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Improved convergence rate for Reflected BSDEs by penalization method

Emmanuel GOBET* Wanqing WANG[†] Mingyu XU[‡]

March 8, 2023

Abstract

We investigate the convergence of numerical solution of Reflected Backward Stochastic Differential Equations (RBSDEs) using the penalization approach in a general non-Markovian framework. We prove the convergence between the continuous penalized solution and the reflected one, in full generality, at order $1/2$ as a function of the penalty parameter; the convergence order becomes 1 when the increasing process of the RBSDE has a bounded density, which is a mild condition in practice. The convergence is analyzed in *a.s.*-sense and \mathbb{L}^p -sense ($p \geq 2$). To achieve these new results, we have developed a refined analysis of the behavior of the process close to the barrier. Then we propose an implicit scheme for computing the discrete solution of the penalized equation and we derive that the global convergence order is $\frac{3}{8}$ as a function of time discretization under mild regularity assumptions. This convergence rate is verified in the case of American Put options and some numerical tests illustrate these results.

KEYWORDS: reflected BSDE, penalization, error bound, convergence rate

MSC2020: 60H10; 60H35; 60G40.

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1 Introduction

Context and model. The Backward Stochastic Differential Equations, BSDEs in short, were first introduced in the case of linear generator by Bismut in [Bis73] as the adjoint equations of some stochastic control problem and later in the general case by Pardoux and Peng [PP92]. It was then widely studied in stochastic finance (see [EPM97]) in view of its natural formulation to model the problem of option pricing and hedging. See [MY99], [PR14], [Zha17] for a broad overview on BSDE and applications in stochastic control. The Reflected BSDE is one of the most important branch of BSDE, which was first introduced by El Karoui et al. [EKP⁺97] as follows: given a terminal condition ξ , a generator f and a barrier $(S_t)_{0 \leq t \leq T}$, the RBSDE is written as

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \mathbf{d}s + K_T - K_t - \int_t^T Z_s \mathbf{d}B_s, & \forall 0 \leq t \leq T, \\ Y_t \geq S_t, & 0 \leq t \leq T, \\ \int_0^T (Y_t - S_t) \mathbf{d}K_t = 0 \end{cases} \quad (1.1)$$

where the process $(K_t)_{0 \leq t \leq T}$ is continuous and non-decreasing, $K_0 = 0$. Here, Y, K are scalar processes and Z is a d -dimensional process (written as a row vector). The triple of processes $\{(Y_t, Z_t, K_t), 0 \leq t \leq T\}$ is solution of RBSDE with respect to the data (ξ, f, S) where the relative assumptions will be clarified later.

The RBSDE topic was quickly developed since it provides a new framework to study stochastic control problems and optimal stopping problems involving one obstacle (the barrier). After the pioneer work [EKP⁺97], the Dynkin game and its related doubly RBSDE is studied in [JK96]; the switching problem is discussed in [HZ10]; the starting and stopping problem is addressed in [HJ07]. But among all of its various applications, the pricing of American option – which writes as an optimal stopping problem in the probabilistic language or as a variational inequality in the PDE terminology – is considered as one of the most fundamental applications. Similarly to the link between BSDE solutions and European option prices, there also exists a relation between RBSDE and American option: when the market is complete without market imperfection, the relation is standard using a linear generator. But this relation can also be extended in several cases with modified equation or even reformulated problem, accounting for imperfections, or for market incompleteness or for other financial specificities. For example, when the market is complete, different settings have been studied for the pricing problem: using a convex risk measure in [ES13]; with asymmetric information in [EI18]; in the presence of 2 interest rates (for borrowing and lending) and default in [DQS18]. Moreover, when the market is incomplete, the American option pricing turns out to be a problem of superhedging (see [EPQ97]) leading to the design of a special RBSDE. The equation, with superlinear and quadratic f is presented in [KLQT02], with an extra constraint in [SX10], or with imperfect market and default in [GQS21].

Besides, regarding the mathematical analysis of existence and uniqueness of solutions, there are many theoretical results for RBSDE under either standard or general assumptions. Here we refer to the hypotheses in [EKP⁺97] as the standard ones, i.e. with Lipschitz generator f , continuous barrier S and under \mathbb{L}^p ($p \geq 2$) integrability properties. First, under standard assumptions, the existence and uniqueness are usually deduced either by the Picard's fixed point argument or by the penalization method which builds a monotone convergent sequence of approximations. The Penalized BSDE (PBSDE in short) related to (1.1) writes as, for the penalty parameter $\lambda > 0$,

$$\begin{cases} Y_t^\lambda &= \xi + \int_t^T f(s, Y_s^\lambda, Z_s^\lambda) \mathbf{d}s + \lambda \int_t^T (Y_s^\lambda - S_s)^- \mathbf{d}s - \int_t^T Z_s^\lambda \mathbf{d}B_s, \\ K_t^\lambda &= \lambda \int_0^t (Y_s^\lambda - S_s)^- \mathbf{d}s. \end{cases} \quad (1.2)$$

The variational inequality is also studied in [BCFE02]. Second, under more general assumptions, the well-posedness is widely studied, using the similar approaches based on fixed point argument or penalization. The reader can find the well-posedness of RBSDE via penalization, with càdlàg S in [LX05], with driving Lévy process in [EO09], with \mathbb{L}^p ($p \in [1, 2)$) integrability in [HP12]. The well-posedness via fixed point argument of RBSDE, with superlinear and quadratic f is identified in [KLQT02], the case with monotone f is addressed in [LMX05], the model with resistance is given in [QX18]. As we can see, the penalization method is usually applied as a tool for solving some existence problem, and seldom for effective numerical approximations. In this paper, we focus on the numerical solution of the one barrier's RBSDEs under standard assumptions, via the penalization method, and our goal is to establish some convergence rate results.

State of the art on numerical methods. Before we introduce our contributions to the penalization approach, let us have a look at the existing results in the literature. Due to the important and various applications of RBSDE, particularly in optimal stopping, it is essential to investigate effective numerical schemes. However, in comparison with the numerous contributions to the theoretical aspects of RBSDE, there are only a few available results about the numerical methods. Indeed, because of the barrier and as a big difference with BSDE, the RBSDE can hardly admit an explicit representation even in the case with linear generator. As a consequence, designing an efficient numerical solution turns out to be quite challenging.

Among all the available approaches, presumably the most usual one is the Snell envelop, consisting in taking the maximum between the barrier and a conditional expectation, the latter being the main difficulty regarding computations. In the Markovian framework, Bally and Pagès [BP03, Theorem 4] proposed a quantization tree algorithm for RBSDE with f not depending on z and proved the discrete approximation \hat{Y} converges to the continuous solution Y with a \mathbb{L}^p error of order $h^{1/2} + h^{-(1+1/d)}N^{-1/d}$ where N denotes the size of quantization grid and h denotes the time step; Bouchard and Touzi [BT04, Theorem 7.2] proposed a kernel regression algorithm and deduced a rate in \mathbb{L}^p of order $h^{1/2} + h^{-(1+d/(4p))}M^{-1/(2p)}$ for RBSDE with f not depending on Z , where M denotes the number of Monte-Carlo simulations. Ma and Zhang [MZ05, Theorem 4.2 and Corollary 3.4] provided a semi-explicit representation for Z , using Malliavin calculus, and derived that, if the barrier $S := \phi(X)$ with $\phi \in C^2$ and the forward process X is an uniformly elliptic diffusion, their pseudo time-discretization (Y^h, Z^h, K^h) converges in \mathbb{L}^2 to the continuous solution (Y, Z, K) is at rate $h^{1/4}$. Bouchard and Chassagneux [BC08, Theorem 4.1] extended the previous result without uniform ellipticity condition, and improved the rate to $h^{1/2}$ (for Y only) if $\phi \in C^2$ with a Lipschitz second derivative. Gobet and Lemor [GL08, Theorem 4] investigated empirical regression algorithms for solving discretized RBSDEs.

The regularization approach is an alternative to Snell Envelop techniques. Bally et al. [BCFE02] designed a new way to simulate the process K , more precisely, K admits a semi-explicit density of the form $\alpha_t \kappa_t$ where α is an unknown process valued in $[0, 1]$ and κ is a known function of (t, S_t) . An empirical regression method based on this principle is investigated in [GL08], and gives good numerical results, but without theoretical guarantee.

Last, in another series of works, numerical methods based on penalization like (1.2) have been developed. Mémin et.al [MPX08] developed a discrete random walk solution with penalization, they showed $(Y^\lambda, Z^\lambda) \rightarrow (Y, Z)$ and $(Y^{\lambda,h}, Z^{\lambda,h}) \rightarrow (Y^h, Z^h)$ at $\lambda^{-1/4}$ in \mathbb{L}^2 , the convergence rate with respect to the time step h is lacking. Martinez et al. [MSMT11, Theorem 1 & 2] proposed a numerical random walk scheme combining Picard iteration and penalization and they proved the convergence $(\hat{Y}^{\lambda,h,i}, \hat{Z}^{\lambda,h,i}) \rightarrow (Y, Z)$ in \mathbb{L}^2 without identifying convergence order, where i denotes i -th Picard iteration. Nevertheless, the complexity of random walk schemes is known to increase quite fast with the Brownian motion dimension and with the number of time steps, and in many situations, we may prefer to avoid random walk approximations. As an alternative, it is reasonable to take into consideration the Monte-Carlo method (a.k.a. empirical regression methods), which is one of the ingredients of this work. For BSDE without reflection, see [LGW06, GT16, GLTV16], for an account on empir-

ical regression methods, including error analyses. Last, Bernhart et al. [BPTW12, Theorem 3.1] presented a discrete scheme for BSDE with constrained jump and proved the quadratic convergence w.r.t. λ and h , but the global rate is unfortunately quite slow (logarithmic rate). Last, for the sake of having a complete state of the art, let us briefly discuss the use of penalization approach to derive schemes for other equations. Let us start with Reflected Forward SDEs (RSDEs in short). For RSDE in a convex set, Menaldi [Men83, Remark 3.1] deduced that the penalized continuous solution converges in \mathbb{L}^2 to the reflected one at rate $\lambda^{-1/2+\varepsilon}$ for any $\varepsilon > 0$. Liu [Liu93] showed the penalized discrete solution converges to the continuous reflected one at $h^{1/4-\varepsilon}$ if $\lambda \sim O(h^{-1/2})$. Petterson [Pet97, Theorem 3.1] derived the rate $(h \log \frac{1}{h})^{1/4}$ in \mathbb{L}^2 provided that $\lambda \sim O(1/h)$ and Slominski [Sł01, Theorem 4.2] proved the same rate but uniformly in time. These results show that the convergence order is more or less 1/4 w.r.t. the time step h for appropriate choice of penalization parameter λ , this shows also the necessary entanglement between h and λ ; however, though inspiring, establishing error bounds for backward equation is usually more difficult than for forward one, due to the adaptedness condition. In this paper, we will show that, if the barrier is a generalized Itô process under mild conditions, our backward discrete penalized solution converges in \mathbb{L}^2 to the reflected continuous one at rate $h^{3/8}$ if $\lambda \sim ch^{-3/8}$: this is strictly better than for forward equation.

Regarding PDEs with variational inequality (related to optimal stopping problem), the penalty approach is also commonly used when it comes to design numerical schemes. This is restricted to Markovian cases, as a difference with our general setting. The discretization of PDE is both in space (with a mesh size Δx) and in time (with a time step h), but as a usual difference with stochastic equations, the spatial mesh is deterministic. Zvan et.al. [ZFV98, Section 5.3] designed a θ -scheme using penalty method for identifying the early exercise region in a stochastic volatility model for the forward component and showed at each time step, the penalization term where the approximation is smaller than the barrier is at $O(\frac{1}{\lambda})$, *a.e.* Forsyth and Vetzal [FV02, Theorem 4.1] discussed the quadratic convergence of θ -schemes for pricing Put option as $h, \Delta x \rightarrow 0$ for a fixed λ and proved a similar result to [ZFV98] but with the discretization constraint $h = o(\Delta x)$. They also showed numerically that the uniform discretization can not ensure the quadratic convergence near the exercise boundary, hence advocating for a time step selector. d'Halluin et.al. [dFL04, Theorem 4.2] designed an iterative scheme for pricing American derivatives in jump models and proved the global convergence of the iterative procedure as the number of iteration goes to infinity at each time step for a fixed λ and time discretization. Howison et.al [HRW13, Theorem 4.3 & Section 6] showed in the convex Markovian barrier, the penalized PDE converges to the exact solution at $1/\lambda$ and presented a detailed error analysis about the convergence order of the penalized approximation to the penalized PDE solution, which is at $\sqrt{\lambda}(h + (\Delta x)^2)$ in $\mathbb{L}^2([0, T] \times \mathbb{R})$.

Our contributions. In this paper, since the PBSDE provides a natural continuous approach to the RBSDE, we first concentrate on the rate of $Y^\lambda \rightarrow Y$ and we show that this convergence can hold at $1/\lambda$ a.s. or $1/\sqrt{\lambda}$ in $\mathbb{L}^p(\Omega)$ depending on the behavior of (ξ, S) , see Theorem 3.5 and Theorem 3.2 for a precise statement. As far as we know (see the previ-

ous state-of-the-art), this is the first time that the rate of continuous PBSDE converging to RBSDE is investigated. Second, when f does not depend on Z , we design an implicit discrete representation for $Y^{\lambda,h}$ and show theoretically that the discretization error, $Y^{\lambda,h} - Y^\lambda$, converges to 0 at order $h^{1/2} + \lambda h^{3/4}$ in \mathbb{L}^2 , see Theorem 4.1. Finally, we conclude that the global discretization error between $Y^{\lambda,h}$ and Y is at $h^{3/8}$ if $\lambda \sim ch^{-3/8}$, which is a significant improvement compared to [BPTW12]. The order $h^{3/8}$ is faster than $h^{1/4}$ in [BC08] for another RBSDE scheme obtained when $S = \phi(X)$ with $\phi \in C^2$, which is stronger than our mild Hölder-continuity condition (\mathbf{H}_2 -ii); but our rate is lower than $h^{1/2}$ in [BC08] obtained under the even stronger assumption Φ is C^3 . The numerical experiment based on Monte-Carlo empirical regression is attached in the end and we show numerically the simulation of $Y^{\lambda,h}$ may converge quicker to Y than the rate upper bound, at least in the example of pricing one-dimensional Put option in the linear market.

Plan of the paper. In Section 2, we define the model, state standing assumptions and exemplify some applications. In Section 3, a priori estimates in $\mathbb{L}^p(\Omega), p \geq 2$ and the rate of $(Y_t^\lambda - S_t)^- \rightarrow 0$ are given. The discretization error of $Y^{\lambda,h} - Y$ in $\mathbb{L}^2(\Omega)$ is well investigated in Section 4. The numerical results are presented at the end of Section 4.

2 Model, assumptions, examples

2.1 Notation

- For a vector $x \in \mathbb{R}^d$, we simply denote its transpose by x^\top and its i -th component by x^i . Its Euclidean norm is denoted by $|x|$.
- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space where a d -dimensional Brownian motion $(B_t)_{0 \leq t \leq T}$ is defined and $(\mathcal{F}_t)_{0 \leq t \leq T}$ is the canonical filtration of $(B_t)_{0 \leq t \leq T}$ where \mathcal{F}_0 contains all \mathbb{P} -null sets of \mathcal{F} . Denote the expectation under \mathbb{P} by \mathbb{E} and the conditional expectation given \mathcal{F}_t by \mathbb{E}_t .
- The following set of random variables and stochastic processes are useful for our subsequent study. Let $p \geq 2$:

$$\mathbb{L}^\infty = \left\{ \text{random variables } \zeta \text{ s.t. } \operatorname{ess\,sup}_{\omega \in \Omega} |\zeta(\omega)| < \infty \right\},$$

$$\mathbb{L}^p = \left\{ \text{random variables } \zeta \text{ s.t. } \mathbb{E}[|\zeta|^p] < \infty \right\},$$

$$\mathcal{S}^p = \left\{ \text{stochastic processes } \phi = \{\phi_t, 0 \leq t \leq T\} \text{ that are adapted and continuous with} \right.$$

$$\left. \mathbb{E} \left[\sup_{0 \leq t \leq T} |\phi_t|^p \right] < \infty \right\},$$

$$\mathbb{H}^p = \left\{ \text{stochastic processes } \phi = \{\phi_t, 0 \leq t \leq T\} \text{ that are predictable with} \right.$$

$$\mathbb{E} \left[\left(\int_0^T |\phi_t|^2 dt \right)^{p/2} \right] < \infty \}.$$

The above random objects take values in an Euclidean space \mathbb{R} or \mathbb{R}^d whose dimension is not indicated for the sake of simplicity, this will be clear in the context.

- The set of $(\mathcal{F}_t)_{0 \leq t \leq T}$ stopping times larger than t and bounded by T is defined by:

$$\mathcal{T}_{t,T} = \{ \tau; \tau \text{ is a stopping time s.t. } t \leq \tau \leq T \}.$$

2.2 Reflected BSDE

We work on the model of RBSDE (1.1) and its penalized version (1.2).

2.3 Assumptions

In our paper, we always consider $p \geq 2$.

(H₁): General Assumptions of Equation (1.1).

(H₁-i) The scalar random variable ξ is \mathcal{F}_T -measurable and $\xi \in \mathbb{L}^p$;

(H₁-ii) The generator $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is such that $\forall (y, z) \in \mathbb{R} \times \mathbb{R}^d, f(\cdot, y, z) \in \mathbb{H}^p$;

(H₁-iii) The generator f is C_{Lip}^f -Lipschitz continuous with respect to (y, z) uniformly in (ω, t) , i.e. $\forall y, y' \in \mathbb{R}, z, z' \in \mathbb{R}^d$, a.s.

$$|f(t, y, z) - f(t, y', z')| \leq C_{Lip}^f (|y - y'| + |z - z'|);$$

(H₁-iv) The barrier $S := \{S_t, 0 \leq t \leq T\}$ is a \mathcal{F}_t -adapted scalar process s.t. $S_T \leq \xi$ a.s. and $S \in \mathcal{S}^p$.

From [EKP⁺97, Theorem 5.2], for $p = 2$, under **(H₁)**, the equation (1.1) admits an unique solution $\{(Y_t, Z_t, K_t), 0 \leq t \leq T\}$ in the space $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathcal{S}^2$. The case when $p > 2$ is proved in Appendix A.4.1.

We now justify that, up to a simple change of variable, we can still assume that the generator satisfies, in addition to **(H₁-iii)**, the monotone property **(H_m)**, the latter will be widely used in our work (especially for better controlling the penalization term, see Theorem 3.4, Theorem 3.5 and Section 4).

(H_m) The generator f is non-increasing with respect to y i.e. $\forall (\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R}^d$,

$$f(t, y_1, z) \leq f(t, y_2, z), \quad \forall y_1 \geq y_2.$$

Indeed, under (\mathbf{H}_1) , for each $\nu \in \mathbb{R}$, define

$$\xi^\nu := \xi e^{\nu T}, \quad f^\nu(t, y, z) := e^{\nu t} f(t, e^{-\nu t} y, e^{-\nu t} z) - \nu y, \quad S_t^\nu := e^{\nu t} S_t.$$

The assumption \mathbf{H}_1 is also fulfilled by (ξ^ν, f^ν, S^ν) , then the solution of RBSDE (Y^ν, Z^ν, K^ν) associated with (ξ^ν, f^ν, S^ν) is also well and uniquely defined in $(\mathcal{S}^p, \mathbb{H}^p, \mathcal{S}^p)$ and it can be expressed as following,

$$(Y_t^\nu, Z_t^\nu, K_t^\nu)_{0 \leq t \leq T} = \left(e^{\nu t} Y_t, e^{\nu t} Z_t, \int_0^t e^{\nu s} \mathbf{d}K_s \right)_{0 \leq t \leq T}. \quad (2.1)$$

We now justify that a proper choice of ν allows f^ν to fulfill (\mathbf{H}_m) .

Proposition 2.1. *Assume $(\mathbf{H}_1\text{-iii})$ holds for f : for any $\nu \geq C_{Lip}^f$, f^ν satisfies (\mathbf{H}_m) .*

Proof. Let ν as above, then for all $y_1 \geq y_2$ and $(\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R}^d$,

$$\begin{aligned} f^\nu(t, y_1, z) &= e^{\nu t} f(t, e^{-\nu t} y_1, e^{-\nu t} z) - \nu y_1 \leq C_{Lip}^f (y_1 - y_2) + e^{\nu t} f(t, e^{-\nu t} y_2, e^{-\nu t} z) - \nu y_1 \\ &\leq \nu (y_1 - y_2) + e^{\nu t} f(t, e^{-\nu t} y_2, e^{-\nu t} z) - \nu y_1 \\ &= f^\nu(t, y_2, z). \end{aligned}$$

□

Hence, for each f satisfying $(\mathbf{H}_1\text{-iii})$, one can always identify f by f^ν and assume (\mathbf{H}_m) for f without loss of generality.

(H₂): Extra assumptions for the barrier S

(H₂-i) The barrier S admits a generalized semi-martingale decomposition as

$$S_t = S_0 + \int_0^t U_s \mathbf{d}s + \int_0^t V_s \mathbf{d}B_s + A_t$$

where $U \in \mathcal{S}^p, V \in \mathcal{S}^p, A$ is continuous, non-decreasing and $A_T \in \mathbb{L}^p, A_0 = 0$.

(H₂-ii) The barrier S satisfies a 1/2-Hölder continuity condition:

$$C_{(2.2)} = \mathbb{E} \left[\sup_{0 \leq t < s \leq T} \left(\frac{\mathbb{E}_t [|S_s - S_t|]}{\sqrt{s - t}} \right)^p \right] < \infty. \quad (2.2)$$

We know that, from [EKP⁺97, Proposition 4.2 and Remark 4.3], if $(\mathbf{H}_2\text{-i})$ is satisfied by S , then K is absolutely continuous with respect to Lebesgue measure, namely,

$$\begin{cases} \mathbf{d}K_t \leq \kappa_t \mathbf{1}_{\{Y_t = S_t\}} \mathbf{d}t, \\ \kappa_t := (f(t, S_t, V_t) + U_t)^-. \end{cases} \quad (2.3)$$

The process $(\kappa_t)_{0 \leq t \leq T}$ is essential to derive the rate of $Y^\lambda \rightarrow Y$ a.s. and in $\mathbb{L}^p(\Omega)$.

Lemma 2.2. Assume the barrier S is of the form $S_t = g(t, X_t)$, where $(X_t)_{0 \leq t \leq T}$ is an Itô process valued in \mathbb{R}^d satisfying the following SDE:

$$dX_t = \mu_t dt + \sigma_t dB_t,$$

with predictable, bounded stochastic processes (μ, σ) , and where $g : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ is $\frac{1}{2}$ -Hölder continuous in t and locally Lipschitz continuous in x :

$$|g(t, x) - g(t', x')| \leq C (|t - t'|^{1/2} + |x - x'| (e^{C|x|} + e^{C|x'|})),$$

with some finite constant C . Then $(\mathbf{H}_2\text{-ii})$ is fulfilled by S .

Proof. See Appendix A.1. □

Remark 2.3. The immediate application in finance is American option pricing problem. If we set the barrier as the payoff of American Call (resp. Put) Option and X is the asset log-price process (Itô process as in Lemma 2.2), i.e.

$$S_t = (e^{X_t} - K^{stk})^+ \text{ (resp. } S_t = (K^{stk} - e^{X_t})^+)$$

with K^{stk} represents the strike price, then it satisfies both $(\mathbf{H}_2\text{-i})$ and $(\mathbf{H}_2\text{-ii})$. In fact, applying Itô-Tanaka formula on S_t :

$$\begin{aligned} dS_t &= \mathbf{1}_{\{e^{X_t} > K^{stk}\}} \left(\mu_t + \frac{1}{2} |\sigma_t|^2 \right) e^{X_t} dt + \mathbf{1}_{\{e^{X_t} > K^{stk}\}} \sigma_t e^{X_t} dB_t + \frac{1}{2} dL_t^{K^{stk}}(e^X) \\ \text{(resp. } dS_t &= -\mathbf{1}_{\{e^{X_t} \leq K^{stk}\}} \left(\mu_t + \frac{1}{2} |\sigma_t|^2 \right) e^{X_t} dt - \mathbf{1}_{\{e^{X_t} \leq K^{stk}\}} \sigma_t e^{X_t} dB_t + \frac{1}{2} dL_t^{K^{stk}}(e^X)) \end{aligned}$$

where $L^a(e^X)$ is the local time process of e^X at level a .

Thanks to the boundedness of μ, σ and $\mathbb{E} \left[\sup_{0 \leq t \leq T} e^{p|X_t|} \right] < \infty, \forall p \geq 2$ as in the proof of Lemma 2.2, we can easily deduce that the corresponding processes $(U, V) \in \mathcal{S}^p$.

3 Penalization

In this section, we present our main result about the rate of convergence $Y^\lambda \rightarrow Y$ with respect to the penalty parameter λ , where the order relies critically on the convergence rate of penalization term, namely, $(Y_t^\lambda - S_t)^- \rightarrow 0$. Using several times the absolute continuity as (2.3), we first introduce a priori estimate for all general \mathcal{S}^p barrier and derive the first bound for $\lambda \int_0^T (Y_t^\lambda - S_t)^- dt$ in \mathbb{L}^p which gives a order 1/2 for $Y^\lambda \rightarrow Y$ in \mathcal{S}^p . Then we focus on the uniform boundedness in t of $(Y_t^\lambda - S_t)^-$ in \mathbb{L}^∞ and in \mathcal{S}^p where we can derive $Y_t^\lambda \rightarrow Y_t$ at order 1 in λ .

3.1 The convergence rate in continuous case

We first investigate the convergence order between the PBDSE and the RBSDE, this is based on the following a priori estimate.

Theorem 3.1. *Under (\mathbf{H}_1) , let (Y, Z, K) be the solution of (1.1) in $\mathcal{S}^p \times \mathbb{H}^p \times \mathcal{S}^p$ and (Y^λ, Z^λ) be the solution of (1.2) in $\mathcal{S}^p \times \mathbb{H}^p$. First, the norm of $(Y^\lambda, Z^\lambda, K^\lambda)$ in $\mathcal{S}^p \times \mathbb{H}^p \times \mathcal{S}^p$ is bounded uniformly in λ , in particular*

$$\sup_{\lambda > 0} \mathbb{E} \left[\left(\lambda \int_0^T (Y_s^\lambda - S_s)^- \mathbf{d}s \right)^p \right] < +\infty. \quad (3.1)$$

Second, we have the following a priori estimate

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t - Y_t^\lambda|^p + \left(\int_0^T |Z_t - Z_t^\lambda|^2 \mathbf{d}t \right)^{p/2} + \sup_{0 \leq t \leq T} |K_t - K_t^\lambda|^p \right] \\ \leq C \mathbb{E} \left[\left(\int_0^T (Y_s^\lambda - S_s)^- \mathbf{d}K_s \right)^{p/2} \right], \end{aligned} \quad (3.2)$$

where C is a constant depending only on ξ, f, S, T, p .

The proof is postponed to Appendix A.4. Whenever necessary, a priori \mathbb{L}^p -estimates on (Y, Z, K) are available in Proposition A.2.

Theorem 3.2. *Under (\mathbf{H}_1) , let (Y, Z, K) be the solution of (1.1) in $\mathcal{S}^p \times \mathbb{H}^p \times \mathcal{S}^p$ and (Y^λ, Z^λ) be the solution of (1.2) in $\mathcal{S}^p \times \mathbb{H}^p$. If the barrier S satisfies $(\mathbf{H}_2\text{-i})$, i.e.*

$$S_t = S_0 + \int_0^t U_s \mathbf{d}s + \int_0^t V_s \mathbf{d}B_s + A_t,$$

and $f(\cdot, 0, 0) \in \mathcal{S}^p$, then

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} (Y_t - Y_t^\lambda)^p + \left(\int_0^T |Z_t - Z_t^\lambda|^2 \mathbf{d}t \right)^{p/2} + \sup_{0 \leq t \leq T} (K_t - K_t^\lambda)^p \right] \leq \frac{C}{\lambda^{p/2}}, \quad (3.3)$$

where C is a constant depending on ξ, f, S, T, p .

Proof. Under $(\mathbf{H}_2\text{-i})$, taking into account the absolute continuity of K in (2.3), we get the following inequality which is a consequence of Theorem 3.1, in particular the bound (3.1):

$$\mathbb{E} \left[\left(\int_0^T (Y_s^\lambda - S_s)^- \mathbf{d}s \right)^p \right] \leq \frac{C}{\lambda^p}.$$

Then

$$\mathbb{E} \left[\left(\int_0^T (Y_s^\lambda - S_s)^- \mathbf{d}K_s \right)^{p/2} \right] \leq \mathbb{E} \left[\left(\sup_{0 \leq t \leq T} \kappa_t \int_0^T (Y_s^\lambda - S_s)^- \mathbf{d}s \right)^{p/2} \right]$$

$$\leq \sqrt{\frac{C}{\lambda^p}} \sqrt{\mathbb{E} \left[\sup_{0 \leq t \leq T} \kappa_t^p \right]}.$$

Thanks to (2.3), we easily verify that κ is bounded in \mathcal{S}^p under $(\mathbf{H}_2\text{-i})$ and $f(\cdot, 0, 0) \in \mathcal{S}^p$. Indeed, write $\kappa_t \leq |f(t, 0, 0)| + C_{Lip}^f(|S_t| + |V_s|) + |U_t|$. The proof of (3.3) is complete. \square

3.2 Advanced estimation for Y^λ below the semi-martingale barrier

In the previous subsection, we used the convergence of integral of penalization term, namely, $\int_0^T (Y_t^\lambda - S_t)^- dt \rightarrow 0$ in \mathbb{L}^p . In this subsection, we focus on the uniform convergence in t i.e. $\sup_{0 \leq t \leq T} (Y_t^\lambda - S_t)^- \rightarrow 0$, a.s. and in \mathbb{L}^p .

Theorem 3.3. *Under (\mathbf{H}_1) , let (Y^λ, Z^λ) be the solution of PBSDE (1.2) in $\mathcal{S}^p \times \mathbb{H}^p$, if the barrier S satisfies $(\mathbf{H}_2\text{-ii})$, then there exists a constant C which depends only on ξ, f, S, T, p such that*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |(Y_t^\lambda - S_t)^-|^p \right]^{1/p} \leq \frac{C}{\sqrt{\lambda}}.$$

Proof. Let $(\tilde{Y}^\lambda, \tilde{Z}^\lambda)$ be the solution (in $\mathcal{S}^p \times \mathbb{H}^p$) of the linear BSDE

$$\begin{aligned} \tilde{Y}_t^\lambda &= \xi + \int_t^T f(s, Y_s^\lambda, Z_s^\lambda) \mathbf{d}s + \lambda \int_t^T (S_s - \tilde{Y}_s^\lambda) \mathbf{d}s - \int_t^T \tilde{Z}_s^\lambda \mathbf{d}B_s \\ &= \mathbb{E}_t \left[e^{-\lambda(T-t)} \xi + \int_t^T e^{-\lambda(s-t)} f(s, Y_s^\lambda, Z_s^\lambda) \mathbf{d}s + \lambda \int_t^T e^{-\lambda(s-t)} S_s \mathbf{d}s \right]. \end{aligned} \quad (3.4)$$

By the classical comparison theorem as [EPM97, Theorem 2.2], for $0 \leq t \leq T$, $Y_t^\lambda \geq \tilde{Y}_t^\lambda$, a.s., it suffices to prove

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |(\tilde{Y}_t^\lambda - S_t)^-|^p \right]^{1/p} \leq \frac{C}{\sqrt{\lambda}}.$$

\triangleright *Step 1:* We rewrite the formula of $\tilde{Y}_t^\lambda - S_t$ with term $e^{-\lambda(T-t)} S_T$,

$$\begin{aligned} \tilde{Y}_t^\lambda - S_t &= \mathbb{E}_t \left[e^{-\lambda(T-t)} (\xi - S_T) + e^{-\lambda(T-t)} (S_T - S_t) + \int_t^T e^{-\lambda(s-t)} f(s, Y_s^\lambda, Z_s^\lambda) \mathbf{d}s \right] \\ &\quad + \lambda \mathbb{E}_t \left[\int_t^T e^{-\lambda(s-t)} (S_s - S_t) \mathbf{d}s \right]. \end{aligned}$$

Taking negative part on both side, and applying the Jensen inequality, it gives

$$\begin{aligned} (\tilde{Y}_t^\lambda - S_t)^- &\leq \mathbb{E}_t [e^{-\lambda(T-t)} (\xi - S_T)^-] + \mathbb{E}_t [e^{-\lambda(T-t)} |S_T - S_t|] \\ &\quad + \mathbb{E}_t \left[\int_t^T e^{-\lambda(s-t)} |f(s, Y_s^\lambda, Z_s^\lambda)| \mathbf{d}s \right] + \lambda \mathbb{E}_t \left[\int_t^T e^{-\lambda(s-t)} |S_s - S_t| \mathbf{d}s \right], \end{aligned}$$

since $\xi \geq S_T$, we have

$$\begin{aligned} (\tilde{Y}_t^\lambda - S_t)^- &\leq \mathbb{E}_t [e^{-\lambda(T-t)} |S_T - S_t|] + \lambda \mathbb{E}_t \left[\int_t^T e^{-\lambda(s-t)} |S_s - S_t| \mathbf{d}s \right] \\ &\quad + \mathbb{E}_t \left[\int_t^T e^{-\lambda(s-t)} |f(s, Y_s^\lambda, Z_s^\lambda)| \mathbf{d}s \right]. \end{aligned} \quad (3.5)$$

▷ *Step 2:* We take supremum on these conditional expectations in the right side of (3.5), and consider the \mathbb{L}^p -norm of each term, which follows

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq t \leq T} \mathbb{E}_t [e^{-\lambda(T-t)} |S_T - S_t|]^p \right]^{1/p} &= \mathbb{E} \left[\sup_{0 \leq t \leq T} \mathbb{E}_t \left[e^{-\lambda(T-t)} \sqrt{T-t} \frac{|S_T - S_t|}{\sqrt{T-t}} \right]^p \right]^{1/p} \\ &\leq \mathbb{E} \left[\sup_{0 \leq t \leq T} \left(e^{-\lambda(T-t)} \sqrt{T-t} \right)^p \times \sup_{0 \leq t \leq T} \left(\frac{\mathbb{E}_t [|S_T - S_t|]}{\sqrt{T-t}} \right)^p \right]^{1/p} \\ &\leq \frac{1}{\sqrt{\lambda}} \sup_{x \geq 0} (e^{-x} \sqrt{x}) C_{(2.2)}^{1/p}. \end{aligned} \quad (3.6)$$

Taking again the advantage of Hölder continuity of S in Assumption (\mathbf{H}_2 -ii), we get

$$\begin{aligned} &\lambda \mathbb{E} \left[\sup_{0 \leq t \leq T} \mathbb{E}_t \left[\int_t^T e^{-\lambda(s-t)} |S_s - S_t| \mathbf{d}s \right]^p \right]^{1/p} \\ &= \mathbb{E} \left[\sup_{0 \leq t \leq T} \mathbb{E}_t \left[\int_t^T \lambda e^{-\lambda(s-t)} \sqrt{s-t} \frac{|S_s - S_t|}{\sqrt{s-t}} \mathbf{d}s \right]^p \right]^{1/p} \\ &\leq \mathbb{E} \left[\sup_{0 \leq t < s \leq T} \mathbb{E}_t \left[\frac{|S_s - S_t|}{\sqrt{s-t}} \right]^p \left(\int_t^T \lambda e^{-\lambda(s-t)} \sqrt{s-t} \mathbf{d}s \right)^p \right]^{1/p} \\ &\leq C_{(2.2)}^{1/p} \frac{\left(\int_0^{+\infty} e^{-x} \sqrt{x} \mathbf{d}x \right)}{\sqrt{\lambda}}. \end{aligned} \quad (3.7)$$

Moreover, by Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\mathbb{E} \left[\sup_{0 \leq t \leq T} \mathbb{E}_t \left[\int_t^T e^{-\lambda(s-t)} |f(s, Y_s^\lambda, Z_s^\lambda)| \mathbf{d}s \right]^p \right]^{1/p} \\ &\leq \mathbb{E} \left[\sup_{0 \leq t \leq T} \left(\sqrt{\int_t^T e^{-2\lambda(s-t)} \mathbf{d}s} \times \mathbb{E}_t \left[\sqrt{\int_t^T |f(s, Y_s^\lambda, Z_s^\lambda)|^2 \mathbf{d}s} \right] \right)^p \right]^{1/p} \\ &\leq \frac{1}{\sqrt{2\lambda}} \mathbb{E} \left[\sup_{0 \leq t \leq T} \mathbb{E}_t \left[\sqrt{\int_0^T |f(s, Y_s^\lambda, Z_s^\lambda)|^2 \mathbf{d}s} \right]^p \right]^{1/p}. \end{aligned} \quad (3.8)$$

▷ *Step 3:* Set $M_T := \sqrt{\int_0^T |f(s, Y_s^\lambda, Z_s^\lambda)|^2 \mathbf{d}s}$. Under $(\mathbf{H}_1\text{-ii}), (\mathbf{H}_1\text{-iii})$ and combined with λ -uniform a priori estimate for (Y^λ, Z^λ) in (A.3), we claim that $M_T \in \mathbb{L}^p$ uniformly in λ . Indeed,

$$\begin{aligned} \mathbb{E}[|M_T|^p] &= \mathbb{E}\left[\left(\int_0^T |f(s, Y_s^\lambda, Z_s^\lambda)|^2 \mathbf{d}s\right)^{p/2}\right] \\ &\leq C_{p,f,T} \mathbb{E}\left[\sup_{0 \leq t \leq T} |Y_t^\lambda|^p + \left(\int_0^T |Z_s^\lambda|^2 \mathbf{d}s\right)^{p/2} + \left(\int_0^T |f(s, 0, 0)|^2 \mathbf{d}s\right)^{p/2}\right] < \infty, \end{aligned}$$

for some constant $C_{p,f,T}$ which does not depend on λ . Hence, $M_t := \mathbb{E}_t[M_T]$ defines a martingale in \mathbb{L}^p and the right-hand side of (3.8) becomes

$$\frac{1}{\sqrt{2\lambda}} \mathbb{E}\left[\sup_{0 \leq t \leq T} |M_t|^p\right]^{1/p} \leq \frac{1}{\sqrt{2\lambda}} \binom{p}{p-1} \mathbb{E}[|M_T|^p]^{1/p}$$

in view of the Doob inequality. All in all, the right-hand side of (3.8) is of order $1/\sqrt{\lambda}$. Finally, combine (3.5), (3.6), (3.7), (3.8), we have the desired result. \square

Theorem 3.4. *Suppose $(\mathbf{H}_1), (\mathbf{H}_2\text{-i})$ are fulfilled; in addition, suppose (\mathbf{H}_m) holds true, i.e. $f(t, y, z)$ is non-increasing on y , assume*

$$\bar{\kappa}_\infty := \operatorname{ess\,sup}_{(t,\omega) \in [0,T] \times \Omega} (f(t, S_t, V_t) + U_t)^- = \operatorname{ess\,sup}_{(t,\omega) \in [0,T] \times \Omega} \kappa_t(\omega) < \infty \quad (3.9)$$

then the penalization term is upper bounded as follows

$$(Y_t^\lambda - S_t)^- \leq \frac{\bar{\kappa}_\infty}{\lambda}, \quad \forall 0 \leq t \leq T, \quad a.s..$$

Proof. The strategy of proof relies on comparison arguments between BSDEs but the fact that S can not be decomposed as a standard BSDE, forbids us to apply standard comparison results (where drivers are compared only along one solution, see [EPM97, Theorem 2.2]). Instead, we could invoke a comparison between RBSDEs but it usually requires to compare the generator everywhere ([EKP⁺97, Theorem 4.1]) which is a too stringent requirement in our case. Hence, we proceed with a direct proof.

Write the dynamic of $S_t - \frac{1}{\lambda} \bar{\kappa}_\infty$ and the one of Y_t^λ as follows:

$$\begin{aligned} S_t - \frac{1}{\lambda} \bar{\kappa}_\infty &= S_T - \frac{1}{\lambda} \bar{\kappa}_\infty - \int_t^T U_s \mathbf{d}s - \int_t^T \mathbf{d}A_s - \int_t^T V_s \mathbf{d}B_s, \\ Y_t^\lambda &= \xi + \int_t^T f^\lambda(s, Y_s^\lambda, Z_s^\lambda) \mathbf{d}s - \int_t^T Z_s^\lambda \mathbf{d}B_s, \end{aligned}$$

where $f^\lambda(s, y, z) := f(s, y, z) + \lambda(y - S_s)^-$. Set $\Delta Y_t = S_t - \frac{1}{\lambda} \bar{\kappa}_\infty - Y_t^\lambda$ and $\Delta Z_t = V_t - Z_t^\lambda$: we have

$$\Delta Y_t = \Delta Y_T - \int_t^T (U_s + f^\lambda(s, Y_s^\lambda, Z_s^\lambda)) \mathbf{d}s - \int_t^T \mathbf{d}A_s - \int_t^T \Delta Z_s \mathbf{d}B_s$$

$$\begin{aligned}
 &= \Delta Y_T - \int_t^T (U_s + f^\lambda(s, S_s - \frac{1}{\lambda} \bar{\kappa}_\infty, V_s) + \Delta_y f^\lambda(s) \Delta Y_s + \Delta_z f^\lambda(s) \cdot \Delta Z_s) \mathbf{d}s \\
 &\quad - \int_t^T \mathbf{d}A_s - \int_t^T \Delta Z_s \mathbf{d}B_s,
 \end{aligned}$$

here $\Delta_y f^\lambda(s) = \frac{f^\lambda(s, Y_s^\lambda, Z_s^\lambda) - f^\lambda(s, S_s - \frac{1}{\lambda} \bar{\kappa}_\infty, Z_s^\lambda)}{\Delta Y_s} \mathbf{1}_{\Delta Y_s \neq 0}$ and similarly for $\Delta_z f^\lambda(s)$ (see e.g. [EPM97, Proof of Theorem 2.5]). Because f is Lipschitz in y and z , the coefficients $\Delta_y f^\lambda(s)$ and $\Delta_z f^\lambda(s)$ are bounded. Thus, using $\Gamma_t^T := \exp(\int_t^T (\Delta_y f^\lambda(s) - \frac{1}{2} |\Delta_z f^\lambda(s)|^2) \mathbf{d}s + \int_t^T \Delta_z f^\lambda(s) \mathbf{d}B_s)$ and Itô formula, we easily get

$$\Delta Y_t = \mathbb{E}_t \left[\Gamma_t^T \Delta Y_T - \int_t^T \Gamma_t^s (U_s + f^\lambda(s, S_s - \frac{1}{\lambda} \bar{\kappa}_\infty, V_s)) \mathbf{d}s - \int_t^T \Gamma_t^s \mathbf{d}A_s \right].$$

We claim that

$$U_s + f^\lambda(s, S_s - \frac{1}{\lambda} \bar{\kappa}_\infty, V_s) \geq 0, \mathbf{d}s \otimes \mathbf{d}\mathbb{P} \text{ a.e. } . \quad (3.10)$$

Since $\Delta Y_T = S_T - \frac{1}{\lambda} \bar{\kappa}_\infty - \xi \leq -\frac{1}{\lambda} \bar{\kappa}_\infty \leq 0$ and A is non-decreasing, we get

$$\Delta Y_t \leq 0,$$

which implies the desired result.

To justify (3.10), we write

$$\begin{aligned}
 U_s + f^\lambda(s, S_s - \frac{1}{\lambda} \bar{\kappa}_\infty, V_s) &= U_s + f(s, S_s - \frac{1}{\lambda} \bar{\kappa}_\infty, V_s) + \lambda (S_s - \frac{1}{\lambda} \bar{\kappa}_\infty - S_s)^- \\
 &\geq U_s + f(s, S_s, V_s) + \bar{\kappa}_\infty \geq 0,
 \end{aligned}$$

where, in the last line, we used that f is non-increasing on y together with the definition of $\bar{\kappa}_\infty$. \square

Theorem 3.5. *Assume the same hypotheses hold as in Theorem 3.4, we have*

$$0 \leq Y_t - Y_t^\lambda \leq \frac{\bar{\kappa}_\infty}{\lambda}, \quad \forall 0 \leq t \leq T, \mathbb{P} - a.s.$$

where $\bar{\kappa}_\infty$ is defined as in Theorem 3.4.

Proof. The left hand side of inequality is an immediate result from comparison theorem (see [EKP⁺97, Section 6]). For the right hand side, our strategy is to apply again a comparison result to Y and $Y' := Y^\lambda + \frac{\bar{\kappa}_\infty}{\lambda}$. Denote

$$K_t^\lambda = \lambda \int_0^t (Y_s^\lambda - S_s)^- \mathbf{d}s.$$

Then $(Y', Z^\lambda, K^\lambda)$ can be considered as the solution of the following BSDE with barrier S and increasing process K^λ ,

$$\begin{cases} Y'_t = \xi' + \int_t^T f(s, Y'_s - \frac{\bar{\kappa}_\infty}{\lambda}, Z_s^\lambda) \mathbf{d}s + \int_t^T \mathbf{d}K_s^\lambda - \int_t^T Z_s^\lambda \mathbf{d}B_s, \\ Y'_t \geq S_t, \\ \int_0^T (Y'_t - S_t) \mathbf{d}K_t^\lambda \geq 0 \end{cases} \quad (3.11)$$

with $\xi' := \xi + \frac{\bar{\kappa}_\infty}{\lambda}$. The second and the third conditions of (3.11) hold thanks to Theorem 3.4 where the third can't be replaced by the equality, which means that (3.11) is not a real RBSDE because the Skorokhod condition is not fulfilled by (Y', S, K^λ) .

Nevertheless, following the arguments of [EKP⁺97, Theorem 4.1], we are going to establish a comparison theorem dedicated to our specific setting. Apply Itô formula on $|(Y_t - Y'_t)^+|^2$, we get, since $\xi' \geq \xi$,

$$\begin{aligned} & \mathbb{E} [|(Y_t - Y'_t)^+|^2] + \mathbb{E} \left[\int_t^T \mathbf{1}_{\{Y_s > Y'_s\}} |Z_s - Z_s^\lambda|^2 \mathbf{d}s \right] \\ & \leq 2\mathbb{E} \left[\int_t^T (Y_s - Y'_s)^+ [f(s, Y_s, Z_s) - f(s, Y'_s - \frac{\bar{\kappa}_\infty}{\lambda}, Z_s^\lambda)] \mathbf{d}s \right] \\ & \quad + 2\mathbb{E} \left[\int_t^T (Y_s - Y'_s)^+ (\mathbf{d}K_s - \mathbf{d}K_s^\lambda) \right]. \end{aligned}$$

It still holds that

$$\int_t^T (Y_s - Y_s^\lambda - \frac{\bar{\kappa}_\infty}{\lambda})^+ (\mathbf{d}K_s - \mathbf{d}K_s^\lambda) = - \int_t^T (Y_s - Y_s^\lambda - \frac{\bar{\kappa}_\infty}{\lambda})^+ \mathbf{d}K_s^\lambda \leq 0$$

because on the set $\{s : Y_s = S_s\}$ where K increases, $Y_s - Y_s^\lambda - \frac{\bar{\kappa}_\infty}{\lambda} = S_s - Y_s^\lambda - \frac{\bar{\kappa}_\infty}{\lambda} \leq 0$ thanks to Theorem 3.4. Moreover, since $\bar{\kappa}_\infty \geq 0$ and f is non-increasing in y , we get, $\forall (y, z) \in \mathbb{R} \times \mathbb{R}^d$,

$$f\left(t, y - \frac{\bar{\kappa}_\infty}{\lambda}, z\right) \geq f(t, y, z), \mathbf{d}t \otimes \mathbf{d}\mathbb{P}, \text{ a.e. .}$$

Hence,

$$\begin{aligned} & \mathbb{E} [|(Y_t - Y'_t)^+|^2] + \mathbb{E} \left[\int_t^T \mathbf{1}_{\{Y_s > Y'_s\}} |Z_s - Z_s^\lambda|^2 \mathbf{d}s \right] \\ & \leq 2\mathbb{E} \left[\int_t^T (Y_s - Y'_s)^+ [f(s, Y_s, Z_s) - f(s, Y'_s, Z_s^\lambda)] \mathbf{d}s \right] \\ & \leq (2C_{Lip}^f + 2(C_{Lip}^f)^2) \mathbb{E} \left[\int_t^T |(Y_s - Y'_s)^+|^2 \mathbf{d}s \right] + \mathbb{E} \left[\int_t^T \mathbf{1}_{\{Y_s > Y'_s\}} |Z_s - Z_s^\lambda|^2 \mathbf{d}s \right]. \end{aligned}$$

It is easy to conclude that $|(Y_t - Y'_t)^+|^2 = 0, 0 \leq t \leq T, a.s..$ using Gronwall's lemma. \square

Remark 3.6. Theorems 3.4 and 3.5 are valid without assuming (\mathbf{H}_m) up to modifying the upper bounds in their statements. Namely, if $\bar{\kappa}_\infty$ in (3.9) is finite, then we have

$$(Y_t^\lambda - S_t)^- \leq \frac{\bar{\kappa}_\infty e^{C_{Lip}^f T}}{\lambda}, \quad \forall 0 \leq t \leq T, \text{ a.s.}, \quad (3.12)$$

$$0 \leq Y_t - Y_t^\lambda \leq \frac{\bar{\kappa}_\infty e^{C_{Lip}^f T}}{\lambda}, \quad \forall 0 \leq t \leq T, \text{ a.s.} \quad (3.13)$$

Indeed, due to the change of variable in Proposition 2.1, we can always assume the generator to be non-increasing in y . For $\nu = C_{Lip}^f$, with (Y^ν, Z^ν, K^ν) as defined in (2.1), the generator f^ν satisfies (\mathbf{H}_m) , then Theorems 3.4 and 3.5 yield

$$\begin{aligned} e^{\nu t} (Y_t^\lambda - S_t)^- &= (Y_t^{\lambda, \nu} - S_t^\nu)^- \leq \frac{\bar{\kappa}_\infty^\nu}{\lambda}, \\ e^{\nu t} (Y_t - Y_t^\lambda) &= Y_t^\nu - Y_t^{\nu, \lambda} \leq \frac{\bar{\kappa}_\infty^\nu}{\lambda}, \end{aligned}$$

where after a few computations, we easily justify

$$\bar{\kappa}_\infty^\nu = \text{ess sup}_{(t, \omega) \in [0, T] \times \Omega} e^{\nu t} (f(t, S_t, V_t) + U_t)^- \leq e^{\nu T} \bar{\kappa}_\infty.$$

Then we can conclude with (3.12) and (3.13).

Remark 3.7. Let us exemplify the above results in the case of the pricing of an American put in finance with two interest rates (r_t for lending, R_t for borrowing), see [EPM97, Example 1.1]. In this case, $f(t, y, z) = -r_t y - z \sigma_t^{-1} (\mu_t + \frac{\sigma_t^2}{2} - r_t) + (R_t - r_t)(y - z \sigma_t^{-1})^-$ and $S_t = (K^{stk} - e^{X_t})^+$ where the log-asset price X is defined in Lemma 2.2 with $d = 1$ and for non-zero volatility σ_t . From Remark 2.3, a direct computation leads to $\kappa_t = r_t^+ K^{stk} \mathbf{1}_{\{e^{X_t} \leq K^{stk}\}}$. Obviously, if we assume r is upper bounded, then $\bar{\kappa}_\infty < +\infty$ and therefore

$$0 \leq Y_t - Y_t^\lambda \leq \frac{C \bar{\kappa}_\infty}{\lambda}$$

for some constant C .

On the other hand, if $r_t \leq 0$ for any t then $\bar{\kappa}_\infty = 0$: combining (3.12) and (3.13) gives $Y_t = Y_t^\lambda \geq S_t$; thus we retrieve that the increasing process K is zero, i.e. American and European put prices coincide when the interest rate remains non-positive.

4 Error of implicit scheme

In this section, we aim at assessing the discretization error of the numerical solution of PBSDE using the estimations from the previous section. When applying a time discretization to Y^λ in (1.2), the difficulty lies in the fact that the Lipschitz constant of the generator part f^λ goes to infinity as $\lambda \rightarrow \infty$. To overcome this, we use an implicit scheme $Y^{\lambda, h}$ defined in (4.1). In our analysis, we first quantify the error of $Y^{\lambda, h} - Y^\lambda$ w.r.t. the time step h and λ ; and second, we use the order 1-bound of $Y - Y^\lambda$ in Theorem 3.5 to get a global convergence rate of $Y^{\lambda, h} \rightarrow Y$. Since we do not have a tight enough rate for $Z - Z^\lambda$ (order $\frac{1}{2}$ in \mathbb{L}^p norm, see Theorem 3.2), we restrict our study to a case where the generator f does not depend on z .

4.1 Main result

Consider the equidistant time discretization $0 = t_0 < t_1 < \dots < t_N = T$ with $t_i = i\frac{T}{N}$, $\forall 0 \leq i \leq N$. Let $h := \frac{T}{N}$. For all $0 \leq i \leq N - 1$, we propose the piecewise implicit scheme for the discrete solution of Y^λ as follows,

$$\begin{aligned} Y_{t_i}^{\lambda,h} &= \mathbb{E}_{t_i} \left[Y_{t_{i+1}}^{\lambda,h} + f^\lambda(t_i, Y_{t_i}^{\lambda,h})h \right], \\ Y_t^{\lambda,h} &= Y_{t_i}^{\lambda,h}, \forall t \in [t_i, t_{i+1}), \end{aligned} \quad (4.1)$$

where $f^\lambda(t, y) := f(t, y) + \lambda(y - S_t)^-$ and $\mathbb{E}_{t_i}[\cdot] := \mathbb{E}[\cdot | \mathcal{F}_{t_i}]$. The continuous solution of (1.2) Y^λ , at each t_i , can be represented as,

$$Y_{t_i}^\lambda = \mathbb{E}_{t_i} \left[Y_{t_{i+1}}^\lambda + f^\lambda(t_i, Y_{t_i}^\lambda)h + \mathcal{E}_i^{f,\lambda} \right],$$

where the perturbation is defined by

$$\mathcal{E}_i^{f,\lambda} := \int_{t_i}^{t_{i+1}} f^\lambda(s, Y_s^\lambda) \mathbf{d}s - f^\lambda(t_i, Y_{t_i}^\lambda)h. \quad (4.2)$$

Before presenting the result of discretization error of PBSDE, let us denote D the Malliavin derivative operator (see [Nua06]) and define the following integral space (see [EPM97, p.58]):

$$\begin{aligned} \mathbb{D}^{1,2} &:= \left\{ \text{random variables } \zeta \text{ that are Malliavin differentiable with} \right. \\ &\quad \left. \zeta \in \mathbb{L}^2, \mathbb{E} \left[|\zeta|^2 + \int_0^T |D_\theta \zeta|^2 \mathbf{d}\theta \right] < \infty \right\}, \\ \mathbb{H}^{1,2} &:= \left\{ \text{scalar predictable process } \phi = \{\phi_t, 0 \leq t \leq T\} \text{ s.t.} \right. \\ &\quad \text{for a.e. } t \in [0, T], \quad \phi_t \in \mathbb{D}^{1,2}, \\ &\quad t \in [0, T] \mapsto D\phi_t \text{ has a progressively measurable version in } L_2([0, T], \mathbb{R}^d), \\ &\quad \left. \mathbb{E} \left[\int_0^T |\phi_t|^2 \mathbf{d}t + \int_0^T \int_0^T |D_\theta \phi_t|^2 \mathbf{d}\theta \mathbf{d}t \right] < \infty \right\}. \end{aligned}$$

Theorem 4.1. *Suppose (\mathbf{H}_1) , (\mathbf{H}_m) and (\mathbf{H}_2) are satisfied. In addition, assume*

- *the terminal condition ξ satisfies $\xi \in \mathbb{D}^{1,2}$ and $\text{ess sup}_{(t,\omega) \in [0,T] \times \Omega} \mathbb{E}_t[|D_t \xi|] < \infty$;*
- *the generator f satisfies:*
 - a. $f(t, y, z) = f(t, y)$;
 - b. $f(\cdot, 0) \in \mathcal{S}^2$;
 - c. $\forall t \in [0, T], f(t, \cdot)$ *is differentiable with uniformly bounded and continuous derivatives;*

d. $f(\cdot, y)$ is $\frac{1}{2}$ -Hölder continuous s.t.

$$\mathbb{E} \left[\sup_{\substack{0 \leq t < s \leq T \\ y \in \mathbb{R}}} \frac{|f(s, y) - f(t, y)|^2}{s - t} \right] < \infty;$$

e. $\forall y \in \mathbb{R}, f(\cdot, y) \in \mathbb{H}^{1,2}$ and $\operatorname{ess\,sup}_{(t, \omega) \in [0, T] \times \Omega} \mathbb{E}_t \left[\sup_{(s, y) \in [t, T] \times \mathbb{R}} |D_t f(s, y)|^2 \right] < \infty;$

f. $y \mapsto D_\theta f(t, y)$ is Lipschitz continuous uniformly on $t, \theta, \omega;$

• the barrier S satisfies:

a. $\operatorname{ess\,sup}_{(t, \omega) \in [0, T] \times \Omega} |V_t| < \infty;$

b. $S \in \mathbb{H}^{1,2}$ and $\operatorname{ess\,sup}_{(t, \theta, \omega) \in [0, T] \times [0, T] \times \Omega} |D_\theta S_t| < \infty;$

• $\bar{\kappa}_\infty$ defined by (3.9) is finite.

Then, we have

$$\sup_{0 \leq i \leq N-1} \mathbb{E} \left[(Y_{t_i}^{\lambda, h} - Y_{t_i}^\lambda)^2 \right] = O(h + \lambda^2 h^{3/2}).$$

Remark 4.2. Notice that if $A = 0$ in $(\mathbf{H}_2\text{-i})$, then $V_t = D_t S_t$, so the hypothesis of the boundedness of V on the barrier S can be removed.

Combining the above Theorem 4.1 with Theorem 3.5, we immediately deduce a bound on the global error.

Corollary 4.3. Under same hypotheses hold as Theorem 4.1, let $\lambda = h^{-3/8}$, we get the estimation of global error

$$\sup_{0 \leq i \leq N-1} \mathbb{E} \left[(Y_{t_i}^{h, \lambda} - Y_{t_i}^\lambda)^2 \right]^{1/2} = O(h^{3/8}).$$

Notice that this convergence is better than that in [MZ05] (order 1/4) and compared with [BC08], although this convergence holds at a slightly slower rate (order 3/8 instead of 1/2): however, instead of assuming high regularity of the Markovian barrier, we require milder regularity conditions for the barrier and for generator, which allows for a wider scope of applications in option pricing. The tightness of the upper bound in Corollary 4.3 is an open question: numerical tests on American Put option in the following section shows that the rate may be faster than $h^{3/8}$ in some situations.

Corollary 4.4. Assume the same assumptions as Theorem 4.1 except $\bar{\kappa}_\infty = +\infty$, then

$$\sup_{0 \leq i \leq N-1} \mathbb{E} \left[(Y_{t_i}^{\lambda, h} - Y_{t_i}^\lambda)^2 \right] = O(\lambda h + \lambda^{5/2} h^2 + \lambda^2 h^{3/2}).$$

Letting $\lambda = h^{-1/2}$, the global error is upper bounded as follows:

$$\sup_{0 \leq i \leq N-1} \mathbb{E} \left[(Y_{t_i}^{h,\lambda} - Y_{t_i})^2 \right]^{1/2} = O(h^{1/4}).$$

We retrieve the global rate from [MZ05], but using this time a penalized scheme.

4.2 Numerical experiments

In this section, we provide a numerical test built for American Put in a risk neutral setting. Precisely, we lock the parameters of Put as following: the underlying asset is under Black-Scholes model with the constant interest rate $r = 3\%$ and volatility $\sigma = 0.2$, the strike price is fixed at $K = 100$, the maturity $T = 1$. About the approximation approach, we apply Monte-Carlo method with empirical regression as in [GLW05, GT16]. The number of trajectories $M = 10^4$, the number of discretization times $N = 10^3$, the degree $l = 7$ of global polynomials as basis functions.

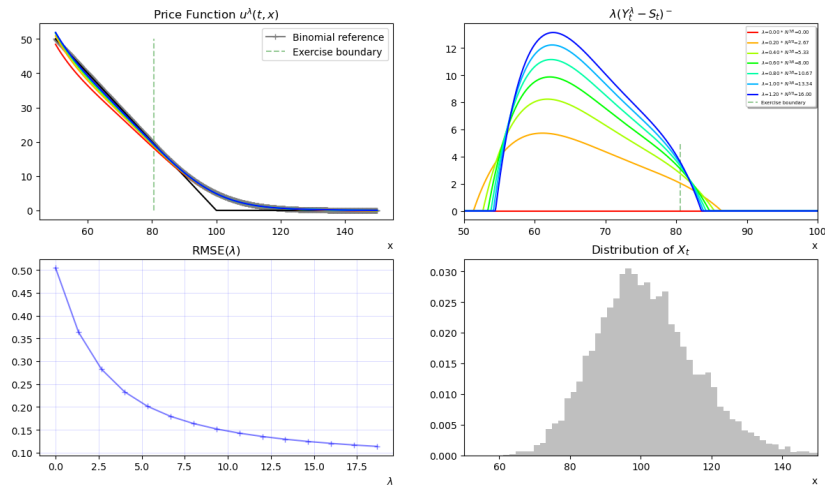
In the first subsection, we focus on simulating the penalized price function $u^\lambda(t, \cdot)$ given by $u^\lambda(t, X_t) = Y_t^{\lambda, h}$ and the behavior of penalization error with increasing λ . We also give 2 different illustrations with constant initial process value $X_0 = 100$ and random one $X_0 \sim \mathcal{U}(50, 150)$.

In the second subsection, we show that, with fixed order of penalty $\lambda = O(h^{-3/8})$, changing only the time discretization step h , the error of MC simulation decreases in h more rapidly than the rate proved in Corollary 4.3. The reference value is given by a 1000-steps Binomial tree.

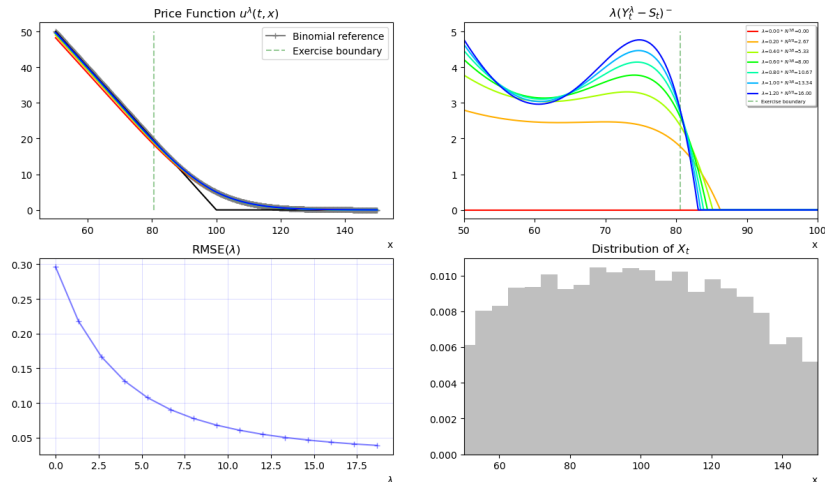
4.2.1 Price function of American Put

In Figure 1a, the global simulation of u^λ is very close to the reference value for large λ . Theorems 3.5 and 3.4 are verified numerically in the following sense: the RMSE at bottom-left shows effectively the tracking error decreases as λ increases; at the top-right, the penalization term remains bounded (with small oscillations around the exercise boundary), it explodes for small x but this is due to the lack of samples in that region (see the sample distribution at bottom-right).

In Figure 1b, we report the same quantities but with random X_0 : in comparison with Figure 1a, we observe that the penalization term behaves much better, for a wide range of values of x , and we have noticed that the simulation scheme is globally more robust.



(a) The 4 panels are as following: at $t = 0.5$, the price function $u^\lambda(t, \cdot)$ at top-left, the penalization term $\lambda[u^\lambda(t, \cdot) - (K - \cdot)^+]^-$ at top-right, the RMSE (compared with binomial reference) as a function of λ at bottom-left, the distribution of MC samples of X_t at $t = 0.5$ at bottom-right.



(b) The same parameters as in Figure 1a except the process X is simulated by starting from $X_0 \sim \mathcal{U}(50, 100)$ independently from the Brownian motion.

4.2.2 Discretization error

We take $X_0 = 100$, $M = 10^4$, $\lambda = h^{-3/8}$, in Figure 2; we monitor the simulation error by MC regression, namely, we track

$$\sup_{0 \leq i \leq N} \frac{1}{M} \sum_{m=1}^M (\hat{Y}_{t_i}^{h, \lambda(h), m} - Y_{t_i}^{Bin, m})^2 \quad (4.3)$$

as a function of h . The numerical result shows, in this example, our penalized scheme converges to the reference value more rapidly than what we have proved theoretically in Corollary 4.3; the error turns out to be roughly constant below $h = 0.01$ due to the limit of capacity of other chosen simulation parameters (e.g. M and l). On the one hand, these experiments show that our numerical scheme is efficient and accurate. On the other hand, the convergence order seems to be close to 1 in this specific case, which can not be explained by our current analysis. Further improvement is left to future works.

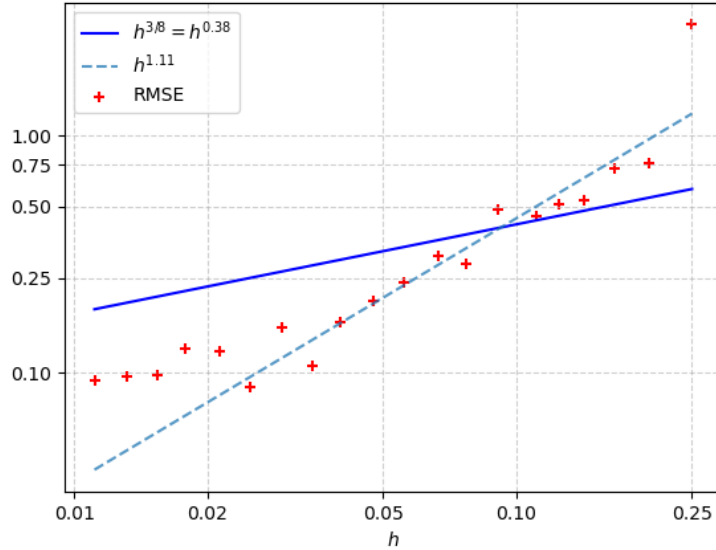


Figure 2: For $4 \leq N \leq 100$, we report the estimation error (4.3). The solid blue line represents the expected rate $h^{3/8}$ and the dashed blue line is fitted by polynomial regression based on the tracked errors, the later is at around $h^{1.11}$.

4.3 Proof of Theorem 4.1

In this subsection, we focus on the error w.r.t. the time step h , while the penalty parameter λ is fixed. To alleviate notations, we remove the notation λ in the processes, i.e. we write $Y, Z, Y^h, \mathcal{E}_i^f$ instead of $Y^\lambda, Z^\lambda, Y^{\lambda,h}, \mathcal{E}_i^{f,\lambda}$, it will be clear in the context. Besides, it is essential to keep track of the impact of λ on the different constants arising in the error analysis. For this, we will use a C as a generic constant (changing from line to line), whose values do depend on T, f, ξ, S and other universal parameters, but not on λ . W.l.o.g, we assume in the following $h \leq 1$.

▷ *Step 1:* For any $0 \leq i \leq N - 1$, define the function

$$\mathcal{V}_i : \Omega \times \mathbb{R} \rightarrow \mathbb{R}, y \mapsto y - f^\lambda(t_i, y)h.$$

To avoid any confusion, we will keep writing f^λ to insist on the dependence of λ . Observed that \mathcal{V}_i is a \mathcal{F}_{t_i} -adapted stochastic mapping since the generator f^λ is random. Then one has

$$\mathcal{V}_i(Y_{t_i}^h) = \mathbb{E}_{t_i} \left[Y_{t_{i+1}}^h \right], \quad \mathcal{V}_i(Y_{t_i}) = \mathbb{E}_{t_i} \left[Y_{t_{i+1}} + \mathcal{E}_i^f \right],$$

$$Y_{t_i}^h = \mathcal{V}_i^{-1} \left(\mathbb{E}_{t_i} \left[Y_{t_{i+1}}^h \right] \right), \quad Y_{t_i} = \mathcal{V}_i^{-1} \left(\mathbb{E}_{t_i} \left[Y_{t_{i+1}} + \mathcal{E}_i^f \right] \right). \quad (4.4)$$

The invertibility of \mathcal{V}_i (in the y -variable) is justified in the following lemma.

Lemma 4.5. *Suppose f fulfills $(\mathbf{H}_1$ -iii) and (\mathbf{H}_m) , then \mathcal{V}_i is invertible. In addition, $\forall h \leq \frac{1}{2C_{Lip}^f}$, the function \mathcal{V}_i^{-1} is $(1 + 2C_{Lip}^f h)$ -Lipschitz continuous uniformly in i , namely $\forall 0 \leq i \leq N-1$ and $\forall w, w' \in \mathbb{R}$,*

$$|\mathcal{V}_i^{-1}(w) - \mathcal{V}_i^{-1}(w')| \leq (1 + 2C_{Lip}^f h)|w - w'|.$$

Proof. We first consider the simple case where $f \equiv 0$ and denote $\mathcal{V}_i^0(y) = y - \lambda(y - S_{t_i})^- h$. Then $(\mathcal{V}_i^0)^{-1}$ is well defined since \mathcal{V}_i^0 is continuously increasing in y . Without loss of generality, suppose $y \geq y'$:

$$0 \leq \mathcal{V}_i^0(y) - \mathcal{V}_i^0(y') = \begin{cases} y - y', & \text{if } y, y' \geq S_{t_i}, \\ (1 + \lambda h)(y - y'), & \text{if } y, y' < S_{t_i}, \\ y - (1 + \lambda h)y' + \lambda h S_{t_i}, & \text{if } y \geq S_{t_i} \geq y'. \end{cases}$$

In the third case, $y - (1 + \lambda h)y' + \lambda h S_{t_i} \geq y - (1 + \lambda h)y' + \lambda h y' = y - y'$. It follows that

$$\mathcal{V}_i^0(y) - \mathcal{V}_i^0(y') \geq y - y', \forall y \geq y',$$

so $(\mathcal{V}_i^0)^{-1}$ is 1-Lipschitz uniformly in λ .

Now let $f^\lambda(t, y) = f(t, y) + \lambda(y - S_{t_i})^-$ with C_{Lip}^f -Lipschitz continuous f . Under (\mathbf{H}_m) , \mathcal{V}_i preserves the same monotonicity as \mathcal{V}_i^0 , so \mathcal{V}_i^{-1} is also well defined. Observe that the advertised result is equivalent to prove that for any $y, y' \in \mathbb{R}$,

$$(1 + 2C_{Lip}^f h)|\mathcal{V}_i(y) - \mathcal{V}_i(y')| \geq |y - y'|.$$

We start by writing the decomposition of \mathcal{V}_i :

$$\begin{aligned} |\mathcal{V}_i(y) - \mathcal{V}_i(y')| &= |y - f(t_i, y)h - \lambda(y - S_{t_i})^- h - (y' - f(t_i, y')h - \lambda(y' - S_{t_i})^- h)| \\ &\geq |\mathcal{V}_i^0(y) - \mathcal{V}_i^0(y')| - |f(t_i, y) - f(t_i, y')| h \\ &\geq (1 - C_{Lip}^f h)|y - y'|. \end{aligned}$$

Now it's easy to show that, $(1 + 2C_{Lip}^f h)(1 - C_{Lip}^f h) \geq 1$, for all $h \leq \frac{1}{2C_{Lip}^f}$. \square

\triangleright *Step 2:* Recall Gronwall lemma in discrete case: for any non-negative sequences $(a_k)_{0 \leq k \leq N}$, $(b_k)_{0 \leq k \leq N}$ and $(c_k)_{0 \leq k \leq N}$ satisfying, with $\gamma \geq 0$,

$$a_{k-1} + c_{k-1} \leq (1 + \gamma h)a_k + b_{k-1},$$

then one has

$$a_k + \sum_{i=k}^{N-1} c_i \leq e^{\gamma(T-t_k)} \left(a_N + \sum_{i=k}^{N-1} b_i \right).$$

From (4.4) and the previous lemma, the sequence $(\mathbb{E} [(Y_{t_i}^h - Y_{t_i})^2])_{0 \leq i \leq N}$ satisfies

$$\begin{aligned} \mathbb{E} [(Y_{t_i}^h - Y_{t_i})^2] &= \mathbb{E} \left[\left(\mathcal{V}_i^{-1} \left(\mathbb{E}_{t_i} [Y_{t_{i+1}}^h] \right) - \mathcal{V}_i^{-1} \left(\mathbb{E}_{t_i} [Y_{t_{i+1}} + \mathcal{E}_i^f] \right) \right)^2 \right] \\ &\leq (1 + Ch)^2 \mathbb{E} \left[\left(\mathbb{E}_{t_i} [Y_{t_{i+1}}^h - Y_{t_{i+1}}] - \mathbb{E}_{t_i} [\mathcal{E}_i^f] \right)^2 \right] \\ &\leq (1 + Ch)^2 \left\{ (1 + h) \mathbb{E} [|Y_{t_{i+1}}^h - Y_{t_{i+1}}|^2] + \left(1 + \frac{1}{h} \right) \mathbb{E} [\mathbb{E}_{t_i} [\mathcal{E}_i^f]^2] \right\} \\ &\leq (1 + Ch) \mathbb{E} [(Y_{t_{i+1}}^h - Y_{t_{i+1}})^2] + \frac{C}{h} \mathbb{E} [\mathbb{E}_{t_i} [\mathcal{E}_i^f]^2] \end{aligned}$$

where we have used Young inequality at the third line and $h \leq 1 \wedge \frac{1}{2C_{Lip}^f}$ at the last line. Gronwall lemma combined with $Y_{t_N}^h = Y_{t_N} = \xi$ gives,

$$\mathbb{E} [(Y_{t_i}^h - Y_{t_i})^2] \leq C \sum_{j=i}^{N-1} \frac{1}{h} \mathbb{E} [\mathbb{E}_{t_j} [\mathcal{E}_j^f]^2].$$

Hence, for h small enough,

$$\sup_{0 \leq i \leq N} \mathbb{E} [(Y_{t_i}^h - Y_{t_i})^2] \leq C \sum_{j=0}^{N-1} \frac{1}{h} \mathbb{E} [\mathbb{E}_{t_j} [\mathcal{E}_j^f]^2]. \quad (4.5)$$

▷ *Step 3:* In this step, we prove the above upper bound is $O(h + \lambda^2 h^{3/2})$. Globally, we will separate the sum into several parts and then investigate each of them. The general integrals are easy to deal with. The term with $(Y_s - S_s)^-$ will give rise to a local time contribution which is the hardest to analyze. We first divide the perturbation \mathcal{E}_i^f into I_i and II_i : from (4.2),

$$\begin{aligned} \mathcal{E}_i^f &= \int_{t_i}^{t_{i+1}} (f(s, Y_s) - f(t_i, Y_{t_i})) \mathbf{d}s + \lambda \int_{t_i}^{t_{i+1}} ((Y_s - S_s)^- - (Y_{t_i} - S_{t_i})^-) \mathbf{d}s \\ &=: I_i + \lambda II_i. \end{aligned}$$

In the following calculus, we will deduce that,

$$\sum_{i=0}^{N-1} \frac{1}{h} \mathbb{E} [\mathbb{E}_{t_i} [I_i]^2] \leq O(h) \quad (4.6)$$

and

$$\sum_{i=0}^{N-1} \frac{1}{h} \mathbb{E} [\mathbb{E}_{t_i} [II_i]^2] \leq O(h^{3/2}). \quad (4.7)$$

With (4.6) and (4.7) at hand and in view of (4.5), we directly complete the proof of Theorem 4.1.

▷ *Step 4: Estimation for I_i .* In fact, we have

$$I_i = \int_{t_i}^{t_{i+1}} [f(t_i, Y_s) - f(t_i, Y_{t_i})] \mathbf{d}s + \int_{t_i}^{t_{i+1}} [f(s, Y_s) - f(t_i, Y_s)] \mathbf{d}s.$$

For all $s \in [t_i, t_{i+1})$,

$$\begin{aligned} & |f(t_i, Y_s) - f(t_i, Y_{t_i})| \\ & \leq C_{Lip}^f \left(\int_{t_i}^s |f(u, Y_u)| \mathbf{d}u + \lambda \int_{t_i}^s (Y_u - S_u)^- \mathbf{d}u + \left| \int_{t_i}^s Z_u \mathbf{d}B_u \right| \right) \\ & \leq C \left(\sup_{t_i \leq s < t_{i+1}} (|f(s, 0)| + |Y_s| + \lambda(Y_s - S_s)^-) h + \sup_{t_i \leq s < t_{i+1}} \left| \int_{t_i}^s Z_u \mathbf{d}B_u \right| \right). \end{aligned}$$

Thanks to the bound of $f(\cdot, 0)$, the estimate (A.3) and Theorem 3.4, we have

$$\sup_{\lambda > 0} \mathbb{E} \left[\sup_{0 \leq s \leq T} (|f(s, 0)| + |Y_s| + \lambda(Y_s - S_s)^-)^2 + \int_0^T |Z_s|^2 \mathbf{d}s \right] < \infty. \quad (4.8)$$

Therefore, by Burkholder-Davis-Gundy (BDG in short) inequality, we obtain

$$\begin{aligned} \sum_{i=0}^{N-1} \frac{1}{h} \mathbb{E} \left[\mathbb{E}_{t_i} \left[\int_{t_i}^{t_{i+1}} |f(t_i, Y_s) - f(t_i, Y_{t_i})| \mathbf{d}s \right]^2 \right] & \leq C \sum_{i=0}^{N-1} h \mathbb{E} \left[\sup_{t_i \leq s < t_{i+1}} |f(t_i, Y_s) - f(t_i, Y_{t_i})|^2 \right] \\ & \leq C \sum_{i=0}^{N-1} h \left(h^2 + \mathbb{E} \left[\int_{t_i}^{t_{i+1}} |Z_u|^2 \mathbf{d}u \right] \right) \\ & \leq Ch \left(1 + \mathbb{E} \left[\int_0^T |Z_t|^2 \mathbf{d}t \right] \right) = O(h). \end{aligned}$$

In addition, thanks to $\frac{1}{2}$ -Hölder continuous in t of f , clearly

$$\sum_i \frac{1}{h} \mathbb{E} \left[\mathbb{E}_{t_i} \left[\int_{t_i}^{t_{i+1}} |f(s, Y_s) - f(t_i, Y_s)| \mathbf{d}s \right]^2 \right] = O(h).$$

We have proved (4.6).

▷ *Step 5: Estimation for II_i .* We apply Tanaka's formula and denote L the local time of $Y - S$ at level 0, by taking the conditional expectation, it follows

$$\begin{aligned} \mathbb{E}_{t_i} [(Y_s - S_s)^- - (Y_{t_i} - S_{t_i})^-] & = \mathbb{E}_{t_i} \left[\int_{t_i}^s \mathbf{1}_{\{Y_u \leq S_u\}} (f(u, Y_u) + U_u) \mathbf{d}u + \lambda \int_{t_i}^s (Y_u - S_u)^- \mathbf{d}u \right] \\ & \quad + \mathbb{E}_{t_i} \left[\frac{1}{2} [L_s - L_{t_i}] + \int_{t_i}^s \mathbf{1}_{Y_u \leq S_u} \mathbf{d}A_u \right] \end{aligned}$$

where A is the non-decreasing process of S in $(\mathbf{H}_2\text{-i})$. Consider the integral on small interval $[t_i, t_{i+1})$, we get:

$$\begin{aligned} \mathbb{E}_{t_i} \left[\int_{t_i}^{t_{i+1}} [(Y_s - S_s)^- - (Y_{t_i} - S_{t_i})^-] \mathbf{d}s \right] &\leq h \mathbb{E}_{t_i} \left[\int_{t_i}^{t_{i+1}} \mathbf{1}_{\{Y_u \leq S_u\}} |f(u, Y_u) + U_u| \mathbf{d}u \right] \\ &\quad + \lambda h \mathbb{E}_{t_i} \left[\int_{t_i}^{t_{i+1}} (Y_u - S_u)^- \mathbf{d}u \right] \\ &\quad + \frac{h}{2} \mathbb{E}_{t_i} [\Delta L_i] + h \mathbb{E}_{t_i} [\Delta A_i], \end{aligned}$$

where $\Delta L_i := L_{t_{i+1}} - L_{t_i}$, $\Delta A_i := A_{t_{i+1}} - A_{t_i}$. We denote, for $0 \leq i \leq N-1$,

$$F_i := \int_{t_i}^{t_{i+1}} \mathbf{1}_{\{Y_u \leq S_u\}} |f(u, Y_u) + U_u| \mathbf{d}u, \quad E_i := \int_{t_i}^{t_{i+1}} (Y_u - S_u)^- \mathbf{d}u.$$

All in all, we get

$$\begin{aligned} \sum_{i=0}^{N-1} \frac{1}{h} \mathbb{E} [\mathbb{E}_{t_i} [II_i]^2] &\leq 4 \sum_{i=0}^{N-1} \frac{1}{h} \left\{ h^2 \mathbb{E} [\mathbb{E}_{t_i} [F_i]^2] + \lambda^2 h^2 \mathbb{E} [\mathbb{E}_{t_i} [E_i]^2] \right. \\ &\quad \left. + \frac{h^2}{4} \mathbb{E} [\mathbb{E}_{t_i} [\Delta L_i]^2] + h^2 \mathbb{E} [\mathbb{E}_{t_i} [\Delta A_i]^2] \right\}. \end{aligned} \quad (4.9)$$

For the first term of right hand side, by Jensen's inequality and Cauchy-Schwarz inequality, it follows

$$\begin{aligned} \sum_{i=0}^{N-1} \frac{1}{h} h^2 \mathbb{E} [\mathbb{E}_{t_i} [F_i]^2] &\leq Ch^2 \sum_{i=0}^{N-1} \mathbb{E} \left[\int_{t_i}^{t_{i+1}} \mathbf{1}_{\{Y_u \leq S_u\}} [|f(u, 0)|^2 + |Y_u|^2 + |U_u|^2] \mathbf{d}u \right] \\ &\leq Ch^2 (\|f(\cdot, 0)\|_{\mathbb{H}^2}^2 + \|Y\|_{\mathbb{H}^2}^2 + \|U\|_{\mathbb{H}^2}^2) \leq Ch^2 \end{aligned} \quad (4.10)$$

using again (4.8). For the second sum in (4.9), we use again the bound in Theorem 3.4:

$$\mathbb{E} [\mathbb{E}_{t_i} [E_i]^2] \leq h^2 \mathbb{E} \left[\left(\sup_{u \in [t_i, t_{i+1}]} (Y_u - S_u)^- \right)^2 \right] \leq h^2 \frac{\bar{\kappa}_\infty^2}{\lambda^2},$$

which implies

$$\sum_{i=0}^{N-1} \frac{1}{h} \lambda^2 h^2 \mathbb{E} [\mathbb{E}_{t_i} [E_i]^2] = O(h^2). \quad (4.11)$$

Now we deal with the third term and the fourth in sum (4.9), applying Lemma B.3:

$$\sum_{i=0}^{N-1} \frac{1}{h} h^2 \mathbb{E} [\mathbb{E}_{t_i} [\Delta L_i]^2] + \sum_{i=0}^{N-1} \frac{1}{h} h^2 \mathbb{E} [\mathbb{E}_{t_i} [\Delta A_i]^2]$$

$$\leq \sqrt{6}h\mathbb{E}[L_T^2]^{1/2}\mathbb{E}\left[\sup_{0\leq i\leq N-1}\mathbb{E}_{t_i}[\Delta L_i]^2\right]^{1/2} + \sqrt{6}h\mathbb{E}[A_T^2]^{1/2}\mathbb{E}\left[\sup_{0\leq i\leq N-1}\mathbb{E}_{t_i}[\Delta A_i]^2\right]^{1/2}. \quad (4.12)$$

A easy computation leads to $\mathbb{E}[L_T^2 + A_T^2] < \infty$ thanks to $f(\cdot, 0), U, V, S \in \mathcal{S}^2$ and Theorem 3.1. Thus it remains to estimate $\mathbb{E}\left[\sup_{0\leq i\leq N-1}\mathbb{E}_{t_i}[|\Delta L_i|]^2\right]^{1/2} + \mathbb{E}\left[\sup_{0\leq i\leq N-1}\mathbb{E}_{t_i}[|\Delta A_i|]^2\right]^{1/2}$. Denote $\Delta(Y_i - S_i) := Y_{t_{i+1}} - S_{t_{i+1}} - (Y_{t_i} - S_{t_i})$. From Itô-Tanaka formula, we have, $\forall 0 \leq i \leq N-1$,

$$\begin{aligned} \frac{1}{2}\mathbb{E}_{t_i}[\Delta L_i] &= \mathbb{E}_{t_i}[\Delta(Y_i - S_i)^-] - \mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}}\mathbf{1}_{Y_u \leq S_u}[f(u, Y_u) + U_u]\mathbf{d}u\right] \\ &\quad - \lambda\mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}}(Y_u - S_u)^-\mathbf{d}u\right] - \mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}}\mathbf{1}_{Y_u \leq S_u}\mathbf{d}A_u\right] \\ &\leq \mathbb{E}_{t_i}[|\Delta(Y_i - S_i)|] - \mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}}\mathbf{1}_{Y_u \leq S_u}[f(u, Y_u) + U_u]\mathbf{d}u\right], \\ \Delta(Y_i - S_i) &= -\int_{t_i}^{t_{i+1}}(f(s, Y_s) + U_s)\mathbf{d}s - \lambda\int_{t_i}^{t_{i+1}}(Y_s - S_s)^-\mathbf{d}s - \Delta A_i + \int_{t_i}^{t_{i+1}}(Z_s - V_s)\mathbf{d}B_s. \end{aligned}$$

So, using Theorem 3.4, we deduce

$$\mathbb{E}_{t_i}[\Delta L_i] \leq C\left\{h\mathbb{E}_{t_i}\left[\sup_{0\leq t\leq T}(|f(t, 0)| + |Y_t| + |U_t|)\right] + h + \mathbb{E}_{t_i}[\Delta A_i] + \mathbb{E}_{t_i}\left[\left|\int_{t_i}^{t_{i+1}}(Z_s - V_s)\mathbf{d}B_s\right|\right]\right\}.$$

Thanks to the assumptions $f(\cdot, 0), U, \xi, S \in \mathcal{S}^2$, the random variable $\zeta := \sup_{0\leq t\leq T}(|f(t, 0)| + |Y_t| + |U_t|)$ belongs to \mathbb{L}^2 and $\{\mathbb{E}_{t_i}[\zeta], 0 \leq i \leq N-1\}$ is a \mathbb{L}^2 -martingale, so

$$\begin{aligned} \mathbb{E}\left[\sup_{0\leq i\leq N-1}\mathbb{E}_{t_i}[\Delta L_i]^2\right] &\leq C\left\{h^2 + h^2\mathbb{E}\left[\sup_{0\leq i\leq N-1}\mathbb{E}_{t_i}[\zeta]^2\right] + \mathbb{E}\left[\sup_{0\leq i\leq N-1}\mathbb{E}_{t_i}[\Delta A_i]^2\right]\right. \\ &\quad \left.+ \mathbb{E}\left[\sup_{0\leq i\leq N-1}\mathbb{E}_{t_i}\left[\left|\int_{t_i}^{t_{i+1}}(Z_s - V_s)\mathbf{d}B_s\right|^2\right]\right]\right\}. \quad (4.13) \end{aligned}$$

We handle the terms at the right-hand side of (4.13) separately. First, applying Doob's inequality and Jensen's inequality,

$$\mathbb{E}\left[\sup_{0\leq i\leq N-1}\mathbb{E}_{t_i}[\zeta]^2\right] \leq 4\sup_{0\leq i\leq N-1}\mathbb{E}[\mathbb{E}_{t_i}[\zeta]^2] \leq 4\mathbb{E}[\zeta^2] < \infty. \quad (4.14)$$

Second, denote $\Delta S_i := S_{t_{i+1}} - S_{t_i}$, then $\Delta A_i = \Delta S_i - \int_{t_i}^{t_{i+1}}U_s\mathbf{d}s - \int_{t_i}^{t_{i+1}}V_s\mathbf{d}B_s$. With $(\mathbf{H}_2\text{-ii})$, we get similarly,

$$\mathbb{E}\left[\sup_{0\leq i\leq N-1}\mathbb{E}_{t_i}[\Delta A_i]^2\right] \leq 2\left(\mathbb{E}\left[\sup_{0\leq i\leq N-1}\mathbb{E}_{t_i}[|\Delta S_i|]^2\right] + h^2\mathbb{E}\left[\sup_{0\leq i\leq N-1}\mathbb{E}_{t_i}\left[\sup_{0\leq t\leq T}|U_t|\right]^2\right]\right)$$

$$\leq C(h + h^2 \|U\|_{\mathcal{L}^2}). \quad (4.15)$$

Third, introduce the following notation of regularized PBSDE solution:

For any fixed $\lambda > 0$, assume $(Y^{\lambda,\varepsilon}, Z^{\lambda,\varepsilon})$ be the solution of (1.2) with respect to $(\xi, f^{\lambda,\varepsilon}, S)$ and

$$f^{\lambda,\varepsilon}(t, y, z) = f(t, y) + \lambda \delta_\varepsilon(y - S_t) \quad (4.16)$$

where $\varepsilon > 0$, $\delta_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^+$ s.t. $\delta_\varepsilon(x) = \frac{(x-\varepsilon)^2}{4\varepsilon}$ for $x \in [-\varepsilon, \varepsilon]$ and $\delta_\varepsilon(x) = x^-$ for $|x| > \varepsilon$.

In the following context, we use $(Y^\varepsilon, Z^\varepsilon)$ to denote $(Y^{\lambda,\varepsilon}, Z^{\lambda,\varepsilon})$ for the fixed λ . Notice that the bound of Z^ε in Proposition B.4 holds uniformly in $t, \omega, \lambda, \varepsilon$, thus applying BDG inequality we obtain

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} \left[\left\| \int_{t_i}^{t_{i+1}} (Z_s - V_s) \mathbf{d}B_s \right\|^2 \right] \right] &= \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} \left[\left\| \int_{t_i}^{t_{i+1}} (Z_s^\varepsilon - V_s) \mathbf{d}B_s \right\|^2 \right] \right] \\ &\leq C \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} \left[\left(\int_{t_i}^{t_{i+1}} |Z_s^\varepsilon - V_s|^2 \mathbf{d}s \right)^{1/2} \right]^2 \right] \leq Ch. \end{aligned} \quad (4.17)$$

So, with (4.13), (4.14), (4.15), (4.17), we get,

$$\mathbb{E} \left[\sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [\Delta L_i]^2 \right] + \mathbb{E} \left[\sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [\Delta A_i]^2 \right] \leq Ch.$$

Back to (4.12), we finally have

$$\sum_{i=0}^{N-1} \frac{1}{h} h^2 \mathbb{E} [\mathbb{E}_{t_i} [\Delta L_i]^2] + \sum_{i=0}^{N-1} \frac{1}{h} h^2 \mathbb{E} [\mathbb{E}_{t_i} [\Delta A_i]^2] \leq Ch^{3/2}. \quad (4.18)$$

All in all, combining with (4.10), (4.11), (4.18), we prove successfully (4.7). Wrapping up our arguments, the proof of Theorem 4.1 is finished. \square

Proof of Corollary 4.4. Using the estimation $\mathbb{E} \left[\sup_{0 \leq t \leq T} |(Y_t^\lambda - S_t)^-|^2 \right] = O(\frac{1}{\lambda})$ in Theorem 3.3, the estimation of I_i in (4.6) becomes

$$\sum_{i=0}^{N-1} \frac{1}{h} \mathbb{E} [\mathbb{E}_{t_i} [I_i]^2] = O(\lambda h).$$

Regarding (4.7) we obtain

$$\sum_{i=0}^{N-1} \frac{1}{h} \mathbb{E} [\mathbb{E}_{t_i} [II_i]^2] \leq O(\sqrt{\lambda} h^2 + h^{3/2})$$

because the estimation in (4.11) becomes

$$\sum_{i=0}^{N-1} \frac{1}{h} \lambda^2 h^2 \mathbb{E} [\mathbb{E}_{t_i} [E_i]^2] = O(\sqrt{\lambda} h^2);$$

and (4.18) turns to

$$\sum_{i=0}^{N-1} \frac{1}{h} h^2 \mathbb{E} [\mathbb{E}_{t_i} [\Delta L_i]^2] + \sum_{i=0}^{N-1} \frac{1}{h} h^2 \mathbb{E} [\mathbb{E}_{t_i} [\Delta A_i]^2] = O(\sqrt{\lambda} h^2 + h^{3/2}).$$

□

A Appendix

A.1 Proof of Lemma 2.2

Let us denote by C any finite positive constant occurring in the proof: it will depend only on T , the bounds on μ, σ and the regularity constant of g . For any $0 \leq t < s \leq T$,

$$\mathbb{E}_t [|S_s - S_t|] \leq C \left(|s - t|^{\frac{1}{2}} + \mathbb{E}_t [(e^{C|X_s|} + e^{C|X_t|}) |X_s - X_t|] \right).$$

Since μ and σ are bounded, it is standard to show that

$$\mathbb{E}_t \left[\sup_{t \leq r \leq T} e^{C|X_r|} \right] \leq C e^{C|X_t|},$$

with a possible increase of C . Besides,

$$\left(\mathbb{E}_t [|X_t - X_s|^2] \right)^{\frac{1}{2}} \leq C |t - s|^{\frac{1}{2}}$$

for some new constant C . Therefore, using Cauchy-Schwarz inequality gives

$$\sup_{0 \leq t < s \leq T} \frac{\mathbb{E}_t [|S_s - S_t|]}{\sqrt{s - t}} \leq C \sup_{0 \leq t \leq T} e^{C|X_t|}.$$

Taking the power p and the expectation gives the advertised result. □

A.2 A priori estimates for general BSDEs

In this subsection, we will give some result of a priori estimate for general BSDEs.

Proposition A.1 ([PR14, Proposition 5.2, p.358, with $p \geq 2$]). *Suppose $(Y, Z) \in \mathcal{S}^2 \times \mathbb{H}^2$ satisfying the scalar equation*

$$Y_t = \xi + \int_t^T \mathbf{d}F_s - \int_t^T Z_s \mathbf{d}B_s$$

where $\xi \in \mathbb{L}^2$, $F \in \mathcal{S}^2$ and F is a scalar process with a.s. bounded variation. Let $p \geq 2$, we assume there exist stochastic processes (D, R, N, Υ) satisfying

$$\mathbf{d}D_t + Y_t \mathbf{d}F_t \leq \mathbf{d}R_t + |Y_t| \mathbf{d}N_t + |Y_t|^2 \mathbf{d}\Upsilon_t + \frac{\eta}{2} |Z_t|^2 \mathbf{d}t, \quad (\text{A.1})$$

where

- D, R, N are progressively measurable, increasing, continuous processes with $D_0 = R_0 = N_0 = 0$;
- Υ is a progressively measurable process with bounded variation and $\Upsilon_0 = 0$;
- $\eta < 1$.

If $\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t e^{\Upsilon_t}|^p \right] < \infty$, then one has the a priori estimate,

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq t \leq T} |e^{\Upsilon_t} Y_t|^p \right] + \mathbb{E} \left[\left(\int_0^T e^{2\Upsilon_s} \mathbf{d}D_s \right)^{p/2} \right] + \mathbb{E} \left[\left(\int_0^T e^{2\Upsilon_s} |Z_s|^2 \mathbf{d}s \right)^{p/2} \right] \\ & \leq C_{p,\eta} \mathbb{E} \left[|e^{\Upsilon_T} Y_T|^p + \left(\int_0^T e^{2\Upsilon_s} \mathbf{d}R_s \right)^{p/2} + \left(\int_0^T e^{\Upsilon_s} \mathbf{d}N_s \right)^p \right], \end{aligned}$$

for some constant $C_{p,\eta}$ depending only on p, η .

A.3 A priori \mathbb{L}^p -estimates for RBSDE

In this subsection, we present some estimation results for Reflected BSDEs.

Proposition A.2 (A priori estimate for $p \geq 2$). *Let $p \geq 2$ and let $\{(Y_t, Z_t, K_t), 0 \leq t \leq T\}$ be a solution in $\mathcal{S}^p \times \mathbb{H}^p \times \mathcal{S}^p$ of the RBSDE w.r.t. (ξ, f, S) . If (\mathbf{H}_1) is fulfilled, then there exists a constant C (depending only on T, f, p) such that*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t|^p + \left(\int_0^T |Z_t|^2 \mathbf{d}t \right)^{p/2} + K_T^p \right] \leq C \mathbb{E} \left[|\xi|^p + \left(\int_0^T |f(t, 0, 0)| \mathbf{d}t \right)^p + \sup_{0 \leq t \leq T} (S_t^+)^p \right].$$

Proof. The proof will be done in several steps.

Step 1: Apply Proposition A.1 on (1.1) using obvious notations, we get

$$\begin{aligned} Y_t \mathbf{d}F_t &= Y_t f(t, Y_t, Z_t) \mathbf{d}t + Y_t \mathbf{d}K_t \\ &\leq S_t \mathbf{d}K_t + |Y_t| \left(|f(t, 0, 0)| + C_{Lip}^f |Y_t| + C_{Lip}^f |Z_t| \right) \mathbf{d}t \\ &\leq (S_t)^+ \mathbf{d}K_t + |Y_t| |f(t, 0, 0)| \mathbf{d}t + (C_{Lip}^f + (C_{Lip}^f)^2) |Y_t|^2 \mathbf{d}t + \frac{1}{4} |Z_t|^2 \mathbf{d}t, \end{aligned}$$

so (A.1) holds with

$$D = 0, \quad \mathbf{d}R_t = (S_t)^+ \mathbf{d}K_t, \quad \mathbf{d}N_t = |f(t, 0, 0)| \mathbf{d}t, \quad \mathbf{d}\Upsilon_t = (C_{Lip}^f + \frac{1}{2}(C_{Lip}^f)^2) \mathbf{d}t, \quad \eta = \frac{1}{2}.$$

For the rest of proof, let us denote by C any generic constant which value may change from line to line, but it depends only on T, f, p . Therefore, from Proposition A.1, we get the intermediate a priori estimate for Y and Z , with $p \geq 2$, and for any $\varepsilon > 0$,

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t|^p \right] + \mathbb{E} \left[\left(\int_0^T |Z_t|^2 \mathbf{d}t \right)^{p/2} \right] \\ & \leq C \mathbb{E} \left[|\xi|^p + \left(\int_0^T |f(t, 0, 0)| \mathbf{d}t \right)^p + \left(\int_0^T (S_t)^+ \mathbf{d}K_t \right)^{p/2} \right] \\ & \leq C \mathbb{E} \left[|\xi|^p + \left(\int_0^T |f(t, 0, 0)| \mathbf{d}t \right)^p \right] + \frac{C^2}{4\varepsilon} \mathbb{E} \left[\sup_{0 \leq t \leq T} |S_t^+|^p \right] + \varepsilon \mathbb{E} [|K_T|^p]. \end{aligned} \quad (\text{A.2})$$

Step 2: We now estimate K_T , simply by rewriting the formulae,

$$\begin{aligned} K_T &= Y_0 - \xi - \int_0^T f(t, Y_t, Z_t) \mathbf{d}t + \int_0^T Z_t \mathbf{d}B_t \\ &\leq \int_0^T |f(t, 0, 0)| \mathbf{d}t + C \sup_{0 \leq t \leq T} |Y_t| + C \int_0^T |Z_t| \mathbf{d}t + \left| \int_0^T Z_t \mathbf{d}B_t \right|. \end{aligned}$$

Then we take the power p and the expectation on both sides then get

$$\mathbb{E} [|K_T|^p] \leq C \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t|^p \right] + C \mathbb{E} \left[\left(\int_0^T |Z_t|^2 \mathbf{d}t \right)^{p/2} \right] + C \mathbb{E} \left[\left(\int_0^T |f(t, 0, 0)| \mathbf{d}t \right)^p \right],$$

where we use Burkholder-Davis-Gundy (BDG in short) inequality at the last step. Combining with (A.2) and the above, and taking ε small enough, we get the desired result. \square

A.4 Proof of Theorem 3.1

In this part, we present the proof of Theorem 3.1, which gives an important estimate result.

A.4.1 Existence, uniqueness for (Y, Z, K) and a priori estimates for $(Y^\lambda, Z^\lambda, K^\lambda)$

We follow a standard routine to prove the existence of penalized BSDE with increasing penalty parameter λ . The case of $p = 2$ is given in [EKP⁺97, Section 6] and the case of $1 \leq p < 2$ is stated in [HP12]. Here we extend their results to $p \geq 2$.

\triangleright *Step 0:* Define $f^\lambda(t, y, z) := f(t, y, z) + \lambda(y - S_t)^-$. The existence and uniqueness of (Y^λ, Z^λ) in \mathbb{L}^p is obvious thanks to the Lipschitz generator f^λ . The difficult part is related

to prove λ -uniform bounds. We could try to use estimates from [BDH⁺03]: on the one hand, f^λ satisfies the monotonicity condition [BDH⁺03, Assumption (H3)]

$$(y - y')(f^\lambda(t, y, z) - f^\lambda(t, y', z)) \leq C_{Lip}^f (y - y')^2$$

uniformly in λ , but on the other hand, the condition [BDH⁺03, Assumption (A)] writes as

$$\text{Sign}(y)f^\lambda(t, y, z) \leq |f^\lambda(t, 0, 0)| + C_{Lip}^f |y| + C_{Lip}^f |z|$$

where $|f^\lambda(t, 0, 0)|$ does strongly depend on the penalty parameter λ . So although the monotonicity condition is satisfied, we can not get an uniform upper bound in λ for PBSDE using the argument of [BDH⁺03].

Instead, we establish a simple proof as in Proposition A.2 using Proposition A.1.

▷ *Step 1:* Given $(Y^\lambda, Z^\lambda, K^\lambda)$ the solution of (1.2), since

$$Y_t^\lambda \mathbf{d}K_t^\lambda \leq S_t \mathbf{d}K_t^\lambda$$

(K^λ increases only when $Y_t^\lambda < S_t$) we can use again the argument in Proposition A.2 by replacing the process K by K^λ . Note that all generic constants C do not depend in λ . Then we get the desired a priori estimate

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^\lambda|^p + \left(\int_0^T |Z_s^\lambda|^2 \mathbf{d}s \right)^{p/2} + |K_T^\lambda|^p \right] \leq c, \quad (\text{A.3})$$

where c depends only on f, S, ξ, T, p .

From the comparison theorem for standard BSDE(see [EPM97, Theorem 2.2]), we have $Y_t^\lambda \leq Y_t^{\lambda'}, 0 \leq t \leq T, a.s.$ for any $0 < \lambda \leq \lambda'$; so there exists a process Y s.t. $Y_t^\lambda \rightarrow Y_t$, as $\lambda \rightarrow \infty, 0 \leq t \leq T, a.s..$ By Fatou's lemma and estimation (A.3), we easily get

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t|^p \right] \leq C$$

using again the convention of generic constant C independent on λ . Then, from the dominated convergence theorem, we have $\mathbb{E} \left[\int_0^T |Y_t^\lambda - Y_t|^p \mathbf{d}t \right] \rightarrow 0$ as $\lambda \rightarrow +\infty$.

▷ *Step 2:* Let us justify that $(Y^\lambda, Z^\lambda, K^\lambda)$ is a Cauchy sequence in $\mathcal{S}^p \times \mathbb{H}^p \times \mathcal{S}^p$. We apply again Proposition A.1 on $Y^{\lambda'} - Y^\lambda$. In fact, define

$$\Delta Y_t^\lambda := Y_t^{\lambda'} - Y_t^\lambda, \Delta Z_t^\lambda := Z_t^{\lambda'} - Z_t^\lambda, \Delta f_t := f(t, Y_t^{\lambda'}, Z_t^{\lambda'}) - f(t, Y_t^\lambda, Z_t^\lambda), \Delta K_t^\lambda = K_t^{\lambda'} - K_t^\lambda,$$

then, we have

$$\begin{aligned} \Delta Y_t^\lambda (\Delta f_t \mathbf{d}t + \mathbf{d}\Delta K_t^\lambda) &\leq \left(C_{Lip}^f + (C_{Lip}^f)^2 \right) |\Delta Y_t^\lambda|^2 \mathbf{d}t + \frac{1}{4} |\Delta Z_t^\lambda|^2 \mathbf{d}t \\ &\quad + (Y_t^\lambda - S_t)^- \mathbf{d}K_t^{\lambda'} + (Y_t^{\lambda'} - S_t)^- \mathbf{d}K_t^\lambda. \end{aligned} \quad (\text{A.4})$$

Let $\eta = 1/2$ and

$$D = 0, \quad \mathbf{d}R_t = (Y_t^\lambda - S_t)^- \mathbf{d}K_t^{\lambda'} + (Y_t^{\lambda'} - S_t)^- \mathbf{d}K_t^\lambda, \quad N = 0, \quad \mathbf{d}\Upsilon_t = \left(C_{Lip}^f + (C_{Lip}^f)^2 \right) \mathbf{d}t.$$

Then we deduce the a priori estimate for $(\Delta Y_t^\lambda, \Delta Z_t^\lambda)$:

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq t \leq T} |\Delta Y_t^\lambda|^p \right] + \mathbb{E} \left[\left(\int_0^T |\Delta Z_t^\lambda|^2 \mathbf{d}t \right)^{p/2} \right] \\ & \leq C \mathbb{E} \left[\left(\int_0^T (Y_t^\lambda - S_t)^- \mathbf{d}K_t^{\lambda'} + (Y_t^{\lambda'} - S_t)^- \mathbf{d}K_t^\lambda \right)^{p/2} \right] \\ & \leq C \sqrt{\mathbb{E} [|K_T^\lambda|^p]} \sqrt{\mathbb{E} \left[\sup_{0 \leq t \leq T} |(Y_t^{\lambda'} - S_t)^-|^p \right]} \\ & \quad + C \sqrt{\mathbb{E} [|K_T^{\lambda'}|^p]} \sqrt{\mathbb{E} \left[\sup_{0 \leq t \leq T} |(Y_t^\lambda - S_t)^-|^p \right]}. \end{aligned} \quad (\text{A.5})$$

Assume for a while that the following limit holds, its proof is postponed afterwards.

Lemma A.3. *Under the assumptions of Theorem 3.1, we have, for $p \geq 2$,*

$$\lim_{\lambda \rightarrow \infty} \mathbb{E} \left[\sup_{0 \leq t \leq T} |(Y_t^\lambda - S_t)^-|^p \right] = 0.$$

Then, from (A.5), we get, as $\lambda, \lambda' \rightarrow \infty$, the following convergence:

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |\Delta Y_t^\lambda|^p \right] + \mathbb{E} \left[\left(\int_0^T |\Delta Z_t^\lambda|^2 \mathbf{d}t \right)^{p/2} \right] \rightarrow 0.$$

Moreover, set

$$\Delta K_t^\lambda = \Delta Y_0^\lambda - \Delta Y_t^\lambda - \int_0^t \Delta f_s \mathbf{d}s + \int_0^t \Delta Z_s^\lambda \mathbf{d}B_s,$$

we have the following estimation for ΔK^λ ,

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |\Delta K_t^\lambda|^p \right] \leq C \left\{ \mathbb{E} \left[\sup_{0 \leq t \leq T} |\Delta Y_t^\lambda|^p \right] + \mathbb{E} \left[\left(\int_0^T |\Delta Z_t^\lambda|^2 \mathbf{d}t \right)^{p/2} \right] \right\}. \quad (\text{A.6})$$

So we deduce that $(Y^\lambda, Z^\lambda, K^\lambda)_\lambda$ is a Cauchy sequence in $\mathcal{S}^p \times \mathbb{H}^p \times \mathcal{S}^p$, denote by (Y, Z, K) its limit (for Y , it coincides with the previous monotone limit).

▷ *Step 3:* it remains to verify (Y, Z, K) is the solution to (1.1), this can be done by applying exactly the same arguments as [EKP⁺97, p.722]. We are done with the first result (3.1) of Theorem 3.1.

A.4.2 Proof of the bound (3.2) on penalization error

We use again Proposition A.1 as for Inequality (A.4). Before, we analyzed the difference $Y^{\lambda'} - Y^\lambda$; now we handle $Y - Y^\lambda$, the decomposition of which is really similar just doing as if $\lambda' = +\infty$ in the algebra of equations. Replacing $(Y^{\lambda'}, Z^{\lambda'}, K^{\lambda'})$ by (Y, Z, K) in the formulas (A.4) of $(\Delta Y^\lambda, \Delta Z^\lambda, \Delta K^\lambda)$, we get

$$\Delta Y_t^\lambda (\Delta f_t \mathbf{d}t + \mathbf{d}\Delta K_t^\lambda) \leq \left(C_{Lip}^f + (C_{Lip}^f)^2 \right) |\Delta Y_t^\lambda|^2 \mathbf{d}t + \frac{1}{4} |\Delta Z_t^\lambda|^2 \mathbf{d}t + (Y_t^\lambda - S_t)^- \mathbf{d}K_t.$$

Then, applying Proposition A.1 with $\eta = 1/2$ and $D = 0$, $\mathbf{d}R_t = (Y_t^\lambda - S_t)^- \mathbf{d}K_t$, $N = 0$, $\mathbf{d}\Upsilon_t = \left(C_{Lip}^f + (C_{Lip}^f)^2 \right) \mathbf{d}t$, we get

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |\Delta Y_t^\lambda|^p \right] + \mathbb{E} \left[\left(\int_0^T |\Delta Z_t^\lambda|^2 \mathbf{d}t \right)^{p/2} \right] \leq C \mathbb{E} \left[\left(\int_0^T (Y_t^\lambda - S_t)^- \mathbf{d}K_t \right)^{p/2} \right]$$

for some constant C independent on λ . We obtain a similar estimate on ΔK^λ by considering the analogous of (A.6). The proof of (3.2) is complete. \square

A.4.3 Proof of Lemma A.3

We extend the proof of [EKP⁺97, Lemma 6.1] when $p = 2$ to the case $p \geq 2$. As in the proof of Theorem 3.3, let $(\tilde{Y}^\lambda, \tilde{Z}^\lambda)$ be the solution (in $\mathcal{S}^p \times \mathbb{H}^p$) of the linear BSDE (3.4) which has the explicit representation

$$\tilde{Y}_t^\lambda = \mathbb{E}_t \left[e^{-\lambda(T-t)} \xi + \int_t^T e^{-\lambda(s-t)} f(s, Y_s^\lambda, Z_s^\lambda) \mathbf{d}s + \lambda \int_t^T e^{-\lambda(s-t)} S_s \mathbf{d}s \right].$$

\triangleright *Step 1:* We first prove that, as $\lambda \rightarrow \infty$, $\tilde{Y}_\tau^\lambda \rightarrow \xi \mathbf{1}_{\{\tau=T\}} + S_\tau \mathbf{1}_{\{\tau < T\}}$ in \mathbb{L}^p , for any $\tau \in \mathcal{T}_{0,T}$. In fact, we consider the following convergences, which hold in both *a.s.* and in \mathbb{L}^p ,

$$\lim_{\lambda \rightarrow +\infty} e^{-\lambda(T-\tau)} \xi = \xi \mathbf{1}_{\{\tau=T\}}, \quad (\text{A.7})$$

$$\lim_{\lambda \rightarrow +\infty} \lambda \int_\tau^T e^{-\lambda(t-\tau)} S_t \mathbf{d}t = S_\tau \mathbf{1}_{\{\tau < T\}}. \quad (\text{A.8})$$

The first identity (A.7) is trivial since $\xi \in \mathbb{L}^p$. To prove (A.8), write

$$\begin{aligned} S_\tau &= (1 - e^{-\lambda(T-\tau)}) S_\tau + e^{-\lambda(T-\tau)} S_\tau = S_\tau \lambda \int_\tau^T e^{-\lambda(t-\tau)} \mathbf{d}t + e^{-\lambda(T-\tau)} S_\tau, \\ \lambda \int_\tau^T e^{-\lambda(t-\tau)} S_t \mathbf{d}t - S_\tau \mathbf{1}_{\{\tau < T\}} &= \lambda \int_\tau^T e^{-\lambda(t-\tau)} (S_t - S_\tau) \mathbf{d}t + e^{-\lambda(T-\tau)} S_\tau \mathbf{1}_{\{\tau < T\}}. \end{aligned}$$

Obviously, $\lim_{\lambda \rightarrow +\infty} e^{-\lambda(T-\tau)} S_\tau \mathbf{1}_{\{\tau < T\}} \rightarrow 0$ holds *a.s.*; since $S \in \mathcal{S}^p$, from the dominated convergence theorem, this convergence holds also in \mathbb{L}^p . Moreover, on $\{\tau < T\}$, we have

$$\lambda \int_\tau^T e^{-\lambda(t-\tau)} |S_t - S_\tau| \mathbf{d}t \leq \int_0^\infty e^{-s} |S_{T \wedge (\tau+s/\lambda)} - S_\tau| \mathbf{d}s.$$

Therefore, thanks to the continuity of S , letting $\lambda \rightarrow +\infty$, the right-hand converges to 0 *a.s.*; thus, from the dominated convergence theorem, with $S \in \mathcal{S}^p$, we get

$$\lim_{\lambda \rightarrow +\infty} \lambda \int_{\tau}^T e^{-\lambda(t-\tau)} |S_t - S_{\tau}| dt = 0, \text{ in } \mathbb{L}^p.$$

We have proved (A.7)-(A.8). Hence, Jensen inequality ensures that the conditional expectation of $e^{-\lambda(T-\tau)}\xi + \lambda \int_{\tau}^T e^{-\lambda(t-\tau)} S_t dt$ given \mathcal{F}_{τ} converges to $\xi \mathbf{1}_{\{\tau=T\}} + S_{\tau} \mathbf{1}_{\{\tau < T\}}$ in \mathbb{L}^p .

To complete our proof, we only have to justify that $\int_{\tau}^T e^{-\lambda(t-\tau)} f(t, Y_t^{\lambda}, Z_t^{\lambda}) dt$ converges to 0 in \mathbb{L}^p . Indeed, from the Hölder inequality,

$$\mathbb{E} \left[\left| \int_{\tau}^T e^{-\lambda(t-\tau)} f(t, Y_t^{\lambda}, Z_t^{\lambda}) dt \right|^p \right] \leq \left(\frac{1}{q\lambda} \right)^{p/q} \mathbb{E} \left[\int_0^T |f(t, Y_t^{\lambda}, Z_t^{\lambda})|^p dt \right]$$

with $\frac{1}{p} + \frac{1}{q} = 1$. The expectation on the right-hand side is bounded uniformly in λ , because of the uniform bounds (A.3) and the assumptions on f ; thus the expectation on the left-hand goes to 0 as announced.

▷ *Step 2:* From [DM75, p. 220], we pass from " $Y_{\tau} \geq S_{\tau}$, *a.s.* for any stopping time τ " to " $Y_t \geq S_t$, for all times $t \in [0, T]$, *a.s.*"

Notice that from the comparison theorem [EPM97, Theorem 4.1], we have $Y_t \geq Y_t^{\lambda} \geq \tilde{Y}_t^{\lambda}$, $0 \leq t \leq T$, *a.s.*, so $(Y_t^{\lambda} - S_t)^- \downarrow 0$ *a.s.*; Dini's theorem yields the convergence uniformly in t , and owing to the dominated convergence theorem, we obtain the statement of Lemma A.3. \square

B Proof of Section 4

We first introduce some results from [EPM97, Proposition 5.3] and we restrict to the case where f does not depend on z .

Lemma B.1. *Assume $\xi \in \mathbb{D}^{1,2}$, f is continuously differentiable in y , with uniformly bounded and continuous derivatives and such that*

- for each $y \in \mathbb{R}$, $f(\cdot, y)$ is in $\mathbb{H}^{1,2}$;
- $\int_0^T \mathbb{E} \left[\int_0^T |D_{\theta} f(t, Y_t)|^2 dt \right] d\theta < \infty$ and there exists a constant C s.t. for any $\theta \in [0, T]$ *a.e.*, for any $t \in [0, T]$ and any $(y_1, y_2) \in \mathbb{R}^2$,

$$|D_{\theta} f(t, \omega, y_1) - D_{\theta} f(t, \omega, y_2)| \leq C|y_1 - y_2|.$$

Then for $\forall 1 \leq i \leq d$, a version of the Malliavin derivatives $\{(D_{\theta}^i Y_t, D_{\theta}^i Z_t)\}_{0 \leq \theta, t \leq T}$ satisfies the following linear BSDE,

$$\begin{cases} D_{\theta}^i Y_t = D_{\theta}^i \xi + \int_t^T [\partial_y f(u, Y_u) D_{\theta}^i Y_u + D_{\theta}^i f(u, Y_u)] du \\ \quad - \int_t^T D_{\theta}^i Z_u dB_u, \quad 0 \leq \theta \leq t \leq T; \\ D_{\theta}^i Y_t = 0, \quad D_{\theta}^i Z_t = 0, \quad 0 \leq t < \theta \leq T. \end{cases} \quad (\text{B.1})$$

Moreover, $\{D_t Y_t\}_{0 \leq t \leq T}$ defined by (B.1) gives a version of $\{Z_t\}_{0 \leq t \leq T}$.

From [EPM97, Proposition 2.2], (B.1) has closed form.

Lemma B.2. *Let (β, γ) be bounded $(\mathbb{R}, \mathbb{R}^d)$ -valued predictable processes, ϕ be an element of \mathbb{H}^2 , and ξ be an element of \mathbb{L}^2 . Then the linear BSDE*

$$-dY_t = [\phi_t + Y_t\beta_t + Z_t\gamma_t]dt - Z_t dB_t, \quad Y_T = \xi,$$

has a unique solution (Y, Z) in $\mathbb{H}^2 \times \mathbb{H}^2$ and Y_t is given by the following formula

$$Y_t = \mathbb{E} \left[\xi \Gamma_T^t + \int_t^T \Gamma_s^t \phi_s ds \middle| \mathcal{F}_t \right], \text{ a.s.},$$

where Γ_s^t is the adjoint process defined for $s \geq t$ by the linear FSDE

$$d\Gamma_s^t = \Gamma_s^t [\beta_s ds + \gamma_s^\top dB_s], \quad \Gamma_t^t = 1.$$

Lemma B.3. *Given an adapted and non-decreasing process \mathcal{L} starting from $\mathcal{L}_0 = 0$, for some time discretization $0 = t_0 < t_1 < \dots < t_N = T$, define $\Delta\mathcal{L}_i := \mathcal{L}_{t_{i+1}} - \mathcal{L}_{t_i}, \forall 0 \leq i \leq N-1$. Assume that \mathcal{L}_T is square integrable. Then we have*

$$\sum_{i=0}^{N-1} \mathbb{E} [\mathbb{E}_{t_i} [\Delta\mathcal{L}_i]^2] \leq \sqrt{6} \mathbb{E} [\mathcal{L}_T^2]^{1/2} \mathbb{E} \left[\sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [\Delta\mathcal{L}_i]^2 \right]^{1/2}.$$

Proof. Because \mathcal{L} is non-decreasing, the square integrability of \mathcal{L}_T easily propagates to each \mathcal{L}_{t_i} . By Cauchy-Schwarz inequality, the above left hand side satisfies

$$\begin{aligned} \sum_{i=0}^{N-1} \mathbb{E} [\mathbb{E}_{t_i} [\Delta\mathcal{L}_i]^2] &\leq \mathbb{E} \left[\left(\sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [\Delta\mathcal{L}_i] \right) \left(\sum_{i=0}^{N-1} \mathbb{E}_{t_i} [\Delta\mathcal{L}_i] \right) \right] \\ &\leq \mathbb{E} \left[\sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [\Delta\mathcal{L}_i]^2 \right]^{1/2} \mathbb{E} \left[\left(\sum_{i=0}^{N-1} \mathbb{E}_{t_i} [\Delta\mathcal{L}_i] \right)^2 \right]^{1/2}. \end{aligned}$$

From Jensen's inequality and Young's inequality, and using that \mathcal{L} is non-decreasing, we get

$$\begin{aligned} \mathbb{E} \left[\left(\sum_{i=0}^{N-1} \mathbb{E}_{t_i} [\Delta\mathcal{L}_i] \right)^2 \right] &= \mathbb{E} \left[\sum_{i=0}^{N-1} \mathbb{E}_{t_i} [\Delta\mathcal{L}_i]^2 \right] + 2\mathbb{E} \left[\sum_{i=0}^{N-1} \sum_{j=i+1}^{N-1} \mathbb{E}_{t_i} [\Delta\mathcal{L}_i] \mathbb{E}_{t_j} [\Delta\mathcal{L}_j] \right] \\ &\leq \mathbb{E} \left[\sum_{i=0}^{N-1} (\Delta\mathcal{L}_i)^2 \right] + 2 \sum_{i=0}^{N-1} \sum_{j=i+1}^{N-1} \mathbb{E} [\mathbb{E}_{t_i} [\Delta\mathcal{L}_i] \Delta\mathcal{L}_j] \\ &\leq \mathbb{E} \left[\left(\sum_{i=0}^{N-1} \Delta\mathcal{L}_i \right)^2 \right] + 2\mathbb{E} \left[\sum_{i=0}^{N-1} \mathbb{E}_{t_i} [\Delta\mathcal{L}_i] (\mathcal{L}_T - \mathcal{L}_{t_{i+1}}) \right] \\ &\leq \mathbb{E} [\mathcal{L}_T^2] + 2\mathbb{E} \left[\mathcal{L}_T \left(\sum_{i=0}^{N-1} \mathbb{E}_{t_i} [\Delta\mathcal{L}_i] \right) \right] \end{aligned}$$

$$\leq 3\mathbb{E} [\mathcal{L}_T^2] + \frac{1}{2}\mathbb{E} \left[\left(\sum_{i=0}^{N-1} \mathbb{E}_{t_i} [\Delta \mathcal{L}_i] \right)^2 \right]$$

from where we get the desired result. \square

Proposition B.4. *Under (\mathbf{H}_1) and (\mathbf{H}_2) , let $(Y_t^\lambda, Z_t^\lambda)$ be solution of PBSDE (1.2) and $(Y^{\lambda,\varepsilon}, Z^{\lambda,\varepsilon})$ be the solution of BSDE with $f^{\lambda,\varepsilon}$ as defined in (4.16). First, we have*

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^\lambda - Y_t^{\lambda,\varepsilon}|^2 + \int_0^T |Z_t^\lambda - Z_t^{\lambda,\varepsilon}|^2 dt \right] = 0. \quad (\text{B.2})$$

Second, for any fixed $\lambda > 0$ and for all $0 \leq i \leq N-1$, denote $M_i^\lambda := \int_{t_i}^{t_{i+1}} (Z_s^\lambda - V_s) \mathbf{d}B_s$ and $M_i^{\lambda,\varepsilon} := \int_{t_i}^{t_{i+1}} (Z_s^{\lambda,\varepsilon} - V_s) \mathbf{d}B_s$, then we have

$$\mathbb{E} \left[\sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [|M_i^\lambda|^2] \right] = \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [|M_i^{\lambda,\varepsilon}|^2] \right]. \quad (\text{B.3})$$

Moreover, suppose the same assumptions as Theorem 4.1, then there exists a constant C which depends only on ξ, f, T s.t.

$$\sup_{\lambda > 0} \sup_{\varepsilon > 0} \operatorname{ess\,sup}_{(t,\omega) \in [0,T] \times \Omega} |Z_t^{\lambda,\varepsilon}| \leq C. \quad (\text{B.4})$$

Proof. Since λ is fixed, we simplify the notations $(Y^\lambda, Z^\lambda), (Y^{\lambda,\varepsilon}, Z^{\lambda,\varepsilon}), M^\lambda, M^{\lambda,\varepsilon}$ into $(Y, Z), (Y^\varepsilon, Z^\varepsilon), M, M^\varepsilon$.

Proof of (B.2). The existence and uniqueness of $(Y^\varepsilon, Z^\varepsilon) \in (\mathcal{S}^2, \mathbb{H}^2)$ come from the Lipschitz continuity of $f^{\lambda,\varepsilon}$ since the regularization does not change the bound of derivatives (δ_ε is 1-Lipschitz continuous). The proof of (B.2) is straightforward from a priori estimate (see [EPM97, Proposition 2.1]), combined with the easy bound $\sup_{x \in \mathbb{R}} |\delta_\varepsilon(x) - x^-| \leq \frac{\varepsilon}{4}$: it gives

$$\begin{aligned} \|Y - Y^\varepsilon\|_{\mathcal{S}^2}^2 + \|Z - Z^\varepsilon\|_{\mathbb{H}^2}^2 &\leq C_\lambda \mathbb{E} \left[\int_0^T \lambda^2 |(Y_t - S_t)^- - \delta_\varepsilon(Y_t - S_t)|^2 dt \right] \\ &\leq \lambda^2 C_\lambda \varepsilon^2 \end{aligned}$$

with changing constants C_λ depending on λ and T, f, S, ξ .

Proof of (B.3). We have

$$\begin{aligned} &\left| \mathbb{E} \left[\sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [|M_i|^2] - \sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [|M_i^\varepsilon|^2] \right] \right| \\ &\leq \mathbb{E} \left[\sup_{0 \leq i \leq N-1} \left| \mathbb{E}_{t_i} [|M_i|^2] - \mathbb{E}_{t_i} [|M_i^\varepsilon|^2] \right| \right] \\ &\leq \mathbb{E} \left[\sup_{0 \leq i \leq N-1} \left\{ \mathbb{E}_{t_i} [|M_i - M_i^\varepsilon|] \mathbb{E}_{t_i} [|M_i| + |M_i^\varepsilon|] \right\} \right] \end{aligned}$$

$$\leq \mathbb{E} \left[\sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} \left[\left\| \int_{t_i}^{t_{i+1}} (Z_s - Z_s^\varepsilon) \mathbf{d}B_s \right\|^2 \right]^{1/2} \right] \mathbb{E} \left[\sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [|M_i| + |M_i^\varepsilon|]^2 \right]^{1/2}. \quad (\text{B.5})$$

To the first factor in (B.5), apply BDG's inequality and Doob's inequality, it gives

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} \left[\left\| \int_{t_i}^{t_{i+1}} (Z_s - Z_s^\varepsilon) \mathbf{d}B_s \right\|^2 \right] \right] &\leq C \mathbb{E} \left[\sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} \left[\left(\int_{t_i}^{t_{i+1}} |Z_s - Z_s^\varepsilon|^2 \mathbf{d}s \right)^{1/2} \right]^2 \right] \\ &\leq C \mathbb{E} \left[\sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} \left[\left(\int_0^T |Z_s - Z_s^\varepsilon|^2 \mathbf{d}s \right)^{1/2} \right]^2 \right] \\ &\leq C \mathbb{E} \left[\int_0^T |Z_s - Z_s^\varepsilon|^2 \mathbf{d}s \right] \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \end{aligned}$$

We now handle the second factor in (B.5), by showing that it is uniformly bounded in ε . We only need to consider the term M_i^ε . Proceeding with similar arguments as before, combined with (B.4) and (A.3), we get

$$\limsup_{\varepsilon \rightarrow 0} \mathbb{E} \left[\sup_{0 \leq i \leq N-1} \mathbb{E}_{t_i} [|M_i^\varepsilon|]^2 \right] \leq C \limsup_{\varepsilon \rightarrow 0} \mathbb{E} \left[\int_0^T |Z_s^\varepsilon - V_s|^2 \mathbf{d}s \right] < +\infty$$

from where we deduce (B.3).

Proof of (B.4). Apply Lemma B.1 and Lemma B.2 to $(Y^\varepsilon, Z^\varepsilon)$, so we can solve Z_t^ε in an explicit form. Observe that the formula is similar for each coordinate i , thus we can suppose $d = 1$. We have

$$\begin{aligned} \partial_y f^{\lambda, \varepsilon}(u, Y_u^\varepsilon) &= \partial_y f(u, Y_u^\varepsilon) + \lambda \delta_\varepsilon'(Y_u^\varepsilon - S_u), \\ D_\theta f^{\lambda, \varepsilon}(u, y) &= D_\theta f(u, y) + \lambda D_\theta \delta_\varepsilon(y - S_u) = D_\theta f(u, y) - \lambda \delta_\varepsilon'(y - S_u) D_\theta S_u. \end{aligned}$$

Let $\beta_u := \partial_y f^{\lambda, \varepsilon}(u, Y_u^\varepsilon)$, $\phi_u^\theta := D_\theta f^{\lambda, \varepsilon}(u, Y_u^\varepsilon)$ and let

$$\Gamma_s^t = \exp \left\{ \int_t^s [\partial_y f(u, Y_u^\varepsilon) + \lambda \delta_\varepsilon'(Y_u^\varepsilon - S_u)] \mathbf{d}u \right\},$$

then

$$Z_t^\varepsilon = D_t Y_t^\varepsilon = \mathbb{E}_t \left[\Gamma_T^t D_t \xi + \int_t^T \Gamma_s^t \phi_s^t \mathbf{d}s \right].$$

Under the assumption $\text{ess sup}_{(t, \omega)} |\partial_y f(t, Y_t)| < \infty$, since $\forall x \in \mathbb{R}, \delta_\varepsilon'(x) \leq 0$, Γ_s^t is bounded uniformly in λ, t, s, ω . In the following, without dedicated explanation, the generic constant C can vary from line to line and it does not depend on λ . Moreover, as for $\int_t^T \Gamma_s^t \phi_s^t \mathbf{d}s$, we first write

$$0 \leq \int_t^T -\lambda \delta_\varepsilon'(Y_s^\varepsilon - S_s) e^{\int_t^s \lambda \delta_\varepsilon'(Y_u^\varepsilon - S_u) \mathbf{d}u} \mathbf{d}s = 1 - e^{\int_t^T \lambda \delta_\varepsilon'(Y_u^\varepsilon - S_u) \mathbf{d}u} \leq 1,$$

then we get,

$$\begin{aligned} \left| \mathbb{E}_t \left[\int_t^T \Gamma_s^t \phi_s^t \mathbf{d}s \right] \right| &\leq \mathbb{E}_t \left[\int_t^T |D_t f(s, Y_s^\varepsilon) - \lambda \delta_\varepsilon'(Y_s^\varepsilon - S_s) D_t S_s| e^{\int_t^s [\partial_y f(u, Y_u^\varepsilon) + \lambda \delta_\varepsilon'(Y_u^\varepsilon - S_u)] \mathbf{d}u} \mathbf{d}s \right] \\ &\leq C \left(\mathbb{E}_t \left[\sup_{(s,y) \in [t,T] \times \mathbb{R}} |D_t f(s, y)| \right] + \mathbb{E}_t \left[\sup_{s \in [t,T]} |D_t S_s| \right] \right). \end{aligned}$$

In addition, we have

$$\left| \mathbb{E}_t [\Gamma_T^t D_t \xi] \right| \leq \mathbb{E}_t \left[e^{\int_t^T \partial_y f(u, Y_u^\varepsilon) \mathbf{d}u} |D_t \xi| \right] \leq C \mathbb{E}_t [|D_t \xi|].$$

As a consequence, Z_t^ε is dominated by some bounded terms independent on λ , thus

$$\sup_{\lambda > 0} \sup_{\varepsilon > 0} \operatorname{ess\,sup}_{(t,\omega) \in [0,T] \times \Omega} |Z_t^\varepsilon| < \infty.$$

□

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