim. Otherwise, we have $\text{trdeg } K > \aleph_0$ and there is an embedding $L(t) \subset K$ for some subfield L with $\text{trdeg } L = \text{trdeg } K$. Since $\text{trdeg } L = \text{Card } L = \text{Card } K$, the claim is proved.

It follows from Corollary 1 that

$$\text{Aut}_{U(K)} \mathbb{A}_K^2 \simeq \text{Aut}_{U(K)} \mathbb{A}_L^2 \subset \text{SL}(2, L(t)) \subset \text{SL}(2, K),$$

therefore $\text{Aut}_1 \mathbb{A}_K^2$ and $\text{Aut}_{U(K)} \mathbb{A}_K^2$ are subgroups of $\text{SL}(2, K)$. \square

5.5 A Corollary

For a finite field K , $\text{Aut}_1 \mathbb{A}_K^2$ has finite index in $\text{Aut } \mathbb{A}_K^2$, hence

Corollary 4. *For a finite field K , the group $\text{Aut } \mathbb{A}_K^2$ is linear over $K(t)$.*

6 The Linear Representation of $\text{SAut}_0^{<n} \mathbb{A}_K^2$

For $n \geq 3$, let $\text{SAut}_0^{<n} \mathbb{A}_K^2$ be the subgroup of $\text{SAut}_0 \mathbb{A}_K^2$ generated by all automorphisms $\phi \in \text{SAut}_0 \mathbb{A}_K^2$ of degree $< n$. In this section, *we will assume that K has characteristic zero*, in order to show that the group $\text{SAut}_0^{<n} \mathbb{A}_K^2$ is linear, what proves Theorem C.2. Unfortunately, our approach does not extend to fields of finite characteristics.

6.1 The group $\text{SAut}_0^{<n} \mathbb{A}_K^2$ is a mixed product with trivial core

Let $\delta \in \mathbb{P}^1$ with projective coordinates $(a; b)$. Let $E_\delta^{<n}(K) \subset E_\delta(K)$ be the subgroup of all automorphisms of the form $(x, y) \mapsto (x, y) + f(bx - ay)(a, b)$ where $f(t) \in t^2 K[t]$ and $\deg f(t) < n$.

Lemma 22. *For any $n \geq 3$, the group $\text{SAut}_0^{<n} \mathbb{A}_K^2$ is isomorphic to the nondegenerate mixed product*

$$\Gamma := \text{SL}(2, K) \rtimes *_{\delta \in \mathbb{P}_K^1} E_\delta^{<n}(K).$$

Moreover $\text{Core}_\Gamma(\text{SL}(2, K))$ is trivial.

Proof. By induction over n , one easily prove that any reduced decomposition $u = u_1 \dots u_n$ in the free product $*_{\delta \in \mathbb{P}_K^1} E_\delta(K)$, where $u_i \in E_{\delta_i}(K)$, has degree $N := \prod \deg u_i$ and its highest degree component is of the form $(x, y) \mapsto (bx - ay)^N(c, d)$, where $(c; d)$ and $(a; b)$ are the coordinates of δ_1 and δ_n .

By Lemma 7, $\text{SAut}_0 \mathbb{A}_K^2$ is isomorphic to the mixed product

$$\text{SL}(2, K) \rtimes *_{\delta \in \mathbb{P}_K^1} E_\delta(K).$$

Hence $\text{SAut}_0^{<n}$ is generated by $\text{SL}(2, K)$ and the subgroups $E_\delta^{<n}(K)$. It follows that

$$\text{SAut}_0^{<n} \mathbb{A}_K^2 \simeq \text{SL}(2, K) \times *_{\delta \in \mathbb{P}_K^1} E_\delta^{<n}(K).$$

It has been noticed that $\text{Core}_\Gamma(\text{SL}(2, K))$ acts trivially on \mathbb{P}_K^1 , hence it is included in $\{\pm 1\}$. Set $\tau(x, y) := (x, y + x^2)$. Since $\tau^{-\text{id}}(x, y) = (x, y - x^2) \neq \tau(x, y)$ or $\text{ch } K = 2$, it follows that $\text{Core}_\Gamma(\text{SL}(2, K))$ is trivial. \square

6.2 The square root η of e

Let ϵ be an odd variable. For an integer $N \geq 1$, let $L(N) \subset K[x, y]$ and $\hat{L}(N) \subset K[x, y, \epsilon]$ be the subspaces of homogenous polynomials of degree N . Let (e, h, f) be the usual basis of $\mathfrak{sl}(2, K)$. As an $SL(2, K)$ -module, we have $\hat{L}(N) = L(N) \oplus L(N - 1)$ and e acts as the derivation $x \frac{\partial}{\partial y}$. Set

$$\eta = x \frac{\partial}{\partial \epsilon} + \epsilon \frac{\partial}{\partial y}.$$

It is clear that $\eta^2 = e$. Indeed $\hat{L}(N)$ is a representation of the Lie superalgebra $\mathfrak{osp}(1, 2)$, and $\eta \in \mathfrak{osp}(1, 2)$ is an odd element such that $\eta^2 = e$.

For any $\delta \in \mathbb{P}_K^1$ with projective coordinates $(a; b)$, set $L_\delta := K \cdot (ax + by)^N$ and $L_\delta^* = L_\delta \setminus \{0\}$. Let $\delta_0 \in \mathbb{P}_K^1$ be the point with projective coordinate $(0; 1)$. Since $(x \frac{\partial}{\partial y})^N (ax + by)^N = N! (bx)^N$, it follows that

Lemma 23. *Let $\delta \neq \delta_0$ be another point in \mathbb{P}_K^1 . Then for any $v \in L_\delta^*$, the element $\eta^{2N} \cdot v = e^N \cdot v$ belongs to $L_{\delta_0}^*$.*

6.3 The representation ρ_N of $\text{SAut}_0 \mathbb{A}_K^2$ on $\hat{L}(N) \otimes K[t]$

We will extend the natural representation of $\text{SL}(2, K)$ on $\hat{L}(N) \otimes K[t]$ to $\text{SAut}_0 \mathbb{A}_K^2$ as follows. For any automorphism $\tau \in E(K)$, set

$$\rho_N(\tau) = \exp(t\eta f(\eta)),$$

if $\tau(x, y) = (x, y + f(x))$, where $f(x) \in x^2 K[x]$. Since $[e, \eta] = 0$ and $[h, \eta] = \eta$, the homomorphism ρ_N is $B(K)$ -equivariant. By Lemma 3, ρ_N extends to a $K[t]$ -linear action of $\text{SAut}_0 \mathbb{A}_K^2$.

Lemma 24. *Assume that $2N$ is divisible by $\text{lcm}(1, 2, \dots, n)$.*

Then the restriction of ρ_N to $\text{SAut}_0^{<n} \mathbb{A}_K^2$ is faithful.

Proof. For any $\delta \in \mathbb{P}_K^1$, set

$$\Omega_\delta = \{v(t) \in \hat{L}(N) \otimes K[t] \setminus \{0\} \mid \text{hc}(v(t)) \in L_\delta^*\}.$$

First step. We claim that

$$(E_{\delta_0}^{<n}(K) \setminus \{1\}) \cdot \Omega_\delta \subset \Omega_{\delta_0},$$

for any $\delta \in \mathbb{P}_K^1$ with $\delta \neq \delta_0$. Let $\tau(x, y) = (x, y + f(x))$ be in $E_{\delta_0}^{<n}(K) \setminus \{1\}$. We have $f(x) = ax^k + \text{higher terms}$, for some $a \in K^*$ and some k with $2 \leq k < n$. By definition, we have

$$\rho_N(\tau) = \exp t\eta f(\eta) = \sum_{m \geq 0} \frac{\eta^m f(\eta)^m}{m!} t^m.$$

Since $\eta f(\eta)$ is divisible by η^{k+1} and $\eta^{2N+1} = 0$, it follows that $\eta^m f(\eta)^m = 0$ for $m > 2N/(k+1)$. Since $k+1$ divides $2N$, $\rho_N(\tau)$ is a polynomial of degree exactly $d := 2N/(k+1)$ and we have

$$\text{hc}(\rho_N(\tau)) = \frac{a^d}{d!} \eta^{2N}.$$

Let $v(t) \in \Omega_\delta$. By Lemma 23, $\text{hc}(\rho_N(\tau)).\text{hc}(v(t))$ is nonzero and belongs to L_{δ_0} . It follows that $\rho_N(\tau).\Omega_\delta \subset \Omega_{\delta_0}$, what proves the claim.

Second step: use of the ping-pong lemma. Using the previous result and the fact that the groups $E_\delta^{<n}(K)$ are conjugate under $\text{SL}(2, K)$, it follows that

$$(E_\delta^{<n}(K) \setminus \{1\}).\Omega_{\delta'} \subset \Omega_\delta, \text{ for any distinct } \delta, \delta' \in \mathbb{P}_K^1.$$

Therefore, the restriction of ρ_N to $\text{SAut}_0^{<n} \mathbb{A}_K^2$ is faithful by Lemma 18. \square

6.4 Proof of Theorem C.2

Since $\dim \hat{L}(N) = 2N + 1$, Lemma 24 implies that

Theorem C.2. *For any $n \geq 3$, there is an embedding*

$$\text{SAut}_0^{<n} \mathbb{A}_K^2 \subset \text{SL}(1 + \text{lcm}(1, 2, \dots, n), K(t)).$$

In particular, any f.g. subgroup of $\text{SAut}_0 \mathbb{A}_K^2$ is linear over $K(t)$.

7 Semi-algebraic Characters

Let $\Lambda \subset K^*$ be a subgroup. For any $n \geq 1$, let $K_n \subset K$ be the subfield generated by Λ^n . Let L be an algebraically closed field, which contains at least one subfield isomorphic to K_1 .

For $n \geq 1$, a group homomorphism $\chi : \Lambda \rightarrow L^*$ is called a *semi-algebraic character* of degree n if $\chi(z) = \mu(z^n)$ for some field embedding $\mu : K_n \rightarrow L$. Let $\mathcal{X}_n(\Lambda)$ be the set of all semi-algebraic characters of Λ of degree n . The degree of a semi-algebraic character is *not uniquely defined*. Given $n \neq m$, we will show a criterion for the disjointness of $\mathcal{X}_n(\Lambda)$ and $\mathcal{X}_m(\Lambda)$.

7.1 The invariant $I_n(\Lambda)$

Let $\mathbb{F}[\Lambda]$ be the group algebra of Λ over the ground field \mathbb{F} of K . For a field $E \supset \mathbb{F}$, any group homomorphism $\chi : \Lambda \rightarrow E^*$ extends to an algebra homomorphism $\hat{\chi} : \mathbb{F}[\Lambda] \rightarrow E$. One should not confuse $\text{Ker } \chi$ with

$$\text{Ker } \hat{\chi} := \{ \sum_{\lambda} a_{\lambda} \lambda \in \mathbb{F}[\Lambda] \mid \sum_{\lambda} a_{\lambda} \chi(\lambda) = 0 \}.$$

For $n \geq 1$, let χ_n be the homomorphism $\chi_n : \lambda \in \Lambda \mapsto \lambda^n \in K_n^*$. Set

$$I_n(\Lambda) = \text{Ker } \hat{\chi}_n.$$

Lemma 25. *A group homomorphism $\chi : \Lambda \rightarrow L^*$ is a semi-algebraic character of degree n iff $\text{Ker } \hat{\chi} = I_n(\Lambda)$.*

In particular, we have $\mathcal{X}_n(\Lambda) = \mathcal{X}_m(\Lambda)$ or $\mathcal{X}_n(\Lambda) \cap \mathcal{X}_m(\Lambda) = \emptyset$, for any positive integers $n \neq m$.

Proof. By definition the fraction field of the prime ring $\mathbb{F}[\Lambda]/I_n(\Lambda)$ is K_n . Hence $\hat{\chi}$ factors through K_n , i.e. $\hat{\chi} = \mu \circ \hat{\chi}_n$ for some field embedding $\mu : K_n \rightarrow L$. The first point follows, as well as the second. \square

7.2 Minimal subgroups of K^*

Let $\Lambda \subset K^*$ be a subgroup. By definition, the *transcendental degree* of Λ is $\text{trdeg } \Lambda := \text{trdeg } K_1$ and its *rank* is $\text{rk } \Lambda := \dim \Lambda \otimes \mathbb{Q}$. Both are cardinals, and we have $\text{trdeg } \Lambda \leq \text{rk } \Lambda$. We say that Λ is a *good subgroup* of K^* if $\text{trdeg } \Lambda' = \text{rk } \Lambda'$, for any f.g. subgroup Λ' of Λ and a *bad subgroup* otherwise.

Assume now that Λ is a free abelian group of rank $r < \infty$, with basis x_1, \dots, x_r . The ring $\mathbb{F}[\Lambda]$ is isomorphic to the ring $\mathbb{F}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$ of Laurent polynomials. For $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{Z}^r$, set $x^{\alpha} = x_1^{\alpha_1} \dots x_r^{\alpha_r}$. The *support* of a Laurent polynomial $P = \sum_{\alpha \in \mathbb{Z}^r} a_{\alpha} x^{\alpha}$ is the set

$$\text{Supp } P := \{ \alpha \in \mathbb{Z}^r \mid a_{\alpha} \neq 0 \}.$$

Assume now that $\text{trdeg } \Lambda = r - 1$. Since $\mathbb{F}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$ is a unique factorization domain, $I_1(\Lambda)$ is a principal ideal, and let P be one its generator. The other generators are the polynomials $ax^{\gamma}P$, for $a \in \mathbb{F}^*$ and $\gamma \in \mathbb{Z}^r$. Hence the subgroup $X(\Lambda) \subset \mathbb{Z}^r$ generated by $\alpha - \beta$ for $\alpha, \beta \in \text{Supp } P$ only depends on Λ . Moreover, if $0 \in \text{Supp } P$, then we have $X(\Lambda) = \langle \text{Supp } P \rangle$.

A subgroup $\Lambda \subset K^*$ is called *minimally bad* if

- (i) Λ is a f.g. free group, and
- (ii) we have $r := \text{rk } \Lambda = 1 + \text{trdeg } \Lambda$, and $X(\Lambda) = \mathbb{Z}^r$.

Lemma 26. *Let $\Lambda \subset K^*$ be a bad subgroup of K^* .*

Then Λ contains a minimally bad subgroup Λ' .

Proof. By definition, Λ contains a f.g. bad subgroup Λ_0 . Moreover, we can assume that Λ_0 is torsion free.

Let \mathcal{C} be the set of all subgroups $\Pi \subset \Lambda_0$ such that $\text{rk } \Pi > \text{trdeg } \Pi$. Let us pick one element of \mathcal{C} , say Π' , of minimal rank. It is clear that $r := \text{rk } \Pi' =$

$1 + \text{trdeg } \Pi'$. Let x_1, \dots, x_r be a basis of Π' and let $P = \sum_{x \in \mathbb{Z}^r} a_x x^\alpha$ be a generator of the ideal $I_1(\Pi')$ of $\mathbb{F}[\Pi'] \simeq \mathbb{F}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$ such that $a_0 \neq 0$.

It is clear that $\Lambda' := \langle x^\alpha \mid \alpha \in \text{Supp } P \rangle$ is a minimally bad subgroup. \square

7.3 Newton polygones of P_n

Let $r \geq 1$ be an integer. Let P be one generator of a principal ideal I of $\mathbb{F}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$. The *Newton polygone* $\text{Newton}(P)$ of P is the convex closure of $\text{Supp } P$ in $\mathbb{R} \otimes \mathbb{Z}^r$, and let $\text{Ext}(P)$ be its set of extremal points. Up to translation by \mathbb{Z}^r , $\text{Ext}(P)$ is an invariant of I . Hence the largest integer $e(P)$ such that $\alpha - \beta \in e(P) \cdot \mathbb{Z}^r$ for any $\alpha, \beta \in \text{Ext}(P)$ only depends on I .

It will be convenient to choose an ordering of \mathbb{Z}^r . A Laurent polynomial $P \in \mathbb{F}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$ is *normalized* if $a_0 = 1$ and any $\alpha \in \text{Supp } P$ is nonnegative. Any principal ideal I of $\mathbb{F}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$ has a unique normalized generator P . Since 0 belongs to $\text{Ext}(P)$, we have $\text{Ext}(P) \subset e(P) \cdot \mathbb{Z}^r$.

Let $\Lambda \subset K^*$ be a minimal subgroup and let x_1, \dots, x_r be a basis of Λ . For any $n \geq 1$, let P_n be the normalized generator of $I_n(\Lambda)$.

Lemma 27. *Assume that $n \geq 1$ is not divisible by $\text{ch } K$. Then we have*

$$\text{Newton}(P_n) = \frac{n^{r-1}}{f_n} \text{Newton}(P_1),$$

for some integer f_n dividing $e(P_1)$.

Proof. In $\overline{\mathbb{F}}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$ the polynomial P_1 decomposes uniquely as

$$P_1 = Q_1 \dots Q_k$$

where Q_1, Q_2, \dots are normalized irreducible polynomials in $\overline{\mathbb{F}}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$. Since they are permuted by $\text{Gal}(\overline{\mathbb{F}})$, we have $\text{Supp } Q_1 = \text{Supp } Q_2, \dots$, hence

- (i) $\text{Supp } Q_1$ generates \mathbb{Z}^r , and
- (ii) $\text{Newton}(Q_1) = \frac{1}{k} \text{Newton}(P_1)$ and k divides $e(P_1)$.

Let $\mu_n \subset \overline{\mathbb{F}}^*$ be the group of all n th roots of one. For various $(\zeta_1, \dots, \zeta_r) \in \mu_n^r$, the normalized polynomials $Q_1(\zeta_1 x_1, \dots, \zeta_r x_r)$ are pairwise distinct by the assertion (i). Thus the polynomial $R = \prod_{\zeta_1, \dots, \zeta_r \in \mu_n^r} Q_1(\zeta_1 x_1, \dots, \zeta_r x_r)$ is irreducible in $\overline{\mathbb{F}}[x_1^{\pm n}, \dots, x_r^{\pm n}]$.

Set $G_1 = \{\sigma \in \text{Gal}(\overline{\mathbb{F}}) \mid Q_1^\sigma = Q_1\}$ and $G = \{\sigma \in \text{Gal}(\overline{\mathbb{F}}) \mid R^\sigma = R\}$. Since R is normalized, the polynomials R^σ for $\sigma \in \text{Gal}(\overline{\mathbb{F}})/G$ are pairwise distinct, and therefore $S = \prod_{\sigma \in \text{Gal}(\overline{\mathbb{F}})/G} R^\sigma$ is a normalized irreducible polynomial in $\overline{\mathbb{F}}[x_1^{\pm n}, \dots, x_r^{\pm n}]$. It follows that

$$P_n(x_1^n, \dots, x_r^n) = S(x_1, \dots, x_r).$$

Since $[\text{Gal}(\overline{\mathbb{F}}) : G] = k$, the integer $f_n := [G : G_1]$ divides $e(P_1)$, and

$$\text{Newton}(P_n) = \frac{n^{r-1}}{f_n} \text{Newton}(P_1). \quad \square$$

7.4 The criterion for disjointness of $\mathcal{X}_n(\Lambda)$ and $\mathcal{X}_m(\Lambda)$

Lemma 28. *Let $\Lambda \subset K^*$ be a bad subgroup of K^* . Then there is an integer $e \geq 1$ such that*

$$\mathcal{X}_n(\Lambda) \cap \mathcal{X}_m(\Lambda) = \emptyset,$$

whenever the integers $n \neq m$ are coprime to e .

Proof. By Lemma 26, Λ contains a minimal subgroup Λ' . Moreover by Lemma 25, it is enough to show the lemma for Λ' . Therefore we can assume that Λ itself is minimal. For any $n \geq 1$, let P_n be the normalized generator of $I_n(\Lambda)$, relative to some order of Λ .

Proof when $\text{rk } \Lambda = 1$. Thus $\mathbb{F} = \mathbb{Q}$ and a generator x of Λ is an algebraic number of infinite order. Let $n, m \geq 1$ be two integers such that $P_n = P_m$. Since $P_n(\lambda^m) = 0$, there is $\sigma \in \text{Gal}(\mathbb{Q})$ such that $\lambda^m = \sigma(\lambda^n)$. Let $k \geq 1$ be an integer such that $\sigma^k(\lambda) = \lambda$. Therefore, we have

$$\lambda^{n^k} = \sigma^k(\lambda^{n^k}) = \lambda^{m^k}.$$

Since λ has infinite order, it follows that $n = m$. In this case, the lemma is proved for $e = 1$.

Proof when $r := \text{rk } \Lambda \geq 2$. Set $e = pe(P_1)$ if $\text{ch } K = p$ and $e = e(P_1)$ otherwise. Let $n, m \geq 1$ be two integers prime to e such that $P_n = P_m$. By Lemma 27, we have $\frac{n^{r-1}}{f_n} = \frac{m^{r-1}}{f_m}$ for some integers f_n and f_m dividing e . It follows that $n = m$. \square

8 A Nonlinearity Criterion for $\text{Aut}_S \mathbb{A}_K^2$

Given a subgroup S_0 of $B(K)$, we first investigate a necessary condition for the existence of a given embedding

$$\rho : S_0 \times E(K) \rightarrow \text{GL}(V),$$

where V is a finite-dimensional vector space over an algebraically closed field L . Let $W(\rho) \subset \text{End } V$ be the linear subspace generated by $\rho(E(K))$.

The commutative group structure on $E(K)$ will be denoted additively. Indeed $E(K)$ has a natural structure of a graded vector space over K , namely $E(K) = \bigoplus_{n \geq 3} K.T_n$ where $T_n(x, y) = (x, y + x^{n-1})$.

For any $g \in B(K)$, set $\chi_B(g) = \lambda$ if $g(x, y) = (\lambda^{-1}x, \lambda y + tx)$, for some $\lambda \in K^*$ and $t \in K$. The set of all eigenvalues of elements in S_0 is the group $\Lambda := \chi_B(S_0) \subset K^*$. We have $gT_n g^{-1} = \chi_B(g)^n T_n$. Since the action of S_0 on $E(K)$ factors through Λ , it follows that $E(K)$ and $W(\rho)$ are Λ -modules.

8.1 An obvious estimate

A integer $n \geq 1$ is called a *divisor* of Λ if Λ contains a primitive n th root of one. Let $d(\Lambda)$ be the number, finite or infinite, of divisors of Λ .

Lemma 29. *We have $d(\Lambda) \leq 2 + (\dim V)^2$.*

Proof. For each divisor n with $n \geq 3$, set $t_n = \rho(T_n)$. Since $gt_n g^{-1} = t_n$ iff $\chi(g)^n = 1$, it follows that the elements t_n are linearly independent. Thus we have $\dim \text{End } V \geq d(\Lambda) - 2$, from which the assertion follows. \square

8.2 Unipotent representations

Lemma 30. (i) *Assume that $\text{ch } K = 0$ and $\text{rk } \Lambda \geq 1$. Then $\text{ch } L = 0$ and $\rho(E(K))$ is a unipotent group.*

(ii) *Assume that $\text{ch } K = p$. Then $\text{ch } L = p$ and $\rho(E(K))$ is a unipotent group.*

Proof. First let us prove Assertion (i). Let $V = \bigoplus_{\chi \in \Omega} V_{(\chi)}$ be the generalized weight decomposition of the $E(K)$ -module V , where Ω is the set of group homomorphisms $\chi : E(K) \rightarrow L^*$ such that $V_{(\chi)} \neq 0$.

Let $\chi \in \Omega$. Since Ω is finite, the group $S'_0 = \{s \in S_0 \mid \chi^s = \chi\}$ has finite index in S_0 . There is some $s \in S'_0$ such that $\chi_B(g)$ has infinite order. It follows that the map $e \in E(K) \mapsto e - ses^{-1} \in E(K)$ is invertible. Therefore χ is trivial and $\rho(E(K))$ is a unipotent group.

Since $E(K)$ is torsion-free, L has characteristic 0.

The proof of Assertion (ii) follows from the fact that $E(K)$ is an elementary p -group of infinite rank. \square

8.3 A linearity criterion for $S_0 \times E(K)$

Lemma 31. *Assume again that ρ is a faithful representation of $S_0 \times E(K)$.*

Then Λ is a good subgroup of K^ .*

Proof. By Lemma 30, K and \mathbb{F} have the same ground field \mathbb{F} . Moreover, for any $n \geq 3$, $\rho(E_n)$ is a unipotent group, where $E_n = K.T_n$.

We claim that, for any $n \geq 3$, there is a $L[\Lambda]$ -submodule $W' \subset W(\rho)$ such that $W(\rho)/W'$ contains a $\mathbb{F}[\Lambda]$ -submodule X isomorphic to $\mathbb{F}[\Lambda]/I_n(\Lambda)$.

Proof for $\mathbb{F} = \mathbb{Q}$. In that case $W' = \{0\}$ and $X := \log \rho(\mathbb{Q}[\Lambda].T_n)$ is a $\mathbb{Q}[\Lambda]$ -submodule of $W(\rho)$ isomorphic to $\mathbb{Q}[\Lambda]/I_n(\Lambda)$.

Proof for $\mathbb{F} = \mathbb{F}_p$. As a substitute for the log, set $\theta(a) = 1 - \rho(a)$ for any $a \in E_n$. Let $M \subset W(\rho)$ be the linear space generated by $\theta(E_n)$. Since

$$\theta(a)\theta(b) = \theta(a) + \theta(b) - \theta(a+b),$$

M is a nonunital algebra and $M^p = \{0\}$. Since ρ is injective, we have $\theta^{-1}(M^p) = \{0\}$. Thus there exists a unique integer $m < p$ such that

$$Y := \theta^{-1}(M^m) \neq \{0\} \text{ but } \theta^{-1}(M^{m+1}) = \{0\}.$$

Set $W' = M^{m+1}$. It follows from the previous formula that Y is a subgroup of E_n and the induced map $\bar{\theta} : Y \rightarrow W(\rho)/W'$ is additive. Thus $\bar{\theta}$ is a homomorphism of $\mathbb{F}_p[\Lambda]$ -modules. Since $\text{Ker } \bar{\theta}$ is trivial, $\bar{\theta}$ is injective. Any cyclic $\mathbb{F}_p[\Lambda]$ -submodule of Y is isomorphic to $\mathbb{F}_p[\Lambda]/I_n(\Lambda)$, therefore $W(\rho)/W'$ contains a $\mathbb{F}_p[\Lambda]$ -submodule X isomorphic to $\mathbb{F}_p[\Lambda]/I_n(\Lambda)$.

We claim now that, for any $n \geq 3$, $W(\rho)_{(\chi)} \neq 0$ for some $\chi \in \mathcal{X}_n(\Lambda)$.

Let Z be the L -vector space generated by X , and let $Z = \bigoplus_{\chi \in \Omega_Z} Z_{(\chi)}$ be the decomposition of the $L[\Lambda]$ -module Z into generalized weight spaces, where Ω_Z is the set of group homomorphisms $\chi : \Lambda \rightarrow L^*$ such that $Z_{(\chi)} \neq 0$. For each $\chi \in \Omega_Z$, let I_χ be the annihilator in $\mathbb{F}[\Lambda]$ of $Z_{(\chi)}$. It follows that

$$I_n(\Lambda) = \bigcap_{\chi \in \Omega_Z} I_\chi.$$

Since Ω_Z is finite and $I_n(\Lambda)$ is a prime ideal, we have $I_n(\Lambda) = I_\chi$, for some $\chi \in \Omega_Z$. Moreover the radical of I_χ is $\text{Ker } \hat{\chi}$, hence $I_n(\Lambda) = \text{Ker } \hat{\chi}$. Thus by Lemma 25, χ is a semi-algebraic character of degree n . Moreover we have $W(\rho)_{(\chi)} \neq 0$, what proves the claim.

End of the proof. Assume otherwise, namely that Λ is bad subgroup of K^* . Then by Lemma 28, there is an infinite set T of integers $n \geq 3$ such that the family $(\mathcal{X}_n(\Lambda))_{n \in T}$ consists of mutually disjoint sets. This would contradict that the finite set of generalized weights of $W(\rho)$ intersects each of them. \square

8.4 Proof of the Nonlinearity Criterion

Let S be a subgroup of $\text{SL}(2, K)$. For $\delta \in \mathbb{P}_K^1$, let $\Lambda_\delta \subset K^*$ be the subgroup of all eigenvalues of elements $g \in S_\delta$.

Nonlinearity Criterion. *Assume one of the following two hypotheses*

- (i) Λ_δ is a bad subgroup of K^* for some $\delta \in \mathbb{P}_K^1$, or
- (ii) the function $\delta \in \mathbb{P}_K^1 \rightarrow d(\Lambda_\delta)$ is unbounded.

Then the group $\text{Aut}_S \mathbb{A}_K^2$ is not linear, even over a ring.

Proof. By contraposition, we will assume that $\text{Aut}_S \mathbb{A}_K^2$ is linear over a ring, and we will show that neither Assertion (i) nor Assertion (ii) holds.

By Lemma 7, $\text{Aut}_S \mathbb{A}_K^2$ is the mixed product $\Gamma := S \rtimes *_{\delta \in \mathbb{P}_K^1} E_\delta(K)$. Since $\bigcap_{\delta \in \delta} S_\delta \subset \{\pm 1\}$ and $T_3^{-\text{id}} = -T_3$, it follows that $\text{Core}_\Gamma(\bigcap_{\delta \in \delta} S_\delta)$ is trivial. Hence by Corollary 3, $\text{Aut}_S \mathbb{A}_K^2$ is linear over a field. Let $\rho : \text{Aut}_S \mathbb{A}_K^2 \rightarrow \text{GL}(n, L)$ be an injective homomorphism, for some algebraically closed field L and some positive integer n .

Since ρ provides a faithful representation of $B_\delta \rtimes E_\delta(K)$, it follows from Lemma 31 that Λ_δ is a good subgroup of K^* . Moreover by Lemma 29, we have $d(\Lambda_\delta) \leq 2 + n^2$.

Therefore neither Assertion (i) nor Assertion (ii) holds. \square

8.5 Proof of the Theorem A.1 of the introduction.

Theorem A.1. *For any infinite field, $\text{SAut}_0 \mathbb{A}_K^2$ is not linear, even over a field.*

Proof. With the previous notations, we have $\text{Aut}_0 \mathbb{A}_K^2 = \text{Aut}_{\text{SL}(2, K)} \mathbb{A}_K^2$, and $\Lambda_\delta = K^*$, for any $\delta \in \mathbb{P}_K^1$. Therefore it is enough to check that K^* satisfies one of the two assertions of the Nonlinearity Criterion.

If K is an infinite subfield of $\overline{\mathbb{F}}_p$, then $d(K^*)$ is infinite. Otherwise K contains \mathbb{Q} or a transcendental element t . For the subgroup $\Lambda := \langle 2 \rangle$ in the first case or $\Lambda := \langle t, t+1 \rangle$ in the second case, it is clear that $\text{rk } \Lambda > \text{trdeg } \Lambda$. Hence K^* itself is a bad group. \square

8.6 Comparison with Cornulier's Theorem

Let G_{Cor} be the group of all automorphisms of \mathbb{Q}^2 of the form

$$(x, y) \mapsto (x + u, y + f(x)), \text{ for some } u \in K \text{ and } f(t) \in \mathbb{Q}[t].$$

The group G_{Cor} is locally nilpotent but not nilpotent, hence [9]

Cornulier Theorem. *Neither G_{Cor} nor $\text{Aut } \mathbb{A}_{\mathbb{Q}}^2$ is linear over a field.*

Set $R := \mathbb{Q}[[t]] \oplus I$, where $I := \mathbb{Q}((t))/\mathbb{Q}[[t]]$ is a square-zero ideal. Let Γ be the subgroup of $\text{SL}(2, R)$ generated by the matrices

$$\begin{pmatrix} 1+t & 0 \\ 0 & (1+t)^{-1} \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \text{ when } a \text{ runs over } I.$$

The group Γ is isomorphic to $G_{\text{Cor}} \simeq \mathbb{Q} \rtimes \mathbb{Q}[x]$, where \mathbb{Q} acts by translation on $\mathbb{Q}[x]$, hence G_{Cor} is linear over R . Similarly, the subgroup $\text{SElem}_0(K)$ is isomorphic to $K^* \rtimes x^2 K[x]$, where K^* acts on $x^2 K[x]$ by multiplication. Since $\text{SElem}_0(K)$ embeds into $\prod_{n \geq 2} (K^* \rtimes Kx^n)$, it is a subgroup of $\text{GL}(2, K^\infty)$.

Therefore both G_{Cor} and $\text{SElem}_0(K)$ (for K infinite) are linear over some rings but not over a field. This explains the motivation of section 3.

9 Two Linearity Criteria for $\text{Aut}_S \mathbb{A}_K^2$

For a subgroup S of $\text{SL}(2, K)$, there are two linearity criteria for $\text{Aut}_S \mathbb{A}_K^2$. The second one, stronger, is only proved for a field of characteristic zero.

Let $\Lambda \subset K^*$ be a subgroup. For any $\mathbb{F}[\Lambda]$ -module M and any $n \geq 1$, set $M^{(n)} = (\rho_n)_* M$, where ρ_n is the group homomorphism $x \in \Lambda \mapsto x^n \in \Lambda$.

9.1 The standard module for torsion-free good subgroups of K^*

Assume now that Λ is a torsion-free good subgroup of K^* . By definition, the standard $\mathbb{F}[\Lambda]$ -module is K_1 , and it is denoted by $\text{St}(\Lambda)$.

Given a K -vector space V and an integer $n \geq 1$, it is clear that $V^{(n)}$ is a direct sum of standard modules, and its multiplicity is the cardinal

$$[V^{(n)} : \text{St}(\Lambda)] = [\Lambda : \Lambda^n] \dim_{K_1} V = \dim_{K_n} V.$$

9.2 The First Linearity Criterion

Let S be a subgroup of $\text{SL}_S(2, K)$ and let Λ_δ be the set of all eigenvalues of S_δ , for any $\delta \in \mathbb{P}_K^1$. Set $\text{SL}_S(2, K[t]) := \{G \in \text{SL}(2, K[t]) \mid G(1) \in S\}$.

Linearity Criterion 1. *Assume that Λ_δ is a torsion-free good subgroup of K^* , for any $\delta \in \mathbb{P}_K^1$. Then, for some field extension L of K , there is an embedding*

$$\text{Aut}_S \mathbb{A}_K^2 \subset \text{SL}(2, L(t)).$$

Moreover if $\text{rk } \Lambda_\delta \leq \aleph_0$ for any $\delta \in \mathbb{P}_K^1$, then we have

$$\text{Aut}_S \mathbb{A}_K^2 \simeq \text{SL}_S(2, K[t]) \subset \text{SL}(2, K(t)).$$

Proof. Set $M = \text{Sup rk } \Lambda_\delta$, where δ runs over \mathbb{P}_K^1 . There exists a field extension $L \supset K$, which satisfies one of the following two hypotheses

- (\mathcal{I}_1) $[L : K] \geq M$ if $M > \aleph_0$, or
- (\mathcal{I}_2) $L = K$ if $M \leq \aleph_0$

It follows from Lemmas 7 and Lemma 20 that

$$\begin{aligned} \text{Aut}_S \mathbb{A}_K^2 &= S \times *_{\delta \in \mathbb{P}_K^1} E_\delta(K), \text{ and} \\ \text{SL}_S(2, L[t]) &= S \times *_{\delta \in \mathbb{P}_L^1} U_\delta(zL[z]) \supset S \times *_{\delta \in \mathbb{P}_K^1} U_\delta(zL[z]). \end{aligned}$$

Let $\delta \in \mathbb{P}_K^1$. The $\mathbb{F}[\Lambda_\delta]$ -modules $E_\delta(K)$ and $U_\delta(zL[z])$ are copies of standard $\mathbb{F}[\Lambda]$ -module. Since $E_\delta(K) \simeq \bigoplus_{n \geq 3} K^{(n)}$, we have

$$[E_\delta(K) : \text{St}(\Lambda)] = \sum_{n \geq 3} [\Lambda : \Lambda^n][K : K_1].$$

On the other hand, $U_\delta(zL[z])$ is isomorphic to \aleph_0 copies of $L^{(2)}$, therefore we have

$$[U_\delta(zL[z]) : \text{St}(\Lambda)] = \aleph_0 [\Lambda : \Lambda^2][L : K][K : K_1].$$

Hence (\mathcal{I}_1) implies the existence of a S_δ -equivariant embedding

$$\psi_\delta : U_\delta(zL[z]) \rightarrow E_\delta(K),$$

and (\mathcal{I}_2) implies the existence a S_δ -equivariant isomorphism

$$\psi_\delta : U_\delta(zK[z]) \rightarrow E_\delta(K).$$

Therefore by Lemma 3, (\mathcal{I}_1) implies the existence of a an embedding

$$\text{Aut}_S \mathbb{A}_K^2 \subset S \times *_{\delta \in \mathbb{P}_K^1} U_\delta(zL[z]) \subset \text{SL}_S(2, L[t]) \subset \text{SL}(2, L(t)),$$

and (\mathcal{I}_2) implies the existence an isomorphism

$$\text{Aut}_S \mathbb{A}_K^2 \simeq S \times *_{\delta \in \mathbb{P}_K^1} U_\delta(zK[z]) \simeq \text{SL}_S(2, K[t]) \subset \text{SL}(2, L(t)). \quad \square$$

Theorem D. *Let R be a f.g. subring of K and let $S \subset \text{SL}(2, R)$.*

If $\text{rk } \Lambda_\delta = \text{trdeg } \Lambda_\delta$ for any $\delta \in \mathbb{P}_K^1$, then $\text{Aut}_S \mathbb{A}_K^2$ is linear over $K(t)$.

Otherwise, $\text{Aut}_S \mathbb{A}_K^2$ is not linear, even over a ring.

Proof. The second assertion follows from the Nonlinearity Criterion.

In order to prove the first one, assume now that $\text{rk } \Lambda_\delta = \text{trdeg } \Lambda_\delta$ for any $\delta \in \mathbb{P}_K^1$. Let L be the field of fraction of R and set $F = \overline{\mathbb{F}} \cap L$. Since R is a f.g. ring, F is a finite extension of \mathbb{F} . Therefore the set $\mu \subset K$ of roots of unity in F or in a quadratic extension of F is finite.

Let \mathfrak{m} be a maximal ideal of R such that $p := \text{ch } R/\mathfrak{m}$ is coprime to $\text{Card } \mu$. Indeed if $\mathbb{F} = \mathbb{Q}$ the characteristic of R/\mathfrak{m} is arbitrarily large, while in the opposite case, $\text{ch } R/\mathfrak{m}$ is automatically coprime to $\text{Card } \mu$.

Set $S' = \{g \in S \mid g \equiv \text{id} \pmod{\mathfrak{m}}\}$. For any $\delta \in \mathbb{P}_K^1$ let Λ'_δ be the eigenvalues of S'_δ . Since S' is residually p -group, Λ'_δ is torsion-free. Since $[S : S'] < \infty$, we can assume that $S = S'$.

We claim that Λ_δ is f.g. for any $\delta \in \mathbb{P}_K^1$. If δ is defined over a field F' , where $F' = F$ or F' is a quadratic extension of F , then $\Lambda_\delta \subset \overline{R}^*$, where \overline{R} is the integral closure of R in F' . Since \overline{R}^* is f.g., so is Λ_δ . Otherwise, Λ_δ is trivial. Hence Λ_δ is a torsion-free good subgroup of K^* for any $\delta \in \mathbb{P}_K^1$.

The Linearity Criterion 1 implies that $\text{Aut}_S \mathbb{A}_K^2$ is linear over $K(t)$. \square

9.3 The standard modules for finite-torsion good subgroups of K^*

From now on, K is a field of characteristic zero. Let Λ be a good subgroup of K^* , such that $\text{Card } \Lambda \cap \mu_\infty = n$ for some $n < \infty$.

The $\mathbb{Q}[\Lambda]$ -module $\text{St}_d(\Lambda) := K_d$, where d is a divisor of n , are called the *standard $\mathbb{Q}[\Lambda]$ -modules*. By Baer Theorem [1], Λ is isomorphic to $\mu_n \times \overline{\Lambda}$, where $\overline{\Lambda} = \Lambda/\mu_n$ is torsion-free. It follows that $\text{St}_d(\Lambda) \simeq \mathbb{Q}(\mu_{n/d}) \otimes \mathbb{Q}(\overline{\Lambda}^d)$.

Given a K -vector space V and an integer $m \geq 1$, it is clear that $V^{(m)}$ is a direct sum of the standard module $\text{St}_{\text{gcd}(n,m)}(\Lambda)$, and its multiplicity is the cardinal

$$[V^{(m)} : \text{St}_{\gcd(n,m)}(\Lambda)] = [\bar{\Lambda} : \bar{\Lambda}^n] \phi(n) / \phi(\gcd(n, m)) \dim_{K_1} V.$$

Lemma 32. *Let S_0 be a subgroup of B such that $\chi_B(S_0) = \Lambda$ is a good subgroup of K^* such that $n := \text{Card } \Lambda \cap \mu_\infty$ is finite. Let $l > n$ be a prime number and let $L \supset K$ be a field such that $[L : K] \geq \aleph_0 \text{rk}(\Lambda)$.*

Then there is a S_0 -equivariant embedding $E(K) \subset E^{<2l}(L)$.

Proof. Let D be the set of divisors of n . The $\mathbb{Q}[\Lambda]$ -module $E(K)$ and $E^{<2l}(L)$ are direct sums of standard modules, therefore we have

$$\begin{aligned} E(K) &= \bigoplus_{d \in D} \text{St}_d(\Lambda)^{m_d}, \text{ and} \\ E^{<2l}(L) &= \bigoplus_{d \in D} \text{St}_d(\Lambda)^{n_d}, \end{aligned}$$

where the multiplicities m_d and n_d are cardinals. Therefore it is enough to prove that $n_d \geq m_d$ for any $d \in D$.

Since $E(K) = \bigoplus_{m \geq 3} K.T_m$, it is clear that

$$m_d := [E(K)(d) : \text{St}_d(\Lambda)] \leq \aleph_0 \text{rk}(\Lambda)[K : K_1].$$

Similarly, we have $E^{<2l}(L) = \bigoplus_{3 \leq m \leq 2l} L.T_m$. Let $d \in D$. If $d \geq 3$, $L.T_d$ is a direct sum of standard modules $\text{St}_d(\Lambda)$ and it is clear that $[L.T_d : \text{St}_d(\Lambda)] \geq [L : K][K : K_1] \geq \aleph_0 \text{rk}(\Lambda)[K : K_1]$, and therefore $n_d \geq m_d$. Since l is coprime to n , then $L.T_l$ is a direct sum of standard modules $\text{St}_1(\Lambda)$, and we have $n_1 \geq m_1$.

If n is odd, the assertion is proved. Otherwise, $L.T_{2l}$ is a direct sum of standard modules $\text{St}_2(\Lambda)$, and similarly we have $n_2 \geq m_2$. \square

9.4 The Second Linearity Criterion

Linearity Criterion 2. *Let K be a field of characteristic zero. Assume that*

- (i) Λ_δ is a good subgroup of K^* , for any $\delta \in \mathbb{P}_K^1$, and
- (ii) the function $\delta \mapsto \text{Card } \Lambda \cap \mu_\infty$ is bounded.

Then $\text{Aut}_S \mathbb{A}_K^2$ is a linear group over some field extension L of K .

Proof. Let $L \supset K$ be a field such that $[L : K] \geq \aleph_0 M$, where $M = \text{Sup rk } \Lambda_\delta$, and let $Q \subset \mathbb{P}_K^1$ be a set of representatives of \mathbb{P}_K^1/S_0 . By Lemma 32 there is a S_δ -embedding $\psi_\delta : E_\delta(K) \rightarrow E_\delta^{<2l}(L)$ where $l > \text{Max Card } \Lambda \cap \mu_\infty$ is a prime number. Therefore we get some embeddings

$$\text{Aut}_S \mathbb{A}_K^2 \simeq S \times *_{\delta \in \mathbb{P}_K^1} E_\delta(K) \subset S \times *_{\delta \in \mathbb{P}_K^1} E_\delta^{<2l}(L) \subset \text{Aut}_0^{<2l} \mathbb{A}_L^2.$$

So, the strong version of Theorem C.2 implies that $\text{Aut}_S \mathbb{A}_K^2$ is linear. \square

10 Examples of Linear or Nonlinear $\text{Aut}_S \mathbb{A}_K^2$

We provide three examples using the Linearity/Nonlinearity Criteria.

9.1 Example A, with $S = \text{SO}(q)$ and K infinite

Example A. Let q be a quadratic form on K^2 and $S = \text{SO}(q)$.

If q is anisotropic, $\text{Aut}_S \mathbb{A}_K^2$ is linear over a field extension of K .

Otherwise $\text{Aut}_S \mathbb{A}_K^2$ is not linear, even over a ring.

As we will see, the proofs in characteristic p and characteristic 0 are very different. E.g. the group $S = \text{SO}(2, \mathbb{R})$ has no subgroups of finite index. Therefore the proof cannot be reduced to the Linearity Criterion 1.

Proof. If q is isotropic or degenerate, we have $\Lambda_\delta = K^*$ for some $\delta \in \mathbb{P}_K^1$. The proof of Theorem A.1 shows that K^* itself is bad. Hence by the Nonlinearity Criterion, $\text{Aut}_S \mathbb{A}_K^2$ is not linear, even over a ring.

Assume now that q is anisotropic. Let $L \supset K$ be the quadratic extension splitting q . Then $\text{SO}(q)$ is isomorphic to $S := \{z \in L^* \mid N_{L/K}(z) = 1\}$. Let S^∞ be the subgroup of all $s \in S$ of order a power of 2.

1. *Proof for $\text{ch } K = p$.* We claim that S^∞ is finite. So we can assume that $\text{Card } S^\infty \geq 4$. Since $\sqrt{-1} \in S$ and $S \cap K^* = \{\pm 1\}$, it follows that $L = K(\sqrt{-1})$. Therefore $p \equiv -1 \pmod{4}$, and $L = K.\mathbb{F}_{p^2}$. Let $s \in S^\infty$ of order > 2 . There is an integer $n \geq 1$ such that $s \in \mathbb{F}_{p^{2n}}$ where $\mathbb{F}_{p^n} \subset K$. Since $\mathbb{F}_{p^2} \not\subset K$, the integer n is odd. Since $\text{Card } \mathbb{F}_{p^{2n}}^* / \mathbb{F}_{p^2}^*$ is odd, s belongs to $\mathbb{F}_{p^2}^*$. Hence $S^\infty \subset \mathbb{F}_{p^2}^*$ is finite.

By Baer's theorem [1], we have $S = S^\infty \times S'$ for some subgroup $S' \subset S$. Since $S'_\delta = \{1\}$ for any $\delta \in \mathbb{P}_K^1$, the group $\text{Aut}_{S'} \mathbb{A}_K^2$ embeds into $\text{SL}(2, K(t))$ by the Linearity Criterion 1. Since

$$[\text{Aut}_{\text{SO}(q)} \mathbb{A}_K^2 : \text{Aut}_{S'} \mathbb{A}_K^2] < \infty,$$

the group $\text{Aut}_{\text{SO}(q)} \mathbb{A}_K^2$ is also linear over $K(t)$.

2. *Proof for $\text{ch } K = 0$.* Since $\text{SO}(q)_\delta = \{\pm 1\}$ for any $\delta \in \mathbb{P}_K^1$, the group $\text{Aut}_{\text{SO}(q)}$ is linear over a field extension of K by the Linearity Criterion 2 \square

9.2 A preparatory lemma for the example B

We did not found a reference for the next well-known result. The proof that $l(\gamma_a)$ is arbitrarily large is due to Y. Benoist.

Lemma 33. *There are lattices $S \subset \mathrm{SL}(2, \mathbb{R})$ such that $\mathrm{Tr} \gamma$ is a transcendental number for any infinite order element $\gamma \in S$.*

Proof. Let $g \geq 2$. Let Σ_g be the oriented Riemann surface of genus g , let $T(\Sigma_g)$ be its Teichmüller space and let Π_g be the group presented by

$$\langle \alpha_1, \beta_1 \dots \alpha_g, \beta_g \mid \prod_{1 \leq k \leq g} (\alpha_k, \beta_k) = 1 \rangle.$$

Let γ be a conjugacy class in Π_g . Any point $a \in T(\Sigma_g)$ determines a group homomorphism $\rho_a : \Pi_g \rightarrow \pi_1(\Sigma_g)$ and an hyperbolic metric g_a on Σ_g , up to some equivalence [15], ch.5. Hence $\rho_a(\gamma)$ is represented by a unique closed g_a -geodesic $\gamma_a : S^1 \rightarrow \Sigma_g$, i.e. a geodesic relative to the metric g_a .

In elementary terms, γ is represented, modulo $\mathrm{PSL}(2, \mathbb{R})$ -conjugacy, by a hyperbolic element $h_a \in \mathrm{PSL}(2, \mathbb{R})$. The complete geodesic $\Gamma \subset \mathbb{H}$ whose the extreme points in $\partial\mathbb{H}$ are the fixed points of h_a , is the locus of minima for the function $z \in \mathbb{H} \mapsto d_{\mathbb{H}}(z, h_a.z)$. The g_a -geodesic γ_a , up to reparametrization, is the image in Σ_g of any segment $[z, h_a.z]$ of Γ .

We claim that the length function $a \mapsto l(\gamma_a)$ is not constant. Let us pick another conjugacy class δ in Π_g such that $\rho_a(\delta)$ is represented by a simple geodesic δ_a which meets γ_a transversally (since $T(\Sigma_g)$ is connected, this condition is independent of a). By a corollary of the collar theorem,

$$\mathrm{sh} \frac{l(\gamma_a)}{2} \mathrm{sh} \frac{l(\delta_a)}{2} > 1,$$

see 4.1.2 in [7]. Since $l(\delta_a)$ can be arbitrarily small, $l(\gamma_a)$ is arbitrarily large.

Each $a \in T(\Sigma_g)$ determines an homomorphism $\pi_1(\Sigma_g) \rightarrow \mathrm{PSL}(2, \mathbb{R})$ up to some equivalence, which induces a homomorphism

$$\tilde{\rho}_a : \Pi_g \rightarrow \mathrm{PSL}(2, \mathbb{R})$$

defined up to conjugacy by $\mathrm{PSL}(2, \mathbb{R})$.

For $g \in \mathrm{PSL}(2, \mathbb{R})$, $\mathrm{Tr} g^2$ is well-defined. We have

$$\mathrm{Tr} \tilde{\rho}_a(\gamma)^2 = 2 \mathrm{ch} l(\gamma_a).$$

Let \mathcal{I} be the set of irreducible polynomials in $\mathbb{Q}[t]$. For $P \in \mathcal{I}$, set

$$\Omega(\gamma, P) = \{a \in T(\Sigma_g) \mid P(\mathrm{Tr} \tilde{\rho}_a(\gamma)^2) \neq 0\}.$$

Since the function $a \mapsto P(\mathrm{Tr} \tilde{\rho}_a(\gamma)^2)$ is nonconstant and analytic, $\Omega(\gamma, P)$ is a dense open subset of $T(\Sigma_g)$. By the Baire Theorem

$$\Omega := \bigcap_{\gamma \neq 1, P \in \mathcal{I}} \Omega(\gamma, P)$$

is dense. For any $a \in \Omega$, the lattice

$$S := \{s \in \mathrm{SL}(2, \mathbb{R}) \mid s \bmod \pm 1 \text{ belongs to } \tilde{\rho}_a(\Pi_g)\}$$

satisfies the required condition. \square

9.3 Example B, where S is a lattice

Example B. For some lattices $S \subset \mathrm{SL}(2, \mathbb{R})$, $\mathrm{Aut}_S \mathbb{A}_{\mathbb{C}}^2$ is linear over \mathbb{C} .

For any lattice $S \subset \mathrm{SL}(2, \mathbb{C})$, $\mathrm{Aut}_S \mathbb{A}_{\mathbb{C}}^2$ is not linear, even over a ring.

Proof. Let $S \subset \mathrm{SL}(2, \mathbb{R})$ be a lattice as in Lemma 33. Then for any $\delta \in \mathbb{P}_{\mathbb{C}}^1$, it is clear that Λ_{δ} has rank one and contains a transcendental element, or it is finite. Hence $\mathrm{Aut}_S \mathbb{A}_{\mathbb{C}}^2$ is linear over \mathbb{C} by Theorem E.

Let S be a lattice of $\mathrm{SL}(2, \mathbb{C})$, let $g \in S$ be of infinite order and let $\delta \in \mathbb{P}_{\mathbb{C}}^1$ be a fixed point of g . By the Garland-Raghunathan rigidity theorem 0.11 of [14], the eigenvalues of g are algebraic numbers. Since $\mathrm{rk} \Lambda_{\delta} > \mathrm{trdeg} \Lambda_{\delta}$, Theorem E implies that $\mathrm{Aut}_S \mathbb{A}_{\mathbb{C}}^2$ is not linear, even over a ring. \square

9.4 A preparatory lemma for example C

Lemma 34. Let R be a prime normal ring with fraction field k and let $m \geq 1$. Let B be the integral closure of $R[t_1, \dots, t_m]$ in some quadratic extension $L_1 \subset k((t_1, \dots, t_m))$ of $k(t_1, \dots, t_m)$.

Then B^*/R^* is isomorphic to $\{1\}$ or \mathbb{Z} .

Proof. Since R is normal, we have $B \cap k = R$, hence the map $B^*/R^* \rightarrow (k \otimes B)^*/k^*$ is one to one. So we can assume that $R = k$. Set $C = \mathrm{Spec} B$. There is a unique normal compactification \overline{C} of C such that the finite map $C \rightarrow \mathbb{A}_k^m = \mathrm{Spec} k[t_1, \dots, t_m]$ extends to a finite map $\pi : \overline{C} \rightarrow \mathbb{P}_k^m$.

Set $Z := \pi^{-1}(\mathbb{P}_k^{m-1})$, where $\mathbb{P}_k^{m-1} := \mathbb{P}_k^m \setminus \mathbb{A}_k^m$. For any irreducible divisor D in \overline{C} , let v_D be the corresponding valuation.

If Z is irreducible, then $v_Z(f) \leq 0$ for any $f \in B$. Hence $v_Z(f) = 0$ for any $f \in B^*$, and therefore $B^* = k^*$.

Otherwise, Z is the union of two divisors Z_1 and Z_2 . For any $f \in B^* \setminus k^*$, either $v_{Z_1}(f) < 0$ or $v_{Z_2}(f) < 0$. Hence the homomorphism $f \in B^* \mapsto (v_{Z_1}(f), v_{Z_2}(f)) \in \mathbb{Z}^2$ embeds B^*/k^* in a free \mathbb{Z} -module of rank ≤ 1 . \square

9.5 Example 1, where $\mathrm{rk} \Lambda_{\delta}$ is not constant

Let A be a torsion-free additive group of any rank, let $d, m > 0$ be integers with d square-free and let \mathcal{O} be the ring of integers of $k := \mathbb{Q}(\sqrt{d})$. Set $K = k(A)((t_1, \dots, t_m))$ where $k(A)$ is the field of fractions of $k[A]$, and

$$S := \mathrm{SL}(2, \mathcal{O}[A][t_1, \dots, t_m]).$$

The Example C of the introduction is the case $A = \mathbb{Z}$ and $m = 1$.

Example C. If $d < 0$, then $\mathrm{Aut}_S \mathbb{A}_K^2$ is linear over a field extension of K .

Otherwise, $\mathrm{Aut}_S \mathbb{A}_K^2$ is not linear, even over a ring.

Proof. Set $L_0 := k(A)(t_1, \dots, t_m)$.

Proof if $d > 0$. Then $\mathbb{Q}(\sqrt{d})$ is a real field, we have $\text{rk } \mathcal{O}^* = 1 > \text{trdeg } \mathcal{O}^* = 0$. For $\delta \in \mathbb{P}_{L_0}^1$, the group $\Lambda_\delta = \mathcal{O}^* \times A$ is a bad subgroup of K^* . So, by the Nonlinearity Criterion, $\text{Aut}_S \mathbb{A}_K^2$ is not linear, even over a ring.

Proof if $d < 0$. Let $\delta \in \mathbb{P}_L^1$.

If δ belongs to $\mathbb{P}_{L_0}^1$, we have $\Lambda_\delta = \mathcal{O}^* \times A$. Since $\mathbb{Q}(\sqrt{d})$ is an imaginary field, $\Lambda_\delta = \mathcal{O}^* \times A$ is a good subgroup of K^* and $\text{Card } \Lambda_\delta \cap \mu_\infty \leq 6$.

Assume now that δ belongs to $\mathbb{P}_{L_1}^1 \setminus \mathbb{P}_{L_0}^1$, where L_1 is a quadratic extension of L_0 . Let B be the algebraic closure of $\mathcal{O}[A][t_1, \dots, t_m]$ in L_1 and let $N := \{z \in L_1 \mid N_{L_1/L_0}(z) = 1\}$ be the norm group. It is clear that

$$\Lambda_\delta = N \cap B^*.$$

By Lemma 34, we have $\Lambda_\delta = \{\pm 1\}$ or $\Lambda_\delta = \{\pm 1\} \times \mathbb{Z}$. Since \mathcal{O} is algebraically closed in L , it should be noted that when $\text{rk } \Lambda_\delta = 1$, we also have $\text{trdeg } \Lambda_\delta = 1$. Thus Λ_δ is a good subgroup of K^* and $\text{Card } \Lambda_\delta \cap \mu_\infty = 2$.

Otherwise, we have $\Lambda_\delta = \{\pm 1\}$ and the same conclusion holds.

Therefore, by the Linearity Criterion 2, $\text{Aut}_S \mathbb{A}_K^2$ is linear over some field extension of K . \square

11 Nonlinearity of Finite-Codimensional Subgroups of $\text{Aut } \mathbb{A}_K^3$

The Theorem A.2 shows that $\text{Aut } \mathbb{A}_K^2$ contains some finite-codimensional subgroups, which are linear as abstract groups. However, this result does not extend to \mathbb{A}_K^n , for $n \geq 3$, as it will be shown in this section.

For our purpose, the case $n = 3$ is enough. Unlike in the introduction, *it will be convenient to use the coordinates (z, x, y) for \mathbb{A}_K^3* . Let $\text{TAut } \mathbb{A}_K^3$ be the the subgroup of tame automorphisms of \mathbb{A}_K^3 , see 10.5 for the definition. By the famous result of Shestakov and Urmibaev [22], $\text{TAut } \mathbb{A}_K^3$ is a *proper* subgroup of $\text{Aut } \mathbb{A}_K^3$.

Let \mathfrak{m} be a finite-codimensional ideal in $K[z, x, y]$. Let $\text{Aut}_{\mathfrak{m}} \mathbb{A}_K^3$ be the group of all polynomial automorphisms ϕ of the form

$$(z, x, y) \mapsto (z + f, x + g, y + h),$$

where f, h and g belongs to \mathfrak{m} . Set

$$\text{TAut}_{\mathfrak{m}} \mathbb{A}_K^3 = \text{TAut } \mathbb{A}_K^3 \cap \text{Aut}_{\mathfrak{m}} \mathbb{A}_K^3.$$

The nonlinearity result for $n = 3$, valid even if K is finite, is unrelated with the existence of wild automorphisms in \mathbb{A}_K^3 , as shown by

Theorem B. *For any finite-codimensional ideal \mathfrak{m} of $K[z, x, y]$, the groups $\text{Aut}_{\mathfrak{m}} \mathbb{A}_K^3$ and $\text{TAut}_{\mathfrak{m}} \mathbb{A}_K^3$ are not linear, even over a ring.*

The proof uses the folklore embedding Φ , likely known by Nagata, and used in [22]. The simplest obstruction for linearity is due to a nonnilpotent locally nilpotent subgroup. In characteristic zero, the proof is easy, and it follows the line of [9] together with Corollary 2. In characteristic p , the proof involves the strange formula of Lemma 37.

11.1 Nilpotency class of some p -groups

The *nilpotency class* of a nilpotent group is the length of its ascending central series. Let p be a prime integer, and let E be an elementary p -group of rank r . Note that E acts by translation on $\mathbb{F}_p[E]$ and set $G(r) = E \ltimes \mathbb{F}_p[E]$.

Set $E = D_1 \times \cdots \times D_r$, where each D_i has rank 1. For each i , the socle filtration of the D_i -module $\mathbb{F}_p[D_i]$ has length p and we have $\mathbb{F}_p[E] = \mathbb{F}_p[D_1] \otimes \cdots \otimes \mathbb{F}_p[D_r]$. Hence the socle filtration of the E -module $\mathbb{F}_p[E]$ has length $1 + (p-1)r$. It follows that

Lemma 35. *The nilpotency class of the nilpotent group $G(r)$ is $1 + (p-1)r$.*

Let M be a cyclic $\mathbb{F}_p[E]$ -module generated by some $f \in M$.

Lemma 36. *If $\sum_{u \in E} u.f \neq 0$, then the $\mathbb{F}_p[E]$ -module M is free of rank one.*

Proof. Set $N = \sum_{u \in E} e^u$, where $(e^u)_{u \in E}$ is the usual basis of $\mathbb{F}_p[E]$. Note that $\mathbb{F}_p.N = H^0(E, \mathbb{F}_p[E])$, hence any nonzero ideal of $\mathbb{F}_p[E]$ contains N . Since $N.f \neq 0$, M is freely generated by f . \square

11.2 A formula

Lemma 37. *Let A be a commutative \mathbb{F}_p -algebra and let $E \subset A$ be a linear subspace of dimension r . Then we have*

$$\sum_{u \in E} u^{p^r-1} = \prod_{u \in E \setminus \{0\}} u.$$

Proof. It is enough to prove the claim for $A = \mathbb{F}_p[x_1, \dots, x_r]$ and $E = \oplus_i \mathbb{F}_p.x_i$. Set $P(x_1, \dots, x_r) := \sum_{u \in E} u^{p^r-1}$, set $H = \mathbb{F}_p.x_2 \oplus \mathbb{F}_p.x_3 \cdots \oplus \mathbb{F}_p.x_r$ and for $v \in H$, set $Q_v := \sum_{\lambda \in \mathbb{F}_p} (\lambda x_1 + v)^{p^r-1}$. For any integer $n \geq 0$, we have $\sum_{\lambda \in \mathbb{F}_p} \lambda^n = 0$ except if n is a positive multiple of $p-1$. Hence

$$Q_v = \sum_{n > 0} c_n x_1^{n(p-1)} v^{p^r-1-n(p-1)},$$

for some $c_n \in \mathbb{F}_p$ and the polynomial Q_v is divisible by x_1^{p-1} . Since

$$P(x_1, \dots, x_n) = \sum_{v \in H} Q_v,$$

the polynomial $P(x_1, \dots, x_n)$ is divisible by x_1^{p-1} . By $\text{GL}(r, \mathbb{F}_p)$ -invariance, $P(x_1, \dots, x_n)$ is divisible by u_1^{p-1} for all $u \in E \setminus \{0\}$. Hence it is divisible by $\prod_{u \in E \setminus \{0\}} u$. Since both polynomials have degree $p^r - 1$, it follows that

$$P(x_1, \dots, x_n) = c \prod_{u \in E \setminus \{0\}} u,$$

for some $c \in \mathbb{F}_p$.

As it is a universal constant, we can compute c for $A = E = \mathbb{F}_{p^r}$. Since

$$\sum_{\lambda \in \mathbb{F}_p^r} \lambda^{p^r-1} = -1, \text{ and } \prod_{\lambda \in \mathbb{F}_p^r} \lambda = -1,$$

it follows that $c = 1$. □

11.3 The locally nilpotent group $G(I)$

Let \mathbb{F} be a prime field. For any ideal I of a commutative \mathbb{F} -algebra A , let us consider the semi-direct product $G(I) := I \rtimes I[t]$, where I acts by translation on the space $I[t]$ of polynomials with coefficients in I .

Let $E \subset I$ be an additive subgroup, let $f(t) \in I[t] \setminus \{0\}$ and let M be the additive subgroup generated by all polynomials $f(t+u)$ when u runs over E . The group $E \rtimes M$, which is a subgroup of $G(I)$, is obviously nilpotent.

Lemma 38. *Assume that the algebra A is prime.*

- (i) *If $\mathbb{F} = \mathbb{Q}$, the nilpotency class of $E \rtimes M$ is $1 + \deg f$.*
- (ii) *Assume that $\mathbb{F} = \mathbb{F}_p$, that $\dim_{\mathbb{F}_p} E = r$ and that $f(t) = at^{p^r-1}$ for some $a \in I \setminus \{0\}$. Then the group $E \rtimes M$ has nilpotency class $1 + r(p-1)$.*
- (iii) *If $\dim_{\mathbb{F}_p} I = \infty$, the group $G(I)$ is locally nilpotent but not nilpotent.*

Proof. For $\mathbb{F} = \mathbb{Q}$ the proof is obvious. From now on, we will assume that $\mathbb{F} = \mathbb{F}_p$ and we will prove the second assertion.

Set $g(t) = \sum_{u \in E} f(t+u)$. We have $g(0) = \sum_{u \in E} au^{p^r-1} = a \prod_{u \in E \setminus \{0\}} u$ by Lemma 37. Since $g(0) \neq 0$, it follows from Lemma 36 that the $\mathbb{F}_p[E]$ -module M is free of rank one, and $E \rtimes M$ is isomorphic to $G(r)$. Thus its nilpotency class is $1 + r(p-1)$ by Lemma 35. □

11.4 The amalgamated product $\text{Aff}(2, I) *_{B_{\text{Aff}(I)}} \text{Elem}(I)$

From now on, let I be a proper nonzero ideal in $K[z]$.

Lemma 39. *The group $\text{Elem}(I)$ is not linear over a field.*

Proof. Since I is a proper ideal, $\text{Elem}(I)$ is the group of all automorphisms

$$\phi : (x, y) \mapsto (x + u, y + f(x)),$$

for some $u \in I$ and $f \in I[x]$. It follows that $\text{Elem}(I)$ is isomorphic to the group $G(I)$, and therefore $\text{Elem}(I)$ contains subgroups of arbitrarily large nilpotency class by Lemma 38. Hence $\text{Elem}(I)$ is not linear over a field. \square

Lemma 40. *The group $\text{Aff}(2, I) *_{B_{\text{Aff}}(I)} \text{Elem}(I)$ is not linear, even over a ring.*

Proof. By Lemma 39, the group $\text{Elem}(I)$ is not linear over a field. Therefore by Corollary 1, it is enough to show that the amalgamated product $\Gamma := \text{Aff}(2, I) *_{B_{\text{Aff}}(I)} \text{Elem}(I)$ satisfies

$$\text{Core}_{\Gamma}(B_{\text{Aff}}(I)) = \{1\}.$$

In order to do so, we first define two specific automorphisms γ and ϕ as follows. Let $r \in I \setminus \{0\}$ and let $n \geq 3$ be an integer coprime to $\text{ch } K$. Let $\gamma \in \text{Aff}(2, I)$ be the linear map $(x, y) \mapsto (x + ry, y)$ and let $\phi \in \text{Elem}(I)$ be the polynomial automorphisms $(x, y) \mapsto (x, y + rx^n)$.

Let g be an arbitrary element of $B_{\text{Aff}}(I) \setminus \{1\}$. By definition, g is an affine map $(x, y) \mapsto (x + u, y + v + wx)$ for some $u, v, w \in I$.

If $w \neq 0$, the linear part of g^γ is not lower triangular, therefore g^γ is not in $B_{\text{Aff}}(I)$. If $w = 0$ but $u \neq 0$, then the leading term g^ϕ , which is $(x, y) \mapsto (0, nrux^{n-1})$, has degree ≥ 2 . Therefore g^ϕ is not in $B_{\text{Aff}}(I)$. Last if $u = w = 0$, then v is not equal to zero. It follows that the leading term of $g^{\gamma\phi}$, which is $(x, y) \mapsto (0, nr^2vx^{n-1})$, has degree ≥ 2 . Therefore $g^{\gamma\phi}$ is not in $B_{\text{Aff}}(I)$.

Hence, for any $g \in B_{\text{Aff}}(I) \setminus \{1\}$ at least one of the three elements g^γ , g^ϕ or $g^{\gamma\phi}$ is not in $B_{\text{Aff}}(I)$. Therefore $\text{Core}_{\Gamma}(B_{\text{Aff}}(I))$ is trivial. \square

11.5 Proof of Theorem B

The group $\text{TAut } \mathbb{A}_K^3 \subset \text{Aut } \mathbb{A}_K^3$ of tame automorphisms of \mathbb{A}_K^3 is

$$\text{TAut } \mathbb{A}_K^3 = \langle \text{Aff}(3, K), T(3, K) \rangle,$$

where $\text{Aff}(3, K)$ is the group of affine automorphisms of \mathbb{A}_K^3 and $T(3, K)$ is the group of all triangular automorphisms

$$(z, x, y) \mapsto (z, x + f(z), y + g(z, x)),$$

where f and g are polynomials. Note that $\text{Aut } \mathbb{A}_{K[z]}^2 = \text{Aut}_{K[z]} K[z, x, y]$ is obviously the subgroup of $\text{Aut } \mathbb{A}_K^3 = \text{Aut}_K K[z, x, y]$ of all automorphisms of the form

$$(z, x, y) \mapsto (z, f(z, x, y, z), g(z, x, z)),$$

where f and g are polynomials.

It is easy to see that the groups $\text{Elem}(K[z])$ and $\text{Aff}(2, K[z])$ are subgroups of $\text{TAut } \mathbb{A}_K^3$. Therefore using the van der Kulk Theorem applied to the field $K(z)$ and Lemma 2, one obtains the folklore embedding [22]

$$\Phi : \text{Aff}(2, K[z]) *_B(K[z]) \text{Elem}(K[z]) \rightarrow \text{Aut } \mathbb{A}_{K[z]}^2 \cap \text{TAut } \mathbb{A}_K^3,$$

which was likely known by Nagata. The hard and beautiful result of [22] states that Φ is onto, a result which is not needed here.

Now, we prove the Theorem B.

Proof. Without loss of generality, we can assume that the ideal \mathfrak{m} is also a proper ideal. Hence the ideal $I := \mathfrak{m} \cap K[z]$ is nonzero and proper.

The previously defined morphism Φ induces an embedding

$$\text{Aff}(2, I) *_B(\text{Aff}(I)) \text{Elem}(I) \rightarrow \text{Aut}_{\mathfrak{m}} \mathbb{A}_K^3.$$

Hence by Lemma 40, the group $\text{TAut}_{\mathfrak{m}} \mathbb{A}_K^3$ is not linear, even over a ring. \square

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