

# Tensor and coupled decompositions in block terms: uniqueness and irreducibility

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## Abstract

Decomposition of a single tensor in a sum of rank-1 terms is a basic and prevalent tool in data analysis. However, in various scenarios, a decomposition in a sum of terms of rank other than one is more faithful to the underlying structure of the data. For this aim, different types of tensor block term decomposition (BTD) have been proposed. When applicable, these models provide advantages such as higher precision and computational efficiency over their rank-1 counterparts.

A fundamental concept in decompositions in a sum of terms of rank other than one is (ir)reducibility. In this context, irreducibility means that each term in the sum cannot be further factorized into several distinct terms of smaller rank. Clearly, irreducibility is irrelevant to decompositions in sum of rank-1 terms.

Earlier work on the uniqueness of BTD has focused on properties of the factor matrices. In order to avoid reducibility, it was assumed that the block-diagonal terms, also known as core tensors, were generic. Given this assumption, it was proven in earlier work that BTD were generically unique, under mild conditions on the factor matrices.

In this talk, we enrich our understanding of block decompositions by presenting new uniqueness results that explicitly take into account non-generic core tensors and (ir)reducibility. Furthermore, we explain how our results provide new insights on CPD and BTD, as well as more general coupled decompositions. For this aim, we formulate an algebraic framework for handling irreducibility. Using this approach, we first prove that BTD (specifically, a type-2 decomposition of a tensor in a sum of rank- $(L_r, M_r, \cdot)$  terms) may be non-unique when the core tensors are irreducible yet not generic. We show that if the diagonal blocks have different size, they cannot cause non-uniqueness; this raises the question whether the set of block dimensions in a BTD can be regarded as a type of diversity.

In the second part of this talk, we demonstrate how the more general framework of coupled decompositions sheds new light on irreducibility and on BTD. For this aim, we consider a new type of coupled block decomposition that cannot be represented (in a meaningful way) as a single tensor factorization. First, we show how to extend our analytical framework to characterize its uniqueness, given irreducibility. Then, we present several ways in which this framework subsumes the rank- $(L_r, M_r, \cdot)$  BTD and its uniqueness results. Finally, we show that different block dimensions are not necessarily associated with generic uniqueness.

### Contribution highlights:

- We extend the concept of k-rank (Kruskal rank) from relations among columns of a matrix to relations among core tensors in the rank- $(L_r, M_r, \cdot)$  BTD.
- We present new necessary conditions on the uniqueness of the rank- $(L_r, M_r, \cdot)$  BTD that extend the condition of “k-rank must be larger than two” to core tensors.
- We present a new conjecture for a sufficient condition on the uniqueness on the rank- $(L_r, M_r, \cdot)$  BTD by extending the well-known Kruskal condition (inequality) to include also our newly-proposed “core rank”:

**Conjecture (Lahat and Jutten, 2018):**

The rank- $(L_r, M_r, \cdot)$  BTD of a tensor  $\mathcal{T}$  is unique if each rank- $(L_r, M_r, \cdot)$  term is irreducible, and

$$k'_{\mathbf{A}} + k'_{\mathbf{B}} + k_{\mathcal{C}}^{\text{core}} \geq 2R + 2.$$

See presentation slides for details.

- Our conjecture allows more relaxed conditions on the rank- $(L_r, M_r, \cdot)$  BTD than proposed until now, especially for factor matrices that do not have full column rank.
- Our conjecture explains earlier results by other authors, who observed uniqueness of the rank- $(L_r, M_r, \cdot)$  BTD for factor matrices that do not have full column rank.
- We conjecture that these results can be extended to the uniqueness of the coupled rank- $(L_r, M_r, \cdot)$  BTD associated with joint independent subspace analysis (JISA) [1].

### References

- [1] D. Lahat and C. Jutten, “Joint independent subspace analysis using second-order statistics,” *IEEE Trans. Signal Process.*, vol. 64, no. 18, pp. 4891–4904, Sep. 2016.