Calculation details

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1 Magnetic models

We use the Kenelly convention. The reference material law is:

$$\boldsymbol{h} = \nu_0(\boldsymbol{b} - \boldsymbol{m}_b(\boldsymbol{b})), \tag{1}$$

with $\boldsymbol{h}$ the magnetic field (A/m), ν_0 the reluctivity of void (m/H), $\boldsymbol{b}$ the flux density (T) and $\boldsymbol{m}_b$ the magnetic polarization, also called J in the litterature (T).

Let's assume we know the anhysteretic BH curve of a material *, i.e.,* a scalar function $h(b)$ that gives the norm of the magnetic field $h = |\boldsymbol{h}|$ depending on the norm of the flux density $b = |\boldsymbol{b}|$. We can define some magnetic models using this scalar function only.

1.a Non-polarized soft magnetic materials

In this type of material, $\boldsymbol{b}$ and $\boldsymbol{h}$ are colinear, so $\boldsymbol{m}_b$ can be expressed simply as :

$$\boldsymbol{m}_{\mathrm{FeSi}}(\boldsymbol{b}) = m_{\mathrm{FeSi}}(|\boldsymbol{b}|).\,\boldsymbol{u}_b(\boldsymbol{b}), \tag{2}$$

where $m_{\mathrm{FeSi}}(b) = b - \mu_0 h(b)$ is a scalar function that can be extracted directly from the BH curve, and $\boldsymbol{u}_b = \frac{\boldsymbol{b}}{|\boldsymbol{b}|}$ is the unit vector in the direction of $\boldsymbol{b}$. In practice, we can use $\max(|b|, \varepsilon)$ with ε a small positive number (such as 5×10^{-324} that is the smallest strictly positive number in Matlab) instead of $|b|$ to prevent division by 0.

1.b Polarized magnetic materials

We assume that for a polarized material :

$$\boldsymbol{m}_{\mathrm{pm}}(\boldsymbol{b}) = m_{\mathrm{pm}}(\boldsymbol{b}.\,\boldsymbol{u}_{\mathrm{pm}}).\,\boldsymbol{u}_{\mathrm{pm}}, \tag{3}$$

where $m_{\mathrm{pm}}(b) = b - \mu_0 h(b)$ is the same scalar function based on the BH curve of the magnet along its d-axis, given by $\boldsymbol{u}_{\mathrm{pm}}$ the fixed unit vector along the orientation of the magnet. Note that we consider only the influence of $\boldsymbol{b}$ in the direction of $\boldsymbol{u}_{\mathrm{pm}}$, and we fix the magnetization direction, which are strong assumptions. It also means that the magnet behaves like air in the transverse direction. Other more elaborated models, such as vectorized Jiles-Atherton, are possible.

2 Derivatives

We want to calculate $\frac{\mathrm{d}\boldsymbol{m}}{\mathrm{d}\boldsymbol{b}} = \begin{pmatrix} \frac{\mathrm{d}m_x}{\mathrm{d}b_x} & \frac{\mathrm{d}m_x}{\mathrm{d}b_y} \\ \frac{\mathrm{d}m_y}{\mathrm{d}b_x} & \frac{\mathrm{d}m_y}{\mathrm{d}b_y} \end{pmatrix}$ for both models. We assume that the scalar functions m_{FeSi} and m_{pm} are known as well as their derivatives m'_{FeSi} and m'_{pm}.

2.a Non-polarized model

Written in full form, Equation 2 reads:

$$\boldsymbol{m}_{\mathrm{FeSi}}(\boldsymbol{b}) = \begin{pmatrix} m_{\mathrm{FeSi}}(N_2(\boldsymbol{b}))\frac{b_x}{N_2(\boldsymbol{b})} \\ m_{\mathrm{FeSi}}(N_2(\boldsymbol{b}))\frac{b_y}{N_2(\boldsymbol{b})} \end{pmatrix},$$

with $N_2(\boldsymbol{b}) = \sqrt{b_x^2 + b_y^2}$ the modulus of $\boldsymbol{b}$. The derivative reads:

$$\frac{\mathrm{d}\boldsymbol{m}_{\mathrm{FeSi}}}{\mathrm{d}\boldsymbol{b}} = \begin{pmatrix} \frac{\mathrm{d}\boldsymbol{m}_{\mathrm{FeSi}}}{\mathrm{d}b_x} & \frac{\mathrm{d}\boldsymbol{m}_{\mathrm{FeSi}}}{\mathrm{d}b_y} \end{pmatrix}$$

The derivative with respect to b_x is:

$$\frac{\mathrm{d}\boldsymbol{m}_{\mathrm{FeSi}}}{\mathrm{d}b_x} = {m'}_{\mathrm{FeSi}}(N_2(\boldsymbol{b}))\frac{\mathrm{d}N_2}{\mathrm{d}b_x}.\begin{pmatrix} \frac{b_x}{N_2(\boldsymbol{b})} \\ \frac{b_y}{N_2(\boldsymbol{b})} \end{pmatrix} + m_{\mathrm{FeSi}}(N_2(\boldsymbol{b})).\begin{pmatrix} \frac{N_2(\boldsymbol{b}) - b_x\frac{\mathrm{d}N_2}{\mathrm{d}b_x}}{N_2^2(\boldsymbol{b})} \\ -\frac{b_y}{N_2^2(\boldsymbol{b})}\frac{\mathrm{d}N_2}{\mathrm{d}b_x} \end{pmatrix}.$$

Because $\frac{\mathrm{d}N_2}{\mathrm{d}b_x} = \frac{b_x}{N_2(\boldsymbol{b})}$ and $\frac{\mathrm{d}N_2}{\mathrm{d}b_y} = \frac{b_y}{N_2(\boldsymbol{b})}$ we obtain :

$$\frac{\mathrm{d}\boldsymbol{m}_{\mathrm{FeSi}}}{\mathrm{d}b_x} = {m'}_{\mathrm{FeSi}}(|\boldsymbol{b}|)\frac{b_x}{|\boldsymbol{b}|^2}.\boldsymbol{b} + \frac{m_{\mathrm{FeSi}}(|\boldsymbol{b}|)}{|\boldsymbol{b}|^3}.\begin{pmatrix} \overbrace{N_2(\boldsymbol{b})^2 - b_x^2}^{b_y^2} \\ -b_y b_x \end{pmatrix}.$$

We do the same calculations for d_{b_y} :

$$\frac{\mathrm{d}\boldsymbol{m}_{\mathrm{FeSi}}}{\mathrm{d}b_y} = {m'}_{\mathrm{FeSi}}(|\boldsymbol{b}|)\frac{b_y}{|\boldsymbol{b}|^2}.\boldsymbol{b} + \frac{m_{\mathrm{FeSi}}(|\boldsymbol{b}|)}{|\boldsymbol{b}|^3}.\begin{pmatrix} -b_x b_y \\ b_x^2 \end{pmatrix}.$$

So all together as a matrix, we have :

$$\frac{\mathrm{d}\boldsymbol{m}_{\mathrm{FeSi}}}{\mathrm{d}\boldsymbol{b}} = \frac{{m'}_{\mathrm{FeSi}}(|\boldsymbol{b}|)}{|\boldsymbol{b}|^2}\underbrace{\begin{pmatrix} b_x^2 & b_y b_x \\ b_x b_y & b_y^2 \end{pmatrix}}_{\boldsymbol{b}.\boldsymbol{b}^T = \boldsymbol{b}\otimes\boldsymbol{b}} + \frac{m_{\mathrm{FeSi}}(|\boldsymbol{b}|)}{|\boldsymbol{b}|^3}\underbrace{\begin{pmatrix} b_y^2 & -b_x b_y \\ -b_x b_y & b_x^2 \end{pmatrix}}_{(R(\frac{\pi}{2}).\boldsymbol{b})\otimes(R(\frac{\pi}{2}).\boldsymbol{b})}, \tag{4}$$

with $R(\frac{\pi}{2}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ the rotation matrix by a $\frac{\pi}{2}$ angle; which is the formula (28) given the paper.

2.b Polarized model

Equation 3 reads in full form :

$$\boldsymbol{m}_{\mathrm{pm}}(\boldsymbol{b}) = m_{\mathrm{pm}}\left(b_x u_{\mathrm{pm}_x} + b_y u_{\mathrm{pm}_y}\right).\begin{pmatrix} u_{\mathrm{pm}_x} \\ u_{\mathrm{pm}_y} \end{pmatrix}, \tag{5}$$

so :

$$\frac{\mathrm{d}\boldsymbol{m}_{\mathrm{pm}}}{\mathrm{d}b_x} = u_{\mathrm{pm}_x}{m'}_{\mathrm{pm}}\left(b_x u_{\mathrm{pm}_x} + b_y u_{\mathrm{pm}_y}\right)\begin{pmatrix} u_{\mathrm{pm}_x} \\ u_{\mathrm{pm}_y} \end{pmatrix},$$

$$\frac{\mathrm{d}\boldsymbol{m}_{\mathrm{pm}}}{\mathrm{d}b_y} = u_{\mathrm{pm}_y}{m'}_{\mathrm{pm}}\left(b_x u_{\mathrm{pm}_x} + b_y u_{\mathrm{pm}_y}\right)\begin{pmatrix} u_{\mathrm{pm}_x} \\ u_{\mathrm{pm}_y} \end{pmatrix},$$

which combined as a matrix, gives:

$$\frac{\mathrm{d}\boldsymbol{m}_{\mathrm{pm}}}{\mathrm{d}\boldsymbol{b}} = {m'}_{\mathrm{pm}}(\boldsymbol{b}.\boldsymbol{u}_{\mathrm{pm}}).\underbrace{\begin{pmatrix} u_{\mathrm{pm}_x}^2 & u_{\mathrm{pm}_x}u_{\mathrm{pm}_y} \\ u_{\mathrm{pm}_x}u_{\mathrm{pm}_y} & u_{\mathrm{pm}_y}^2 \end{pmatrix}}_{\boldsymbol{u}_{\mathrm{pm}}\otimes\boldsymbol{u}_{\mathrm{pm}}}. \tag{6}$$

which is the formula (29) in the paper. We note that Equation 4 and Equation 6 are symmetric.