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A primitive formulation (v, ψ) for incompressible flows based on velocity and vector potential of the acceleration

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Abstract

An original primitive formulation of the equation of motion for viscous flows, based on two variables, the velocity and the vector potential of the acceleration, ensures the strict incompressibility of the flow. This equation, which reflects the conservation of the total energy, states that the intrinsic acceleration of the material medium is equal to the sum of the two terms of a Helmholtz-Hodge decomposition, one divergence-free and the second curl-free; the velocity field is itself decomposed into a solenoidal and an irrotational part. The incompressibility approximation of a flow leads to retain only the solenoidal components of all the terms of the equation of motion, viscous, inertial and vector potential of the acceleration; thus, all the terms which are expressed in the form of a dual curl are free of divergence. If, at a given moment, the velocity field is solenoidal, it will remain so both on a trajectory and in the whole domain. This decomposition allows us to state dual Bernoulli's theorem which applies to rotational flows as the equivalent of Bernoulli's theorem associated to irrotational flows.

Keywords

Navier-Stokes equation; Discrete Mechanics; Helmholtz-Hodge Decomposition; Conservation of Acceleration; Noether Theorem

1 Introduction

There are many formulations of the equations of fluid mechanics and each has advantages and disadvantages that are described in many textbooks in this area of physics. They apply to a variety of equations, but the most important of these is the Navier-Stokes equation, which is defined as a partial differential equation with a mass conservation equation and possibly an energy conservation equation. The system of equations is closed by constitutive laws.

For simplicity, two families of formulations are used, the formulations in primitive variables and those where a derivation operator is applied to the Navier-Stokes equation. The first one is symbolized by the couple (p, \mathbf{V}) , the pressure and the velocity. It is a natural formulation that has many qualities and is legitimately used to simulate all types of flows. A large number of computational methodologies are based on this form, the projection methods [1, 2], the artificial compressibility method [3] or its implicit form, the augmented Lagrangian method [4], to mention only these techniques.

The second family of formulations of the Navier-Stokes equations is based on the application of a differential operator to conserve or eliminate the pressure. The divergence of the equation leads to a pressure equation in the form of a Poisson equation with a relatively complex right-hand side. The curl operator is applied if the pressure has to be eliminated; this is the case for the rotational-streamfunction (ω, ψ) formulation in two space dimensions or (Ω, Ψ) in rotational-vector potential in three dimensions where velocity equal to $\mathbf{V} = \nabla \times \Psi$. This second family of methods has serious drawbacks, in particular when the physical properties are not constant. Moreover, it leads, in dimension three, to increase the number of equations to solve. However, it has the advantage of strictly safeguarding the incompressibility in the case where it is necessary.

The problem posed here is to propose a formulation which preserves the interests of a formulation in primitive variables and those of the second family where the constraint $\nabla \cdot \mathbf{V} = 0$ is strictly maintained naturally. In the framework of classical mechanics this question does not seem to arise because it is the pressure p , representative of the effects of compressibility, which maintains the mechanical equilibrium; it is the scalar potential of the intrinsic acceleration $\boldsymbol{\gamma} \equiv d\mathbf{V}/dt$ or rather the ratio $\phi = p/\rho$ where ρ is the density of the medium. The equivalent of the scalar potential is the acceleration vector potential, $\boldsymbol{\psi}$; it represents for viscous terms what pressure is for compression effects. In fact, for incompressible flows, the pressure is only a Lagrangian with no real physical existence, it is only used to reach the mechanical equilibrium and to maintain an acceptable level of divergence during a simulation. The discrete mechanics equation, whose basics are briefly recalled in the following sections, states that the proper acceleration of a material medium is equal to the sum of the gradient of the scalar potential ϕ and the dual curl of the vector potential $\boldsymbol{\psi}$. From then on, the solution of the problem consists in replacing the scalar potential $-\nabla\phi$ of acceleration by its dual curl $\nabla \otimes \boldsymbol{\psi}$, these two quantities of different variables being equal to a constant. Thus, the identity $\nabla \cdot (\nabla \otimes \boldsymbol{\psi}) = 0$ ensures strict incompressibility.

All formulations in primitive variables containing the pressure used to ensure the incompressibility of the flow are confronted with this paradoxical difficulty, one must know a non-zero divergence at any time of the solution to obtain $\nabla \cdot \mathbf{V} = 0$. From a practical point of view, obtaining the pressure in the codes for solving the Navier-Stokes equation constitutes a very important part of the total solution time.

The starting point of the presentation devoted to incompressible flows is Newton's second dynamic law in discrete form which states that the acceleration is equal to the sum of the accelerations corresponding to the compression and rotation accelerations $\boldsymbol{\gamma} = -\nabla\phi + \nabla \otimes \boldsymbol{\psi}$. The attribution of a particular status by Galileo to mass (density) is not a necessity, the Equivalence Principle is only linked to accelerations, that of the proper inertia of the medium and that of gravitation. Newton's second law is thus transformed into a balance of accelerations.

2 General framework

2.1 Primal and dual structures

An important distinction must first be made about the notion of velocity; \mathbf{V} is the velocity vector of \mathbb{R}^3 of continuum mechanics whereas \boldsymbol{v} is the scalar assigned to an oriented segment of discrete media mechanics. Similarly, one should not confuse $\mathbf{V} = \nabla \times \boldsymbol{\Psi}$ where $\boldsymbol{\Psi}$ is the vector potential of velocity and that of acceleration $\boldsymbol{\psi}^o = \nu \nabla \times \boldsymbol{v}$ where ν is the kinematic viscosity.

The notion of discrete medium allows us to introduce the basic elements of differential geometry which are used to define the interactions between the different quantities of physics. The physical domain is composed of two structures called primal and dual which can be interchanged *a priori*, they are represented schematically on Figure 1. The primal geometrical structure of Figure 1(a) is composed of rectilinear segments Γ oriented by a unit vector \mathbf{t} ; the extremities (a, b) of each segment define its length dh named discrete horizon on which is fixed the scalar potential of the acceleration ϕ . The dual structure of Figure 1(b) is formed by the closed contour Δ which carries the vector potential of the acceleration positively oriented by the unit vector \mathbf{n} . The primal contour Γ can be open or closed and when the family Γ^* of segments Γ forms a closed circuit, the two structures are nested. Thus, the primal contour can be open or closed but the dual contour Δ is necessarily closed. The Stokes theorem allows then to transform a vector circulation on a contour into a curl on the surface bounded by Γ ; a second application of this theorem on Δ allows to project the result on the initial contour. This discrete approach

can be easily transformed into a local approach by considering a homothetic reduction, $dh \rightarrow 0$, of the structures where the angles are conserved. Contrary to the notion of continuous medium, the result of this reduction is not a point. The local reference frame created by the assembly of the two structures does not require the creation of the global reference frame of continuum mechanics, the classical reference frame $\mathbb{R}^3(x, y, z)$ does not exist. The interactions between the different quantities are realized according to the principle of causality adopted in the present context.

The discrete operators for transforming a vector into its dual or inverse are (i) the divergence of velocity, $\nabla \cdot \mathbf{v}$ which assigns the result to a vertex of the primal structure, (ii) the primal curl $\nabla \times \mathbf{v}$ which fixes the result on the barycenter of the primal facets, (iii) the gradient of the scalar is represented by $\nabla \phi$, a quantity that is uniquely defined on the segment Γ oriented by the unit vector \mathbf{t} and finally (iv) the dual curl $\nabla \otimes \boldsymbol{\psi}$ that reprojects the result of the circulation on the dual contour Δ on the segment Γ . This dual curl operator does not exist in the context of the continuous medium, it is replaced by the nature of the vector to which it applies, a polar vector or an axial vector still called pseudo-vector. Several notations exist in the field of differential geometry in index or exponent of the operator ∇ but, given the importance of this operator, it is appropriate to give it a symbol which completes in a coherent way the first three definitions, it is thus the symbol $\nabla \otimes$ which is chosen. This means that the curl is computed directly by the circulation of a vector on the closed contour Δ by the Stokes theorem. Classically, in continuum mechanics, it represents the tensor product, but in discrete mechanics, tensors do not exist and the vectors themselves are scalars on oriented segments. From a physical point of view, the entanglement of the two geometrical structures of the Figure 1(b) corresponds to electromagnetic phenomena in circuits.

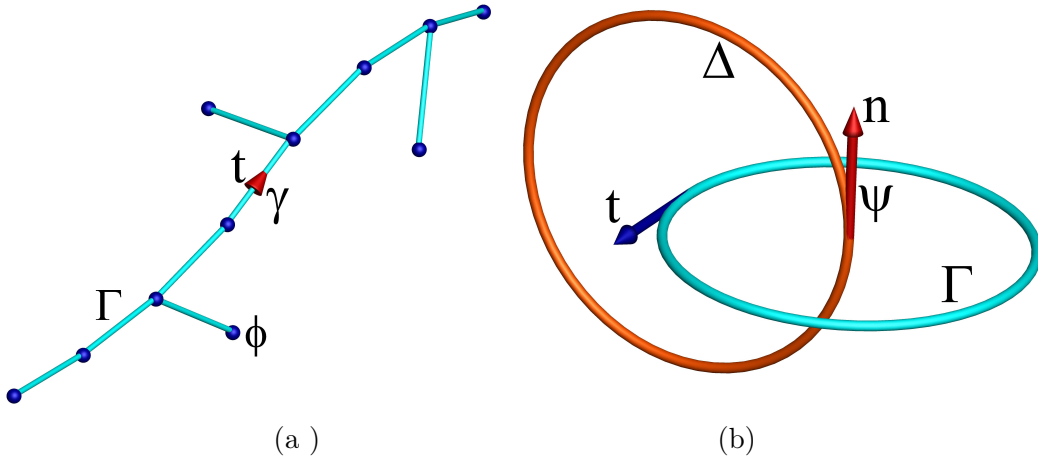


Figure 1. Primal (a) and dual (b) structures defined respectively by the contours Γ and Δ . The primal contour Γ consists of segments of length $dh = [a, b]$ oriented by the unit vector \mathbf{t} ; the scalar potential ϕ is positioned on the vertices of the segments. The closed dual contour Δ is oriented by the unit vector \mathbf{n} where the condition $\mathbf{n} \cdot \mathbf{t} = 0$ is realized by construction; the vector potential $\boldsymbol{\psi}$ is carried by the vector \mathbf{n} . The primal contour Γ can be open (a) or closed (b). The acceleration and velocities are expressed on the segment Γ .

The properties of this geometric description are remarkable, in particular the classical identities of the continuous medium,

$$\nabla \cdot (\nabla \otimes \boldsymbol{\psi}) = 0, \quad \nabla \times (\nabla \phi) = 0, \quad (1)$$

are verified exactly whatever the choice of the polygonal or polyhedral planar surfaces which

tessellate the physical domain. In a continuous medium, these relations are $\nabla \cdot (\nabla \times \mathbf{V}) = 0$ and $\nabla \times (\nabla f) = 0$ where \mathbf{V} and f are differentiable vector and scalar functions respectively. If these relations are mathematically correct, they have a physical representation flaw. In discrete mechanics, the divergence of a curl applies only to the vector potential $\boldsymbol{\psi}$ and the curl of gradient applies only to the scalar potential ϕ . If \mathbf{v} is the vector carried by the segment Γ we get the following expressions:

$$\nabla \cdot (\nabla \otimes (\nabla \times \mathbf{v})) = 0; \quad \nabla \times (\nabla (\nabla \cdot \mathbf{v})) = 0 \quad (2)$$

Indeed, the scalar potential is characterized by the divergence of the velocity and the vector potential is defined from the curl of the velocity. The scalar and vector potentials of the acceleration vector are at the same level and are expressed in the same units. In continuum mechanics and electromagnetism, there is still a confusion between the potential acceleration vector and the potential velocity vector. To obtain a velocity field at zero divergence, it seems natural to pose $\mathbf{V} = \nabla \times \boldsymbol{\Psi}$ such that $\nabla \cdot \nabla \times \boldsymbol{\Psi} = 0$ where $\boldsymbol{\Psi}$ is the velocity vector potential. Similarly, in electromagnetism, the scalar potential of the acceleration is, e , the electric potential but the vector potential \mathbf{A} is not that of the acceleration but that of the induction field \mathbf{B} . The induction field is conservative if we fix $\mathbf{B} = \nabla \times \mathbf{A}$ where \mathbf{A} is the vector potential of the induction. This additional derivation to obtain a field with zero divergence leads to harmful and useless effects both in fluid mechanics and in electromagnetism.

In fluid mechanics, the condition $\nabla \cdot \mathbf{V} = 0$ is not an absolute necessity, it is only an approximation. The vector potential of the velocity $\nabla \times \boldsymbol{\Psi}$ exists only for a strictly solenoidal field. This is not the case for the vector potential of the acceleration $\boldsymbol{\psi}$ which exists under all circumstances for any flow. The potential $\boldsymbol{\psi}$ is constructed from $\nabla \times \mathbf{v}$ which has the effect of making the divergence-free component of the velocity of \mathbf{v} disappear and $\boldsymbol{\psi}$ is therefore a conservative field. Similarly, the scalar potential ϕ is defined by $\nabla \cdot \mathbf{v}$ which removes the irrotational component of \mathbf{v} . The intrinsic acceleration then takes a structure of the form $\boldsymbol{\gamma} = -\nabla (\nabla \cdot \mathbf{v}) + \nabla \otimes (\nabla \times \mathbf{v})$. According to the chosen approximation, some terms are naturally removed.

The law of motion is an evolution equation in time that uniquely addresses a time interval dt between t^o which defines mechanical equilibrium at the previous instant and a current time $t = t^o + dt$ where mechanical equilibrium is also required. The time interval is chosen so that the discrete horizon dh is travelled at a velocity equal to or less than celerity of medium such that $dt \leq h/c$. At time t^o , the velocity is equal to \mathbf{v}^o and the potentials at ϕ^o and $\boldsymbol{\psi}^o$ are called retarded potentials [5] in reference to those in electromagnetism. The velocity \mathbf{v}^o is a symbolic notion because the integration in time can be realized by a scheme of any order.

2.2 Equation of motion

The law of motion in terms of accelerations developed in the last few years is presented as an alternative to the Navier-Stokes equation; it allows to simulate flows of any nature by strictly finding the classical solutions of fluid mechanics [6]. Several important differences distinguish the proposed formulation from the Navier-Stokes equation. First of all, the concept of continuous medium is abandoned, as well as the one-point derivation, the integration and, in general, the mathematical analysis. The concept of a global frame of reference is replaced by the existence of local frames of reference where interactions are realized from cause to effect. The concept of mass or density is discarded, the Equivalence Principle reflects only the equivalence of inertial effects and gravitation. An essential modification concerns the inertial terms which are written, like all the other terms of the equation of motion, in the form of a Helmholtz-Hodge decomposition [7]. The equation of motion conceived both in discrete and continuous form is written:

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla \left(\phi^o + \frac{1}{2} |\mathbf{v}|^2 - c_t^2 dt \nabla \cdot \mathbf{v} \right) + \nabla \otimes \left(\boldsymbol{\psi}^o + \frac{1}{2} |\mathbf{v}|^2 \mathbf{n} - c_t^2 dt \nabla \times \mathbf{v} \right) + \mathbf{h}_s \quad (3)$$

where $\mathbf{h}_s = \nabla\phi_s - \nabla \otimes \psi_s$ is a source term also expressed as a Helmholtz-Hodge decomposition.

Solving the equation (3) allows us to find the current solution at time $t^o + dt$, \mathbf{v} from its value \mathbf{v}^o and the equilibrium potentials ϕ^o and ψ^o at time t^o . The divergence and curl of \mathbf{v} allow the updating of the potentials in an explicit form:

$$\text{updates} \begin{cases} \alpha_l \phi^o - c_l^2 dt \nabla \cdot (\mathbf{v} - \mathbf{v}^o) \mapsto \phi^o \\ \alpha_t \psi^o - c_t^2 dt \nabla \times (\mathbf{v} - \mathbf{v}^o) \mapsto \psi^o \\ \mathbf{u} + \mathbf{v} dt \mapsto \mathbf{u}^o \end{cases} \quad (4)$$

The symbol \mapsto corresponds to this update of $\nabla \cdot (\mathbf{v} - \mathbf{v}^o)$ and of $\nabla \times (\mathbf{v} - \mathbf{v}^o)$. It is finally possible to update the displacement \mathbf{u} . The parameters α_l and α_t are attenuation factors for longitudinal and transverse waves respectively. For example, if the longitudinal waves are not attenuated, $\alpha_l = 1$ and, similarly, if the transverse waves are completely dissipated, $\alpha_t = 0$. Mechanical dissipation in the form of heat by attenuation of compressional and shear waves completes the conservation of total energy fixed by the equation of motion. The dissipated energy between times t^o and $t^o + dt$ is therefore $(1 - \alpha_l) \phi^o + (1 - \alpha_t) \psi^o$.

In the case where part of the compressional or shear energy is lost by viscous dissipation, the equation (3) must retain the full energy between the times t^o and $t^o + dt$ where the velocity evolves between the equilibrium state \mathbf{v}^o and the current state \mathbf{v} hence the notation $(\mathbf{v} - \mathbf{v}^o)$ in updates. For example, a rigid rotating flow, where the velocity of this incompressible flow is $\mathbf{v}_r = \boldsymbol{\Omega} \times \mathbf{r}$, does not correspond to energy dissipation since its primary curl is equal to $\nabla \times \mathbf{v}_r = cte$; this curl cannot be accumulated in time within the vector potential.

The law of motion (3) is a wave equation that applies to the motion of fluids or solids as well as to waves of any kind, such as swell, acoustic waves or light. It is necessary to distinguish and separate once and for all the motions of the media to which they apply:

- the motion can be (i) non-zero rotational and of non-zero divergence, (ii) irrotational and compressible, (iii) non-zero rotational and solenoidal, and (iv) curl-free and divergence-free; there is, *a priori*, no other choice;
- the propagation medium can be a fluid, a solid or a vacuum; the equation (3) contains no constitutive law, state law or rheological law. The medium is defined only by its propagation properties, the longitudinal c_l and transverse c_t celerities and the respective attenuations of these waves, α_l and α_t .

This second part of the classification of a medium is not of particular interest, because fluids can be any, Newtonian, viscoelastic, viscoplastic, with threshold, time-dependent properties, etc. Equations that incorporate the constitutive laws of media or the characteristics of these media are confusing and lead to a misinterpretation of physical phenomena. This is the case of some equations which introduces physical characteristics such as density and dynamic viscosity which mix the two notions presented by considering, for example, that a constant density leads to an incompressible flow, or that a zero viscosity leads to an irrotational motion. The hyperbolic law of motion based on waves must contain only longitudinal and transverse wave propagation properties.

The form of the law (3) suggests that the motion can be decomposed into a divergence-free part and a curl-free part by separating the two contributions to the acceleration, $\boldsymbol{\gamma} = \boldsymbol{\gamma}_\phi + \boldsymbol{\gamma}_\psi$ where $\boldsymbol{\gamma}_\phi = -\nabla\phi$ and $\boldsymbol{\gamma}_\psi = \nabla \otimes \psi$. The velocity then naturally decomposes into an equivalent form, $\mathbf{v} = \mathbf{v}_\phi + \mathbf{v}_\psi$ but the scalar and vector potentials of the velocity will not be useful or sought. This splitting of the equation of motion is however not completely possible because of the nonlinear terms; indeed, the inertial potential $|\mathbf{v}|^2/2$ can be written as:

$$\frac{1}{2} |\mathbf{v}|^2 = \frac{1}{2} |\mathbf{v}_\phi|^2 + \frac{1}{2} |\mathbf{v}_\psi|^2 + \mathbf{v}_\phi \cdot \mathbf{v}_\psi \quad (5)$$

The quadratic terms $|\mathbf{v}_\phi|^2/2$ and $|\mathbf{v}_\psi|^2/2$ remain in the strategy of decoupling compression and rotation effects and can be associated to Bernoulli potentials $\phi_B^o = \phi^o + |\mathbf{v}_\psi|^2/2$ and $\psi_B^o = \psi + |\mathbf{v}_\psi|^2/2$. The equation (3) then reads as follows:

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + \nabla (\mathbf{v}_\phi \cdot \mathbf{v}_\psi) - \nabla \otimes (\mathbf{v}_\phi \cdot \mathbf{v}_\psi \mathbf{n}) = & -\nabla \left(\phi^o + \frac{1}{2} |\mathbf{v}_\phi|^2 - c_l^2 dt \nabla \cdot \mathbf{v}_\phi \right) \\ & + \nabla \otimes \left(\psi^o + \frac{1}{2} |\mathbf{v}_\psi|^2 \mathbf{n} - c_t^2 dt \nabla \times \mathbf{v}_\psi \right) + \nabla \phi_s - \nabla \otimes \psi_s \end{aligned} \quad (6)$$

On the other hand, the terms $(\mathbf{v}_\phi \cdot \mathbf{v}_\psi)$ prevent any possibility of separating the irrotational and solenoidal components, unless one of the components is zero. In the general case, the compressional and rotational effects are intertwined and the equation (6) must be solved without simplification. An attempt by D.D. Joseph [8] to decouple the irrotational terms from the solenoidal terms of the Navier-Stokes equation shows that this Helmholtz-Hodge decomposition is legitimate to obtain a solution suitable for the type of solution sought; however, the Navier-Stokes equation has characteristics that prevent this operation.

Let us now consider the cases where the equation (6) can be disassembled; there are only two possibilities, (i) the one corresponding to a velocity curl-free, $\nabla \times \mathbf{v} = 0$ where the velocity field reduces to its only irrotational component $\mathbf{v} = \mathbf{v}_\phi$ and (ii) the one where it is the divergence of the velocity which is null $\nabla \cdot \mathbf{v} = 0$ and where the velocity field is solenoidal $\mathbf{v} = \mathbf{v}_\psi$. The case where the solution is harmonic, i.e. without curl or divergence, does not exist. As [9] shows, these solutions with zero or constant divergence or zero or constant curl are structurally discarded from the solutions of the equation (3).

When $\nabla \times \mathbf{v} = 0$, the equation of motion can be reduced to finding its unique irrotational solution, \mathbf{v}_ϕ , it is written:

$$\begin{cases} \frac{\partial \mathbf{v}_\phi}{\partial t} = -\nabla \left(\phi^o + \frac{1}{2} |\mathbf{v}_\phi|^2 - dt c_l^2 \nabla \cdot \mathbf{v}_\phi \right) - \nabla \phi_s \\ \alpha_l \phi^o - dt c_l^2 \nabla \cdot (\mathbf{v}_\phi - \mathbf{v}_\phi^o) \mapsto \phi^o \end{cases} \quad (7)$$

This equation (7) is equivalent to the Euler equation of motion where the fluid is ideal and the flow compressible where its solution is of the form (\mathbf{v}, ϕ) . However, the differences are important: (i) the presented formulation has no constitutive law, the state law is contained in the scalar potential $\phi = p/\rho$, (ii) the thermodynamic evolution is integrated in the longitudinal celerity c_l which can reflect an isothermal or isentropic evolution, (iii) it does not require an associated energy equation because the equation (7) already expresses the conservation of the compression energy and where ϕ represents the mechanical energy of compression per unit of mass. Of course, the fluid considered cannot be viscous nor conduct heat, it is necessarily an ideal fluid. This formalism can potentially represent compressible flows without shocks or in presence of shocks. In the latter case, a term representing the discontinuity of the form $\nabla(\xi \phi)$, where ξ is a phase function, must be added to the equation of motion. Figure 1(a) schematizes the associated primary structure where the flow is basically directional although the overall domain is three-dimensional. The velocity divergence $\nabla \cdot \mathbf{v}_\phi$ reflects only the cause-and-effect interactions from one Γ segment to the next, the interlocking of direct and induced currents has been broken and information propagates only through the vertices.

When $\nabla \cdot \mathbf{v} = 0$, the solenoidal solution of the equation of motion can be found directly from its component \mathbf{v}_ψ :

$$\begin{cases} \frac{\partial \mathbf{v}_\psi}{\partial t} = \nabla \otimes \left(\psi^o + \frac{1}{2} |\mathbf{v}_\psi|^2 \mathbf{n} - dt c_t^2 \nabla \times \mathbf{v}_\psi \right) - \nabla \otimes \psi_s \\ \alpha_t \psi^o - dt c_t^2 \nabla \times (\mathbf{v}_\psi - \mathbf{v}_\psi^o) \mapsto \psi^o \end{cases} \quad (8)$$

where α_t is the attenuation factor of transverse waves. The formulation can be used without a constitutive law, it is $dt c_t^2 = \nu$, the kinematic viscosity which defines the viscous interactions within the fluid; it can be any function of another variable, for example the temperature or directly the shear energy ψ^o . It can depend on the direction considered since this transverse propagation term allows to take into account anisotropic and polarizable media. In the case of an isotropic and homogeneous medium such as a Newtonian viscous fluid, the attenuation of transverse waves is fully and $\alpha_t = 0$, the instantaneous shear stress is equal to $\boldsymbol{\psi}^o = \nu \nabla \times (\mathbf{v}_\psi - \mathbf{v}_\psi^o)$, the energy lost by viscous dissipation and transformed into heat is thus equal to $(1 - \alpha_t) \psi^o$.

The solutions of the equation (3) are also those of the Navier-Stokes equation when $\nabla \cdot \mathbf{V} = 0$ is imposed with sufficient accuracy, many simulations show it clearly [6]. In this case, the mechanical equilibrium is ensured by the term $\nabla \phi^o$. What changes in the equation (8) is that the equilibrium is assured by the term $\nabla \otimes \boldsymbol{\psi}^o$ but the solution of the problem is strictly the same. Thus the relation $-\nabla \phi^o = \nabla \otimes \boldsymbol{\psi}^o$ makes it possible to adapt the equilibrium term to the problem posed, a flow with zero divergence without the presence of the scalar potential (pressure) reserved for compressible flows. The boundary conditions of the problem posed are not formulated in the same way as in continuum mechanics, in the form of partial derivatives on the edge, but in terms of accelerations directly integrated in the equation of motion [10]. They can therefore be absolutely arbitrary since they correspond to fluid flows or to localized shear stresses for each part of the domain boundary.

The law of motion (3) has remarkable properties related to its structure which is a decomposition of the intrinsic acceleration. It is naturally relativistic [9] because it is a non-linear wave equation; it satisfies Lorentz invariance without using the factor $\gamma = 1/\sqrt{1 - \mathbf{v}^2/c_0^2}$, c_0 being here the celerity of light. In classical applications, the velocity is negligible compared to c_0 , but the theory of relativity cannot be ignored in the derivation of any new physical law.

Another event that marked the science of the 20th century is Noether's theorem [11] which states that if a physical law can be expressed in the form of a Lagrangian or Hamiltonian, then it has the properties of invariance and conservation of some quantities. The discrete law is formed by two Lagrangians associated respectively with longitudinal and transverse effects; each of these Lagrangians has two actions represented by the potential and kinetic energies. Thus, although the equation (3) is formulated in terms of accelerations, the time invariance is related to the conservation of the total energy, the translational invariance to the conservation of the compressive energy and the rotational invariance to the conservation of the angular acceleration (angular momentum in continuum mechanics).

2.3 Primal and dual local form of Bernoulli's theorem

The two-term decomposed form, without divergence and without curl, of the equation of motion suggests a local form on a segment Γ equivalent to Bernoulli's theorem for an ideal fluid. In discrete mechanics, the local form of the theorem is directly deduced from the compression energy per unit mass defined by the gradient of the scalar potential ϕ of the acceleration. Indeed, the integral along the segment Γ of the primal structure of the acceleration represents the compression energy:

$$\int_{\Gamma} \boldsymbol{\gamma}_\phi \cdot \mathbf{t} dl = - \int_{\Gamma} \nabla \phi \cdot \mathbf{t} dl \quad (9)$$

When the length of the segment tends to zero, we obtain the equivalent local form of Bernoulli's theorem:

$$\frac{\partial \mathbf{v}_\phi}{\partial t} = -\nabla \left(\phi^o + \frac{1}{2} |\mathbf{v}_\phi|^2 - dt c_t^2 \nabla \cdot \mathbf{v}_\phi - \phi_s \right) \quad \text{along the path } \Gamma \text{ and in } \Omega \quad (10)$$

where ϕ_s is a scalar potential, for example $\phi_s = g z$ for gravity.

When the motion is irrotational, it is possible to traverse the whole Ω domain from one vertex of the primary structure to the other by a discrete integration on the Γ segment. The path once defined allows to cover the whole physical domain. If the first two terms are identical, the differences between this form (10) and the classical form of Bernoulli's theorem are multiple [12, 13, 14], (i) the flow can be unsteady and (ii) the flow can be compressible with a term $-dt c_t^2 \nabla \cdot \mathbf{v}_\phi$ which is expressed as a function of the velocity itself.

The statement of Bernoulli's theorem for irrotational flows applies equally well to a Lagrangian and an Eulerian description:

Primal Bernoulli theorem: *If, at a given time, an irrotational flow of an ideal fluid is compressible, inertial and derived from a scalar potential, it will remain so during its evolution in time.*

An equivalent statement can be deduced to represent the evolution of the compression energy per unit mass on the segment Γ :

$$\phi^o + \frac{1}{2} |\mathbf{v}_\phi|^2 - dt c_t^2 \nabla \cdot \mathbf{v}_\phi - \phi_s = cte \text{ along the path } \Gamma \quad (11)$$

where the term $dt c_t^2 \nabla \cdot \mathbf{v}_\phi$, also expressed on the segment, corresponds to the transformation of kinetic energy into potential energy symbolized by ϕ^o .

Bernoulli's law (10) is naturally relativistic [9] without applying *a priori* the Lorentz transformation. The most important consequence concerns the limit velocity that a fluid can reach by following a rectilinear trajectory where it cannot exceed the celerity of the medium, $\mathbf{v} < c_0$ for light in vacuum or $\mathbf{v} < c_l$ for fluids. The most significant example is that of the Sod's tube [15, 16] where the velocity of the gas contained in a channel cannot exceed the local celerity of sound; a shock wave materializes the propagation of the wave front in the pipe.

In the presented formalism, ϕ and ψ are the acceleration potentials, two energies per unit mass. In the discrete incompressible formulation, it is the energy ψ associated with the induced effects that must be considered and not the velocity:

$$\int_{\Gamma} \gamma_\psi \cdot \mathbf{t} dl = \int_{\Gamma} \nabla \otimes \psi \cdot \mathbf{t} dl \quad (12)$$

or a local form:

$$\frac{\partial \mathbf{v}_\psi}{\partial t} = \nabla \otimes \left(\psi^o + \frac{1}{2} |\mathbf{v}_\psi|^2 \mathbf{n} - dt c_t^2 \nabla \times \mathbf{v}_\psi - \psi_s \right) \text{ along the path } \Gamma \text{ and in } \Omega \quad (13)$$

This relation (13) applies with a minimum of assumptions; the flow can be unsteady, inertial, no-dissipative and derive from a vector source potential ψ_s . An equivalent statement can be deduced to represent the evolution of the rotation energy per unit mass on closed contour Δ :

$$\psi^o + \frac{1}{2} |\mathbf{v}_\psi|^2 \mathbf{n} - dt c_t^2 \nabla \times \mathbf{v}_\psi - \psi_s = cte \text{ along the closed path } \Delta \quad (14)$$

where the term $dt c_t^2 \nabla \times \mathbf{v}_\psi$, also expressed on the contour Δ , corresponds to the transformation of transverse kinetic energy in potential energy or its inverse. The statement of Bernoulli's theorem in its dual form is therefore the following:

Dual Bernoulli theorem: *If, at a given time, a solenoidal flow of an incompressible fluid is inertial and derived from a vector potential, it will remain so during its evolution in time.*

The equations (10) and (13) have a perfectly symmetric structure. The conditions of application of the equation (10) are simple, it is sufficient that $\nabla \times \mathbf{v} = 0$ for the motion to be irrotational. For equation (13), it is sufficient that $\nabla \cdot \mathbf{v} = 0$ for the flow to be solenoidal. This result is due to the absence of harmonic motion, both curl-free and divergence-free [9].

By returning to the complete form of the law of motion, it is possible to represent the entanglement of the compressional and rotational effects on a Γ trajectory; this form extends, as for the two Bernoulli theorems, to the whole physical domain. Figure 2 schematizes the entanglement of these irrotational and solenoidal terms.

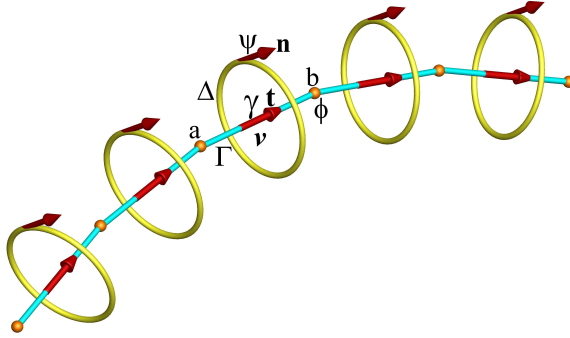


Figure 2. Primal (a) and dual (b) structures defined respectively by contours Γ and Δ mimicking the dynamic entanglement between direct and induced currents; each segment Γ of length $dh = [a, b]$ is oriented by a unit vector \mathbf{t} . The normals to the primal facets are also oriented along \mathbf{n} with $\mathbf{n} \cdot \mathbf{t} = 0$. The scalar potential ϕ is defined on each vertex of this primitive structure and the vector potential $\boldsymbol{\psi}$ is carried by unit vector \mathbf{n} . The acceleration and velocities are expressed on the segment Γ .

This diagram illustrates very well the J.C. Maxwell's inspiration [17] to couple the electrical and magnetic effects in the same system of equations. The link between Bernoulli's primal and dual theorems can be interpreted in the light of Maxwell's remarkable idea of representing direct and induced currents on the same support Γ .

2.4 An equation of motion for incompressible flows of viscous fluid

The Navier-Stokes equation is always accompanied by a conservation of mass constraint, the continuity equation in the general case or $\nabla \cdot \mathbf{V} = 0$ in the incompressible case. If the divergence were zero at any time, it would be impossible to obtain the pressure field; it is therefore the residual divergence $\nabla \cdot \mathbf{V} \approx \varepsilon$ that allows us to find the value of the pressure in order to ensure mechanical equilibrium. Similarly, in discrete mechanics the scalar potential of the acceleration $\phi^o = p/\rho$ is obtained by the accumulation process of the equation (3) from $\nabla \cdot \mathbf{v}$. In both cases, the equilibrium of the equation of motion associated with a zero or smallest possible residue requires the knowledge of $\nabla \cdot \mathbf{v}$.

The proposed formulation consists in noticing that the scalar potential ϕ (or the pressure $p = \rho \phi$) can be replaced by a vector potential of the acceleration $\boldsymbol{\psi}$. Any acceleration can be written as a gradient of a scalar function or as a dual curl of a vector function:

$$-\nabla\phi = \nabla \otimes \boldsymbol{\psi} \quad (15)$$

where ϕ and $\boldsymbol{\psi}$ are two functions generally of different variables. The equation of motion in its

incompressible acceptance is then:

$$\begin{cases} \frac{\partial \mathbf{v}_\psi}{\partial t} = \nabla \otimes \left(\boldsymbol{\psi}^o + \frac{1}{2} |\mathbf{v}_\psi|^2 \mathbf{n} - \nu \nabla \times \mathbf{v}_\psi \right) \\ \boldsymbol{\psi}^o - \nu \nabla \times (\mathbf{v}_\psi - (\mathbf{v}_\psi^o - \mathbf{v}_\psi^o)) \mapsto \boldsymbol{\psi}^o \end{cases} \quad (16)$$

The uniform rotation flow must be filtered beforehand into the energy accumulation so that $\nu \nabla \times (\mathbf{v}_\psi - \mathbf{v}_\psi^o)$ represents only shear and not full rotation. This distinction for Newtonian fluids is required by the complete dissipation of the shear energy between t^o and $t^o + dt$. The partial derivative in time can be evaluated at any order, but assume here that it is represented by a first-order Euler-type scheme; the current velocity \mathbf{v}_ψ is then written as a function of that obtained at the previous instant \mathbf{v}_ψ^o :

$$\mathbf{v}_\psi = \mathbf{v}_\psi^o + dt \nabla \otimes \left(\boldsymbol{\psi}^o + \frac{1}{2} |\mathbf{v}_\psi|^2 \mathbf{n} - \nu \nabla \times \mathbf{v}_\psi \right) \quad (17)$$

Given the identity $\nabla \cdot (\nabla \otimes \boldsymbol{\psi}) = 0$, the velocity \mathbf{v}_ψ remains at zero divergence. However, it is necessary that $\nabla \cdot \mathbf{v}_\psi^o = 0$ is initially at zero divergence, including the boundary conditions attached to it. The simulations performed with the equation (16) show indeed that the divergence remains at the level of the accuracy of the machine throughout the computation. This equation can be treated explicitly, in its form (17) or implicitly by semi-implicating the nonlinear term $|\mathbf{v}_\psi|^2 = \mathbf{v}_\psi \cdot \mathbf{v}_\psi^o$.

Figure 3 represents schematically the process of solving in time the equation (16). From the value of the vector potential $\boldsymbol{\psi}^o$, for example zero at the initial instant, the implicit resolution of the equation provides the velocity \mathbf{v}_ψ ; from the curl of this one the potential $\boldsymbol{\psi}^o$ is explicitly re-actualized and a new cycle allows to progress in time.

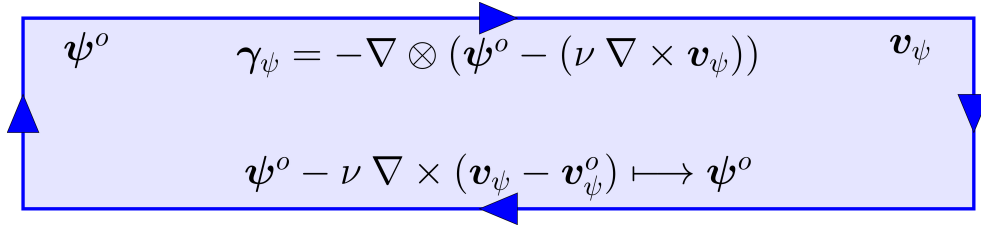


Figure 3. Diagram of the equation solving process (16)

The interest of the presented formulation is obvious, it allows to suppress the projection step on a field of zero divergence of the formulations where the pressure is used as for the Navier-Stokes equation. The resolution of this equation of motion requires the application of a specific method to reduce or even cancel the divergence of the prediction step. This is the case of the family of methods based on a Poisson equation on the pressure [2]. Implicit or semi-implicit methods such as the augmented Lagrange method are also efficient but more expensive in terms of computation time [4]. In any case, the method consisting in imposing a zero or limited divergence represents a very important part of the total time of solution of the Navier-Stokes equation.

Equation (17) is remarkably fast to solve because the field \mathbf{v}_ψ must be searched in an already divergence-free space where the property $\nabla \cdot (\nabla \otimes \boldsymbol{\psi}) = 0$ is satisfied structurally. The equivalent primary operator $\nabla(\nabla \cdot \tilde{\mathbf{v}}) = -\nabla(\nabla \cdot \hat{\mathbf{v}})$ has been used to project a velocity field $\mathbf{v}_\psi = \tilde{\mathbf{v}} + \hat{\mathbf{v}}$ onto a zero divergence field where $\tilde{\mathbf{v}}$ satisfies well $\nabla \cdot \tilde{\mathbf{v}} = -\nabla \cdot \hat{\mathbf{v}}$. This operator developed by several authors from 1999 onwards [18] for the method of vector projection shows that, if the second member is of the same nature as the first, then the solution is obtained by solving the

linear system in a few iterations of a conjugate gradient method, independently of the number of unknowns. This exceptional convergence is explained by the algebraic properties of the linear system and of the spectrum of the discrete operator.

The validation of this approach will require a long validation work in different situations. In order to show the interest of the formulation, only two simple cases are analyzed, the plane Poiseuille flow and a flow in a vertical channel subjected to a temperature difference.

3 Two elementary examples of solenoidal flows

The two proposed examples cover only a small part of the potential of the presented formulation. The corresponding computer simulations have been well performed using the equation (16) and show the expected results, (i) the solution of the Poiseuille problem is obtained exactly (to machine precision) since the method is second order in space and (ii) the solution of the heated channel is second order since the solution corresponds to a polynomial of degree three. Given the theoretical nature of the background, only explanations of the physical model are developed.

3.1 Planar Poiseuille flow

The Poiseuille stationary flow of a viscous fluid in a planar channel is emblematic of those in fluid mechanics for incompressible flows. Let us consider a channel of height $h = \pm 1$ in which a fluid of kinematic viscosity ν flows under the effect of a constraint, a pressure gradient or an imposed velocity, here equal to $V_0 = 1$, the mean velocity in the channel.

In continuum mechanics, the solution is obtained from the Navier-Stokes equation where the inertia term takes several forms, including $\mathbf{V} \cdot \nabla \mathbf{V} = \nabla(\mathbf{V}^2/2) - \mathbf{V} \times \nabla \times \mathbf{V}$:

$$\rho \nabla \left(\frac{1}{2} \mathbf{V}^2 \right) - \rho \mathbf{V} \times \nabla \times \mathbf{V} = -\nabla p + \nabla \times (\mu \nabla \times \mathbf{V}) \quad (18)$$

The solution is represented by the velocity, $\mathbf{V} \cdot \mathbf{e}_x = u(y) = 3/2 (1 - y^2)$ and by pressure $p(x) = -3\mu x$ to a constant or by $p_B(x, y) = p(x) + 1/2 \rho \mathbf{V}^2 = -3\mu x + 9/8 \rho (1 - y^2)^2$, Bernoulli pressure. The projection onto the (x, y) components reads as follows:

$$\frac{9}{2} \rho (1 - y^2) y \mathbf{e}_y - \frac{9}{2} \rho (1 - y^2) y \mathbf{e}_y = -\frac{\partial p(x)}{\partial x} - 3\mu \mathbf{e}_x \quad (19)$$

Thus, the pressure gradient balances the viscous terms and the inertia terms cancel, the Poiseuille flow is non-inertial but only by compensation. Note that in this case the Lamb vector of the form $-\mathbf{V} \times \nabla \times \mathbf{V}$ is a gradient and not a curl. In the general case, its form is indeterminate.

In the discrete formalism the equation of motion reads:

$$\nabla \left(\frac{|\mathbf{v}|^2}{2} \right) - \nabla \otimes \left(\frac{|\mathbf{v}|^2}{2} \mathbf{n} \right) = \nabla \otimes \psi^o - \nabla \otimes (\nu \nabla \times \mathbf{v}) \quad (20)$$

The solution corresponds to the equilibrium of the Poiseuille flow:

$$\frac{9}{2} (1 - y^2) y \mathbf{t} - \frac{9}{2} (1 - y^2) y \mathbf{t} = -\nabla \otimes \psi^o(y) - 3\nu \mathbf{t} \quad (21)$$

or a vector potential equal to $\psi^o(y) = 3\nu y \mathbf{n}$ or $\psi_B^o(y) = 3\nu y \mathbf{n} + 9/8 (1 - y^2)^2 \mathbf{n}$ the Bernoulli vector potential.

These solutions of the continuum mechanics and the discrete formulation lead of course to the same result with scalar and vector potentials of different functions:

$$-\nabla \phi^o(x) = \nabla \otimes \psi^o(y) = 3\nu \mathbf{e}_x \quad (22)$$

The functions $-\nabla\phi^o(x)$ and $\nabla\otimes\psi^o(y)$ of two different variables are both equal to a constant. This is a special case because, in general, these two functions are arbitrary but the equality $-\nabla\phi^o = \nabla\otimes\psi^o$ can always be used to select the appropriate potential.

3.2 Vertical channel differentially heated

Consider a vertical planar channel whose no-slip boundaries define a fluid cavity where $x \in [-h, h]$ and the direction y is undefined; the viscous, expandable fluid is heated differentially with a temperature difference ΔT between the vertical planes. The gravitational force acting on each portion of the fluid is due to the dilatibility of the fluid with temperature. Using the Boussinesq approximation, this force per unit mass is written:

$$\frac{\rho}{\rho_0} \mathbf{g} = - \left(\frac{g \delta}{2} \frac{x}{h} \right) \mathbf{e}_y = -\nabla\phi_h + \nabla\otimes\psi_h \quad (23)$$

where ϕ_h and ψ_h are the Archimedean potentials and $\delta = \beta\Delta T$ where β is the thermal expansion coefficient.

The Navier-Stokes equation or the presented model lead of course to the same solution,

$$\mathbf{v}(x) = \frac{g \delta h^2}{12 \nu} \left(\frac{x}{h} - \frac{x^3}{h^3} \right) \mathbf{e}_y, \quad (24)$$

where, for a low enough Rayleigh number, the inertia has been neglected.

The scalar and vector potentials, defined with respect to a constant, can be written in terms of the coordinates (x, y) as:

$$\begin{cases} \phi_h(x, y) = \frac{g \delta}{2} \frac{x y}{h} \\ \psi_h(x) = -\frac{g \delta}{4} \frac{x^2}{h} \mathbf{e}_z \end{cases} \quad (25)$$

In the case where the scalar potential is chosen to balance the equation of motion, it is expressed as a function of both spatial coordinates. When the formulation is based on the vector potential, it depends on only one coordinate. This difference reflects the consistency of the formulation $(\mathbf{v}, \boldsymbol{\psi})$, as the physical problem posed does not depend on the vertical coordinate at all. It is to be noted that the equation (16) provides, at the same time, \mathbf{v} and $\boldsymbol{\psi}$ with $\boldsymbol{\psi} = \nu\nabla\times\mathbf{v}$.

These two simple examples do not validate the physical model presented, but they allow us to understand why it is more appropriate to use a velocity-vector potential formulation of the acceleration $(\mathbf{v}, \boldsymbol{\psi})$ for an incompressible flow.

4 Discussion

The consequences of separating acceleration and velocity into solenoidal and irrotational components cannot be drawn immediately. For example, the impact on the boundary conditions would deserve further investigation. However, the superposition of direct and induced currents on the same medium suggested by Maxwell's theory leads, for viscous fluids, to a dual Bernoulli theorem where the sum of the divergence-free terms is equal to a constant along a closed path. The equation of motion specific to solenoidal flows benefits from the form of this theorem where its solution remains divergence-free if it is divergence-free at a given time. From a practical point of view, the solution of this equation does not require to project the current solution on a field of zero divergence if the solution at the instant preceding the instant t^o is already of zero divergence. The difficulty of transforming a gradient field into an equivalent curl field is real in fluid mechanics

equations and in particular for the Navier-Stokes equation. The proposed formulation suppresses the projection step in the case where only the solenoidal solution is sought.

A real flow can be both to non-zero divergence and non-zero curl, but it is not necessarily turbulent; it can also be inertial and non-turbulent. One of the questions that can legitimately be asked is: can a strictly incompressible flow be turbulent? The answer is not found among the many simulations of turbulence performed from the Navier-Stokes equation because, when solving it, at certain stages, the flow is not at divergence-free. In general, if the pressure can be calculated, it is because the divergence was not zero at a given time. The constraint $\nabla \cdot \mathbf{V} = 0$ cannot be imposed *a priori* in the Navier-Stokes equation. The presented formulation is strictly solenoidal and the problem remains open.

Under what conditions does a compressible, viscous, inertial flow become turbulent? The turbulence statistics of computationally simulated flows will not tell us either. The respective roles of rotational and compressive effects are not yet known with precision. In particular, the effects of acoustic propagation of longitudinal waves are largely underestimated at present. The separation of divergence-free and curl-free terms in the velocity field is a new way to study physical turbulence. Solving the equation of motion in two components \mathbf{v}_ϕ and \mathbf{v}_ψ will probably allow a finer analysis of the turbulent character of a flow. The inertial terms in \mathbf{v}_ϕ^2 and \mathbf{v}_ψ^2 correspond respectively to the pressure and the shear energy which do not interact with each other and can simply be integrated in the potentials to form the Bernoulli potentials. On the other hand, the inertial terms $(\mathbf{v}_\phi \cdot \mathbf{v}_\psi)$ alone allow the mutual transfer of compressional and rotational energies over time. It seems possible that the product $(\mathbf{v}_\phi \cdot \mathbf{v}_\psi)$ represents the cornerstone of understanding the physics of turbulent flows.

Declaration of competing interest

There are no conflict of interest in this work.

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