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Mohamad Alameddine, Olivier Marchal

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Hamiltonian representation of isomonodromic deformations of twisted rational connections: The Painlevé 1 hierarchy

¹Olivier Marchal*, ²Mohamad Alameddine†

Abstract

In this paper, we build the Hamiltonian system and the corresponding Lax pairs associated to a twisted connection in $\mathfrak{gl}_2(\mathbb{C})$ admitting an irregular and ramified pole at infinity of arbitrary degree, hence corresponding to the Painlevé 1 hierarchy. We provide explicit formulas for these Lax pairs and Hamiltonians in terms of the irregular times and standard $2g$ Darboux coordinates associated to the twisted connection. Furthermore, we obtain a map that reduces the space of irregular times to only g non-trivial isomonodromic deformations. In addition, we perform a symplectic change of Darboux coordinates to obtain a set of symmetric Darboux coordinates in which Hamiltonians and Lax pairs are polynomial. Finally, we apply our general theory to the first cases of the hierarchy: the Airy case ($g = 0$), the Painlevé 1 case ($g = 1$) and the next two elements of the Painlevé 1 hierarchy.

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*Université de Lyon, Université Jean Monnet Saint-Étienne, CNRS UMR 5208, Institut Camille Jordan, Institut Universitaire de France, F-42023 Saint-Etienne, France.

†Université de Lyon, Université Jean Monnet Saint-Étienne, CNRS UMR 5208, Institut Camille Jordan, F-42023 Saint-Etienne, France.

Email Addresses: ¹olivier.marchal@univ-st-etienne.fr, ²mohamad.alameddine@univ-st-etienne.fr

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1 Introduction and summary of the results

Isomonodromic deformations have been studied since the beginning of the twentieth century [29, 18, 21, 28, 20, 30] and is still currently an active domain in modern mathematics. If the initial restriction to Fuchsian singularities is now very well understood, many questions in the case of irregular singularities remain open. If a geometrical understanding of the Hamiltonian representation of the isomonodromic equations for a generic meromorphic connection in $\mathfrak{gl}_d(\mathbb{C})$ with $d \geq 2$ is now well understood [22, 5], the explicit expression for the Hamiltonians and Lax pairs was derived on a case by case basis until some very recent results. In [19], the authors obtained explicit expressions for the Hamiltonians using confluences of isomonodromic deformations of Fuchsian systems. In particular, this method may only obtain results for deformations obtained by confluences of simple poles limiting its range of application for $d \geq 3$. Independently, in [27], the authors proposed a generic construction and some explicit formulas for the Lax pairs and Hamiltonians associated to meromorphic connections in $\mathfrak{gl}_2(\mathbb{C})$ such that all leading orders at each pole are assumed to be diagonalizable.¹ In addition, they proposed an explicit map from the geometric set of irregular times (defined in [8, 9, 10]) to a smaller set of isomonodromic times complemented by a set of trivial times and showed that the Darboux coordinates are independent of the trivial times. Thus, it is natural to wonder if the method of [27] can be extended to the case where a meromorphic connection in $\mathfrak{gl}_2(\mathbb{C})$ exhibits a pole whose leading order cannot be diagonalized.

The main purpose of this article is to provide a positive answer to this question and to obtain some explicit expressions for both the Hamiltonian system and the Lax pairs in the so-called “twisted case”, i.e. for meromorphic connections in $\mathfrak{gl}_2(\mathbb{C})$ such that the leading order of the connection at a pole is non-diagonalizable. Such poles are also referred to as “ramified poles” in the literature. In [27] results indicated that the formulas are independent at each pole so that we focus, without loss of generality, to the case of only one ramified pole at infinity in this paper (the position of the pole playing no role). In the end, combining the results of the present paper and those of [27] completes the study of all meromorphic connections in $\mathfrak{gl}_2(\mathbb{C})$.

Let us emphasize that the “twisted case” requires a specific and non-trivial analysis. Indeed, the underlying geometry, in particular the definition of the irregular times at a ramified pole, is more difficult and less understood than the non-ramified case. Main results in this area are [8, 7, 9, 10] and shall be used throughout the article. One of the main difference in the twisted case is the necessity to introduce a ramified cover around each pole with some associated local coordinates in order to be able to “diagonalize” the singular part of the connection around the pole. Moreover, the definition of irregular times differs since for example the eigenvalues of the leading order of the Lax matrix are necessarily the same (because the matrix is assumed to be non-diagonalizable) so that the dimension of the space of irregular times and associated deformations drastically change. All these important changes require a detailed analysis of the twisted case that we propose in the present paper.

In particular, our main results that can be seen as a summary and plan of the article are:

- For any isomonodromic deformation, characterized by a vector α in the tangent space, we provide an explicit gauge transformation between the geometric Lax pair $(\tilde{L}(\lambda), \tilde{A}_\alpha(\lambda))$

¹For clarity in the exposition, results of [27] are restricted to the stronger assumption that the matrices defining the Lax matrix are assumed to have distinct eigenvalues at all orders, but as remarked in the paper, the results may easily be generalized to the assumption that only leading orders at each pole are assumed to be diagonalizable after a suitable restriction of the deformation space.

and the companion-like Lax pair $(L(\lambda), A_\alpha(\lambda))$ in terms of apparent singularities $(q_i)_{1 \leq i \leq g}$, their dual coordinates $(p_i)_{1 \leq i \leq g}$ and the irregular times \mathbf{t} in Proposition 2.2.

- A general expression of the companion-like Lax pair $(L(\lambda), A_\alpha(\lambda))$ in terms of the Darboux coordinates $(q_i, p_i)_{1 \leq i \leq g}$ is given in Propositions 2.4, 4.2 and 4.3, complemented with equation (4-5). These results follow from the local asymptotics at infinity of the wave matrix obtained in Proposition 2.3 following the geometric construction of the twisted meromorphic connection.
- Explicit expressions of the evolutions of the Darboux coordinates relatively to irregular times and a proof that these evolutions are indeed Hamiltonian with an explicit expression of the latter are given in Theorem 5.1.
- A symplectic change from Darboux coordinates $(q_i, p_i)_{1 \leq i \leq g}$ to the set of symmetric Darboux coordinates $(Q_i, P_i)_{1 \leq i \leq g}$ for which the geometric Lax pair $(\tilde{L}(\lambda), \tilde{A}_\alpha(\lambda))$ and the Hamiltonians are polynomial is provided in Definition 6.2. The explicit polynomial expressions are given in Theorem 6.1 and Propositions 6.3 and 6.4.
- A natural reduction of the space of deformations relatively to irregular times (of dimension $2g + 4$) to a subspace of non-trivial isomonodromic deformations (of dimension g) complemented by a space of trivial deformations is detailed in Section 7. Note in particular that this map is explicit both at the level of the tangent space (Definition 7.1) and at the level of times (Definition 7.2). The terminology “trivial deformations” stands for the fact that the evolutions of the (shifted) Darboux coordinates $(\check{q}_i, \check{p}_i)_{1 \leq i \leq g}$ relatively to these times are proven trivial in Theorem 7.2.
- Simpler formulas in terms of the Darboux coordinates for the companion-like Lax pair $(L(\lambda), A_\alpha(\lambda))$ and Hamiltonians, after a canonical choice of the trivial times (given in Definition 8.1), are provided in Proposition 8.2 and Theorem 8.1.
- Some simpler polynomial expressions in terms of the symmetric Darboux coordinates of the geometric Lax pair $(\tilde{L}(\lambda), \tilde{A}_\alpha(\lambda))$ and Hamiltonians, after a canonical choice of the trivial times (defined in Definition 8.1), are provided in Proposition 8.1 and Theorem 8.1.
- The connection with the quantization of classical spectral curves via the topological recursion of [16] is presented as a by-product in Section 3.

2 Twisted meromorphic connections at infinity

2.1 Twisted meromorphic connections and irregular times

The space of $\mathfrak{gl}_2(\mathbb{C})$ meromorphic connections has been studied from many different perspectives. In the present article, we shall mainly follow the point of view of the Montréal group [1, 2] together with some insight from the work of P. Boalch [8]. Let us first define the space we shall be studying.

Definition 2.1 (Space of meromorphic connections with a pole at infinity). Let $r_\infty \geq 3$ be a given integer. We shall consider

$$F_{\infty, r_\infty} := \left\{ \hat{L}(\lambda) = \sum_{k=1}^{r_\infty-1} \hat{L}^{[\infty, k]} \lambda^{k-1} / \{ \hat{L}^{[\infty, k]} \} \in (\mathfrak{gl}_2(\mathbb{C}))^{r_\infty-1} \right\} / GL_2 \quad (2-1)$$

where GL_2 acts simultaneously by conjugation on all the coefficients $\{\hat{L}^{[\infty,k]}\}_{1 \leq k \leq r_\infty-1}$. The corresponding meromorphic connection is defined by

$$d\hat{\Psi} = \hat{L}(\lambda)d\lambda\hat{\Psi} \Leftrightarrow \partial_\lambda\hat{\Psi} = \hat{L}(\lambda)\hat{\Psi} \quad (2-2)$$

where $\hat{\Psi}$ is referred to as the wave matrix.

The space F_{∞,r_∞} corresponds to the space of meromorphic connections with a pole at infinity whose order is prescribed by the integer r_∞ . Since $r_\infty \geq 3$, the pole at infinity is said to be irregular. The generic case where the leading order $\hat{L}^{[\infty,r_\infty-1]}$ is diagonalizable has been studied in [27] where a complete construction of the associated Hamiltonian systems is provided. In this article, we shall deal with the so-called ‘‘twisted’’ case of [10]. It corresponds to the case where the leading order $\hat{L}^{[\infty,r_\infty-1]}$ is assumed to be non-diagonalizable. In the literature, this case is also referred to as ‘‘ramified’’ at infinity. We introduce the following definition.

Definition 2.2 (Set of twisted meromorphic connections at infinity). Let $r_\infty \geq 3$ be a given integer. We shall consider the subset of $\hat{F}_{\infty,r_\infty}$ defined by

$$\hat{F}_{\infty,r_\infty} = \left\{ \hat{L}(\lambda) = \sum_{k=1}^{r_\infty-1} \hat{L}^{[\infty,k]} \lambda^{k-1} / \{ \hat{L}^{[\infty,k]} \} \in (\mathfrak{gl}_2(\mathbb{C}))^{r_\infty-1} \text{ and } \hat{L}^{[\infty,r_\infty-1]} \text{ is not diagonalizable} \right\} / GL_2 \quad (2-3)$$

F_{∞,r_∞} can be given a Poisson structure inherited from the Poisson structure of a corresponding loop algebra and this space has been intensively studied from the point of view of isospectral and isomonodromic deformations. Following P. Boalch’s works, we can use the Poisson structure on $F_{\mathcal{R},r_\infty}$ in order to describe it as a bundle whose fibers are symplectic leaves obtained by fixing the irregular type and monodromies of $\hat{L}(\lambda) \in \hat{F}_{\infty,r_\infty}$. The general theory for the non-twisted or twisted case has been described in [8, 9, 10]. Let us briefly review this perspective in the twisted setting and use it to define local coordinates on $\hat{F}_{\infty,r_\infty}$ trivializing the fibration.

The main difference when dealing with the twisted case at infinity is that one needs to introduce a two-sheeted cover above infinity and define the local coordinate at infinity by

$$z_\infty(\lambda) = \lambda^{-\frac{1}{2}} \quad (2-4)$$

The general theory of [10] implies the following proposition.

Proposition 2.1. *Let $z \stackrel{\text{def}}{=} \lambda^{\frac{1}{2}}$. For any given $\hat{L}(\lambda)$ in an orbit of $\hat{F}_{\infty,r_\infty}$, there exists a local gauge matrix $G_\infty(z)$ around ∞ such that*

$$G_\infty(z) = G_{\infty,-1}z + G_{\infty,0} + \sum_{k=1}^{\infty} G_{\infty,k}z^{-k} \text{ with } G_{\infty,-1} \text{ of rank 1} \quad (2-5)$$

and

- $\Psi_\infty(z) \stackrel{\text{def}}{=} G_\infty(z)\hat{\Psi}$ is a formal fundamental solution, also known as a Turritin-Levelt fundamental form (or Birkhoff factorization):

$$\Psi_\infty(\lambda) = \Psi_\infty^{(reg)}(z) \text{diag} \left(\exp \left(- \sum_{k=1}^{2r_\infty-2} \frac{t_{\infty,k}}{k} z^k + \frac{1}{2} \ln z \right), \exp \left(- \sum_{k=1}^{2r_\infty-2} (-1)^k \frac{t_{\infty,k}}{k} z^k + \frac{1}{2} \ln z \right) \right)$$

$$= \Psi_\infty^{(reg)}(z) \operatorname{diag} \left(\exp \left(- \sum_{k=1}^{2r_\infty-2} \frac{t_{\infty,k}}{k} \lambda^{\frac{k}{2}} + \frac{1}{4} \ln \lambda \right), \exp \left(- \sum_{k=1}^{2r_\infty-2} (-1)^k \frac{t_{\infty,k}}{k} \lambda^{\frac{k}{2}} + \frac{1}{4} \ln \lambda \right) \right) \quad (2-6)$$

where $\Psi_\infty^{(reg)}(z) \in GL_2[[z^{-1}]]$ is holomorphic at $z = \infty$.

- The associated Lax matrix $L_\infty = G_\infty \hat{L} G_\infty^{-1} + (\partial_\lambda G_\infty) G_\infty^{-1}$ has a diagonal singular part at ∞ :

$$\begin{aligned} L_\infty(\lambda) &= \operatorname{diag} \left(-\frac{1}{2} \sum_{k=1}^{2r_\infty-2} (-1)^k t_{\infty,k} z^{k-2} + \frac{1}{4z^2}, -\frac{1}{2} \sum_{k=1}^{2r_\infty-2} (-1)^k t_{\infty,k} z^{k-2} + \frac{1}{4z^2} \right) + O(1) \\ &= \operatorname{diag} \left(-\frac{1}{2} \sum_{k=1}^{2r_\infty-2} t_{\infty,k} \lambda^{\frac{k}{2}-1} + \frac{1}{4\lambda}, -\frac{1}{2} \sum_{k=1}^{2r_\infty-2} (-1)^k t_{\infty,k} \lambda^{\frac{k}{2}-1} + \frac{1}{4\lambda} \right) + O(1) \end{aligned} \quad (2-7)$$

The complex numbers $(t_{\infty,k})_{1 \leq k \leq 2r_\infty-2}$ define the “irregular times” at infinity that we shall denote $\mathbf{t} = \{(t_{\infty,k})_{1 \leq k \leq 2r_\infty-2}\}$ the irregular type of $\hat{L} \in \hat{F}_{\infty, r_\infty}$.

Remark 2.1. The coefficients $\mathbf{t} = (t_{\infty,k})_{1 \leq k \leq 2r_\infty-2}$ are also referred to as “spectral times” or “KP times” in part of the literature.

Note that the diagonal part could be expressed as $\operatorname{diag}(t_{\infty(1),k}, t_{\infty(2),k})$ with $t_{\infty(2),k} = (-1)^k t_{\infty(1),k}$. This immediately follows from the fact that $\operatorname{Tr} \hat{L} = \operatorname{Tr} L_\infty + O(1)$ and by definition $\operatorname{Tr} \hat{L}$ may only involve even powers of z . In the same way, we also have the following remark.

Remark 2.2. Since $\operatorname{Tr} L_\infty = - \sum_{k=1}^{r_\infty-1} t_{\infty,2k} z^{2k-2} + O(1)$, the relation $\operatorname{Tr} \hat{L} = \operatorname{Tr} L_\infty + O(1)$ shall also provide the diagonal coefficients in (2-13). Similarly, note that the determinant $\det L_\infty$ only involves even powers of z and satisfies

$$\det L_\infty = \frac{1}{4} t_{\infty,2r_\infty-2}^2 \lambda^{2r_\infty-4} + \left(\frac{1}{4} t_{\infty,2r_\infty-3}^2 + \frac{1}{2} t_{\infty,2r_\infty-2} t_{\infty,2r_\infty-4} \right) \lambda^{2r_\infty-5} + O(\lambda^{2r_\infty-6}) \quad (2-8)$$

so that from $\det L_\infty = \det(\hat{L} + G_\infty^{-1} \partial_\lambda G_\infty)$ the coefficients of $\hat{L}^{[r_\infty-1]}$ as well as the upper-right coefficient of $\hat{L}^{[r_\infty-1]}$ (that is necessarily 1) shall be fixed in (2-13).

2.2 Choice of representative normalized at infinity

Fixing the irregular type of $\hat{L}(\lambda)$ does not fix it uniquely. In fact, the space

$$\hat{\mathcal{M}}_{\infty, r_\infty, \mathbf{t}} := \left\{ \hat{L}(\lambda) \in \hat{F}_{\infty, r_\infty} / \hat{L}(\lambda) \text{ has irregular type } \mathbf{t} \right\} \quad (2-9)$$

is a symplectic manifold of dimension

$$\dim \hat{\mathcal{M}}_{\mathcal{R}, \mathbf{r}, \mathbf{t}} = 2r_\infty - 6 = 2g \quad (2-10)$$

where

$$g := r_\infty - 3 \quad (2-11)$$

is the genus of the spectral curve defined by $\det(yI_2 - \hat{L}(\lambda)) = 0$.

For any value of the irregular times, the Montréal group introduced a set of local Darboux coordinates $(q_i, p_i)_{1 \leq i \leq g}$ on $\hat{\mathcal{M}}_{\infty, r_{\infty}, \mathbf{t}}$. Indeed, in each orbit in $\hat{F}_{\infty, r_{\infty}}$, the global action of $GL_2(\mathbb{C})$ implies that we may choose the leading coefficient $\hat{L}^{[r_{\infty}-1]}$ as a lower triangular matrix with identical coefficients on the diagonal (which is the standard form for a non-diagonalizable matrix of size 2). Furthermore, the remaining action allows to fix the coefficients on the diagonal of the subleading order $\hat{L}^{[r_{\infty}-2]}$ at equal values. Combining this choice with Remark 2.2, we obtain the existence of a unique element for which $\hat{L}(\lambda)$ is of the form

$$\hat{L}(\lambda) = \begin{pmatrix} -\frac{1}{2}t_{\infty, 2r_{\infty}-2} & 0 \\ \frac{1}{4}(t_{\infty, 2r_{\infty}-3})^2 & -\frac{1}{2}t_{\infty, 2r_{\infty}-2} \end{pmatrix} \lambda^{r_{\infty}-2} + \begin{pmatrix} -\frac{1}{2}t_{\infty, 2r_{\infty}-4} & 1 \\ X & -\frac{1}{2}t_{\infty, 2r_{\infty}-4} \end{pmatrix} \lambda^{r_{\infty}-3} + O(\lambda^{r_{\infty}-4}). \quad (2-12)$$

One may thus identify $\hat{\mathcal{M}}_{\infty, r_{\infty}, \mathbf{t}}$ with the space of such representatives

$$\hat{\mathcal{M}}_{\infty, r_{\infty}, \mathbf{t}} \sim \left\{ \begin{aligned} \tilde{L}(\lambda) &= \sum_{k=1}^{r_{\infty}-1} \tilde{L}^{[\infty, k]} \lambda^{k-1} / \tilde{L}^{[\infty, r_{\infty}-1]} = \begin{pmatrix} -\frac{1}{2}t_{\infty, 2r_{\infty}-2} & 0 \\ \frac{1}{4}(t_{\infty, 2r_{\infty}-3})^2 & -\frac{1}{2}t_{\infty, 2r_{\infty}-2} \end{pmatrix} \\ \text{and } \tilde{L}^{[\infty, r_{\infty}-2]} &= \begin{pmatrix} -\frac{1}{2}t_{\infty, 2r_{\infty}-4} & 1 \\ \delta_{\infty} & -\frac{1}{2}t_{\infty, 2r_{\infty}-4} \end{pmatrix}, \delta_{\infty} \in \mathbb{C} \end{aligned} \right\}. \quad (2-13)$$

In the following, we shall use the notation $\tilde{L}(\lambda)$ whenever we consider such a representative and call it a representative “normalized at infinity”.

2.3 Darboux coordinates

The work of the Montréal group implies that the space $\hat{\mathcal{M}}_{\infty, r_{\infty}, \mathbf{t}}$ is a symplectic manifold of dimension $2r_{\infty} - 6 = 2g$. Consequently, one may define a set of Darboux coordinates $(q_i, p_i)_{1 \leq i \leq g}$ on $\hat{\mathcal{M}}_{\infty, r_{\infty}, \mathbf{t}}$ that we shall present in this section. Let $\tilde{L}(\lambda) \in \hat{\mathcal{M}}_{\infty, r_{\infty}, \mathbf{t}}$ be a representative of the form described above in eq. (2-13). By definition, the entry $\left[\tilde{L}(\lambda) \right]_{1,2}$ is a monic polynomial function of λ of degree $r_{\infty} - 3 = g$. We thus define $(q_i)_{1 \leq i \leq g}$ as the g zeroes of $\left[\tilde{L}(\lambda) \right]_{1,2}$

$$\forall i \in \llbracket 1, g \rrbracket : \left[\tilde{L}(q_i) \right]_{1,2} = 0. \quad (2-14)$$

This defines half of the spectral Darboux coordinates. The second half is obtained by evaluating the entry $\left[\tilde{L}(\lambda) \right]_{1,1}$ at $\lambda = q_i$,

$$\forall i \in \llbracket 1, g \rrbracket : p_i := \left[\tilde{L}(q_i) \right]_{1,1}. \quad (2-15)$$

Let us remark that, for any $i \in \llbracket 1, g \rrbracket$, the pair (q_i, p_i) is by definition a point on the spectral curve defined by $\det(yI_2 - \tilde{L}(\lambda)) = 0$. In other words, we have

$$\forall i \in \llbracket 1, g \rrbracket : \det(p_i I_2 - \tilde{L}(q_i)) = 0. \quad (2-16)$$

As in the non-twisted case, the previous construction provides a local description of the space $\hat{F}_{\mathcal{R}, r}$ as a trivial bundle $\hat{F}_{\infty, r_{\infty}} \rightarrow B$ where the base $B = \mathbf{t}$ is the set of irregular times. The fiber above a point $\mathbf{t} \in B$ is $\hat{\mathcal{M}}_{\infty, r_{\infty}, \mathbf{t}}$ that we equip with spectral Darboux coordinates $(q_i, p_i)_{1 \leq i \leq g}$.

The space B is a space of isomonodromic deformations meaning that any vector field $\partial_t \in T_{\mathbf{t}}B$ gives rise to a deformation of $\tilde{L}(\lambda)$ preserving its generalized monodromy data. There exist

different equivalent ways to characterize the property of being an isomonodromic vector field. The one that we shall use in this article is the existence of a compatible system of the form

$$\begin{cases} \partial_\lambda \tilde{\Psi}(\lambda, \mathbf{t}) = \tilde{L}(\lambda) \tilde{\Psi}(\lambda, \mathbf{t}) \\ \partial_t \tilde{\Psi}(\lambda, \mathbf{t}) = \tilde{A}_t(\lambda) \tilde{\Psi}(\lambda, \mathbf{t}) \end{cases} \quad (2-17)$$

where $\tilde{A}_t(\lambda)$ is a polynomial function of λ with a pole at infinity lower or equal to $r_\infty - 2$ (the order of the pole at infinity of \tilde{L}). Equations (2-17) are referred to as a Lax pair whose compatibility condition is

$$\partial_\lambda \tilde{A}_t(\lambda) - \partial_t \tilde{L}(\lambda) + [\tilde{L}(\lambda), \tilde{A}_t(\lambda)] = 0. \quad (2-18)$$

2.4 Scalar differential equation and companion gauge

Let us now consider an orbit in $\hat{F}_{\mathcal{R}, \mathbf{r}}$ and a representative $\tilde{L}(\lambda)$ of this orbit normalized at infinity as above. Let $\tilde{\Psi}(\lambda)$ be a wave matrix solution to the linear system

$$\partial_\lambda \tilde{\Psi}(\lambda) = \tilde{L}(\lambda) \tilde{\Psi}(\lambda). \quad (2-19)$$

The differential system $\partial_\lambda \tilde{\Psi}(\lambda) = \tilde{L}(\lambda) \tilde{\Psi}(\lambda)$ may be rewritten into a scalar differential equation for $\tilde{\Psi}_{1,1}$ that is equivalent to a companion like matrix system. More precisely, defining

$$\Psi(\lambda) = G(\lambda) \tilde{\Psi}(\lambda) \text{ with } G(\lambda) = \begin{pmatrix} 1 & 0 \\ \tilde{L}_{1,1} & \tilde{L}_{1,2} \end{pmatrix} \quad (2-20)$$

we get that Ψ is a solution of the companion-like system

$$\partial_\lambda \Psi(\lambda) = L(\lambda) \Psi(\lambda) \text{ with } L(\lambda) = \begin{pmatrix} 0 & 1 \\ L_{2,1} & L_{2,2} \end{pmatrix} \quad (2-21)$$

given by

$$\begin{aligned} L_{2,1} &= -\det \tilde{L} + \partial_\lambda \tilde{L}_{1,1} - \tilde{L}_{1,1} \frac{\partial_\lambda \tilde{L}_{1,2}}{\tilde{L}_{1,2}}, \\ L_{2,2} &= \text{Tr } \tilde{L} + \frac{\partial_\lambda \tilde{L}_{1,2}}{\tilde{L}_{1,2}}. \end{aligned} \quad (2-22)$$

Note in particular that the first line of Ψ and $\tilde{\Psi}$ is obviously the same: $\Psi_{1,1} = \tilde{\Psi}_{1,1} \stackrel{\text{def}}{=} \psi_1$ and $\Psi_{1,2} = \tilde{\Psi}_{1,2} \stackrel{\text{def}}{=} \psi_2$ so that we immediately get

$$\Psi(\lambda) = \begin{pmatrix} \tilde{\Psi}_{1,1}(\lambda) & \tilde{\Psi}_{1,2}(\lambda) \\ \partial_\lambda \tilde{\Psi}_{1,1}(\lambda) & \partial_\lambda \tilde{\Psi}_{1,2}(\lambda) \end{pmatrix} = \begin{pmatrix} \psi_1(\lambda) & \psi_2(\lambda) \\ \partial_\lambda \psi_1(\lambda) & \partial_\lambda \psi_2(\lambda) \end{pmatrix}. \quad (2-23)$$

The companion-like system (2-21) is equivalent to say that ψ_1 and ψ_2 satisfy the linear ODE:

$$\left([\partial_\lambda]^2 - L_{2,2}(\lambda) \partial_\lambda - L_{2,1}(\lambda) \right) \psi_i = 0 \quad (2-24)$$

which is sometimes referred to as the ‘‘quantum curve’’.

2.5 Introduction of a scaling parameter \hbar

In order to make the connection with formal \hbar -transseries appearing in the quantization of classical spectral curves via topological recursion of [16], we shall also introduce a formal \hbar parameter by a simple rescaling of the irregular times.

$$\begin{aligned} t_{\infty,k} &\rightarrow \hbar^{\frac{k}{2}-1} t_{\infty,k}, \quad \forall k \in \llbracket 1, 2r_{\infty} - 2 \rrbracket, \\ \lambda &\rightarrow \hbar^{-1} \lambda \end{aligned} \quad (2-25)$$

This very simple rescaling implies that the differential system reads

$$\hbar \partial_{\lambda} \tilde{\Psi}(\lambda, \hbar) = \tilde{L}(\lambda, \hbar) \tilde{\Psi}(\lambda, \hbar). \quad (2-26)$$

However, for readers uneasy with this additional parameter, **we stress here that \hbar may be fixed to 1 in the rest of the paper** except for Section 3.

2.6 Explicit expressions of the gauge transformation

Using the Darboux coordinates $(q_i, p_i)_{1 \leq i \leq g}$ and the irregular times \mathbf{t} , one may obtain the explicit expression of the gauge transformation relating Ψ and $\tilde{\Psi}$. In order to do so, we shall introduce an intermediate wave matrix $\check{\Psi}$ for the following proposition.

Proposition 2.2. *The matrices $\check{\Psi}$ and Ψ are related by the gauge transformations*

$$\begin{aligned} \tilde{\Psi}(\lambda, \hbar) &= G_1(\lambda, \hbar) \check{\Psi}(\lambda, \hbar) \quad \text{with } G_1(\lambda, \hbar) = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} t_{\infty, 2r_{\infty}-2} \lambda + g_0 & 1 \end{pmatrix} \\ \check{\Psi}(\lambda, \hbar) &= J(\lambda, \hbar) \Psi(\lambda, \hbar) \quad \text{with } J(\lambda, \hbar) = \begin{pmatrix} 1 & 0 \\ \frac{Q(\lambda, \hbar)}{\prod_{j=1}^g (\lambda - q_j)} & \frac{1}{\prod_{j=1}^g (\lambda - q_j)} \end{pmatrix} \end{aligned} \quad (2-27)$$

where Q is the unique polynomial in λ of degree $g-1$ such that (with the convention that empty products are set to 1)

$$Q(q_i, \hbar) = -p_i, \quad \forall i \in \llbracket 1, g \rrbracket \quad (2-28)$$

i.e.

$$Q(\lambda, \hbar) = - \sum_{i=1}^g p_i \prod_{j \neq i} \frac{\lambda - q_j}{q_i - q_j} \quad (2-29)$$

and the coefficient g_0 is given by

$$g_0 = \frac{1}{2} t_{\infty, 2r_{\infty}-4} + \frac{1}{2} t_{\infty, r_{\infty}-2} \sum_{j=1}^g q_j \quad (2-30)$$

Proof. The proof consists in observing that

$$\tilde{G}(\lambda) = (G_1(\lambda, \hbar) J(\lambda, \hbar))^{-1} = \begin{pmatrix} 0 & 1 \\ -Q(\lambda) - \left(\frac{1}{2} t_{\infty, 2r_{\infty}-2} \lambda + g_0\right) \prod_{j=1}^g (\lambda - q_j) & \prod_{j=1}^g (\lambda - q_j) \end{pmatrix} \quad (2-31)$$

recovers the matrix (2-20). Indeed, we first have that $\tilde{G}_{2,2}(\lambda) = \tilde{L}_{1,2}(\lambda)$ and by definition $\tilde{L}_{1,2}$ is a monic polynomial of degree $g = r_{\infty} - 3$ with zeroes given by $(q_i)_{1 \leq i \leq g}$. Similarly, the entry

$\tilde{G}_{2,1}(\lambda)$ is polynomial in λ of degree $g + 1 = r_\infty - 2$. Moreover, it satisfies $\tilde{G}_{2,1}(q_i) = p_i$ for all $i \in \llbracket 1, g \rrbracket$ because of (2-15). Finally its leading coefficients at infinity are

$$\tilde{G}_{2,1}(\lambda) = -\frac{1}{2}t_{\infty,2r_\infty-2}\lambda^{r_\infty-2} + \left(\frac{1}{2}t_{\infty,r_\infty-2} \sum_{j=1}^g q_j - g_0 \right) \lambda^{r_\infty-3} + O(\lambda^{r_\infty-4}) \quad (2-32)$$

so that taking

$$g_0 = \frac{1}{2}t_{\infty,2r_\infty-4} + \frac{1}{2}t_{\infty,r_\infty-2} \sum_{j=1}^g q_j \quad (2-33)$$

provides $\tilde{G}_{2,1}(\lambda) = -\frac{1}{2}t_{\infty,2r_\infty-2}\lambda^{r_\infty-2} - \frac{1}{2}t_{\infty,2r_\infty-4}\lambda^{r_\infty-3} + O(\lambda^{r_\infty-4})$. Hence, with this choice of g_0 , it is equal to $\tilde{L}_{1,1}(\lambda)$. Consequently, $\tilde{G}(\lambda)$ recovers the matrix $G(\lambda)$ of equation (2-20) ending the proof. \square

Remark 2.3. By definition, the matrix $\check{\Psi}(\lambda, \hbar)$ satisfies the Lax system:

$$\begin{aligned} \hbar \partial_\lambda \check{\Psi}(\lambda, \hbar) &= \check{L}(\lambda, \hbar) \check{\Psi}(\lambda, \hbar) \\ \hbar \partial_t \check{\Psi}(\lambda, \hbar) &= \check{A}_t(\lambda, \hbar) \check{\Psi}(\lambda, \hbar) \end{aligned} \quad (2-34)$$

for any irregular time $t \in \mathbf{t}$. In particular, the corresponding Lax matrices $\check{L}(\lambda, \hbar)$ and $\check{A}_t(\lambda, \hbar)$ are given by

$$\begin{aligned} \check{L}(\lambda, \hbar) &= J(\lambda, \hbar) L(\lambda, \hbar) J^{-1}(\lambda, \hbar) + \hbar (\partial_\lambda J(\lambda, \hbar)) J^{-1}(\lambda, \hbar) \\ \check{A}_t(\lambda, \hbar) &= J(\lambda, \hbar) A_t(\lambda, \hbar) J^{-1}(\lambda, \hbar) + \hbar \partial_t (J(\lambda, \hbar)) J^{-1}(\lambda, \hbar) \end{aligned} \quad (2-35)$$

and are polynomial functions of λ with no singularities at $\lambda \in \{q_1, \dots, q_g\}$.

Note that by definition, the entries of \check{L} are related to those of L by

$$\begin{aligned} \check{L}_{1,1}(\lambda, \hbar) &= -Q(\lambda, \hbar), \\ \check{L}_{1,2}(\lambda, \hbar) &= \prod_{j=1}^g (\lambda - q_j), \\ \check{L}_{2,2}(\lambda, \hbar) &= L_{2,2}(\lambda, \hbar) + Q(\lambda, \hbar) - \sum_{j=1}^g \frac{\hbar}{\lambda - q_j}, \\ \check{L}_{2,1}(\lambda, \hbar) &= \frac{\hbar}{\prod_{j=1}^g (\lambda - q_j)} \frac{\partial Q(\lambda, \hbar)}{\partial \lambda} + \frac{L_{2,1}(\lambda, \hbar)}{\prod_{j=1}^g (\lambda - q_j)} - L_{2,2}(\lambda, \hbar) \frac{Q(\lambda, \hbar)}{\prod_{j=1}^g (\lambda - q_j)} - \frac{Q(\lambda, \hbar)^2}{\prod_{j=1}^g (\lambda - q_j)} \end{aligned} \quad (2-36)$$

Similarly, the entries of \tilde{L} are related to those of \check{L} by

$$\begin{aligned} \tilde{L}_{1,1}(\lambda, \hbar) &= \check{L}_{1,1}(\lambda, \hbar) - \left(\frac{1}{2}t_{\infty,2r_\infty-2}\lambda + g_0 \right) \check{L}_{1,2}(\lambda, \hbar) \\ \tilde{L}_{1,2}(\lambda, \hbar) &= \check{L}_{1,2}(\lambda, \hbar) \\ \tilde{L}_{2,1}(\lambda, \hbar) &= \check{L}_{2,1}(\lambda, \hbar) - \left(\frac{1}{2}t_{\infty,2r_\infty-2}\lambda + g_0 \right)^2 \check{L}_{1,2}(\lambda, \hbar) \\ &\quad + \left(\frac{1}{2}t_{\infty,2r_\infty-2}\lambda + g_0 \right) (\check{L}_{1,1}(\lambda, \hbar) - \check{L}_{2,2}(\lambda, \hbar)) + \frac{1}{2}\hbar t_{\infty,2r_\infty-2} \\ \tilde{L}_{2,2}(\lambda, \hbar) &= \check{L}_{2,2}(\lambda, \hbar) + \left(\frac{1}{2}t_{\infty,2r_\infty-2}\lambda + g_0 \right) \check{L}_{1,2}(\lambda, \hbar) \end{aligned} \quad (2-37)$$

2.7 Wronskians and asymptotics of the wave functions

Combining the gauge transformations G_∞ , G_1 and J , we obtain the following proposition.

Proposition 2.3. *The scalar wave functions $\psi_1 = \Psi_{1,1} = \tilde{\Psi}_{1,1} = \check{\Psi}_{1,1}$ and $\psi_2 = \Psi_{1,2} = \tilde{\Psi}_{1,2} = \check{\Psi}_{1,2}$ have the following expansions around ∞ .*

$$\begin{aligned}\psi_1(\lambda) &\stackrel{\lambda \rightarrow \infty}{=} \exp\left(-\frac{1}{\hbar} \sum_{k=1}^{2r_\infty-2} \frac{t_{\infty,k}}{k} \lambda^{\frac{k}{2}} - \frac{1}{4} \ln \lambda + O(1)\right), \\ \psi_2(\lambda) &\stackrel{\lambda \rightarrow \infty}{=} \exp\left(-\frac{1}{\hbar} \sum_{k=1}^{2r_\infty-2} (-1)^k \frac{t_{\infty,k}}{k} \lambda^{\frac{k}{2}} - \frac{1}{4} \ln \lambda + O(1)\right).\end{aligned}\quad (2-38)$$

Proof. The proof is done in Appendix A. □

For convenience, we shall also define the Wronskians associated to the Lax systems and provide their explicit expressions that follow from the previous proposition:

Definition 2.3 (Wronskians). Let us define $W(\lambda, \hbar) = \det \Psi(\lambda, \hbar)$, $\check{W}(\lambda, \hbar) = \det \check{\Psi}(\lambda, \hbar)$ and $\tilde{W}(\lambda, \hbar) = \det \tilde{\Psi}(\lambda, \hbar)$ the Wronskians associated to the corresponding wave matrices. They are given by

$$\begin{aligned}\check{W}(\lambda) &= \check{W}_0 \exp\left(\frac{1}{\hbar} \int_0^\lambda \tilde{P}_1(s) ds\right), \\ \tilde{W}(\lambda) &= \tilde{W}_0 \exp\left(\frac{1}{\hbar} \int_0^\lambda \tilde{P}_1(s) ds\right) \\ W(\lambda) &= W_0 \left(\prod_{i=1}^g (\lambda - q_i)\right) \exp\left(\frac{1}{\hbar} \int_0^\lambda \tilde{P}_1(s) ds\right).\end{aligned}\quad (2-39)$$

where W_0 and \tilde{W}_0 are unknown constants (in the sense independent of λ) and where we have defined

$$\tilde{P}_{\infty,k}^{(1)} = -t_{\infty,2k+2}, \quad \forall k \in \llbracket 0, r_\infty - 2 \rrbracket, \quad (2-40)$$

and regrouped them into the polynomial \tilde{P}_1 :

$$\tilde{P}_1(\lambda) = \sum_{j=0}^{r_\infty-2} \tilde{P}_{\infty,j}^{(1)} \lambda^j = - \sum_{j=0}^{r_\infty-2} t_{\infty,2j+2} \lambda^j. \quad (2-41)$$

Proof. The proof starts with $W(\lambda) = \hbar(\psi_1(\lambda)\partial_\lambda\psi_2(\lambda) - \psi_2(\lambda)\partial_\lambda\psi_1(\lambda))$. From the general construction and Proposition 2.3, $W(\lambda) \exp\left(-\frac{1}{\hbar} \int_0^\lambda \tilde{P}_1(s) ds\right)$ is a polynomial function of λ of degree $r_\infty - 3 = g$. Moreover, since $L_{2,2}(\lambda) = \hbar \frac{\partial_\lambda W(\lambda)}{W(\lambda)}$, we get that the zeroes of $W(\lambda)$ are simple poles of $L_{2,2}(\lambda)$. Since $L_{2,2}(\lambda)$ may only have poles at infinity or at $(q_i)_{1 \leq i \leq g}$, we get that $W(\lambda) \exp\left(-\frac{1}{\hbar} \int_0^\lambda \tilde{P}_1(s) ds\right) = W_0 \prod_{i=1}^g (\lambda - q_i)$ for some constant W_0 (i.e. independent of λ). Formulas for $\check{W}(\lambda)$ and $\tilde{W}(\lambda)$ follow from $W(\lambda)$ and the gauge transformations using the determinants of G_1 and J . □

2.8 Explicit expression for the Lax matrix L

In this section we shall provide an explicit expression for the matrix $L(\lambda)$ in terms of irregular times and Darboux coordinates. Only g coefficients of the matrix shall remain undetermined at this stage. These coefficients will be put in one-to-one correspondence with the upcoming Hamiltonians. In order to write down the Lax matrix in a compact form, we shall introduce the following definition.

Definition 2.4. We define the following quantities:

$$\begin{aligned}\tilde{P}_{\infty,k}^{(2)} &= \frac{1}{4} \sum_{j=2k-2r_{\infty}+6}^{2r_{\infty}-2} (-1)^j t_{\infty,j} t_{\infty,2k-j+4}, \quad \forall k \in \llbracket r_{\infty}-2, 2r_{\infty}-4 \rrbracket \\ \tilde{P}_{\infty,r_{\infty}-3}^{(2)} &= \frac{1}{4} \sum_{j=1}^{2r_{\infty}-3} (-1)^j t_{\infty,j} t_{\infty,2r_{\infty}-j-2}\end{aligned}\tag{2-42}$$

and regroup them into the polynomial function \tilde{P}_2 :

$$\tilde{P}_2(\lambda) = \sum_{k=r_{\infty}-3}^{2r_{\infty}-4} \tilde{P}_{\infty,k}^{(2)} \lambda^k\tag{2-43}$$

We shall also define

$$\hat{P}_2(\lambda) = \tilde{P}_2(\lambda) - \sum_{k=0}^{r_{\infty}-4} H_{\infty,k} \lambda^k\tag{2-44}$$

where the $g = r_{\infty} - 3$ coefficients $(H_{\infty,k})_{0 \leq k \leq r_{\infty}-4}$ remain undetermined at this stage.

Using the previous definition, we obtain the following proposition.

Proposition 2.4. *The Lax matrix $L(\lambda, \hbar)$ is given by*

$$L(\lambda, \hbar) = \begin{pmatrix} 0 & 1 \\ L_{2,1}(\lambda, \hbar) & L_{2,2}(\lambda, \hbar) \end{pmatrix}\tag{2-45}$$

with

$$\begin{aligned}L_{2,2}(\lambda, \hbar) &= \tilde{P}_1(\lambda) + \sum_{j=1}^g \frac{\hbar}{\lambda - q_j} \\ L_{2,1}(\lambda, \hbar) &= -\tilde{P}_2(\lambda) + \sum_{k=0}^{r_{\infty}-4} H_{\infty,k} \lambda^k - \sum_{j=1}^g \frac{\hbar p_j}{\lambda - q_j}\end{aligned}\tag{2-46}$$

Coefficients $(H_{\infty,k})_{0 \leq k \leq r_{\infty}-4}$ shall be determined later in Proposition 5.1.

Proof. The proof is based on the fact that the entries of L are rational functions of λ with poles only at ∞ or at apparent singularities $(q_i)_{1 \leq i \leq g}$. Using the knowledge of the asymptotics expansion at ∞ provides the result. This is detailed in Appendix B. \square

3 Classical spectral curve and connection with topological recursion

Before turning to deformations relatively to the irregular times, let us briefly mention the connection of the present setup with the classical spectral curve and the topological recursion. This section being independent of the others, we stress that readers with no interest in topological recursion or in WKB expansions may skip the content of this section.

Let us first recall how one may obtain the classical spectral curve from a Lax system. When dealing with a Lax system of the form

$$\hbar \partial_\lambda \Psi(\lambda, \hbar) = L(\lambda, \hbar) \Psi(\lambda, \hbar), \quad (3-1)$$

it is standard to define the ‘‘classical spectral curve’’ as $\lim_{\hbar \rightarrow 0} \det(yI_2 - L(\lambda, \hbar)) = 0$. It is important to note that the classical spectral curve is unaffected by the gauge transformations $\Psi(\lambda, \hbar) \rightarrow G(\lambda, \hbar) \Psi(\lambda, \hbar)$ with $G(\lambda, \hbar)$ regular in \hbar . Indeed, the conjugation of the Lax matrix does not change the characteristic polynomial and the additional term $\hbar(\partial_\lambda G)G^{-1}$ disappears in the limit $\hbar \rightarrow 0$. In particular, in our setup, it means that one may compute the classical spectral curve using either \tilde{L} , \check{L} or L :

$$\lim_{\hbar \rightarrow 0} \det(yI_2 - L(\lambda, \hbar)) = \lim_{\hbar \rightarrow 0} \det(yI_2 - \check{L}(\lambda, \hbar)) = \lim_{\hbar \rightarrow 0} \det(yI_2 - \tilde{L}(\lambda, \hbar)). \quad (3-2)$$

In our case, the general expression of the matrix $L(\lambda, \hbar)$ implies that the classical spectral curve is

$$y^2 - \tilde{P}_1(\lambda, \hbar = 0)y + \hat{P}_2(\lambda, \hbar = 0) = 0. \quad (3-3)$$

It defines a Riemann surface Σ of genus $g = r_\infty - 3$ whose coefficients are determined by (2-40) and Definition 2.4. Note that only g coefficients remained undetermined at this stage (i.e. $(H_{\infty, k})_{0 \leq k \leq r_\infty - 4}$) that can be mapped with the so-called filling fractions $(\epsilon_i)_{1 \leq i \leq g}$ naturally associated to the Riemann surface. Moreover, the present twisted case corresponds to the case where infinity is a ramification point of the Riemann surface. In other words, the twisted case happens when a pole of the connection is also a ramification point of the underlying classical spectral curve. The asymptotic expansions of the differential form ydx at each pole is in direct relation with the asymptotics of the wave functions (2-38) since we have

$$\begin{aligned} y_1(z) &\stackrel{z \rightarrow \infty}{=} -\frac{1}{2} \sum_{k=1}^{2r_\infty - 2} t_{\infty, k} x(z)^{\frac{k}{2} - 1} - \frac{1}{4x(z)} + O\left(x(z)^{-\frac{3}{2}}\right) \\ y_2(z) &\stackrel{z \rightarrow \infty}{=} -\frac{1}{2} \sum_{k=1}^{2r_\infty - 2} (-1)^k t_{\infty, k} x(z)^{\frac{k}{2} - 1} - \frac{1}{4x(z)} + O\left(x(z)^{-\frac{3}{2}}\right) \end{aligned} \quad (3-4)$$

where $y_1(z)$ and $y_2(z)$ are the expressions of $y(z)$ in both sheets.

Let us now discuss the connection of the present work with the Chekhov-Eynard-Orantin topological recursion [11, 12, 13, 16, 17] as given in [27]. Recent works [26, 15] have shown how to quantize the classical spectral curve using topological recursion. Indeed, applying the topological recursion to the classical spectral curve (3-3) generates Eynard-Orantin differentials $(\omega_{h, n})_{h \geq 0, n \geq 0}$ that can be regrouped into formal \hbar -transseries to define formal wave functions $(\psi_1^{\text{TR}}, \psi_2^{\text{TR}})$ that satisfy a quantum curve, i.e. a linear ODE of degree 2 with pole singularities at infinity and apparent singularities at $\lambda = q_i$ and whose $\hbar \rightarrow 0$ limit recovers the classical

spectral curve. The construction presented in [26, 15] implies that this ODE is the same as the one defined by the Lax matrix $L(\lambda, \hbar)$ of the present paper so that we get

$$\Psi(\lambda, \hbar) = C \begin{pmatrix} \psi_1^{\text{TR}}(\lambda, \hbar) & \psi_2^{\text{TR}}(\lambda, \hbar) \\ \hbar \partial \psi_1^{\text{TR}}(\lambda, \hbar) & \hbar \partial \psi_2^{\text{TR}}(\lambda, \hbar) \end{pmatrix} \quad (3-5)$$

where C is a constant (independent of λ) matrix. In other words, the topological recursion reconstructs our wave functions ψ_1 and ψ_2 making the classical spectral curve the only necessary object to build the full Lax system. However, the price to pay in this perspective is the mandatory introduction of the formal parameter \hbar to define the formal \hbar -transseries and then $(\psi_1^{\text{TR}}, \psi_2^{\text{TR}})$. As explained in Section 2.5, this formal parameter can be removed by proper rescaling at the level of the Lax system but it is unclear how the topological recursion wave functions may be defined after this rescaling, since there is no more formal parameter to define the series. This issue is in deep relation with the analytical meaning that might be given to the formal \hbar -transseries. In particular, it is presently unclear how to resum analytically the \hbar -transseries to obtain non-formal identities and current works are in progress to tackle this problem.

4 General isomonodromic deformations and auxiliary matrices

4.1 Definition of general isomonodromic deformations

The previous sections provide a natural set of parameters for which we may consider deformations, namely the irregular times $(t_{\infty, k})_{1 \leq k \leq 2r_{\infty} - 2}$. In order to study deformations relatively to these parameters we introduce the following definition.

Definition 4.1. We define the following general deformation operators.

$$\mathcal{L}_{\alpha} = \hbar \sum_{k=1}^{2r_{\infty} - 2} \alpha_{\infty, k} \partial_{t_{\infty, k}} \quad (4-1)$$

where we define the vector $\alpha \in \mathbb{C}^{2r_{\infty} - 2} = \mathbb{C}^{2g+4}$ by

$$\alpha = \sum_{k=1}^{2r_{\infty} - 2} \alpha_{\infty, k} \mathbf{e}_k. \quad (4-2)$$

Deformations defined by Definition 4.1 shall be seen as general isomonodromic deformations in $\hat{F}_{\infty, r_{\infty}}$.

Associated to a vector α are general auxiliary Lax matrices $\tilde{A}_{\alpha}(\lambda)$, $\check{A}_{\alpha}(\lambda)$ and $A_{\alpha}(\lambda)$ defined by

$$\begin{aligned} \tilde{A}_{\alpha}(\lambda) = \mathcal{L}_{\alpha}[\tilde{\Psi}(\lambda)]\tilde{\Psi}^{-1}(\lambda) &\Leftrightarrow \mathcal{L}_{\alpha}[\tilde{\Psi}(\lambda)] = \tilde{A}_{\alpha}(\lambda)\tilde{\Psi}(\lambda) \\ \check{A}_{\alpha}(\lambda) = \mathcal{L}_{\alpha}[\check{\Psi}(\lambda)]\check{\Psi}^{-1}(\lambda) &\Leftrightarrow \mathcal{L}_{\alpha}[\check{\Psi}(\lambda)] = \check{A}_{\alpha}(\lambda)\check{\Psi}(\lambda) \\ A_{\alpha}(\lambda) = \mathcal{L}_{\alpha}[\Psi(\lambda)]\Psi^{-1}(\lambda) &\Leftrightarrow \mathcal{L}_{\alpha}[\Psi(\lambda)] = A_{\alpha}(\lambda)\Psi(\lambda) \end{aligned} \quad (4-3)$$

In particular, $\tilde{A}_{\alpha}(\lambda)$ and $\check{A}_{\alpha}(\lambda)$ are polynomial functions of λ while $A(\lambda)$ may also have additional poles at $\{q_1, \dots, q_g\}$. Note that $(L(\lambda), A_{\alpha}(\lambda))$, $(\check{L}(\lambda), \check{A}_{\alpha}(\lambda))$ and $(\tilde{L}(\lambda), \tilde{A}_{\alpha}(\lambda))$ provide equivalent Lax pairs but expressed in three different gauges. The corresponding compatibility equations are

$$\begin{aligned}
\mathcal{L}_\alpha[L] &= [A_\alpha, L] + \hbar \partial_\lambda A_\alpha \\
\mathcal{L}_\alpha[\check{L}] &= [\check{A}_\alpha, \check{L}] + \hbar \partial_\lambda \check{A}_\alpha \\
\mathcal{L}_\alpha[\tilde{L}] &= [\tilde{A}_\alpha, \tilde{L}] + \hbar \partial_\lambda \tilde{A}_\alpha.
\end{aligned} \tag{4-4}$$

We shall now use the asymptotic expansions of the wave matrices in order to obtain information on the general form of the auxiliary matrices. Then, we shall use the compatibility equations in order to determine the evolutions of the Darboux coordinates under general isomonodromic deformations and prove that these evolutions are Hamiltonian as performed in a similar way for the non-twisted case in [27].

4.2 General form of the auxiliary matrix $A_\alpha(\lambda, \hbar)$

Using compatibility conditions one may easily obtain two of the entries of $A_\alpha(\lambda)$. Indeed, since L is a companion-like matrix, compatibility equations (4-4) imply that

$$\begin{aligned}
[A_\alpha(\lambda)]_{2,1} &= \hbar \partial_\lambda [A_\alpha(\lambda)]_{1,1} + [A_\alpha(\lambda)]_{1,2} L_{2,1}(\lambda), \\
[A_\alpha(\lambda)]_{2,2} &= \hbar \partial_\lambda [A_\alpha(\lambda)]_{1,2} + [A_\alpha(\lambda)]_{1,1} + [A_\alpha(\lambda)]_{1,2} L_{2,2}(\lambda),
\end{aligned} \tag{4-5}$$

so that only the first line of $A_\alpha(\lambda)$ remains unknown at this stage. The other two entries of the compatibility equation (4-4) leads to

$$\begin{aligned}
\mathcal{L}_\alpha[L_{2,1}(\lambda)] &= \hbar^2 \frac{\partial^2 [A_\alpha(\lambda)]_{1,1}}{\partial \lambda^2} + 2\hbar L_{2,1}(\lambda) \partial_\lambda [A_\alpha(\lambda)]_{1,2} + \hbar [A_\alpha(\lambda)]_{1,2} \partial_\lambda L_{2,1}(\lambda) \\
&\quad - \hbar L_{2,2}(\lambda) \partial_\lambda [A_\alpha(\lambda)]_{1,1}, \\
\mathcal{L}_\alpha[L_{2,2}(\lambda)] &= \hbar^2 \frac{\partial^2 [A_\alpha(\lambda)]_{1,2}}{\partial \lambda^2} + 2\hbar \partial_\lambda [A_\alpha(\lambda)]_{1,1} + \hbar L_{2,2}(\lambda) \partial_\lambda [A_\alpha(\lambda)]_{1,2} \\
&\quad + \hbar [A_\alpha(\lambda)]_{1,2} \partial_\lambda L_{2,2}(\lambda)
\end{aligned} \tag{4-6}$$

that shall be used later to determine the evolution equations for $(q_i, p_i)_{1 \leq i \leq n}$. Before studying the compatibility equations, let us observe that the asymptotic expansions of the wave matrix Ψ at infinity allows to determine the general form of the auxiliary matrix $A_\alpha(\lambda, \hbar)$. This leads us to the following results.

Proposition 4.1. *The asymptotic expansion of entry $[A_\alpha(\lambda)]_{1,2}$ at infinity is given by*

$$\forall M \geq 1 : [A_\alpha(\lambda)]_{1,2} \stackrel{\lambda \rightarrow \infty}{\cong} \sum_{i=-1}^M \frac{\nu_{\infty,i}^{(\alpha)}}{\lambda^i} + O(\lambda^{-M-1}). \tag{4-7}$$

Moreover, coefficients $(\nu_{\infty,k}^{(\alpha)})_{-1 \leq k \leq r_\infty-3}$ are determined by

$$M_\infty \begin{pmatrix} \nu_{\infty,-1}^{(\alpha)} \\ \nu_{\infty,0}^{(\alpha)} \\ \vdots \\ \nu_{\infty,r_\infty-3}^{(\alpha)} \end{pmatrix} = \begin{pmatrix} \frac{2\alpha_{\infty,2r_\infty-3}}{(2r_\infty-3)} \\ \frac{2\alpha_{\infty,2r_\infty-5}}{(2r_\infty-5)} \\ \vdots \\ \frac{2\alpha_{\infty,1}}{1} \end{pmatrix} \tag{4-8}$$

where M_∞ is a lower triangular Toeplitz matrix of size $(r_\infty - 1) \times (r_\infty - 1)$ independent of the deformation α :

$$M_\infty = \begin{pmatrix} t_{\infty,2r_\infty-3} & 0 & \dots & 0 \\ t_{\infty,2r_\infty-5} & t_{\infty,2r_\infty-3} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ t_{\infty,3} & & \ddots & \ddots & 0 \\ t_{\infty,1} & t_{\infty,3} & \dots & & t_{\infty,2r_\infty-3} \end{pmatrix}. \quad (4-9)$$

Proof. The proof is presented in Appendix C. \square

The previous proposition may be used to determine the general form of the entry $[A_\alpha(\lambda)]_{1,2}$.

Proposition 4.2. *Entry $[A_\alpha(\lambda)]_{1,2}$ is given by*

$$[A_\alpha(\lambda)]_{1,2} = \nu_{\infty,-1}^{(\alpha)} \lambda + \nu_{\infty,0}^{(\alpha)} + \sum_{j=1}^g \frac{\mu_j^{(\alpha)}}{\lambda - q_j}. \quad (4-10)$$

Coefficients $(\mu_j^{(\alpha)})_{1 \leq j \leq g}$ are determined by the linear system

$$V_\infty \begin{pmatrix} \mu_1^{(\alpha)} \\ \vdots \\ \mu_g^{(\alpha)} \end{pmatrix} = \begin{pmatrix} \nu_{\infty,1}^{(\alpha)} \\ \nu_{\infty,2}^{(\alpha)} \\ \vdots \\ \nu_{\infty,r_\infty-3}^{(\alpha)} \end{pmatrix} \quad (4-11)$$

where V_∞ is a $(r_\infty - 3) \times g$ matrix

$$V_\infty = \begin{pmatrix} 1 & 1 & \dots & \dots & 1 \\ q_1 & q_2 & \dots & \dots & q_g \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ q_1^{r_\infty-4} & q_2^{r_\infty-4} & \dots & \dots & q_g^{r_\infty-4} \end{pmatrix}, \quad (4-12)$$

Proof. We know that $[A_\alpha(\lambda)]_{1,2}$ is rational in λ with only simple poles in $\{q_1, \dots, q_g\}$ and a pole at infinity. Proposition 4.1 provides the asymptotics at infinity so that (4-10) holds. Moreover the expansion at infinity of

$$\sum_{j=1}^g \frac{\mu_j^{(\alpha)}}{\lambda - q_j} = \sum_{k=1}^{\infty} \sum_{j=1}^g \mu_j^{(\alpha)} q_j^{k-1} \lambda^{-k} \quad (4-13)$$

identifies with (4-7) only with (4-11). \square

Note that we may determine coefficients $(\nu_{\infty,k}^{(\alpha)})_{k \geq r_\infty-2}$ by the fact that

$$\sum_{j=1}^g \mu_j^{(\alpha)} \prod_{i \neq j} (\lambda - q_i) = \left([A_\alpha(\lambda)]_{1,2} - \nu_{\infty,-1}^{(\alpha)} \lambda - \nu_{\infty,0}^{(\alpha)} \right) \left(\prod_{i=1}^g (\lambda - q_i) \right)$$

$$\begin{aligned}
&= \left(\sum_{k=1}^{\infty} \nu_{\infty,k}^{(\alpha)} \lambda^{-k} \right) \left(\prod_{i=1}^g (\lambda - q_i) \right) \\
&= \left(\sum_{k=1}^{\infty} \nu_{\infty,k}^{(\alpha)} \lambda^{-k} \right) \left(\sum_{i=0}^g (-1)^{g-i} e_{g-i}(\{q_1, \dots, q_g\}) \lambda^i \right) \quad (4-14)
\end{aligned}$$

where the l.h.s. is a polynomial in λ and $(e_k(\{q_1, \dots, q_g\}))_{0 \leq k \leq g}$ are the elementary symmetric polynomials. Thus, for all $m \geq 1$:

$$0 = \sum_{k=m}^{g+m} (-1)^{g+m-k} \nu_{\infty,k}^{(\alpha)} e_{g+m-k}(\{q_1, \dots, q_g\}) = \nu_{\infty,g+m}^{(\alpha)} + \sum_{k=m}^{g+m-1} (-1)^{g+m-k} \nu_{\infty,k}^{(\alpha)} e_{g+m-k}(\{q_1, \dots, q_g\}) \quad (4-15)$$

so that we obtain the recursive relations

$$\forall m \geq 1 : \nu_{\infty,g+m}^{(\alpha)} = \sum_{k=m}^{g+m-1} (-1)^{g+m-1-k} \nu_{\infty,k}^{(\alpha)} e_{g+m-k}(\{q_1, \dots, q_g\}) \quad (4-16)$$

In particular, we get for $m = 1$:

$$\nu_{\infty,r_{\infty}-2}^{(\alpha)} = \sum_{k=1}^g (-1)^{g-k} \nu_{\infty,k}^{(\alpha)} e_{g+1-k}(\{q_1, \dots, q_g\}) \quad (4-17)$$

Let us now perform similar computation for $[A_{\alpha}(\lambda)]_{1,1}$. We obtain the following proposition.

Proposition 4.3. *The entry $[A_{\alpha}(\lambda)]_{1,1}$ is given by*

$$[A_{\alpha}(\lambda)]_{1,1} = \sum_{i=0}^{r_{\infty}-1} c_{\infty,i}^{(\alpha)} \lambda^i + \sum_{j=1}^g \frac{\rho_j^{(\alpha)}}{\lambda - q_j}. \quad (4-18)$$

with

$$\forall j \in \llbracket 1, n \rrbracket : \rho_j^{(\alpha)} = -\mu_j^{(\alpha)} p_j \quad (4-19)$$

Coefficients $(c_{\infty,k}^{(\alpha)})_{1 \leq k \leq r_{\infty}-1}$ are determined by

$$M_{\infty} \begin{pmatrix} c_{\infty,r_{\infty}-1}^{(\alpha)} \\ \vdots \\ c_{\infty,k}^{(\alpha)} \\ \vdots \\ c_{\infty,1}^{(\alpha)} \end{pmatrix} = \begin{pmatrix} \frac{\alpha_{\infty,2r_{\infty}-3}}{2r_{\infty}-3} t_{\infty,2r_{\infty}-2} - \frac{\alpha_{\infty,2r_{\infty}-2}}{2r_{\infty}-2} t_{\infty,2r_{\infty}-3} \\ \vdots \\ \sum_{m=k}^{r_{\infty}-1} \left(\frac{\alpha_{\infty,2k+2r_{\infty}-2m-3}}{2k+2r_{\infty}-2m-3} t_{\infty,2m} - \frac{\alpha_{2k+2r_{\infty}-2m-2}}{2k+2r_{\infty}-2m-2} t_{\infty,2m-1} \right) \\ \vdots \\ \sum_{m=1}^{r_{\infty}-1} \left(\frac{\alpha_{\infty,2r_{\infty}-2m-1}}{2r_{\infty}-2m-1} t_{\infty,2m} - \frac{\alpha_{2r_{\infty}-2m}}{2r_{\infty}-2m} t_{\infty,2m-1} \right) \end{pmatrix} \quad (4-20)$$

with the matrix M_{∞} given by (4-9).

Proof. The proof is done in Appendix D. \square

Note that $c_{\infty,0}^{(\alpha)}$ is not determined but will play no role in the rest of the paper. In the previous propositions, one may easily observe that M_{∞} , $(\nu_{\infty,k}^{(\alpha)})_{-1 \leq k \leq r_{\infty}-3}$ and $(c_{\infty,k}^{(\alpha)})_{1 \leq k \leq r_{\infty}-1}$ are independent of the Darboux coordinates and depend only on irregular times and the deformation α . On the contrary, $(\mu_j^{(\alpha)})_{1 \leq j \leq g}$ and V_{∞} depend on the Darboux coordinates.

5 General Hamiltonian evolutions

The previous sections provide the general form of the matrices $L(\lambda, \hbar)$ and $A_\alpha(\lambda)$ through Propositions 2.4, 4.1, 4.2, 4.3 and equation (4-5). As we shall see below, inserting the previous knowledge into the compatibility equations (4-6) provides the evolutions of the Darboux coordinates.

The first step is to look at order $(\lambda - q_j)^{-2}$ in $\mathcal{L}_\alpha[L_{2,2}(\lambda)]$. We obtain, for all $j \in \llbracket 1, g \rrbracket$:

$$\mathcal{L}_\alpha[q_j] = 2\mu_j^{(\alpha)} \left(p_j - \frac{1}{2} \tilde{P}_1(q_j) \right) - \hbar \nu_{\infty,0}^{(\alpha)} - \hbar \nu_{\infty,-1}^{(\alpha)} q_j - \hbar \sum_{i \neq j} \frac{\mu_j^{(\alpha)} + \mu_i^{(\alpha)}}{q_j - q_i}. \quad (5-1)$$

The next step is to determine the coefficients $(H_{\infty,j})_{0 \leq j \leq r_\infty - 4}$ that remain unknown in $L_{2,1}(\lambda)$. To achieve this task, we look at order $(\lambda - q_j)^{-2}$ in $\mathcal{L}_\alpha[L_{2,1}(\lambda)]$ using (4-6). We obtain

$$\begin{aligned} -\hbar p_j \mathcal{L}_\alpha[q_j] &= -2\hbar \mu_j^{(\alpha)} \left(-\tilde{P}_2(q_j) + \sum_{k=0}^{r_\infty - 4} H_{\infty,k} q_j^k - \sum_{i \neq j} \frac{\hbar p_i}{q_j - q_i} \right) \\ &\quad + \hbar^2 p_j \left(\nu_{\infty,-1}^{(\alpha)} q_j + \nu_{\infty,0}^{(\alpha)} + \sum_{i \neq j} \frac{\mu_i^{(\alpha)}}{q_j - q_i} \right) - \hbar \mu_j^{(\alpha)} p_j \left(\tilde{P}_1(q_j) + \sum_{i \neq j} \frac{\hbar}{q_j - q_i} \right). \end{aligned} \quad (5-2)$$

Inserting (5-1) provides, for all $j \in \llbracket 1, g \rrbracket$,

$$\sum_{k=0}^{r_\infty - 4} H_{\infty,k} q_j^k = p_j^2 - \tilde{P}_1(q_j) p_j + \tilde{P}_2(q_j) + \hbar \sum_{i \neq j} \frac{p_i - p_j}{q_j - q_i} \quad (5-3)$$

where it is obvious that the r.h.s. is independent of the deformation vector α . The last relation can be rewritten into a matrix form.

Proposition 5.1. *We have*

$$(V_\infty)^t \begin{pmatrix} H_{\infty,0} \\ \vdots \\ H_{\infty,r_\infty - 4} \end{pmatrix} = \begin{pmatrix} p_1^2 - \tilde{P}_1(q_1) p_1 + \tilde{P}_2(q_1) + \hbar \sum_{i \neq 1} \frac{p_i - p_1}{q_1 - q_i} \\ \vdots \\ p_g^2 - \tilde{P}_1(q_g) p_g + \tilde{P}_2(q_g) + \hbar \sum_{i \neq g} \frac{p_i - p_g}{q_g - q_i} \end{pmatrix} \quad (5-4)$$

Finally, in order to obtain the evolution equation for $(p_j)_{1 \leq j \leq g}$ we look at order $(\lambda - q_j)^{-1}$ of the entry $\mathcal{L}_\alpha[L_{2,1}(\lambda)]$. We get, for all $j \in \llbracket 1, g \rrbracket$,

$$\begin{aligned} \mathcal{L}_\alpha[p_j] &= \hbar \sum_{i \neq j} \frac{(\mu_i^{(\alpha)} + \mu_j^{(\alpha)})(p_i - p_j)}{(q_j - q_i)^2} + \mu_j^{(\alpha)} \left(p_j \tilde{P}'_1(q_j) - \tilde{P}'_2(q_j) + \sum_{k=1}^{r_\infty - 4} k H_{\infty,k} q_j^{k-1} \right) \\ &\quad + \hbar \nu_{\infty,-1}^{(\alpha)} p_j + \hbar \sum_{k=1}^{r_\infty - 1} k c_{\infty,k}^{(\alpha)} q_j^{k-1}. \end{aligned} \quad (5-5)$$

Thus, we have obtained the general evolutions for $(p_j, q_j)_{1 \leq j \leq g}$ through (5-1) and (5-5). We may now formulate our first main Theorem showing that these evolutions are Hamiltonian.

Theorem 5.1 (Hamiltonian evolution). *Defining*

$$Ham^{(\alpha)}(\mathbf{q}, \mathbf{p}) = \sum_{k=0}^{r_\infty-4} \nu_{\infty, k+1}^{(\alpha)} H_{\infty, k} - \hbar \sum_{j=1}^g \sum_{k=1}^{r_\infty-1} c_{\infty, k}^{(\alpha)} q_j^k - \hbar \nu_{\infty, 0}^{(\alpha)} \sum_{j=1}^g p_j - \hbar \nu_{\infty, -1}^{(\alpha)} \sum_{j=1}^g q_j p_j, \quad (5-6)$$

the evolutions for $j \in \llbracket 1, g \rrbracket$,

$$\begin{aligned} \mathcal{L}_\alpha[q_j] &= 2\mu_j^{(\alpha)} \left(p_j - \frac{1}{2} \tilde{P}_1(q_j) \right) - \hbar \nu_{\infty, 0}^{(\alpha)} - \hbar \nu_{\infty, -1}^{(\alpha)} q_j - \hbar \sum_{i \neq j} \frac{\mu_j^{(\alpha)} + \mu_i^{(\alpha)}}{q_j - q_i}, \\ \mathcal{L}_\alpha[p_j] &= \hbar \sum_{i \neq j} \frac{(\mu_i^{(\alpha)} + \mu_j^{(\alpha)})(p_i - p_j)}{(q_j - q_i)^2} + \mu_j^{(\alpha)} \left(p_j \tilde{P}'_1(q_j) - \tilde{P}'_2(q_j) + \sum_{k=1}^{r_\infty-4} k H_{\infty, k} q_j^{k-1} \right) \\ &\quad + \hbar \nu_{\infty, -1}^{(\alpha)} p_j + \hbar \sum_{k=1}^{r_\infty-1} k c_{\infty, k}^{(\alpha)} q_j^{k-1} \end{aligned} \quad (5-7)$$

are Hamiltonian in the sense that

$$\forall j \in \llbracket 1, g \rrbracket : \mathcal{L}_\alpha[q_j] = \frac{\partial Ham^{(\alpha)}(\mathbf{q}, \mathbf{p})}{\partial p_j} \quad \text{and} \quad \mathcal{L}_\alpha[p_j] = -\frac{\partial Ham^{(\alpha)}(\mathbf{q}, \mathbf{p})}{\partial q_j}. \quad (5-8)$$

Quantities involved in the Hamiltonian evolution are defined by Propositions 4.1, 4.2, 4.3 and 5.1.

Proof. Proof is done in Appendix E. □

Remark 5.1. Note that there is an alternative expression for the Hamiltonian (5-6):

$$\begin{aligned} Ham^{(\alpha)}(\mathbf{q}, \mathbf{p}) &= -\frac{\hbar}{2} \sum_{\substack{(i,j) \in \llbracket 1, g \rrbracket^2 \\ i \neq j}} \frac{(\mu_i^{(\alpha)} + \mu_j^{(\alpha)})(p_i - p_j)}{q_i - q_j} - \hbar \sum_{j=1}^g (\nu_{\infty, 0}^{(\alpha)} p_j + \nu_{\infty, -1}^{(\alpha)} q_j p_j) \\ &\quad + \sum_{j=1}^g \mu_j^{(\alpha)} \left[p_j^2 - \tilde{P}_1(q_j) p_j + \tilde{P}_2(q_j) \right] - \hbar \sum_{j=1}^g \sum_{k=1}^{r_\infty-1} c_{\infty, k}^{(\alpha)} q_j^k \end{aligned} \quad (5-9)$$

Theorem 5.1 shows that the Hamiltonian expression for a general isomonodromic deformation may be split into several contributions

- A linear combination of the $(H_{\infty, k})_{0 \leq k \leq r_\infty-4}$ whose coefficients are given by $\left(\nu_{\infty, k+1}^{(\alpha)} \right)_{1 \leq k \leq r_\infty-3}$. Note that the coefficients $(H_{\infty, k})_{0 \leq k \leq r_\infty-4}$ do not depend on the isomonodromic deformations and correspond to the unknown coefficients of $L_{2,1}(\lambda)$.
- A linear combination of $\left(\sum_{j=1}^g q_j^k \right)_{1 \leq k \leq r_\infty-1}$ whose coefficients are given by $\left(c_{\infty, k}^{(\alpha)} \right)_{1 \leq k \leq r_\infty-1}$. As we will see in the next sections, these terms will vanish for a suitable choice of non-trivial isomonodromic deformations.
- Two additional terms $\sum_{j=1}^g p_j$ and $\sum_{j=1}^g q_j p_j$ that are respectively proportional to $-\hbar \nu_{\infty, 0}^{(\alpha)}$ and $-\hbar \nu_{\infty, -1}^{(\alpha)}$. These terms shall be removed after a suitable symplectic rescaling of $(q_i, p_i)_{1 \leq i \leq g}$.

6 Expression of the Hamiltonian and Lax matrices in terms of symmetric Darboux coordinates

In this section, we show that we may use the symmetric polynomials $(e_i(\{\check{q}_1, \dots, \check{q}_g\}))_{1 \leq i \leq g}$ to obtain polynomial Hamiltonians and polynomial explicit formulas for the matrices \tilde{L} and $\tilde{A}^{(\alpha\tau)}$.

6.1 Notations and identities regarding symmetric polynomials

In the rest of the paper we need to introduce elementary symmetric polynomials and other basis of symmetric polynomials.

Definition 6.1 (Basis of symmetric polynomials). We shall introduce the following basis of symmetric polynomials:

- Elementary symmetric polynomials are denoted by $(e_i(\{x_1, \dots, x_n\}))_{i \geq 0}$ with the convention that $e_0(\{x_1, \dots, x_n\}) = 1$ and $e_k(\{x_1, \dots, x_n\}) = 0$ if $k > n$. By definition we have:

$$e_k(\{x_1, \dots, x_n\}) = \sum_{1 \leq i_1 < \dots < i_k \leq n} x_{i_1} \dots x_{i_k}, \quad \forall k \in \llbracket 1, n \rrbracket \quad (6-1)$$

- Complete homogeneous symmetric polynomial are denoted by $(h_i(\{x_1, \dots, x_n\}))_{i \geq 0}$ with the convention that $h_0(\{x_1, \dots, x_n\}) = 1$. By definition we have:

$$h_k(\{x_1, \dots, x_n\}) = \sum_{1 \leq i_1 \leq \dots \leq i_k \leq n} x_{i_1} \dots x_{i_k}, \quad \forall k \in \llbracket 1, n \rrbracket \quad (6-2)$$

- k^{th} symmetric power sum polynomials are denoted by $(S_k(\{x_1, \dots, x_n\}))_{k \geq 0}$. By definition, we have:

$$\begin{aligned} S_0(\{x_1, \dots, x_n\}) &= n \\ S_k(\{x_1, \dots, x_n\}) &= \sum_{j=1}^n x_j^k, \quad \forall k \geq 1 \end{aligned} \quad (6-3)$$

$(e_k(\{x_1, \dots, x_n\}))_{0 \leq k \leq n}$, $(h_k(\{x_1, \dots, x_n\}))_{0 \leq k \leq n}$ and $(S_k(\{x_1, \dots, x_n\}))_{0 \leq k \leq n}$ are some basis of symmetric polynomials in the variables $\{x_1, \dots, x_n\}$. We also have the relations

$$\begin{aligned} \prod_{j=1}^n (\lambda - x_j) &= \sum_{k=0}^n (-1)^{n-k} e_{n-k}(\{x_1, \dots, x_n\}) \lambda^k = \sum_{k=0}^n (-1)^k e_k(\{x_1, \dots, x_n\}) \lambda^{n-k} \\ \frac{1}{\prod_{j=1}^n (\lambda - x_j)} &= \sum_{k=0}^{\infty} h_k(\{x_1, \dots, x_n\}) \lambda^{-n-k} \end{aligned} \quad (6-4)$$

The relation between the various sets are given by

$$\begin{aligned} h_0(\{x_1, \dots, x_n\}) &= e_0(\{x_1, \dots, x_n\}) \\ h_k(\{x_1, \dots, x_n\}) &= \sum_{j=1}^k (-1)^j \sum_{\substack{b_1, \dots, b_j \in \llbracket 1, k \rrbracket^j \\ b_1 + \dots + b_j = k}} \prod_{m=1}^j (-1)^{b_m} e_{b_m}(\{x_1, \dots, x_n\}), \quad \forall k \in \llbracket 1, n \rrbracket \end{aligned}$$

(6-5)

and $\forall m \geq 1$:

$$\begin{aligned}
S_m(\{x_1, \dots, x_n\}) &= (-1)^m m \sum_{k=1}^m \frac{1}{k} \hat{B}_{m,k}(-e_1(\{x_1, \dots, x_n\}), \dots, -e_{m-k+1}(\{x_1, \dots, x_n\})) \\
&= (-1)^m m \sum_{\substack{b_1+2b_2+\dots+mb_m=m \\ b_1 \geq 0, \dots, b_m \geq 0}} \frac{(-1)^{b_1+\dots+b_m}}{(b_1+\dots+b_m)} \binom{b_1+\dots+b_m}{b_1, \dots, b_m} \prod_{i=1}^m e_i(\{x_1, \dots, x_n\})^{b_i}
\end{aligned} \tag{6-6}$$

where $(\hat{B}_{m,k})_{m \geq k \geq 0}$ are the ordinary Bell polynomials. Finally, we also have the identities

$$\begin{aligned}
(n-k)e_k(\{x_1, \dots, x_n\}) &= \sum_{i=0}^k (-1)^i e_{k-i}(\{x_1, \dots, x_n\}) S_i(\{x_1, \dots, x_n\}), \quad \forall k \in \llbracket 0, n \rrbracket \\
S_k(\{x_1, \dots, x_n\}) &= \sum_{i=k-n}^{k-1} (-1)^{k-1+i} e_{k-i}(\{x_1, \dots, x_n\}) S_i(\{x_1, \dots, x_n\}), \quad \forall k \geq n
\end{aligned} \tag{6-7}$$

Elementary symmetric polynomials satisfy some useful relations:

Lemma 6.1. *For any $(i, m) \in \llbracket 1, g \rrbracket^2$:*

$$\frac{\partial e_i(\{x_1, \dots, x_g\})}{\partial x_m} = \sum_{j=0}^{i-1} (-1)^j e_{i-1-j}(\{x_1, \dots, x_g\}) x_m^j \tag{6-8}$$

Proposition 6.1. *For any $i \in \llbracket 1, g \rrbracket$, we have*

$$\sum_{k=1}^g \frac{\partial e_i(\{x_1, \dots, x_g\})}{\partial x_k} \prod_{j \neq k} \frac{\lambda - x_j}{x_k - x_j} = \sum_{j=0}^{i-1} (-1)^j e_{i-j-1}(\{x_1, \dots, x_g\}) \lambda^j \tag{6-9}$$

These relations allow to express $Q(\lambda)$.

Corollary 6.1. *We have*

$$Q(\lambda) = \sum_{j=0}^{g-1} (-1)^{j-1} \left(\sum_{i=j+1}^g P_i Q_{i-j-1} \right) \lambda^j \tag{6-10}$$

Moreover, the elementary symmetric polynomials satisfy:

$$\begin{aligned}
\forall (i, j) \in \llbracket 1, n \rrbracket^2 : e_{n-i}(\{x_1, \dots, x_n\} \setminus \{x_j\}) &= \sum_{m=i}^n (-1)^{m-i} e_{n-m}(\{x_1, \dots, x_n\}) x_j^{m-i} \\
\forall j \in \llbracket 1, n \rrbracket : 0 &= \sum_{m=0}^g (-1)^{n-m} e_{n-m}(\{x_1, \dots, x_n\}) x_j^m
\end{aligned} \tag{6-11}$$

so that we obtain

Lemma 6.2. For any $i \in \llbracket 1, n \rrbracket$ and any $M \geq 0$ we have:

$$x_i^M = \sum_{j=1}^n \sum_{m=\text{Max}(j, j+n-1-M)}^n (-1)^{n-m} e_{n-m}(\{x_1, \dots, x_n\}) h_{M+m-j-n+1}(\{x_1, \dots, x_n\}) x_i^{j-1} \quad (6-12)$$

In particular, we may invert the Vandermonde matrix with the following Proposition.

Proposition 6.2. For any $i \in \llbracket 0, n \rrbracket$ and $M \geq 0$ we have:

$$\sum_{j=1}^n \frac{(-1)^{n-i} e_{n-i}(\{x_1, \dots, x_n\} \setminus \{x_j\})}{\prod_{m \neq j} (x_j - x_m)} x_j^M = \sum_{m=\text{Max}(i, i+n-1-M)}^n (-1)^{n-m} e_{n-m}(\{x_1, \dots, x_n\}) h_{M+m-i-n+1}(\{x_1, \dots, x_n\}) \quad (6-13)$$

In particular, for $M \leq n-1$ we get:

$$\forall M \in \llbracket 0, n-1 \rrbracket : \sum_{j=1}^n \frac{(-1)^{n-i} e_{n-i}(\{x_1, \dots, x_n\} \setminus \{x_j\})}{\prod_{m \neq j} (x_j - x_m)} x_j^M = \delta_{i, M+1} \quad (6-14)$$

Proof. For completeness, the proofs are presented in Appendix F. \square

6.2 Symmetric Darboux coordinates

Let us first recall the well-known result from symplectic geometry.

Lemma 6.3. If we define new coordinates $(\tilde{Q}_1, \dots, \tilde{Q}_g, \tilde{P}_1, \dots, \tilde{P}_g)$ from old symplectic coordinates $(Q_1, \dots, Q_g, P_1, \dots, P_g)$ by

$$\begin{aligned} \tilde{Q}_i &= f_i(\alpha^{-1}Q_1 + \beta, \dots, \alpha^{-1}Q_g + \beta) \\ \alpha P_i + h(\alpha^{-1}Q_i + \beta) &= \alpha \sum_{k=1}^g \tilde{P}_k \frac{\partial f_k(\alpha^{-1}Q_1 + \beta, \dots, \alpha^{-1}Q_g + \beta)}{\partial Q_i}, \quad \forall i \in \llbracket 1, g \rrbracket \end{aligned} \quad (6-15)$$

with $\alpha \in \mathbb{C} \setminus \{0\}$ and $\beta \in \mathbb{C}$ two given constants, h any function of class \mathcal{C}^1 and $(f_m(x_1, \dots, x_g))_{1 \leq m \leq g}$ any functions of class \mathcal{C}^2 then the change of coordinates is symplectic.

Proof. For completeness, the proof is done in Appendix G. \square

We may now apply the lemma with the elementary symmetric polynomials $(e_i(\tilde{q}_1, \dots, \tilde{q}_g))_{1 \leq i \leq g}$ which is a basis of the symmetric polynomials in $(\tilde{q}_1, \dots, \tilde{q}_g)$.

Definition 6.2 (Symmetric Darboux coordinates). We define $(Q_1, \dots, Q_g, P_1, \dots, P_g)$ using the elementary symmetric polynomials:

$$\begin{aligned} Q_i &= e_i(q_1, \dots, q_g) = e_i(T_2^{-1}\tilde{q}_1 - T_1, \dots, T_g^{-1}\tilde{q}_g - T_1) \\ p_i &= \sum_{k=1}^g P_k \frac{\partial e_k(q_1, \dots, q_g)}{\partial q_i} = T_2 \tilde{p}_i + \frac{1}{2} \tilde{P}_1 (T_2^{-1}\tilde{q}_i - T_1), \quad \forall i \in \llbracket 1, g \rrbracket \end{aligned} \quad (6-16)$$

We shall denote $(Q_1, \dots, Q_g, P_1, \dots, P_g)$, the symmetric Darboux coordinates.

It is obvious from Lemma 6.3 that the change of coordinates is symplectic. Indeed, for the old variables $(q_1, \dots, q_g, p_1, \dots, p_g)$ this is nothing but an application of Lemma 6.3 with $\alpha = 1$, $\beta = 0$ and $h(x) = 0$ while for the other old variables $(\tilde{q}_1, \dots, \tilde{q}_g, \tilde{q}_1, \dots, \tilde{q}_g)$, this corresponds to an application of Lemma 6.3 with $\alpha = T_2$, $\beta = -T_1$ and $h(x) = \frac{1}{2} \tilde{P}_1(x)$.

6.3 Polynomial expression of the Hamiltonian in the symmetric Darboux coordinates

Since the change of coordinates $(\check{q}_i, \check{p}_i)_{1 \leq i \leq g} \rightarrow (Q_i, P_i)_{1 \leq i \leq g}$ is symplectic, we may compute the Hamiltonian $\text{Ham}(\mathbf{Q}, \dots, \mathbf{P})$ by just replacing the coordinates $(q_i, p_i)_{1 \leq i \leq g}$ in terms of $(Q_i, P_i)_{1 \leq i \leq g}$ in Theorem 5.1.

Theorem 6.1 (Expression of the general Hamiltonian in terms of symmetric Darboux coordinates). *We have:*

$$\begin{aligned}
\text{Ham}^{(\alpha)}(\mathbf{Q}, \mathbf{P}) &= -\hbar \sum_{j=1}^g (\nu_{\infty,0}^{(\alpha)} p_j + \nu_{\infty,-1}^{(\alpha)} q_j p_j) - \hbar \sum_{j=1}^g \sum_{k=1}^{r_\infty-1} c_{\infty,k}^{(\alpha)} q_j^k + \sum_{i=1}^g \nu_{\infty,i}^{(\alpha)} H_{\infty,i+1} \\
&= -\hbar \nu_{\infty,0}^{(\alpha)} \sum_{k=0}^{g-1} (g-k) Q_k P_{k+1} - \hbar \nu_{\infty,-1}^{(\alpha)} \sum_{k=1}^g k Q_k P_k - \hbar \sum_{k=1}^{r_\infty-1} c_{\infty,k}^{(\alpha)} S_k(\{q_1, \dots, q_g\}) \\
&\quad - \hbar \sum_{i=1}^g \nu_{\infty,i}^{(\alpha)} \sum_{k=i+1}^g \left((-1)^i (g-i) P_k Q_{k-1-i} + \sum_{m=i+1}^{k-1} (-1)^m P_k Q_{k-1-m} S_{m-i}(\{q_1, \dots, q_g\}) \right) \\
&\quad + \sum_{i=1}^g \nu_{\infty,i}^{(\alpha)} \sum_{k_1=1}^g \sum_{k_2=1}^g P_{k_1} P_{k_2} \left[(-1)^{i-1} \sum_{r_1=\text{Max}(0, i-k_2)}^{\text{Min}(k_1-1, i-1)} Q_{k_1-1-r_1} Q_{k_2-i+r_1} \right. \\
&\quad \left. + \sum_{\substack{0 \leq r_1 \leq k_1-1 \\ 0 \leq r_2 \leq k_2-1 \\ r_1+r_2 \geq g}} (-1)^{r_1+r_2} Q_{k_1-1-r_1} Q_{k_2-1-r_2} \sum_{m=i}^g (-1)^{g-m} Q_{g-m} h_{r_1+r_2+m-i-g+1}(\{q_1, \dots, q_g\}) \right] \\
&\quad + \sum_{i=1}^g \nu_{\infty,i}^{(\alpha)} \sum_{k=1}^g \left[\sum_{r=0}^{\text{Min}(k-1, i-1)} (-1)^r t_{\infty, 2i-2r} P_k Q_{k-1-r} \right. \\
&\quad \left. + \sum_{r=0}^{k-1} \sum_{s=g-r}^{g+1} \sum_{m=i}^g (-1)^{g+r-m} t_{\infty, 2s+2} P_k Q_{k-1-r} Q_{g-m} h_{r+s+m-i-g+1}(\{q_1, \dots, q_g\}) \right] \\
&\quad + \sum_{i=1}^g \nu_{\infty,i}^{(\alpha)} \sum_{r=g}^{2r_\infty-4} \sum_{m=i}^g (-1)^{g-m} \tilde{P}_{\infty,r}^{(2)} Q_{g-m} h_{r+m-i-g+1}(\{q_1, \dots, q_g\})
\end{aligned} \tag{6-17}$$

where $(\tilde{P}_{\infty,k}^{(2)})_{g \leq k \leq 2g+4}$ are given by Proposition 2.4, $(\nu_{\infty,i}^{(\alpha)})_{-1 \leq i \leq g}$ are given by (4-8), and coefficients $(S_k(\{q_1, \dots, q_g\}))_{k \geq 0}$, $(h_k(\{q_1, \dots, q_g\}))_{k \geq 0}$ are given by Definition 6.1.

The main advantage of the explicit expression (6-17) is that it immediately shows that the **general Hamiltonian is polynomial in** $(Q_i, P_i)_{1 \leq i \leq g}$, i.e. it has the same kind of singularities as the initial connection. In particular, it is quadratic in $(P_i)_{1 \leq i \leq g}$. Note also that the explicit dependence of the Hamiltonian in the irregular times is contained only in $(\nu_{\infty,i}^{(\alpha)})_{-1 \leq i \leq g}$ and $(\tilde{P}_{\infty,k}^{(2)})_{g \leq k \leq 2g+1}$.

Proof. The proof is done in Appendix H. □

6.4 Expressing the Lax matrices with the symmetric Darboux coordinates

Symmetric Darboux coordinates $(Q_1, \dots, Q_g, P_1, \dots, P_g)$ are well-suited for the matrix \tilde{L} given by (2-37) as the following proposition shows

Proposition 6.3. *Entries of the matrix $\tilde{L}(\lambda)$ are given by*

$$\begin{aligned}
\tilde{L}_{1,1}(\lambda) &= -\sum_{j=0}^{g-1} (-1)^{j-1} \left(\sum_{i=j+1}^g P_i Q_{i-j-1} \right) \lambda^j - \left(\frac{1}{2} t_{\infty, 2r_{\infty}-2} \lambda + g_0 \right) \sum_{j=0}^g (-1)^{g-j} Q_{g-j} \lambda^j \\
\tilde{L}_{1,2}(\lambda) &= \sum_{m=0}^g (-1)^{g-m} Q_{g-m} \lambda^m \\
\tilde{L}_{2,2}(\lambda) &= \sum_{j=0}^{g-1} (-1)^{j-1} \left(\sum_{i=j+1}^g P_i Q_{i-j-1} \right) \lambda^j + \left(\frac{1}{2} t_{\infty, 2r_{\infty}-2} \lambda + g_0 \right) \sum_{j=0}^g (-1)^{g-j} Q_{g-j} \lambda^j - \sum_{k=0}^{r_{\infty}-2} t_{\infty, 2k+2} \lambda^k \\
\tilde{L}_{2,1}(\lambda) &= -\sum_{i=0}^{r_{\infty}-2} \sum_{j=g+i}^{2r_{\infty}-4} \left(\tilde{P}_{\infty, j}^{(2)} h_{j-g-i}(\{q_1, \dots, q_g\}) \right) \lambda^i \\
&\quad + \sum_{i=0}^{g-1} \left(\sum_{j=i}^{g-1} \sum_{s=g+i-j}^{g+1} (-1)^{j-1} t_{\infty, 2s+2} \left(\sum_{r=j+1}^g P_r Q_{r-j-1} \right) h_{s+j-i-g} \right) \lambda^i \\
&\quad - \sum_{i=0}^{g-2} \left(\sum_{j_1=i+1}^{g-1} \sum_{j_2=g+i-j_1}^{g-1} (-1)^{j_1+j_2} \left(\sum_{i_1=j_1+1}^g P_{i_1} Q_{i_1-j_1-1} \right) \left(\sum_{i_2=j_2+1}^g P_{i_2} Q_{i_2-j_2-1} \right) h_{j_1+j_2-g-i}(\{q_1, \dots, q_g\}) \right) \lambda^i \\
&\quad - \left(\frac{1}{2} t_{\infty, 2r_{\infty}-2} \lambda + g_0 \right)^2 \sum_{m=0}^g (-1)^{g-m} Q_{g-m} \lambda^m + \left(\frac{1}{2} t_{\infty, 2r_{\infty}-2} \lambda + g_0 \right) \sum_{s=0}^{g+1} t_{\infty, 2s+2} \lambda^s \\
&\quad - \left(\frac{1}{2} t_{\infty, 2r_{\infty}-2} \lambda + g_0 \right) \sum_{j=0}^{g-1} (-1)^{j-1} \left(\sum_{i=j+1}^g P_i Q_{i-j-1} \right) \lambda^j \tag{6-18}
\end{aligned}$$

where $g_0 = \frac{1}{2} t_{\infty, 2r_{\infty}-4} + \frac{1}{2} t_{\infty, 2r_{\infty}-2} Q_1$.

Proof. Only the expression of $Q(\lambda)$ is non-trivial and is given by Corollary 6.1. \square

We remind the reader that $\left(\tilde{P}_{\infty, j}^{(2)} \right)_{g \leq j \leq 2r_{\infty}-4}$ are given by Definition 2.4 while the complete homogenous symmetric polynomials $(h_j(\{q_1, \dots, q_g\}))_{0 \leq j \leq g}$ may be expressed in terms of $(Q_k)_{1 \leq k \leq g}$ using

$$\begin{aligned}
h_0(\{q_1, \dots, q_g\}) &= 1 \\
h_k(\{q_1, \dots, q_n\}) &= \sum_{j=1}^k (-1)^j \sum_{\substack{b_1, \dots, b_j \in \llbracket 1, k \rrbracket^j \\ b_1 + \dots + b_j = k}} \prod_{m=1}^j (-1)^{b_m} Q_{b_m}, \quad \forall k \in \llbracket 1, g \rrbracket \tag{6-19}
\end{aligned}$$

In particular Proposition 6.3 implies that the entries of the **Lax matrix \tilde{L} are also polynomial in the symmetric Darboux coordinates and at most quadratic in $(P_i)_{1 \leq i \leq g}$.**

We may also obtain the entries of \tilde{A}_{α} in terms of the symmetric Darboux coordinates.

Proposition 6.4. *Entries of the matrix $\tilde{A}_{\alpha}(\lambda)$ are given by*

$$\begin{aligned}
[\tilde{A}_{\alpha}(\lambda)]_{1,1} &= \sum_{i=0}^{r_{\infty}-1} c_{\infty, i}^{(\alpha)} \lambda^i - \sum_{i=0}^g \sum_{m=\text{Max}(-1, -i)}^{g-1-i} (-1)^{i+m-1} \nu_{\infty, m}^{(\alpha)} \left(\sum_{r=i+m+1}^g P_r Q_{r-i-m-1} \right) \lambda^i \\
&\quad - \left(\frac{1}{2} t_{\infty, 2r_{\infty}-2} \lambda + g_0 \right) \sum_{i=0}^{g+1} \sum_{m=\text{Max}(-1, -i)}^{g-i} (-1)^{g-i-m} Q_{g-i-m} \nu_{\infty, m}^{(\alpha)} \lambda^i \\
[\tilde{A}_{\alpha}(\lambda)]_{1,2} &= \sum_{j=0}^{g+1} \left(\sum_{m=\text{Max}(-1, -j)}^{g-j} (-1)^{g-j-m} \nu_{\infty, m}^{(\alpha)} Q_{g-j-m} \right) \lambda^j \\
[\tilde{A}_{\alpha}(\lambda)]_{2,2} &= -[\tilde{A}^{(\alpha)}(\lambda)]_{1,1} - \sum_{s=1}^{r_{\infty}-1} \frac{1}{s} \alpha_{\infty, 2s} \lambda^s + 2c_{\infty, 0}^{(\alpha)} + \hbar(g+1) \nu_{\infty, -1}^{(\alpha)} - \sum_{j=0}^{r_{\infty}-2} t_{\infty, 2j+2} \nu_{\infty, j}^{(\alpha)}
\end{aligned}$$

$$\begin{aligned}
[\tilde{A}_\alpha(\lambda)]_{2,1} = & -\frac{\hbar}{2}\nu_{\infty,-1}^{(\alpha)}t_{\infty,2r_\infty-2}\lambda^2 - \hbar\nu_{\infty,-1}^{(\alpha)}g_0\lambda - \hbar\nu_{\infty,-1}^{(\alpha)}(-1)^gP_g \\
& -\frac{\hbar}{2}gt_{\infty,2r_\infty-2}\nu_{\infty,-1}^{(\alpha)}\lambda - \hbar gg_0\nu_{\infty,-1}^{(\alpha)} + \frac{\hbar}{2}t_{\infty,2r_\infty-2}\nu_{\infty,-1}^{(\alpha)}Q_1 - \frac{\hbar}{2}gt_{\infty,2r_\infty-2}\nu_{\infty,0}^{(\alpha)} \\
& + \hbar(r_\infty-1)c_{\infty,r_\infty-1}^{(\alpha)}\lambda + \hbar(r_\infty-1)c_{\infty,r_\infty-1}^{(\alpha)}Q_1 + \hbar(r_\infty-2)c_{\infty,r_\infty-2}^{(\alpha)} + \nu_{\infty,-1}^{(\alpha)}H_{\infty,r_\infty-4} \\
& + \frac{\hbar}{2}\alpha_{\infty,2r_\infty-2} + \frac{\hbar}{2}\alpha_{\infty,2r_\infty-4} + \frac{\hbar}{2}\alpha_{\infty,2r_\infty-2}Q_1 \\
& - \sum_{i=0}^g \sum_{j=\text{Max}(i-1,0)}^{g-1} \sum_{s=g+i-j-1}^{g+1} \sum_{r=j+1}^g \sum_{m=-1}^{s+j-g-i} (-1)^j t_{\infty,2s+2} \nu_{\infty,m}^{(\alpha)} h_{s+j-m-g-i} P_r Q_{r-j-1} \lambda^i \\
& - \sum_{i=0}^{r_\infty} \sum_{j=\text{Max}(g,g+i-1)}^{2r_\infty-4} \sum_{m=-1}^{j-g-i} \nu_{\infty,m}^{(\alpha)} h_{j-g-m-i} \tilde{P}_{\infty,j}^{(2)} \lambda^i \\
& - \sum_{i=0}^g \sum_{j_1=0}^{g-1} \sum_{j_2=0}^{g-1} \sum_{m=-1}^{j_1+j_2-g-i} (-1)^{j_1+j_2} \nu_{\infty,m}^{(\alpha)} h_{j_1+j_2-g-m-i} \sum_{r_1=j_1+1}^g \sum_{r_2=j_2+1}^g P_{r_1} P_{r_2} Q_{r_1-j_1-1} Q_{r_2-j_2-1} \lambda^i \\
& + \left(\frac{1}{2}t_{\infty,2r_\infty-2}\lambda + g_0\right) \sum_{i=0}^{r_\infty-1} \sum_{s=\text{Max}(i-1,0)}^{r_\infty-2} t_{\infty,2s+2} \nu_{\infty,s-i}^{(\alpha)} \lambda^i \\
& - (t_{\infty,2r_\infty-2}\lambda + 2g_0) \sum_{i=0}^g \sum_{j=\text{Max}(i-1,0)}^{g-1} \sum_{r=j+1}^g (-1)^{j-1} \nu_{\infty,j-i}^{(\alpha)} P_r Q_{r-j-1} \lambda^i \\
& - \left(\frac{1}{2}t_{\infty,2r_\infty-2}\lambda + g_0\right) \sum_{i=0}^{2g+1} \sum_{j=\text{Max}(i-1,0)}^g (-1)^{g-j} Q_{g-j} \nu_{\infty,j-i}^{(\alpha)} \lambda^i \tag{6-20}
\end{aligned}$$

with $g_0 = \frac{1}{2}t_{\infty,2r_\infty-4} + \frac{1}{2}t_{\infty,2r_\infty-2}Q_1$. Coefficients $\left(\nu_{\infty,m}^{(\alpha)}\right)_{-1 \leq m \leq g}$ are given by (4-8). Coefficients $\left(c_{\infty,i}^{(\alpha)}\right)_{1 \leq i \leq r_\infty-1}$ given by (4-20) and $\nu_{\infty,r_\infty-2}^{(\alpha)} = \sum_{k=1}^g (-1)^{g-k} \nu_{\infty,k}^{(\alpha)} Q_{g+1-k}$ from (4-17).

Proof. The proof is rather long and done in Appendix I. \square

7 Decomposition and reduction of the space of isomonodromic deformations

The second goal of this paper is now to provide a decomposition of the space of isomonodromic deformations (of dimension $2g+4$) into a subspace of trivial deformations associated to trivial times (i.e. for which rescaled Darboux coordinates are independent) and a subspace of non-trivial deformations of dimension g associated to non-trivial times while providing the corresponding Hamiltonian evolutions.

7.1 Subspaces of trivial and non-trivial deformations

In the previous section we considered general isomonodromic deformations relatively to all irregular times by considering \mathcal{L}_α characterized by a general vector $\alpha \in \mathbb{C}^{2g+4}$. However, as we will see below, there exists a subspace of deformations of dimension $g+4$ for which the evolutions of the Darboux coordinates are trivial, thus leaving only a non-trivial subspace of deformations of dimension g . These non-trivial deformations shall later be mapped to g isomonodromic times whose expressions will be explicit in terms of the initial irregular times. Trivial deformations shall correspond to the fact that only odd irregular times $(t_{\infty,2k-1})_{1 \leq k \leq r_\infty-1}$ are relevant whereas even irregular times $(t_{\infty,2k})_{1 \leq k \leq r_\infty-1}$ do not appear in the Hamiltonians. In other words, **considering meromorphic connections in $\mathfrak{gl}_2(\mathbb{C})$ or in $\mathfrak{sl}_2(\mathbb{C})$ is essentially**

the same at the level of the Hamiltonian systems. This remark shall provide a subspace of trivial deformations of dimension $g + 2$. Finally, the remaining 2 trivial deformations correspond to the remaining two degrees of freedom in the action of the Möbius transformations. As we will see below, this choice encodes the necessity of a symplectic rescaling (translation and dilatation) of the Darboux coordinates. These two degrees of freedom shall be used to fix the first two leading non-trivial coefficients at infinity: $t_{\infty, 2r_{\infty}-3}$ (conventionally set to 2) and $t_{\infty, 2r_{\infty}-5}$ (conventionally set to 0).

Let us first recall that the space of isomonodromic deformations, denoted \mathcal{T} , is given by:

$$\mathcal{L}_{\alpha} = \hbar \sum_{k=1}^{2r_{\infty}-2} \alpha_{\infty, k} \partial_{t_{\infty, k}} \quad (7-1)$$

We make the identification with $\mathbb{C}^{2r_{\infty}-2}$ by identifying an isomonodromic deformation \mathcal{L}_{α} with its vector $\alpha \in \mathbb{C}^{2r_{\infty}-2}$:

$$\mathcal{L}_{\alpha} = \hbar \sum_{k=1}^{2r_{\infty}-2} \alpha_{\infty, k} \partial_{t_{\infty, k}} \Leftrightarrow \alpha = \sum_{k=1}^{2r_{\infty}-2} \alpha_{\infty, k} \mathbf{e}_k \quad (7-2)$$

where we shall denote $(\mathbf{e}_k)_{1 \leq k \leq 2r_{\infty}-2}$ the canonical basis of $\mathbb{C}^{2r_{\infty}-2}$.

Definition 7.1. We define the following vectors of $\mathbb{C}^{2r_{\infty}-2}$ and their corresponding deformations.

$$\begin{aligned} \mathbf{w}_k &= \mathbf{e}_{2k}, \quad \forall k \in \llbracket 1, r_{\infty} - 1 \rrbracket \\ \mathbf{u}_k &= \frac{1}{2} \sum_{m=1}^{r_{\infty}-k-2} (2m-1) t_{\infty, 2m+1+2k} \mathbf{e}_{2m-1} + \frac{1}{2} \sum_{s=1}^{r_{\infty}-k-2} 2s t_{\infty, 2s+2k+2} \mathbf{e}_{2s} \\ &= \frac{1}{2} \sum_{r=1}^{2r_{\infty}-2k-4} r t_{\infty, r+2k+2} \mathbf{e}_r, \quad \forall k \in \llbracket -1, r_{\infty} - 3 \rrbracket \end{aligned} \quad (7-3)$$

and we shall denote:

$$\begin{aligned} \mathcal{U}_{\text{trivial}} &= \text{Span} \{ \mathbf{w}_1, \dots, \mathbf{w}_{r_{\infty}-1}, \mathbf{u}_{-1}, \mathbf{u}_0 \} \\ \mathcal{U}_{\text{iso}} &= \text{Span} \{ \mathbf{u}_1, \dots, \mathbf{u}_{r_{\infty}-3} \} \end{aligned} \quad (7-4)$$

Note in particular that \mathcal{U}_{iso} is of dimension $g = r_{\infty} - 3$ and that $(\mathbf{w}_1, \dots, \mathbf{w}_{r_{\infty}-1}, \mathbf{u}_{-1}, \dots, \mathbf{u}_{r_{\infty}-3})$ is a basis of $\mathbb{C}^{2r_{\infty}-2}$. The choice of basis is such that the following proposition holds.

Proposition 7.1. *We have for all $k \in \llbracket 1, r_{\infty} - 1 \rrbracket$:*

$$\begin{aligned} \nu_j^{(\mathbf{w}_k)} &= 0, \quad \forall j \in \llbracket -1, r_{\infty} - 3 \rrbracket \\ \mu_j^{(\mathbf{w}_k)} &= 0, \quad \forall j \in \llbracket 1, g \rrbracket \\ c_{\infty, j}^{(\mathbf{w}_k)} &= -\frac{1}{2k} \delta_{j, k}, \quad \forall j \in \llbracket 1, r_{\infty} - 1 \rrbracket \end{aligned} \quad (7-5)$$

and for all $k \in \llbracket -1, r_{\infty} - 3 \rrbracket$:

$$\begin{aligned} \nu_j^{(\mathbf{u}_k)} &= \delta_{j, k}, \quad \forall j \in \llbracket -1, r_{\infty} - 3 \rrbracket \\ c_{\infty, j}^{(\mathbf{u}_k)} &= 0, \quad \forall j \in \llbracket 1, r_{\infty} - 1 \rrbracket \end{aligned} \quad (7-6)$$

Proof. The proof is presented in Appendix J. □

Note that Proposition 7.1 and (4-11) imply that

$$\mu_j^{(\mathbf{u}_{-1})} = 0 = \mu_j^{(\mathbf{u}_0)}, \quad \forall j \in \llbracket 1, g \rrbracket \quad (7-7)$$

so that for all $j \in \llbracket 1, g \rrbracket$:

Proposition 7.1 combined with Theorem 5.1 provides the following theorem.

Theorem 7.1. *For any $j \in \llbracket 1, g \rrbracket$, we have:*

$$\begin{aligned} \mathcal{L}_{\mathbf{w}_k}[q_j] &= 0, \quad \forall k \in \llbracket 1, r_\infty - 1 \rrbracket \\ \mathcal{L}_{\mathbf{w}_k}[p_j] &= -\frac{\hbar}{2} q_j^{k-1}, \quad \forall k \in \llbracket 1, r_\infty - 1 \rrbracket \\ \mathcal{L}_{\mathbf{u}_{-1}}[q_j] &= -\hbar q_j \\ \mathcal{L}_{\mathbf{u}_{-1}}[p_j] &= \hbar p_j \\ \mathcal{L}_{\mathbf{u}_0}[q_j] &= -\hbar \\ \mathcal{L}_{\mathbf{u}_0}[p_j] &= 0 \end{aligned} \quad (7-8)$$

Proof. The proof is done in Appendix K. □

Note that $\mathcal{L}_{\mathbf{u}_{-1}}$ and $\mathcal{L}_{\mathbf{u}_0}$ do not act trivially on $(q_j, p_j)_{1 \leq j \leq g}$. As we will see below, one needs to rescale the Darboux coordinates in order to have a trivial action. The purpose of the next section is to define trivial and isomonodromic times that are dual to the previous deformations. However, it is not possible to define some times $(\tau_1, \dots, \tau_{r_\infty-3})$ such that $\hbar \partial_{\tau_k} = \mathcal{L}_{\mathbf{u}_k}$ since the system becomes non compatible for $g \geq 4$.

7.2 Definition of trivial times and isomonodromic times

The split in the tangent space between trivial and non-trivial subspaces may be translated at the level of coordinates. This corresponds to choosing g non-trivial times and $g + 4$ trivial times for which the evolutions of the shifted Darboux coordinates are trivial. In particular, one may then choose the values of these trivial times to any arbitrary values without changing the Hamiltonian evolutions. However, it is important to notice that the choice of trivial times and isomonodromic times is not unique since, for example, one may use any arbitrary combination of isomonodromic times to provide a new one. We propose the following set of trivial and non-trivial times that are particularly convenient in our context.

Definition 7.2 (Trivial and non-trivial deformation times). Let us define the following ‘‘trivial times’’:

$$\begin{aligned} T_{\infty, k} &= t_{\infty, 2k}, \quad \forall k \in \llbracket 1, r_\infty - 1 \rrbracket \\ T_2 &= \left(\frac{1}{2} t_{\infty, 2r_\infty - 3} \right)^{\frac{2}{2r_\infty - 3}} \\ T_1 &= \frac{t_{\infty, 2r_\infty - 5}}{2r_\infty - 5} \left(\frac{1}{2} t_{\infty, 2r_\infty - 3} \right)^{-\frac{2r_\infty - 5}{2r_\infty - 3}} \end{aligned} \quad (7-9)$$

We also define the $g = r_\infty - 3$ ‘‘isomonodromic’’ times $(\tau_k)_{1 \leq k \leq g}$, for all $k \in \llbracket 1, g \rrbracket$, by:

$$\tau_k = \sum_{i=0}^{k-1} \frac{(-1)^i \left(\prod_{s=1}^i (2r_\infty - 2k + 2s - 7) \right) \left(\frac{1}{2} t_{\infty, 2r_\infty - 5} \right)^i \left(\frac{1}{2} t_{\infty, 2r_\infty - 3} \right)^{-\frac{(2r_\infty - 3)i + 2r_\infty - 5 - 2k}{2r_\infty - 3}} \frac{1}{2} t_{\infty, 2r_\infty - 5 - 2k + 2i}}{i! (2r_\infty - 5)^i}$$

$$+ \frac{(-1)^k \left(\prod_{s=1}^k (2r_\infty - 2k + 2s - 7) \right) \left(\frac{1}{2} t_{\infty, 2r_\infty - 5} \right)^{k+1} \left(\frac{1}{2} t_{\infty, 2r_\infty - 3} \right)^{-\frac{(k+1)(2r_\infty - 5)}{2r_\infty - 3}}}{(k+1)(k-1)!(2r_\infty - 5)^k} \quad (7-10)$$

We shall denote $\mathcal{T}_{\text{trivial}}$ the set of trivial times and \mathcal{T}_{iso} the set of isomonodromic times:

$$\mathcal{N}_{\text{trivial}} = \{T_{\infty, 1}, \dots, T_{\infty, 2r_\infty - 2}, T_1, T_2\}, \quad \mathcal{N}_{\text{iso}} = \{\tau_1, \dots, \tau_g\} \quad (7-11)$$

The previous set of trivial and non-trivial times is trivially in one-to-one correspondence with the irregular times $(t_{\infty, k})_{1 \leq k \leq 2r_\infty - 3}$. Moreover, the inverse change of coordinates is given by the following proposition.

Proposition 7.2. *One may recover the irregular times $(t_{\infty, k})_{1 \leq k \leq 2r_\infty - 3}$ from $\mathcal{T}_{\text{trivial}} \cup \mathcal{T}_{\text{iso}}$ with the following formulas:*

$$\begin{aligned} t_{\infty, 2r_\infty - 3} &= 2T_2^{\frac{2r_\infty - 3}{2}} \\ t_{\infty, 2r_\infty - 5} &= (2r_\infty - 5)T_1 T_2^{\frac{2r_\infty - 5}{2}} \\ t_{\infty, 2i} &= T_{\infty, i}, \quad \forall i \in \llbracket 1, r_\infty - 1 \rrbracket \end{aligned} \quad (7-12)$$

and for all $k \in \llbracket 1, r_\infty - 3 \rrbracket$:

$$t_{\infty, 2k-1} = 2T_2^{\frac{2k-1}{2}} \left(\sum_{p=1}^{r_\infty - k - 2} \frac{\prod_{m=p+1}^{r_\infty - k - 2} (2r_\infty - 2m - 5)}{2^{r_\infty - k - p - 2} (r_\infty - k - p - 2)!} T_1^{r_\infty - k - p - 2} \tau_p + T_1^{r_\infty - 1 - k} \frac{\prod_{m=0}^{r_\infty - k - 2} (2r_\infty - 2m - 5)}{2^{r_\infty - 1 - k} (r_\infty - 1 - k)!} \right) \quad (7-13)$$

Proof. The proof is computational and is proposed in Appendix L. \square

The last proposition allows to obtain immediately the expression of derivatives relatively to trivial and non-trivial times using the chain rule.

Proposition 7.3. *For all $k \in \llbracket 1, r_\infty - 3 \rrbracket$, we have:*

$$\partial_{\tau_k} = 2 \sum_{i=1}^{r_\infty - 2 - k} \frac{\prod_{m=k+1}^{r_\infty - i - 2} (2r_\infty - 2m - 5)}{2^{r_\infty - i - k - 2} (r_\infty - i - k - 2)!} T_1^{r_\infty - i - k - 2} T_2^{\frac{2i-1}{2}} \partial_{t_{\infty, 2i-1}} \quad (7-14)$$

and

$$\begin{aligned} \partial_{T_{\infty, i}} &= \partial_{t_{\infty, 2i}}, \quad \forall i \in \llbracket 1, r_\infty - 1 \rrbracket \\ \partial_{T_1} &= (2r_\infty - 5) T_2^{\frac{2r_\infty - 5}{2}} \partial_{t_{\infty, 2r_\infty - 5}} \\ &+ 2 \sum_{k=1}^{r_\infty - 3} T_2^{\frac{2k-1}{2}} \left(\sum_{p=1}^{r_\infty - k - 3} \frac{\prod_{m=p+1}^{r_\infty - k - 2} (2r_\infty - 2m - 5)}{2^{r_\infty - k - p - 2} (r_\infty - k - p - 3)!} T_1^{r_\infty - k - p - 3} \tau_p + T_1^{r_\infty - k - 2} \frac{\prod_{m=0}^{r_\infty - k - 1} (2r_\infty - 2m - 5)}{2^{r_\infty - 1 - k} (r_\infty - k - 2)!} \right) \partial_{t_{\infty, 2k-1}} \\ \partial_{T_2} &= (2r_\infty - 3) T_2^{\frac{2r_\infty - 5}{2}} \partial_{t_{\infty, 2r_\infty - 3}} + \frac{(2r_\infty - 5)^2}{2} T_1 T_2^{\frac{2r_\infty - 7}{2}} \partial_{t_{\infty, 2r_\infty - 5}} \\ &+ \sum_{k=1}^{r_\infty - 3} (2k - 1) T_2^{\frac{2k-3}{2}} \left(\sum_{p=1}^{r_\infty - k - 2} \frac{T_1^{r_\infty - k - p - 2} \tau_p \prod_{m=p+1}^{r_\infty - k - 2} (2r_\infty - 2m - 5)}{2^{r_\infty - k - p - 2} (r_\infty - k - p - 2)!} + \frac{T_1^{r_\infty - 1 - k} \prod_{m=0}^{r_\infty - k - 2} (2r_\infty - 2m - 5)}{2^{r_\infty - 1 - k} (r_\infty - 1 - k)!} \right) \partial_{t_{\infty, 2k-1}} \end{aligned} \quad (7-15)$$

For clarity, we shall denote α^{τ_k} the corresponding vector in the tangent space associated to ∂_{τ_k} for any $k \in \llbracket 1, r_\infty - 3 \rrbracket$. Its entries are given by

$$\begin{aligned} \alpha_{\infty, 2i}^{\tau_k} &= 0, \quad \forall i \in \llbracket 1, r_\infty - 1 \rrbracket \\ \alpha_{\infty, 2r_\infty - 3}^{\tau_k} &= 0 \\ \alpha_{\infty, 2r_\infty - 5}^{\tau_k} &= 0 \\ \alpha_{\infty, 2i-1}^{\tau_k} &= \delta_{1 \leq i \leq r_\infty - k - 2} \frac{2 \prod_{m=k+1}^{r_\infty - i - 2} (2r_\infty - 2m - 5)}{2^{r_\infty - i - k - 2} (r_\infty - i - k - 2)!} T_1^{r_\infty - i - k - 2} T_2^{\frac{2i-1}{2}}, \quad \forall i \in \llbracket 1, r_\infty - 3 \rrbracket \end{aligned} \quad (7-16)$$

Remark 7.1. Note that inserting (7-16) into (4-8) implies that

$$\forall k \in \llbracket 1, g \rrbracket, \quad \forall i \in \llbracket -1, k - 1 \rrbracket : \nu_{\infty, i}^{(\alpha^{\tau_k})} = 0 \quad (7-17)$$

In particular

$$\forall k \in \llbracket 1, g \rrbracket : \nu_{\infty, -1}^{(\alpha^{\tau_k})} = \nu_{\infty, 0}^{(\alpha^{\tau_k})} = 0 \quad (7-18)$$

7.3 Properties of trivial and isomonodromic times

Trivial and non-trivial times are chosen so that they satisfy the following properties.

Proposition 7.4. For all $k \in \llbracket 1, r_\infty - 1 \rrbracket$:

$$\begin{aligned} \mathcal{L}_{\mathbf{w}_k}[T_{\infty, j}] &= \hbar \delta_{j, k}, \quad \mathcal{L}_{\mathbf{u}_{-1}}[T_{\infty, j}] = \hbar j t_{\infty, 2j}, \quad \mathcal{L}_{\mathbf{u}_0}[T_{\infty, j}] = \hbar j t_{\infty, 2j+2}, \quad \forall j \in \llbracket 1, r_\infty - 1 \rrbracket \\ \mathcal{L}_{\mathbf{w}_k}[T_2] &= 0, \quad \mathcal{L}_{\mathbf{u}_{-1}}[T_2] = \hbar T_2, \quad \mathcal{L}_{\mathbf{u}_0}[T_2] = 0 \\ \mathcal{L}_{\mathbf{w}_k}[T_1] &= 0, \quad \mathcal{L}_{\mathbf{u}_{-1}}[T_1] = 0, \quad \mathcal{L}_{\mathbf{u}_0}[T_1] = \hbar T_2 \end{aligned} \quad (7-19)$$

Proof. Results on $(T_{\infty, j})_{1 \leq j \leq r_\infty - 1}$ follow by straightforward computations using the fact that

$$\begin{aligned} \mathcal{L}_{\mathbf{w}_k} &= \hbar \partial_{t_{\infty, 2k}}, \quad \forall k \in \llbracket 1, r_\infty - 1 \rrbracket \\ \mathcal{L}_{\mathbf{u}_{-1}} &= \frac{\hbar}{2} \sum_{r=1}^{2r_\infty - 2} r t_{\infty, r} \partial_{t_{\infty, r}} \\ \mathcal{L}_{\mathbf{u}_0} &= \frac{\hbar}{2} \sum_{r=1}^{2r_\infty - 4} r t_{\infty, r+2} \partial_{t_{\infty, r}} \end{aligned} \quad (7-20)$$

Results on T_2 are also straightforward using the fact that T_2 only depends on $t_{\infty, 2r_\infty - 3}$. Finally since T_1 depends only on $t_{\infty, 2r_\infty - 3}$ and $t_{\infty, 2r_\infty - 5}$, we get that

$$\begin{aligned} \mathcal{L}_{\mathbf{u}_{-1}}[T_1] &= \frac{\hbar}{2} (2r_\infty - 3) t_{\infty, 2r_\infty - 3} \partial_{t_{\infty, 2r_\infty - 3}} [T_1] + \frac{\hbar}{2} (2r_\infty - 5) t_{\infty, 2r_\infty - 5} \partial_{t_{\infty, 2r_\infty - 5}} [T_1] \\ &= \frac{\hbar}{2} (2r_\infty - 3) t_{\infty, 2r_\infty - 3} \left(-\frac{t_{\infty, 2r_\infty - 5} (2r_\infty - 5)}{2r_\infty - 5 (2r_\infty - 3)} \left(\frac{1}{2} \right)^{-\frac{2r_\infty - 5}{2r_\infty - 3}} (t_{\infty, 2r_\infty - 3})^{-\frac{2r_\infty - 5}{2r_\infty - 3} - 1} \right) \\ &\quad + \frac{\hbar}{2} (2r_\infty - 5) t_{\infty, 2r_\infty - 5} \frac{1}{2r_\infty - 5} \left(\frac{1}{2} t_{\infty, 2r_\infty - 3} \right)^{-\frac{2r_\infty - 5}{2r_\infty - 3}} \\ &= 0 \end{aligned} \quad (7-21)$$

and

$$\mathcal{L}_{\mathbf{u}_0}[T_1] = \frac{\hbar}{2} (2r_\infty - 5) t_{\infty, 2r_\infty - 3} \partial_{t_{\infty, 2r_\infty - 5}} [T_1]$$

$$\begin{aligned}
&= \frac{\hbar}{2}(2r_\infty - 5)t_{\infty, 2r_\infty - 3} \frac{1}{2r_\infty - 5} \left(\frac{1}{2}t_{\infty, 2r_\infty - 3} \right)^{-\frac{2r_\infty - 5}{2r_\infty - 3}} \\
&= \hbar \left(\frac{1}{2}t_{\infty, 2r_\infty - 3} \right)^{\frac{2}{2r_\infty - 3}} = \hbar T_2
\end{aligned} \tag{7-22}$$

□

7.4 Shifted Darboux coordinates

Theorem 7.1 indicates that deformations $\mathcal{L}_{\mathbf{u}_{-1}}$ and $\mathcal{L}_{\mathbf{u}_0}$ do not act trivially on the Darboux coordinates $(q_j, p_j)_{1 \leq j \leq g}$. However, since the action is very simple, we may easily perform a symplectic transformation on the Darboux coordinates to obtain “shifted Darboux coordinates” for which the action of $\mathcal{L}_{\mathbf{u}_{-1}}$ and $\mathcal{L}_{\mathbf{u}_0}$ becomes trivial.

Definition 7.3. The shifted Darboux coordinates $(\check{q}_j, \check{p}_j)_{1 \leq j \leq g}$ are defined by

$$\begin{aligned}
\check{q}_j &= T_2 q_j + T_1 \\
\check{p}_j &= T_2^{-1} \left(p_j - \frac{1}{2} \tilde{P}_1(q_j) \right), \quad \forall j \in \llbracket 1, g \rrbracket
\end{aligned} \tag{7-23}$$

Using Theorem 7.1 and Proposition 7.4, we get that the shifted Darboux coordinates satisfy the following proposition.

Proposition 7.5. For all $j \in \llbracket 1, g \rrbracket$:

$$\begin{aligned}
\mathcal{L}_{\mathbf{w}_k}[\check{q}_j] &= \mathcal{L}_{\mathbf{w}_k}[\check{p}_j] = 0, \quad \forall k \in \llbracket 1, r_\infty - 1 \rrbracket \\
\mathcal{L}_{\mathbf{u}_{-1}}[\check{q}_j] &= \mathcal{L}_{\mathbf{u}_{-1}}[\check{p}_j] = 0 \\
\mathcal{L}_{\mathbf{u}_0}[\check{q}_j] &= \mathcal{L}_{\mathbf{u}_0}[\check{p}_j] = 0
\end{aligned} \tag{7-24}$$

In other words, for any $\alpha_0 \in \mathcal{U}_{trivial}$: $\mathcal{L}_{\alpha_0}[\check{q}_j] = \mathcal{L}_{\alpha_0}[\check{p}_j] = 0$, hence the terminology “trivial deformations” and “trivial subspace”.

Proof. The proof directly follows from Theorem 7.1 and Proposition 7.4 but we detail it in Appendix M. □

Note also that the change of coordinates $(q_j, p_j)_{1 \leq j \leq g} \leftrightarrow (\check{q}_j, \check{p}_j)_{1 \leq j \leq g}$ is symplectic in the sense that $\sum_{j=1}^g dq_j \wedge dp_j = \sum_{j=1}^g d\check{q}_j \wedge d\check{p}_j$.

We finally get to our second main theorem.

Theorem 7.2. [Independence of the shifted Darboux coordinates relatively to trivial times] The shifted Darboux coordinates $(\check{q}_j, \check{p}_j)_{1 \leq j \leq g}$ are independent of the trivial times $(T_{\infty, 1}, \dots, T_{\infty, r_\infty - 1}, T_1, T_2)$. They are only functions of isomonodromic times $(\tau_k)_{1 \leq k \leq g}$. Moreover, any function $f(t_{\infty, 1}, t_{\infty, 2}, \dots, t_{\infty, 2r_\infty - 3})$ that is solution of

$$\forall k \in \llbracket 1, r_\infty - 1 \rrbracket : \mathcal{L}_{\mathbf{w}_k}[f] = 0 \text{ and } \mathcal{L}_{\mathbf{u}_{-1}}[f] = 0 \text{ and } \mathcal{L}_{\mathbf{u}_0}[f] = 0 \tag{7-25}$$

is an arbitrary function u of the isomonodromic times: $f(t_{\infty, 1}, t_{\infty, 2}, \dots, t_{\infty, 2r_\infty - 3}) = u(\tau_1, \dots, \tau_g)$.

Proof. The proof is presented in Appendix N. □

Finally let us mention the following observation.

Proposition 7.6. *For any isomonodromic deformations $(\tau_j)_{1 \leq j \leq g}$, associated to vectors α^{τ_j} , the trace of the corresponding matrices $\check{A}_{\alpha^{\tau_j}}$ and $\check{A}_{\alpha^{\tau_j}}$ are independent of λ because of the compatibility equations. Moreover, the matrices $(\check{A}_{\alpha^{\tau_j}})_{1 \leq j \leq g}$ (resp. $(\check{A}_{\alpha^{\tau_j}})_{1 \leq j \leq g}$) can be set traceless simultaneously by the additional gauge transformation $\check{\Psi}_n = \check{G}\check{\Psi}$ (resp. $\check{\Psi}_n = \check{G}\check{\Psi}$) with*

$$\begin{aligned}\check{G} &= \exp \left(-\frac{1}{2} \sum_{j=1}^g \int^{\tau_j} \text{Tr}(\check{A}_{\alpha^{\tau_j}})(s) ds \right) I_2 \\ \check{G} &= \exp \left(-\frac{1}{2} \sum_{j=1}^g \int^{\tau_j} \text{Tr}(\check{A}_{\alpha^{\tau_j}})(s) ds \right) I_2\end{aligned}\tag{7-26}$$

Note that these additional gauge transformations do not change neither \check{L} nor \check{L} .

Proof. For any isomonodromic deformation τ we have $\hbar \partial_\tau[\check{P}_1] = 0$ because the coefficients of \check{P}_1 are trivial times. From the expression of the Wronskians in Definition 2.3, we get that $\text{Tr}\check{L} = \text{Tr}\check{L} = \check{P}_1(\lambda)$. Thus, we get that $\partial_\tau[\text{Tr}\check{L}] = \partial_\tau[\text{Tr}\check{L}] = 0$. The compatibility equation (4-4) implies that $\partial_\lambda \text{Tr}\check{A}_{\alpha^\tau} = 0$. Moreover, for $g \geq 2$, if we denote $(\tau_i)_{1 \leq i \leq g}$ a set of isomonodromic times, the compatibility of the Lax system also gives

$$\partial_{\tau_j}[\check{A}_{\alpha^{\tau_i}}] = \partial_{\tau_i}[\check{A}_{\alpha^{\tau_j}}] + [\check{A}_{\alpha^{\tau_j}}, \check{A}_{\alpha^{\tau_i}}], \quad \forall i \neq j.\tag{7-27}$$

This leads to $\partial_{\tau_j}[\text{Tr}\check{A}_{\alpha^{\tau_i}}] = \partial_{\tau_i}[\text{Tr}\check{A}_{\alpha^{\tau_j}}]$. It is obvious that the additional gauge transformation $\check{\Psi}_{n,1} = \check{G}^{(1)}\check{\Psi}$ with $\check{G}^{(1)} = \exp \left(-\frac{1}{2} \int^{\tau_1} \text{Tr}(\check{A}_{\alpha^{\tau_1}})(s) ds \right) I_2$ defines a gauge in which the corresponding $\check{A}_{\alpha^{\tau_1}}^{(1)}$ is traceless. In this new gauge, (7-27) implies that $\partial_{\tau_1}[\text{Tr}\check{A}_{\alpha^{\tau_i}}^{(1)}] = 0$ for all $i \geq 2$. In particular, a new gauge transformation $\check{\Psi}_{n,2} = \check{G}^{(2)}\check{\Psi}_{n,1}$ with $\check{G}^{(2)} = \exp \left(-\frac{1}{2} \int^{\tau_2} \text{Tr}(\check{A}_{\alpha^{\tau_2}}^{(1)})(s) ds \right) I_2$ does not change the value of $\check{A}_{\alpha^{\tau_1}}^{(1)} = \check{A}_{\alpha^{\tau_1}}^{(2)}$ and the result follows by induction. Finally, it is obvious that a gauge transformation independent of λ and proportional to I_2 does not change neither \check{L} nor L . \square

The last proposition shall be useful when \check{L} and \check{L} are traceless. In this case, it is interesting to perform this additional gauge transformation in order to obtain a Lax pair that belongs to $\mathfrak{sl}_2(\mathbb{C})$ rather than $\mathfrak{gl}_2(\mathbb{C})$. In particular, this is always possible for the canonical choice of trivial times that shall be proposed in Section 8.

8 Canonical choice of trivial times and simplification of the Hamiltonian systems

The purpose of this section is to select some specific values of the trivial times in order to obtain simpler form of the Hamiltonian evolutions of Theorem 5.1. Indeed, the last section indicates (Theorem 7.2) that the shifted Darboux coordinates are independent of the values of the trivial times so that we may choose them without affecting the Hamiltonian evolutions. As it turns out, there exists a natural choice of the trivial times for which the Hamiltonian evolutions drastically simplify.

8.1 Canonical choice of the trivial times and main theorem

Definition 8.1 (Canonical choice of the trivial times). We define the “canonical choice of trivial times” by choosing

$$\begin{aligned} T_{\infty,k} &= 0, \quad \forall k \in \llbracket 0, r_{\infty} - 1 \rrbracket, \\ T_1 &= 0, \\ T_2 &= 1. \end{aligned} \tag{8-1}$$

In the rest of the paper, we shall always set the trivial times to their canonical values. The canonical choice of trivial times implies that

- All even irregular times are set to 0: for all $k \in \llbracket 1, r_{\infty} - 1 \rrbracket$: $t_{\infty,2k} = 0$.
- $t_{\infty,2r_{\infty}-3} = 2$ and $t_{\infty,2r_{\infty}-5} = 0$.
- \tilde{P}_1 is identically null. This is equivalent to say that \tilde{L} and $\tilde{\check{L}}$ are traceless. Hence, Proposition 7.6 implies that under a potential additional trivial gauge transformation, we may choose a gauge in which \tilde{L} , $\tilde{\check{L}}$, $\tilde{A}_{\alpha\tau}$ and $\tilde{\check{A}}_{\alpha\tau}$ are traceless for any isomonodromic time $\tau \in \mathcal{T}_{\text{iso}}$.
- The shifted Darboux coordinates are identical to the initial Darboux coordinates:

$$\forall j \in \llbracket 1, g \rrbracket : \check{q}_j = q_j \text{ and } \check{p}_j = p_j \tag{8-2}$$

- The isomonodromic times τ_k identify with an irregular time:

$$\forall k \in \llbracket 1, g \rrbracket : \tau_k = \frac{1}{2}t_{\infty,2r_{\infty}-2k-5} \Leftrightarrow \frac{1}{2}t_{\infty,2k-1} = \tau_{r_{\infty}-k-2} \tag{8-3}$$

- \tilde{P}_2 reduces to $\tilde{P}_2(\lambda) = -\lambda$ if $r_{\infty} = 3$ or for $r_{\infty} \geq 4$:

$$\begin{aligned} \tilde{P}_2(\lambda) &= -\lambda^{2r_{\infty}-5} - \sum_{k=r_{\infty}-2}^{2r_{\infty}-7} \left(2\tau_{2r_{\infty}-k-6} + \sum_{m=k-r_{\infty}+6}^{r_{\infty}-3} \tau_{r_{\infty}-m-2}\tau_{r_{\infty}-k+m-5} \right) \lambda^k \\ &\quad - \left(2\tau_{r_{\infty}-3} + \sum_{m=3}^{r_{\infty}-3} \tau_{r_{\infty}-m-2}\tau_{m-2} \right) \lambda^{r_{\infty}-3} \end{aligned} \tag{8-4}$$

In other words, for $r_{\infty} \geq 4$, we have

$$\begin{aligned} \tilde{P}_{\infty,2r_{\infty}-4}^{(2)} &= 0 \\ \tilde{P}_{\infty,2r_{\infty}-5}^{(2)} &= -1 \\ \tilde{P}_{\infty,2r_{\infty}-6}^{(2)} &= 0 \\ \tilde{P}_{\infty,k}^{(2)} &= - \sum_{k=r_{\infty}-2}^{2r_{\infty}-7} \left(2\tau_{2r_{\infty}-k-6} + \sum_{m=k-r_{\infty}+6}^{r_{\infty}-3} \tau_{r_{\infty}-m-2}\tau_{r_{\infty}-k+m-5} \right), \quad \forall k \in \llbracket r_{\infty}-2, 2r_{\infty}-7 \rrbracket \\ \tilde{P}_{\infty,r_{\infty}-3}^{(2)} &= - \left(2\tau_{r_{\infty}-3} + \sum_{m=3}^{r_{\infty}-3} \tau_{r_{\infty}-m-2}\tau_{m-2} \right) \end{aligned} \tag{8-5}$$

- Coefficients $\left(c_{\infty,k}^{(\alpha\tau)} \right)_{1 \leq k \leq r_{\infty}-1}$ are vanishing for any isomonodromic deformation $\tau \in \mathcal{T}_{\text{iso}}$.

- The gauge matrices $G_1(\lambda)$ and $J(\lambda)$ of Proposition 2.2 simplifies to

$$G_1(\lambda) = I_2, \quad J(\lambda) = \begin{pmatrix} 1 & 0 \\ -\sum_{i=1}^g \frac{\check{p}_i}{\lambda - \check{q}_i} \prod_{j \neq i} \frac{1}{\check{q}_i - \check{q}_j} & \frac{1}{\prod_{j=1}^g (\lambda - \check{q}_j)} \end{pmatrix} \quad (8-6)$$

In particular, $\check{L}(\lambda) = \tilde{L}(\lambda)$ and for all $\tau \in \mathcal{T}_{\text{iso}}$, $\check{A}(\alpha^\tau)(\lambda) = \tilde{A}(\alpha^\tau)(\lambda)$.

We also get the explicit expression

Proposition 8.1. *Under the canonical choice of trivial times given by Definition 8.1, the Lax matrices L is given by*

$$\begin{aligned} \tilde{L}_{1,1}(\lambda, \hbar) &= -Q(\lambda, \hbar), \\ \tilde{L}_{1,2}(\lambda, \hbar) &= \prod_{j=1}^g (\lambda - \check{q}_j), \\ \tilde{L}_{2,2}(\lambda, \hbar) &= Q(\lambda, \hbar), \\ \tilde{L}_{2,1}(\lambda, \hbar) &= \hbar \frac{\partial \left(\frac{Q(\lambda, \hbar)}{\prod_{j=1}^g (\lambda - \check{q}_j)} \right)}{\partial \lambda} + \frac{L_{2,1}(\lambda, \hbar)}{\prod_{j=1}^g (\lambda - \check{q}_j)} - \frac{Q(\lambda, \hbar)^2}{\prod_{j=1}^g (\lambda - \check{q}_j)} \end{aligned} \quad (8-7)$$

with $L_{2,1}(\lambda, \hbar) = -\tilde{P}_2(\lambda) + \sum_{k=0}^{r_\infty-4} H_{\infty,k} \lambda^k - \sum_{j=1}^g \frac{\hbar \check{p}_j}{\lambda - \check{q}_j}$ and $Q(\lambda, \hbar) = -\sum_{i=1}^g \check{p}_i \prod_{j \neq i} \frac{\lambda - \check{q}_j}{\check{q}_i - \check{q}_j}$ and $L_{2,2}(\lambda, \hbar) = \sum_{j=1}^g \frac{\hbar}{\lambda - \check{q}_j}$. Similarly, the matrix $A_{\alpha^\tau}(\lambda, \hbar)$ is given by

$$\begin{aligned} [A_{\alpha^\tau}(\lambda, \hbar)]_{1,1} &= -\sum_{j=1}^g \frac{\mu_j^{(\alpha^\tau)} \check{p}_j}{\lambda - \check{q}_j} \\ [A_{\alpha^\tau}(\lambda, \hbar)]_{1,2} &= \sum_{j=1}^g \frac{\mu_j^{(\alpha^\tau)}}{\lambda - \check{q}_j} \\ [A_{\alpha^\tau}(\lambda, \hbar)]_{2,1} &= -\hbar \sum_{j=1}^g \sum_{i \neq j} \frac{\mu_j^{(\alpha^\tau)}}{(\lambda - \check{q}_j)(\lambda - \check{q}_i)} + \left(\sum_{j=1}^g \frac{\mu_j^{(\alpha^\tau)}}{\lambda - \check{q}_j} \right) \left(-\tilde{P}_2(\lambda) + \sum_{k=0}^{r_\infty-4} H_{\infty,k} \lambda^k \right) \\ [A_{\alpha^\tau}(\lambda, \hbar)]_{2,2} &= -\sum_{j=1}^g \frac{\mu_j^{(\alpha^\tau)} \check{p}_j}{\lambda - \check{q}_j} + \hbar \sum_{j=1}^g \sum_{i \neq j} \frac{\mu_j^{(\alpha^\tau)}}{(\lambda - \check{q}_j)(\lambda - \check{q}_i)} \end{aligned} \quad (8-8)$$

We may also simplify Propositions 6.3 and 6.4.

Proposition 8.2. *Under the canonical choice of trivial times given by Definition 8.1, the Lax matrices \tilde{L} and \tilde{A}_α may be expressed in terms of symmetric Darboux coordinates as follow. For any $\tau \in \mathcal{T}_{\text{iso}}$:*

$$\begin{aligned} \tilde{L}_{1,1}(\lambda) &= -\sum_{j=0}^{g-1} (-1)^{j-1} \left(\sum_{i=j+1}^g P_i Q_{i-j-1} \right) \lambda^j \\ \tilde{L}_{1,2}(\lambda) &= \sum_{m=0}^g (-1)^{g-m} Q_{g-m} \lambda^m \end{aligned}$$

$$\begin{aligned}
\tilde{L}_{2,2}(\lambda) &= \sum_{j=0}^{g-1} (-1)^{j-1} \left(\sum_{i=j+1}^g P_i Q_{i-j-1} \right) \lambda^j \\
\tilde{L}_{2,1}(\lambda) &= - \sum_{i=0}^{r_\infty-2} \sum_{j=g+i}^{2r_\infty-5} \left(\tilde{P}_{\infty,j}^{(2)} h_{j-g-i}(\{\check{\mathbf{q}}\}) \right) \lambda^i \\
&\quad - \sum_{i=0}^{g-2} \left(\sum_{j_1=i+1}^{g-1} \sum_{j_2=g+i-j_1}^{g-1} (-1)^{j_1+j_2} \left(\sum_{i_1=j_1+1}^g P_{i_1} Q_{i_1-j_1-1} \right) \left(\sum_{i_2=j_2+1}^g P_{i_2} Q_{i_2-j_2-1} \right) h_{j_1+j_2-g-i}(\{\check{\mathbf{q}}\}) \right) \lambda^i \\
[\tilde{A}_{\alpha^\tau}(\lambda)]_{1,1} &= - \sum_{i=0}^{g-2} \sum_{m=1}^{g-1-i} (-1)^{i+m-1} \nu_{\infty,m}^{(\alpha)} \left(\sum_{r=i+m+1}^g P_r Q_{r-i-m-1} \right) \lambda^i \\
[\tilde{A}_{\alpha^\tau}(\lambda)]_{1,2} &= \sum_{j=0}^{g-1} \left(\sum_{m=1}^{g-j} (-1)^{g-j-m} \nu_{\infty,m}^{(\alpha)} Q_{g-j-m} \right) \lambda^j \\
[\tilde{A}_{\alpha^\tau}(\lambda)]_{2,2} &= - [\tilde{A}^{(\alpha)}(\lambda)]_{1,1} \\
[\tilde{A}_{\alpha^\tau}(\lambda)]_{2,1} &= - \sum_{i=0}^g \sum_{j=Max(g,g+i-1)}^{2r_\infty-5} \sum_{m=1}^{j-g-i} \nu_{\infty,m}^{(\alpha)} h_{j-g-m-i}(\{\check{\mathbf{q}}\}) \tilde{P}_{\infty,j}^{(2)} \lambda^i \\
&\quad - \sum_{i=0}^g \sum_{j_1=0}^{g-1} \sum_{j_2=0}^{g-1} \sum_{m=1}^{j_1+j_2-g-i} (-1)^{j_1+j_2} \nu_{\infty,m}^{(\alpha)} h_{j_1+j_2-g-m-i}(\{\check{\mathbf{q}}\}) \sum_{r_1=j_1+1}^g \sum_{r_2=j_2+1}^g P_{r_1} P_{r_2} Q_{r_1-j_1-1} Q_{r_2-j_2-1} \lambda^i
\end{aligned} \tag{8-9}$$

where $\left(\tilde{P}_{\infty,k}^{(2)} \right)_{r_\infty-3 \leq k \leq 2r_\infty-4}$ are determined by (8-5) and $(h_k(\{\check{\mathbf{q}}\}))_{k \geq 0}$ are expressed in terms of symmetric Darboux coordinates by $h_0(\{\check{\mathbf{q}}\}) = 1$ and

$$h_k(\{\check{\mathbf{q}}\}) = \sum_{j=1}^k (-1)^j \sum_{\substack{b_1, \dots, b_j \in [1, k] \\ b_1 + \dots + b_j = k}} \prod_{m=1}^j (-1)^{b_m} Q_{b_m}, \quad \forall k \in [1, g] \tag{8-10}$$

Coefficients $\left(\nu_{\infty,k}^{(\alpha^\tau)} \right)_{1 \leq k \leq r_\infty-3}$ shall be given by Proposition 8.3 depending on the isomonodromic deformation $\tau \in \mathcal{T}_{iso}$ and $\nu_{\infty, r_\infty-2}^{(\alpha)} = \sum_{k=1}^g (-1)^{g-k} \nu_{\infty,k}^{(\alpha)} Q_{g+1-k}$.

We shall now apply Theorem 5.1 for the canonical values of the trivial times and obtain our third main theorem.

Theorem 8.1 (Hamiltonian representation for the canonical choice of trivial times). *The canonical choice of the trivial times given by Definition 8.1 and the definition of trivial times (Definition 7.2) imply that for any isomonodromic time $\tau \in \mathcal{T}_{iso}$:*

$$Ham^{(\alpha^\tau)}(\check{\mathbf{q}}, \check{\mathbf{p}}) = \sum_{k=0}^{r_\infty-4} \nu_{\infty, k+1}^{(\alpha^\tau)} H_{\infty, k} \tag{8-11}$$

In other words, the Hamiltonian is a (time-dependent) linear combination of the isospectral Hamiltonians $(H_{\infty, k})_{0 \leq k \leq r_\infty-4}$ that are determined by

$$\begin{pmatrix} 1 & \check{q}_1 & \dots & \dots & \check{q}_1^{g-1} \\ 1 & \check{q}_2 & \dots & \dots & \check{q}_2^{g-1} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ 1 & \check{q}_g & \dots & \dots & \check{q}_g^{g-1} \end{pmatrix} \begin{pmatrix} H_{\infty, 0} \\ \vdots \\ \vdots \\ H_{\infty, r_\infty-4} \end{pmatrix} = \begin{pmatrix} \check{p}_1^2 + \tilde{P}_2(\check{q}_1) + \hbar \sum_{i \neq 1} \frac{\check{p}_i - \check{p}_1}{\check{q}_1 - \check{q}_i} \\ \vdots \\ \vdots \\ \check{p}_g^2 + \tilde{P}_2(\check{q}_g) + \hbar \sum_{i \neq g} \frac{\check{p}_i - \check{p}_g}{\check{q}_g - \check{q}_i} \end{pmatrix} \tag{8-12}$$

where \tilde{P}_2 is given by (8-4). Coefficients $\left(\nu_{\infty,k}^{(\alpha^\tau)}\right)_{1 \leq k \leq r_{\infty-3}}$ shall be given by Proposition 8.3 depending on the isomonodromic deformation $\tau \in \mathcal{T}_{iso}$. In terms of symmetric Darboux coordinates, the Hamiltonian is given by:

$$\begin{aligned}
Ham^{(\alpha^\tau)}(\mathbf{Q}, \mathbf{P}) = & -\hbar \sum_{i=1}^g \nu_{\infty,i}^{(\alpha^\tau)} \sum_{k=i+1}^g \left((-1)^i (g-i) P_k Q_{k-1-i} + \sum_{m=i+1}^{k-1} (-1)^m P_k Q_{k-1-m} S_{m-i}(\{\check{\mathbf{q}}\}) \right) \\
& + \sum_{i=1}^g \nu_{\infty,i}^{(\alpha^\tau)} \sum_{k_1=1}^g \sum_{k_2=1}^g P_{k_1} P_{k_2} \left[(-1)^{i-1} \sum_{r_1=Max(0,i-k_2)}^{Min(k_1-1,i-1)} Q_{k_1-1-r_1} Q_{k_2-i+r_1} \right. \\
& + \left. \sum_{\substack{0 \leq r_1 \leq k_1-1 \\ 0 \leq r_2 \leq k_2-1 \\ r_1+r_2 \geq g}} (-1)^{r_1+r_2} Q_{k_1-1-r_1} Q_{k_2-1-r_2} \sum_{m=i}^g (-1)^{g-m} Q_{g-m} h_{r_1+r_2+m-i-g+1}(\{\check{\mathbf{q}}\}) \right] \\
& + \sum_{i=1}^g \nu_{\infty,i}^{(\alpha^\tau)} \sum_{r=g}^{2r_{\infty}-5} \sum_{m=i}^g (-1)^{g-m} \tilde{P}_{\infty,r}^{(2)} Q_{g-m} h_{r+m-i-g+1}(\{\check{\mathbf{q}}\}) \quad (8-13)
\end{aligned}$$

where $(S_k(\{\check{\mathbf{q}}\}))_{0 \leq k \leq g}$ and $(h_k(\{\check{\mathbf{q}}\}))_{0 \leq k \leq g}$ are determined by $h_0(\{\check{\mathbf{q}}\}) = 1$, $S_0(\{\check{\mathbf{q}}\}) = g$ and for all $k \in \llbracket 1, g \rrbracket$:

$$\begin{aligned}
h_k(\{\check{\mathbf{q}}\}) &= \sum_{j=1}^k (-1)^j \sum_{\substack{b_1, \dots, b_j \in \llbracket 1, k \rrbracket^j \\ b_1 + \dots + b_j = k}} \prod_{m=1}^j (-1)^{b_m} Q_{b_m}, \quad \forall k \in \llbracket 1, g \rrbracket \\
S_k(\{\check{\mathbf{q}}\}) &= (-1)^k k \sum_{\substack{b_1+2b_2+\dots+kb_k=k \\ b_1 \geq 0, \dots, b_k \geq 0}} \frac{(-1)^{b_1+\dots+b_k}}{(b_1+\dots+b_k)} \binom{b_1+\dots+b_k}{b_1, \dots, b_k} \prod_{i=1}^k Q_i^{b_i} \quad (8-14)
\end{aligned}$$

Proof. The proof is obvious since the canonical choice of trivial times implies that the coefficients $\left(c_{\infty,k}^{(\alpha^\tau)}\right)_{1 \leq k \leq r_{\infty-1}}$ are vanishing for any isomonodromic deformation. Moreover, Remark 7.1 implies that $\nu_{\infty,-1}^{(\alpha^\tau)} = \nu_{\infty,0}^{(\alpha^\tau)} = 0$. \square

Note that only the coefficients $\left(\nu_{\infty,k}^{(\alpha^\tau)}\right)_{1 \leq k \leq r_{\infty-3}}$ of the linear combination depend on the deformation, since the isospectral Hamiltonians are independent of it. We shall now obtain their explicit values from the simplification of (4-8) depending on the choice of isomonodromic time τ_j with $j \in \llbracket 1, g \rrbracket$.

Proposition 8.3 (Expression of $\nu_{\infty,k}^{(\alpha^{\tau_j})}$). *For any $j \in \llbracket 1, g \rrbracket$, the coefficients $\left(\nu_{\infty,k}^{(\alpha^{\tau_j})}\right)_{1 \leq k \leq r_{\infty-3}}$ are determined under the canonical choice of trivial times of Definition 8.1 by*

$$\begin{pmatrix} 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & 1 & 0 & \ddots & & 0 \\ \tau_1 & 0 & 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \tau_{g-2} & \tau_{g-3} & \dots & \tau_1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \nu_{\infty,1}^{(\alpha^{\tau_j})} \\ \vdots \\ \nu_{\infty,r_{\infty-3}}^{(\alpha^{\tau_j})} \end{pmatrix} = \frac{2}{2r_{\infty} - 2j - 5} \mathbf{e}_j \quad (8-15)$$

One may also obtain a simplified expression for $\left(\mu_i^{(\alpha^{\tau_j})}\right)_{1 \leq i \leq g}$ from (4-11). We find for the canonical choice of trivial times:

$$\begin{pmatrix} 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & 1 & 0 & \ddots & & 0 \\ \tau_1 & 0 & 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \tau_{g-2} & \tau_{g-3} & \dots & \tau_1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \dots & \dots & 1 \\ \check{q}_1 & \check{q}_2 & \dots & \dots & \check{q}_g \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \check{q}_1^{r_\infty-4} & \check{q}_2^{r_\infty-4} & \dots & \dots & \check{q}_g^{r_\infty-4} \end{pmatrix} \begin{pmatrix} \mu_1^{(\alpha^{\tau_j})} \\ \vdots \\ \mu_g^{(\alpha^{\tau_j})} \end{pmatrix} = \frac{2}{2r_\infty - 2j - 5} \mathbf{e}_j \quad (8-16)$$

for all $j \in \llbracket 1, g \rrbracket$.

Finally, the evolution equations under this canonical choice reduce to

$$\begin{aligned} \hbar \partial_{\tau_j} \check{q}_m &= 2\mu_m^{(\alpha^{\tau_j})} \check{p}_m - \hbar \sum_{i \neq m} \frac{\mu_m^{(\alpha^{\tau_j})} + \mu_i^{(\alpha^{\tau_j})}}{\check{q}_m - \check{q}_i}, \\ \hbar \partial_{\tau_j} \check{p}_m &= \hbar \sum_{i \neq m} \frac{(\mu_i^{(\alpha^{\tau_j})} + \mu_m^{(\alpha^{\tau_j})})(\check{p}_i - \check{p}_m)}{(\check{q}_m - \check{q}_i)^2} + \mu_m^{(\alpha^{\tau_j})} \left(-\tilde{P}'_2(\check{q}_m) + \sum_{k=1}^{r_\infty-4} k H_{\infty, k} \check{q}_m^{k-1} \right) \end{aligned} \quad (8-17)$$

for all $(j, m) \in \llbracket 1, g \rrbracket^2$.

Let us now underline a few aspects of Theorem 8.1:

- Note that Theorem 8.1 is valid for any choice of the trivial times and not only the canonical choice of Definition 8.1. Indeed $(\check{q}_j, \check{p}_j)_{1 \leq j \leq g}$ are independent of the choice of trivial times so that we may choose any value to compute the Hamiltonian system. However, for other choices of trivial times, the connection between Darboux coordinates and shifted Darboux coordinates and the relation between irregular times and isomonodromic times may be more complex and is given by Definitions 7.3 and 7.2. This observation was already made in the non-twisted case in [27].
- Note that the Hamiltonian system for $(\check{q}_j, \check{p}_j)_{1 \leq j \leq g}$ does not depend on \tilde{P}_1 . Since $\tilde{P}_1 = \text{Tr} \tilde{L}$, this means that the non-trivial isomonodromic evolutions are the same in the study of isomonodromic deformations of twisted connections on $\mathfrak{gl}_2(\mathbb{C})$ or $\mathfrak{sl}_2(\mathbb{C})$. However, in the case of $\mathfrak{gl}_2(\mathbb{C})$, apparent singularities are no longer the right Darboux coordinates and a shift by $-\frac{1}{2}\tilde{P}_1(q_j)$ becomes necessary (Cf. Definition 7.3). This observation was already made in the non-twisted case in [27].
- Hamiltonian evolutions only depend on τ_g and τ_{g-1} through \tilde{P}_2 (because the matrix in Proposition 8.3 does not depend neither on τ_{g-1} nor τ_g) so that the explicit dependence of the Hamiltonians in τ_g and τ_{g-1} is linear. The explicit dependence of the Hamiltonians in $(\tau_i)_{1 \leq i \leq g-2}$ is polynomial and the corresponding degrees are given by (8-22).

8.2 Explicit expressions for the inverse of the matrices

One may invert the Vandermonde matrix in Theorem 8.1 in order to have some explicit expressions for $(H_{\infty,k})_{0 \leq k \leq r_{\infty}-4}$. For all $i \in \llbracket 1, r_{\infty} - 3 \rrbracket$ we find

$$H_{\infty,i-1} = \sum_{m=1}^{r_{\infty}-3} \frac{(-1)^{g-i} e_{g-i}(\{\check{q}_1, \dots, \check{q}_g\} \setminus \{\check{q}_m\})}{\prod_{r \neq m} (\check{q}_m - \check{q}_r)} \left(\check{p}_m^2 + \check{P}_2(\check{q}_m) + \hbar \sum_{s \neq m} \frac{\check{p}_s - \check{p}_m}{\check{q}_m - \check{q}_s} \right) \quad (8-18)$$

where we have defined the elementary symmetric functions by

$$e_m(\{y_1, \dots, y_k\}) = \sum_{1 \leq j_1 < \dots < j_m \leq k} y_{j_1} \dots y_{j_m}, \quad \forall m \geq 0, k \geq 1 \quad (8-19)$$

Similarly, one may invert the lower triangular Toeplitz matrix of Proposition 8.3 in order to have an explicit expression for $\nu_{\infty,k}^{(\alpha^{\tau_j})}$. We find

$$\nu_k^{(\alpha^{\tau_j})} = \frac{2}{2r_{\infty} - 2j - 5} (\delta_{j,k} + F_{j-k-1}(\tau_1, \dots, \tau_{k-j-1}) \delta_{k \leq j-2}), \quad \forall (j, k) \in \llbracket 1, g \rrbracket^2 \quad (8-20)$$

where we have defined:

$$\begin{pmatrix} 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & 1 & 0 & \ddots & & 0 \\ \tau_1 & 0 & 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \tau_{g-2} & \tau_{g-3} & \dots & \tau_1 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & & & 0 & & \dots & \dots & \dots & 0 \\ 0 & 1 & & 1 & & 0 & \ddots & & 0 \\ F_1(\tau_1) & & 0 & 1 & 0 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ F_{g-2}(\tau_1, \dots, \tau_{g-2}) & F_{g-3}(\tau_1, \dots, \tau_{g-3}) & \dots & F_1(\tau_1) & 0 & 1 \end{pmatrix} \quad (8-21)$$

with

$$F_i(\tau_1, \dots, \tau_i) = \sum_{\substack{(b_1, \dots, b_i) \in \mathbb{N}^i \\ \sum_{j=1}^i (j+1)b_j = i+1}} \binom{\sum_{j=1}^i b_j}{b_1, \dots, b_i} (-1)^{\sum_{j=1}^i b_j} \tau_1^{b_1} \dots \tau_i^{b_i}, \quad \forall i \geq 1 \quad (8-22)$$

For example, the first values of $(F_i(\tau_1, \dots, \tau_i))_{1 \leq i \leq 5}$ are

$$\begin{aligned} F_1(\tau_1) &= -\tau_1 \\ F_2(\tau_1, \tau_2) &= -\tau_2 \\ F_3(\tau_1, \tau_2, \tau_3) &= \tau_1^2 - \tau_3 \\ F_4(\tau_1, \tau_2, \tau_3, \tau_4) &= 2\tau_1\tau_2 - \tau_4 \\ F_5(\tau_1, \dots, \tau_5) &= -\tau_1^3 + 2\tau_1\tau_3 + \tau_2^2 - \tau_5 \end{aligned} \quad (8-23)$$

Finally, we may obtain an explicit expression for

$$\mu_i^{(\alpha^{\tau_k})} = \frac{2}{(2r_{\infty} - 2i - 5) \prod_{m \neq i} (\check{q}_i - \check{q}_m)} \left[(-1)^{g-i} e_{g-i}(\{\check{q}_1, \dots, \check{q}_g\} \setminus \{\check{q}_i\}) \right]$$

$$+ \sum_{r=i+2}^{r_\infty-3} (-1)^{g-r} e_{g-r}(\{\tilde{q}_1, \dots, \tilde{q}_g\} \setminus \{\tilde{q}_i\}) \sum_{\substack{(b_1, \dots, b_{r-i-1}) \in \mathbb{N}^{r-i-1} \\ \sum_{s=1}^i (s+1)b_s = r-i}} \binom{\sum_{s=1}^{r-i-1} b_s}{b_1, \dots, b_{r-i-1}} (-1)^{\sum_{s=1}^{r-i-1} b_s} \tau_1^{b_1} \dots \tau_{r-i-1}^{b_{r-i-1}} \quad (8-24)$$

for all $(k, i) \in \llbracket 1, r_\infty - 3 \rrbracket^2$.

9 Topological type property and formal WKB solutions

Starting from twisted meromorphic connections on $\mathfrak{gl}_2(\mathbb{C})$ with a pole at infinity, we have obtained some isomonodromic times $(\tau_j)_{1 \leq j \leq g}$ and some Lax pairs $(\tilde{L}(\lambda, \tau, \hbar), \tilde{A}_{\alpha\tau_1}(\lambda, \tau, \hbar), \dots, \tilde{A}_{\alpha\tau_g}(\lambda, \tau, \hbar))$ corresponding to the compatible differential systems

$$\hbar \partial_\lambda \tilde{\Psi}(\lambda, \tau, \hbar) = \tilde{L}(\lambda, \tau, \hbar), \quad \hbar \partial_{\tau_j} \tilde{\Psi}(\lambda, \tau, \hbar) = \tilde{A}_{\alpha\tau_j}(\lambda, \tau, \hbar) \tilde{\Psi}(\lambda, \tau, \hbar), \quad \forall j \in \llbracket 1, g \rrbracket \quad (9-1)$$

These matrices are expressed in terms of the isomonodromic times and the Darboux coordinates $(\tilde{q}_i, \tilde{p}_i)_{1 \leq i \leq g}$ satisfying some Hamiltonian systems. This construction is independent of the type of solutions, in particular in [26], it is argued that the most general formal solutions are expected to be formal \hbar -transseries. However, one may look for a simpler form of solutions. Of particular interests are formal power series solutions of the Hamiltonian systems:

$$\hat{q}_i(\tau, \hbar) = \sum_{k=0}^{\infty} q_i^{(k)}(\tau) \hbar^k, \quad \hat{p}_i(\tau, \hbar) = \sum_{k=0}^{\infty} p_i^{(k)}(\tau) \hbar^k, \quad \forall i \in \llbracket 1, g \rrbracket \quad (9-2)$$

that equivalently correspond to formal WKB solutions

$$\tilde{\Psi}(\lambda, \tau, \hbar) = \exp \left(\sum_{k=-1}^{\infty} \Psi_k(\lambda, \tau) \hbar^k \right) \quad (9-3)$$

of the Lax system. In [26], the authors proved that, in this formal WKB setup, the Lax systems arising from general isomonodromic deformations (twisted or not) always satisfy the so-called ‘‘Topological Type property’’ of [3]. In particular, the central argument (section 4.2 of [26]) to prove the topological type property is the existence of an isomonodromic time τ (built from isospectral deformations in [26]) for which the auxiliary matrix $\tilde{A}_{\alpha\tau}(\lambda, \hbar)$ is of the form $\tilde{A}_{\alpha\tau}(\lambda, \hbar) = \frac{B_1 \lambda + B_0}{p(\lambda)}$ where B_0 and B_1 are independent of λ and p is a polynomial. Our formalism generates a similar result without using isospectral deformations. Indeed, it is obvious that the isomonodromic time $\tau_{\infty, r_\infty-3}$ provides a matrix $\tilde{A}_{\alpha\tau_{r_\infty-3}}$ that satisfies the condition presented above.

Thus, in the context of formal WKB solutions (or equivalently of formal power series solutions of the Hamiltonian systems), the Lax pair satisfies the topological type property and one may reconstruct the formal correlation functions $(W_n(\lambda_1, \dots, \lambda_n))_{n \geq 1}$ built from ‘‘determinantal formulas’’ (see [4] for definitions) of the differential system $\hbar \partial_\lambda \tilde{\Psi}(\lambda, \hbar) = \tilde{L}(\lambda, \hbar) \tilde{\Psi}(\lambda, \hbar)$ by the formal \hbar -power series of the Eynard-Orantin differentials $(\omega_{k,n})_{k \geq 0, n \geq 0}$ produced by the topological recursion on the classical spectral curve (that always reduces in this formal WKB setup to a genus 0 curve):

$$W_n(\lambda_1, \dots, \lambda_n; \tau, \hbar) = \sum_{k=0}^{\infty} \omega_{k,n}(\lambda_1, \dots, \lambda_n; \tau) \hbar^{n-2+2k}, \quad \forall n \geq 1 \quad (9-4)$$

Moreover, the formal Jimbo-Miwa-Ueno τ -function τ_{JMU} [24, 6] is reconstructed by the free energies $(\omega_{k,0})_{k \geq 0}$ produced by the topological recursion

$$\ln \tau_{\text{JMU}}(\boldsymbol{\tau}, \hbar) = \sum_{k=0}^{\infty} \omega_{k,0}(\boldsymbol{\tau}) \hbar^{2k-2} \quad (9-5)$$

We stress again that the Hamiltonian systems and Lax pairs obtained in this article do not depend on the type of the solutions considered. As explained above, when considering solutions expressed as formal power series or formal WKB series in \hbar , the picture simplifies since the genus of the classical spectral curve drops to 0, the topological type property is verified and one may reconstruct the formal correlation functions or the formal tau-function of the Lax system directly from the topological recursion. Nevertheless, it is presently an open question to prove that the same picture remains valid when considering more general solutions of the Lax system like \hbar -transseries solutions. Even in the formal WKB setup, the issue of giving some analytic meaning to the formal solutions is currently a widely open question.

10 Examples

Let us now apply the general theory to the first cases of the Painlevé 1 hierarchy.

10.1 The Airy case: $r_{\infty} = 3$

The Airy case corresponds to $r_{\infty} = 3$ so that $g = 0$. The canonical choice of trivial times corresponds to $t_{\infty,4} = t_{\infty,2} = 0$, $t_{\infty,3} = 2$ and $t_{\infty,1} = 0$ so that $\tilde{P}_2(\lambda) = -\lambda$. There is no Darboux coordinates and any Hamiltonian evolutions. However, one may still write the Lax matrices L and \tilde{L} . They are given by

$$L(\lambda) = \begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix} = \tilde{L}(\lambda) \quad (10-1)$$

giving the Airy spectral curve: $y^2 = \lambda$. Since $g = 0$, the only interesting result provided by the paper is that the wave function Ψ may be reconstructed by topological recursion after the introduction of the formal parameter \hbar . This is of course in agreement with known results about the Airy spectral curve [25, 17, 14].

10.2 Painlevé 1 case: $r_{\infty} = 4$

Let us consider $r_{\infty} = 4$, i.e. $g = 1$ corresponding to the Painlevé 1 case. In this setup, the canonical choice of trivial times corresponds to $t_{\infty,6} = t_{\infty,4} = t_{\infty,2} = 0$, $t_{\infty,5} = 2$ and $t_{\infty,3} = 0$. The only non-trivial isomonodromic time is $\tau := \tau_1 = \frac{1}{2}t_{\infty,1}$. Since $g = 1$, we shall drop the useless index in this case (i.e. $q := q_1$, $p := p_1$, etc.). Application of the general results to this case is straightforward and give under the choice of trivial times made in Definition 8.1:

$$\begin{aligned} \tilde{P}_2(\lambda) &= -\lambda^3 - 2\tau\lambda \\ H_{\infty,0} &= \check{p}^2 + \tilde{P}_2(\check{q}) = \check{p}^2 - \check{q}^3 - 2\tau\check{q} \\ Q(\lambda, \hbar) &= -\check{p} \\ \nu_{\infty,1}^{(\boldsymbol{\alpha}^{\tau})} &= 2, \mu^{(\boldsymbol{\alpha}^{\tau})} = 2 \end{aligned} \quad (10-2)$$

Thus, we get that the Hamiltonian is

$$\text{Ham}^{(\alpha^\tau)}(\check{q}, \check{p}) = 2H_{\infty,0} = 2\check{p}^2 - 2\check{q}^3 - 4\tau\check{q} \quad (10-3)$$

It corresponds to the ordinary differential equations

$$\hbar\partial_\tau\check{q} = 4\check{p} \quad , \quad \hbar\partial_\tau\check{p} = 6\check{q}^2 + 4\tau \quad (10-4)$$

so that $\check{q}(\tau)$ satisfies a Painlevé 1 like equation:

$$\hbar^2 \frac{d^2\check{q}}{d\tau^2} = 24\check{q}^2 + 16\tau \quad (10-5)$$

The associated Lax pairs are given by

$$\begin{aligned} L(\lambda, \hbar) &= \begin{pmatrix} 0 & 1 \\ \lambda^3 + 2\tau\lambda + \check{p}^2 - \check{q}^3 - 2\tau\check{q} - \frac{\hbar\check{p}}{\lambda-\check{q}} & \frac{\hbar}{\lambda-\check{q}} \end{pmatrix}, \\ A_{\alpha^\tau}(\lambda, \hbar) &= \begin{pmatrix} -\frac{2\check{p}}{\lambda-\check{q}} & \frac{2}{\lambda-\check{q}} \\ \frac{2}{\lambda-\check{q}}(\lambda^3 + 2\tau\lambda + \check{p}^2 - \check{q}^3 - 2\tau\check{q}) & -\frac{2\check{p}}{\lambda-\check{q}} \end{pmatrix} \end{aligned} \quad (10-6)$$

or equivalently

$$\tilde{L}(\lambda, \hbar) = \begin{pmatrix} \check{p} & \lambda - \check{q} \\ \lambda^2 + \check{q}\lambda + \check{q}^2 + 2\tau & -\check{p} \end{pmatrix}, \quad \tilde{A}_{\alpha^\tau}(\lambda, \hbar) = \begin{pmatrix} 0 & 2 \\ 2(\lambda + 2\check{q}) & 0 \end{pmatrix} \quad (10-7)$$

Remark 10.1. If we perform $t = 2^{\frac{6}{5}}\tau$, $\tilde{q} = 2^{-\frac{2}{5}}\check{q}$, $\tilde{p} = 2^{\frac{2}{5}}\check{p}$ we find that $\tilde{q}(t)$ satisfies the normalized Painlevé 1 equation:

$$\hbar^2 \frac{d^2\tilde{q}}{dt^2} = 6\tilde{q}^2 + t \quad (10-8)$$

Moreover, one may recover the Jimbo-Miwa Lax pair (eq. C.2 of [23]):

$$L_{\text{JMU}}(x) = \begin{pmatrix} -z(t) & x^2 + y(t)x + y(t)^2 + \frac{t}{2} \\ 4(x - y(t)) & z(t) \end{pmatrix}, \quad A_{\text{JMU}}(x) = \begin{pmatrix} 0 & \frac{x}{2} + y(t) \\ 2 & 0 \end{pmatrix} \quad (10-9)$$

$$\Psi_{\text{JMU}}(x) = \begin{pmatrix} 0 & 1 \\ 2^{\frac{6}{5}} & 0 \end{pmatrix} \tilde{\Psi}(\lambda), \quad x = 2^{-\frac{2}{5}}\lambda, \quad t = 2^{\frac{6}{5}}\tau, \quad y(t) = \tilde{q}(t), \quad z(t) = \tilde{p}(t) \quad (10-10)$$

10.3 Second element of the Painlevé 1 hierarchy: $r_\infty = 5$

Let us consider $r_\infty = 5$, i.e. $g = 2$ corresponding to the second element of the Painlevé 1 hierarchy. In this setup, the canonical choice of trivial times corresponds to $t_{\infty,8} = t_{\infty,6} = t_{\infty,4} = t_{\infty,2} = 0$, $t_{\infty,7} = 2$ and $t_{\infty,5} = 0$. The only non-trivial isomonodromic times are $\tau_1 = \frac{1}{2}t_{\infty,3}$ and $\tau_2 = \frac{1}{2}t_{\infty,1}$. We have also

$$\begin{aligned} \tilde{P}_2(\lambda) &= -\lambda^5 - 2\tau_1\lambda^3 - 2\tau_2\lambda^2 \\ Q(\lambda, \hbar) &= -\frac{\check{p}_1(\lambda - \check{q}_2)}{\check{q}_1 - \check{q}_2} - \frac{\check{p}_2(\lambda - \check{q}_1)}{\check{q}_2 - \check{q}_1} = \frac{(\check{p}_2 - \check{p}_1)\lambda + \check{p}_1\check{q}_2 - \check{p}_2\check{q}_1}{\check{q}_1 - \check{q}_2} \end{aligned} \quad (10-11)$$

Coefficients $(H_{\infty,0}, H_{\infty,1})$ are determined by (8-12):

$$H_{\infty,0} = \frac{(\check{q}_1\check{p}_2^2 - \check{q}_2\check{p}_1^2)}{(\check{q}_1 - \check{q}_2)^2} - \hbar \frac{(\check{p}_1 - \check{p}_2)}{(\check{q}_1 - \check{q}_2)} + (\check{q}_1 + \check{q}_2)\check{q}_1\check{q}_2(\check{q}_1^2 + \check{q}_2^2 + 2\tau_1) + 2\tau_2\check{q}_1\check{q}_2$$

$$H_{\infty,1} = \frac{\check{p}_1^2 - \check{p}_2^2}{\check{q}_1 - \check{q}_2} - 2\tau_1(\check{q}_1^2 + \check{q}_1\check{q}_2 + \check{q}_2^2) - 2\tau_2(\check{q}_1 + \check{q}_2) - \check{q}_1^4 - \check{q}_1^3\check{q}_2 - \check{q}_1^2\check{q}_2^2 - \check{q}_1\check{q}_2^3 - \check{q}_2^4 \quad (10-12)$$

Coefficients $(\nu_{\infty,1}^{(\alpha^{\tau_1})}, \nu_{\infty,2}^{(\alpha^{\tau_1})}, \nu_{\infty,1}^{(\alpha^{\tau_2})}, \nu_{\infty,2}^{(\alpha^{\tau_2})})$ are determined by Proposition 8.3 whose l.h.s. is trivial for $g = 2$ so that

$$\begin{aligned} \nu_{\infty,1}^{(\alpha^{\tau_1})} &= \frac{2}{3}, \quad \nu_{\infty,2}^{(\alpha^{\tau_1})} = 0 \\ \nu_{\infty,1}^{(\alpha^{\tau_2})} &= 0, \quad \nu_{\infty,2}^{(\alpha^{\tau_2})} = 2 \end{aligned} \quad (10-13)$$

Coefficients $(\mu_1^{(\alpha^{\tau_1})}, \mu_2^{(\alpha^{\tau_1})}, \mu_1^{(\alpha^{\tau_2})}, \mu_2^{(\alpha^{\tau_2})})$ are determined by (8-16):

$$\begin{aligned} \mu_1^{(\alpha^{\tau_1})} &= -\frac{2\check{q}_2}{3(\check{q}_1 - \check{q}_2)}, \quad \mu_2^{(\alpha^{\tau_1})} = \frac{2\check{q}_1}{3(\check{q}_1 - \check{q}_2)} \\ \mu_1^{(\alpha^{\tau_2})} &= \frac{2}{\check{q}_1 - \check{q}_2}, \quad \mu_2^{(\alpha^{\tau_2})} = -\frac{2}{\check{q}_1 - \check{q}_2} \end{aligned} \quad (10-14)$$

The Hamiltonians are

$$\begin{aligned} \text{Ham}^{(\alpha^{\tau_1})}(\check{\mathbf{q}}, \check{\mathbf{p}}) &= \frac{2}{3} \left(\frac{(\check{q}_1\check{p}_2^2 - \check{q}_2\check{p}_1^2)}{(\check{q}_1 - \check{q}_2)^2} - \hbar \frac{(\check{p}_1 - \check{p}_2)}{(\check{q}_1 - \check{q}_2)} + (\check{q}_1 + \check{q}_2)\check{q}_1\check{q}_2(\check{q}_1^2 + \check{q}_2^2 + 2\tau_1) + 2\tau_2\check{q}_1\check{q}_2 \right) \\ &= \frac{2}{3} (P_2^2(Q_1^2 - Q_2) + P_1^2 + 2P_1P_2Q_1 + \hbar P_2) \\ \text{Ham}^{(\alpha^{\tau_2})}(\check{\mathbf{q}}, \check{\mathbf{p}}) &= 2 \left(\frac{\check{p}_1^2 - \check{p}_2^2}{\check{q}_1 - \check{q}_2} - 2\tau_1(\check{q}_1^2 + \check{q}_1\check{q}_2 + \check{q}_2^2) - 2\tau_2(\check{q}_1 + \check{q}_2) - \check{q}_1^4 - \check{q}_1^3\check{q}_2 - \check{q}_1^2\check{q}_2^2 - \check{q}_1\check{q}_2^3 - \check{q}_2^4 \right) \\ &= 2(-P_2^2Q_1 - 2P_1P_2 - Q_1^4 + 3Q_1^3Q_2 - Q_2^2 + 2(Q_2 - Q_1^2)\tau_1 - 2Q_1\tau_2) \end{aligned} \quad (10-15)$$

where

$$Q_1 = \check{q}_1 + \check{q}_2, \quad Q_2 = \check{q}_1\check{q}_2, \quad P_1 = \frac{\check{q}_1\check{p}_1 - \check{q}_2\check{p}_2}{\check{q}_1 - \check{q}_2}, \quad P_2 = -\frac{\check{p}_1 - \check{p}_2}{\check{q}_1 - \check{q}_2} \quad (10-16)$$

The Lax matrices are

$$\begin{aligned} L(\lambda, \hbar) &= \begin{pmatrix} 0 & 1 \\ \lambda^5 + 2\tau_1\lambda^3 + 2\tau_2\lambda^2 + H_{\infty,1}\lambda + H_{\infty,0} & \frac{\hbar}{\lambda - \check{q}_1} + \frac{\hbar}{\lambda - \check{q}_2} \end{pmatrix} \\ [A_{\alpha^{\tau_1}}(\lambda, \hbar)]_{1,1} &= \frac{2\check{p}_1\check{q}_2}{3(\check{q}_1 - \check{q}_2)(\lambda - \check{q}_1)} - \frac{2\check{q}_1\check{p}_2}{3(\check{q}_1 - \check{q}_2)(\lambda - \check{q}_2)} \\ [A_{\alpha^{\tau_1}}(\lambda, \hbar)]_{1,2} &= -\frac{2\check{q}_2}{3(\check{q}_1 - \check{q}_2)(\lambda - \check{q}_1)} + \frac{2\check{q}_1}{3(\check{q}_1 - \check{q}_2)(\lambda - \check{q}_2)} \\ [A_{\alpha^{\tau_1}}(\lambda, \hbar)]_{2,1} &= 2 \left(-\frac{\check{q}_2}{3(\check{q}_1 - \check{q}_2)(\lambda - \check{q}_1)} + \frac{\check{q}_1}{3(\check{q}_1 - \check{q}_2)(\lambda - \check{q}_2)} \right) (\lambda^5 + 2\tau_1\lambda^3 + 2\tau_2\lambda^2 + H_{\infty,1}\lambda + H_{\infty,0}) \\ &\quad - \hbar \left(\frac{2}{3(\lambda - \check{q}_1)(\lambda - \check{q}_2)} \right) \\ [A_{\alpha^{\tau_1}}(\lambda, \hbar)]_{2,2} &= \frac{2\check{q}_2\check{p}_1}{3(\check{q}_1 - \check{q}_2)(\lambda - \check{q}_1)} - \frac{2\check{q}_1\check{p}_2}{3(\check{q}_1 - \check{q}_2)(\lambda - \check{q}_2)} + \hbar \left(\frac{2}{3(\lambda - \check{q}_1)(\lambda - \check{q}_2)} \right) \\ [A_{\alpha^{\tau_2}}(\lambda, \hbar)]_{1,1} &= 2 \left(-\frac{\check{p}_1}{(\check{q}_1 - \check{q}_2)(\lambda - \check{q}_1)} + \frac{\check{p}_2}{(\check{q}_1 - \check{q}_2)(\lambda - \check{q}_2)} \right) \\ [A_{\alpha^{\tau_2}}(\lambda, \hbar)]_{1,2} &= \frac{1}{(\check{q}_1 - \check{q}_2)(\lambda - \check{q}_1)} - \frac{1}{(\check{q}_1 - \check{q}_2)(\lambda - \check{q}_2)} \\ [A_{\alpha^{\tau_2}}(\lambda, \hbar)]_{2,1} &= 2 \left(\frac{1}{(\check{q}_1 - \check{q}_2)(\lambda - \check{q}_1)} - \frac{1}{(\check{q}_1 - \check{q}_2)(\lambda - \check{q}_2)} \right) (\lambda^5 + 2\tau_1\lambda^3 + 2\tau_2\lambda^2 + H_{\infty,1}\lambda + H_{\infty,0}) \\ [A_{\alpha^{\tau_2}}(\lambda, \hbar)]_{2,2} &= 2 \left(-\frac{\check{p}_1}{(\check{q}_1 - \check{q}_2)(\lambda - \check{q}_1)} + \frac{\check{p}_2}{(\check{q}_1 - \check{q}_2)(\lambda - \check{q}_2)} \right) \end{aligned} \quad (10-17)$$

or equivalently

$$\begin{aligned}
\tilde{L}(\lambda) &= \begin{pmatrix} \frac{\check{p}_1 - \check{p}_2}{\check{q}_1 - \check{q}_2} \lambda + \frac{\check{q}_1 \check{p}_2 - \check{q}_2 \check{p}_1}{\check{q}_1 - \check{q}_2} & (\lambda - \check{q}_1)(\lambda - \check{q}_2) \\ \lambda^3 + (\check{q}_1 + \check{q}_2)\lambda^2 + (\check{q}_1^2 + \check{q}_1 \check{q}_2 + \check{q}_2^2 + 2\tau_1)\lambda - \frac{(\check{p}_1 - \check{p}_2)^2}{(\check{q}_1 - \check{q}_2)^2} + (\check{q}_1^2 + \check{q}_2^2 + 2\tau_1)(\check{q}_1 + \check{q}_2) + 2\tau_2 & -\frac{\check{p}_1 - \check{p}_2}{\check{q}_1 - \check{q}_2} \lambda - \frac{\check{q}_1 \check{p}_2 - \check{q}_2 \check{p}_1}{\check{q}_1 - \check{q}_2} \end{pmatrix} \\
&= \begin{pmatrix} -P_2\lambda - Q_1P_2 - P_1 & \lambda^2 - Q_1\lambda + Q_2 \\ \lambda^3 + Q_1\lambda^2 + (Q_1^2 - Q_2 + 2\tau_1)\lambda - P_2^2 + Q_1(Q_1^2 - 2Q_2) + 2Q_1\tau_1 + 2\tau_2 & P_2\lambda + Q_1P_2 + P_1 \end{pmatrix} \quad (10-18)
\end{aligned}$$

$$\begin{aligned}
\tilde{A}_{\alpha^{\tau_1}}(\lambda) &= \begin{pmatrix} \frac{2(\check{p}_1 - \check{p}_2)}{3(\check{q}_1 - \check{q}_2)} & \frac{2}{3}(\lambda - \check{q}_1 - \check{q}_2) \\ \frac{2}{3}(\lambda^2 + (\check{q}_1 + \check{q}_2)\lambda + \check{q}_1^2 + \check{q}_2^2 + 2\tau_1) & -\frac{2(\check{p}_1 - \check{p}_2)}{3(\check{q}_1 - \check{q}_2)} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{2}{3}P_2 & \frac{2}{3}(\lambda - Q_1) \\ \frac{2}{3}(\lambda^2 + Q_1\lambda + Q_1^2 - 2Q_2 + 2\tau_1) & \frac{2}{3}P_2 \end{pmatrix} \\
\tilde{A}_{\alpha^{\tau_2}}(\lambda) &= \begin{pmatrix} 0 & 2 \\ 2(\lambda + 2(\check{q}_1 + \check{q}_2)) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2(\lambda + 2Q_1) & 0 \end{pmatrix} \quad (10-19)
\end{aligned}$$

10.4 Third element of the Painlevé 1 hierarchy: $r_\infty = 6$

Let us consider $r_\infty = 6$, i.e. $g = 3$ corresponding to the third element of the Painlevé 1 hierarchy. In particular, this case is the first case where the Hamiltonians are non-trivial linear combinations of the coefficients $(H_{\infty,k})_{0 \leq k \leq r_\infty - 4}$, this is due to the fact that the matrix of Proposition 8.3 is no longer diagonal. Indeed we have

$$\begin{aligned}
\text{Ham}^{(\alpha^{\tau_1})} &= \frac{2}{5}H_{\infty,0} - \frac{2}{5}\tau_1 H_{\infty,2} \\
\text{Ham}^{(\alpha^{\tau_2})} &= \frac{2}{3}H_{\infty,1} \\
\text{Ham}^{(\alpha^{\tau_3})} &= 2H_{\infty,2} \quad (10-20)
\end{aligned}$$

In this setup, the canonical choice of trivial times corresponds to $t_{\infty,10} = t_{\infty,8} = t_{\infty,6} = t_{\infty,4} = t_{\infty,2} = 0$, $t_{\infty,9} = 2$ and $t_{\infty,7} = 0$. The only non-trivial isomonodromic times are $\tau_1 = \frac{1}{2}t_{\infty,5}$, $\tau_2 = \frac{1}{2}t_{\infty,3}$ and $\tau_3 = \frac{1}{2}t_{\infty,1}$. For compactness, we shall only present results expressed in terms of the symmetric Darboux coordinates $(Q_1, Q_2, Q_3, P_1, P_2, P_3)$. One may recover the expression in terms of shifted Darboux coordinates using

$$\begin{aligned}
Q_1 &= \check{q}_1 + \check{q}_2 + \check{q}_3, \quad Q_2 = \check{q}_1 \check{q}_2 + \check{q}_1 \check{q}_3 + \check{q}_2 \check{q}_3, \quad Q_3 = \check{q}_1 \check{q}_2 \check{q}_3 \\
P_1 &= \frac{\check{q}_1^2(\check{q}_2 - \check{q}_3)\check{p}_1 - \check{q}_2^2(\check{q}_1 - \check{q}_3)\check{p}_2 + \check{q}_3^2(\check{q}_1 - \check{q}_2)\check{p}_3}{(\check{q}_1 - \check{q}_2)(\check{q}_1 - \check{q}_3)(\check{q}_2 - \check{q}_3)} \\
P_2 &= -\frac{\check{q}_1(\check{q}_2 - \check{q}_3)\check{p}_1 - \check{q}_2(\check{q}_1 - \check{q}_3)\check{p}_2 + \check{q}_3(\check{q}_1 - \check{q}_2)\check{p}_3}{(\check{q}_1 - \check{q}_2)(\check{q}_1 - \check{q}_3)(\check{q}_2 - \check{q}_3)} \\
P_3 &= \frac{(\check{q}_2 - \check{q}_3)\check{p}_1 - (\check{q}_1 - \check{q}_3)\check{p}_2 + (\check{q}_1 - \check{q}_2)\check{p}_3}{(\check{q}_1 - \check{q}_2)(\check{q}_1 - \check{q}_3)(\check{q}_2 - \check{q}_3)} \quad (10-21)
\end{aligned}$$

We have

$$\begin{aligned}
\tilde{P}_2(\lambda) &= -\lambda^7 - 2\tau_1\lambda^5 - 2\tau_2\lambda^4 - (\tau_1^2 + 2\tau_3)\lambda^3 \\
Q(\lambda) &= -P_3\lambda^2 + (P_3Q_1 + P_2)\lambda - P_3Q_2 - P_2Q_1 - P_1 \quad (10-22)
\end{aligned}$$

The Hamiltonians are

$$\text{Ham}^{(\alpha^{\tau_1})} = \frac{2}{5} \left[(-Q_1Q_3 + Q_2^2)P_3^2 + 2(Q_1 + Q_2)P_1P_3 + Q_1^2P_2^2 + 2Q_1P_1P_2 + P_1^2 + 2(Q_1Q_2 - Q_3)P_2P_3 - 2Q_1P_1P_3 \right]$$

$$\begin{aligned}
& -Q_3(Q_1^4 - 3Q_1^2Q_2 + 2Q_1Q_3 + Q_2^2) + 2(Q_1^2 - Q_2)\tau_1\tau_2 + 2Q_1\tau_1\tau_3 + Q_1\tau_1^3 + (2Q_1^3 - 4Q_1Q_2 + Q_3)\tau_1^2 \\
& -(-Q_1^5 + 4Q_1^3Q_2 + 2Q_1P_2P_3 + Q_2P_3^2 - Q_1^2Q_3 - 3Q_1Q_2^2 + 2P_1P_3 + P_2^2)\tau_1 - 2Q_1Q_3\tau_2 - 2Q_3\tau_3 \\
& + \hbar(Q_1P_3 + 2P_2) \\
\text{Ham}^{(\alpha^{\tau_2})} &= \frac{2}{3} \left[-2Q_1^2P_2P_3 - 2Q_1P_1P_3 - 2P_1P_2 + (Q_3 - Q_1Q_2)P_3^2 - 2Q_1P_2^2 + 4Q_1Q_2Q_3 + Q_1^4Q_2 - Q_1^2Q_2^2 \right. \\
& \left. - Q_1^3Q_3 + Q_2^3 - Q_3^2 + Q_2\tau_1^2 + 2(Q_1^2Q_2 - Q_1Q_3 - Q_2^2)\tau_1 + 2(Q_1Q_2 - Q_3)\tau_2 + 2Q_2\tau_3 - \hbar P_3 \right] \\
\text{Ham}^{(\alpha^{\tau_3})} &= 2 \left[2Q_1P_2P_3 + Q_2P_3^2 + 2P_1P_3 + P_2^2 - Q_1^5 + 4Q_1^3Q_2 - 3Q_1^2Q_3 - 3Q_1Q_2^2 + 2Q_2Q_3 \right. \\
& \left. - Q_1\tau_1^2 + 2(2Q_1Q_2 - Q_1^3 - Q_3)\tau_1 + 2(Q_2 - Q_1^2)\tau_2 - 2Q_1\tau_3 \right] \tag{10-23}
\end{aligned}$$

The Lax matrices are

$$\begin{aligned}
\tilde{L}_{1,1}(\lambda) &= P_3\lambda^2 - (P_2 + Q_1P_3)\lambda + P_1 + Q_2P_3 + Q_1P_2 \\
\tilde{L}_{1,2}(\lambda) &= \lambda^3 - Q_1\lambda^2 + Q_2\lambda - Q_3 \\
\tilde{L}_{2,1}(\lambda) &= \lambda^4 + Q_1\lambda^3 + (Q_1^2 - Q_2 + 2\tau_1)\lambda^2 + (-P_3^2 + Q_1^3 + Q_3 - 2Q_1Q_2 + 2Q_1\tau_1 + 2\tau_2)\lambda \\
& \quad 2P_2P_3 + Q_1P_3^2 + Q_1^4 - 3Q_2Q_1^2 + 2Q_3Q_1 + Q_2^2 + \tau_1^2 + 2(Q_1^2 - Q_2)\tau_1 + 2Q_1\tau_2 + 2\tau_3 \\
\tilde{L}_{2,2}(\lambda) &= -P_3\lambda^2 + (P_2 + Q_1P_3)\lambda - P_1 - Q_2P_3 - Q_1P_2 \tag{10-24}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{A}_{\alpha^{\tau_1}}(\lambda) &= \begin{pmatrix} \frac{2}{5}(P_3\lambda - Q_1P_3 - P_2) & \frac{2}{5}(\lambda^2 - Q_1\lambda + Q_2 - \tau_1) \\ \frac{2}{5}(\lambda^3 + Q_1\lambda^2 + (Q_1^2 - Q_2 + \tau_1)\lambda - P_3 + Q_1^3 - 2Q_1Q_2 + 2Q_3 + 2\tau_2) & -\frac{2}{5}(P_3\lambda - Q_1P_3 - P_2) \end{pmatrix} \\
\tilde{A}_{\alpha^{\tau_2}}(\lambda) &= \begin{pmatrix} \frac{2}{3}P_3 & \frac{2}{3}(\lambda - Q_1) \\ \frac{2}{3}(\lambda^2 + Q_1\lambda + Q_1^2 - 2Q_2 + 2\tau_1) & -\frac{2}{3}P_3 \end{pmatrix} \\
\tilde{A}_{\alpha^{\tau_3}}(\lambda) &= \begin{pmatrix} 0 & 2 \\ 2(\lambda + 2Q_1) & 0 \end{pmatrix} \tag{10-25}
\end{aligned}$$

11 Outlooks

In this article, we complemented the results of [27] by dealing with the case of twisted meromorphic connections in $\mathfrak{gl}_2(\mathbb{C})$ obtaining explicit expressions of the Hamiltonians and Lax pairs in various sets of Darboux coordinates. Moreover, we provided a reduction of the initial space of irregular times (of dimension $2g + 4$) to a set of non-trivial isomonodromic times of dimension g . In particular, we recover in the twisted case the fact that meromorphic connections in $\mathfrak{gl}_2(\mathbb{C})$ are equivalent at the level of Hamiltonian systems to meromorphic connections in $\mathfrak{sl}_2(\mathbb{C})$, a point that was already raised in [27] in the non-twisted case. The method used in the present article opens the way to several generalizations:

- This article and [27] ends the study of meromorphic connections in $\mathfrak{gl}_2(\mathbb{C})$. Thus, a natural issue is to know if the present setup extends to $\mathfrak{gl}_d(\mathbb{C})$ with $d \geq 2$. In principle, a similar strategy shall be used but it is unclear if all technical issues might be overcome when the dimension is arbitrary, especially in the twisted case where the underlying geometric construction is far less understood. We let this very exciting question for future works.
- This article deals with meromorphic connections in $\mathfrak{gl}_2(\mathbb{C})$. However, one may be interested in a more general abstract setup with any Lie algebra and not only a matrix representation of it. In this case, we believe that most of the results shall be generalized upon the adequate quantities and terminology.
- As mentioned in Section 3 and similarly to [27], this article makes the connection with formal WKB expansions and \hbar -transseries via topological recursion. At the level of isomonodromic deformations, the introduction of the formal parameter \hbar is done by a simple

rescaling of irregular times (Section 2.5). A better understanding of the role of \hbar and its limit to 0 in the context of meromorphic connections would be interesting in order to address the issue of resummation and analytical properties associated to the formal \hbar (trans)series.

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A Proof of Proposition 2.3

As mentioned in Proposition 2.1 there exists a local gauge transformation $G_\infty(z) = G_{\infty,-1}z + G_{\infty,0} + G_{\infty,1}z + \dots$ with $z = \lambda^{\frac{1}{2}}$ and $G_{\infty,-1}$ of rank 1 such that $\Psi_\infty = G_\infty \tilde{\Psi}$ is given by (we recall that we added the extra parameter \hbar by rescaling)

$$\begin{aligned} \Psi_\infty(\lambda) &= \Psi_\infty^{(\text{reg})}(z) \text{diag} \left(\exp \left(-\frac{1}{\hbar} \sum_{k=1}^{2r_\infty-2} \frac{t_{\infty,k}}{k} z^k + \frac{1}{2} \ln z \right), \exp \left(-\frac{1}{\hbar} \sum_{k=1}^{2r_\infty-2} (-1)^k \frac{t_{\infty,k}}{k} z^k + \frac{1}{2} \ln z \right) \right) \\ &= \begin{pmatrix} \left[\Psi_\infty^{(\text{reg})}(z) \right]_{1,1} \exp \left(-\frac{1}{\hbar} \sum_{k=1}^{2r_\infty-2} \frac{t_{\infty,k}}{k} z^k + \frac{1}{2} \ln z \right) & \left[\Psi_\infty^{(\text{reg})}(z) \right]_{1,2} \exp \left(-\frac{1}{\hbar} \sum_{k=1}^{2r_\infty-2} (-1)^k \frac{t_{\infty,k}}{k} z^k + \frac{1}{2} \ln z \right) \\ \left[\Psi_\infty^{(\text{reg})}(z) \right]_{2,1} \exp \left(-\frac{1}{\hbar} \sum_{k=1}^{2r_\infty-2} \frac{t_{\infty,k}}{k} z^k + \frac{1}{2} \ln z \right) & \left[\Psi_\infty^{(\text{reg})}(z) \right]_{2,2} \exp \left(-\frac{1}{\hbar} \sum_{k=1}^{2r_\infty-2} (-1)^k \frac{t_{\infty,k}}{k} z^k + \frac{1}{2} \ln z \right) \end{pmatrix} \end{aligned} \quad (\text{A-1})$$

Since $\tilde{\Psi}$ is normalized at infinity by (2-13), the verification is straightforward when one considers the highest order in the expansion

$$G_{\infty,-1} = \begin{pmatrix} X & 0 \\ X & 0 \end{pmatrix} \quad (\text{A-2})$$

In particular, the expansion of G_∞^{-1} is of the form $G_\infty^{-1} = \hat{G}_{\infty,0} + \hat{G}_{\infty,1}z^{-1} + \hat{G}_{\infty,2}z^{-2} + \dots$ with $\hat{G}_{\infty,0} = \begin{pmatrix} 0 & 0 \\ X & X \end{pmatrix}$.

Thus, multiplying on the left by G_∞^{-1} provides

$$\tilde{\Psi}(z) = \begin{pmatrix} \left[R_\infty^{(\text{reg})}(z) \right]_{1,1} \exp \left(-\frac{1}{\hbar} \sum_{k=1}^{2r_\infty-2} \frac{t_{\infty,k}}{k} z^k - \frac{1}{2} \ln z \right) & \left[R_\infty^{(\text{reg})}(z) \right]_{1,2} \exp \left(-\frac{1}{\hbar} \sum_{k=1}^{2r_\infty-2} (-1)^k \frac{t_{\infty,k}}{k} z^k - \frac{1}{2} \ln z \right) \\ \left[R_\infty^{(\text{reg})}(z) \right]_{2,1} z \exp \left(-\frac{1}{\hbar} \sum_{k=1}^{2r_\infty-2} \frac{t_{\infty,k}}{k} z^k - \frac{1}{2} \ln z \right) & \left[R_\infty^{(\text{reg})}(z) \right]_{2,2} z \exp \left(-\frac{1}{\hbar} \sum_{k=1}^{2r_\infty-2} (-1)^k \frac{t_{\infty,k}}{k} z^k - \frac{1}{2} \ln z \right) \end{pmatrix} \quad (\text{A-3})$$

where $\left(\left[R_\infty^{(\text{reg})}(z) \right]_{i,j} \right)_{(i,j) \in \llbracket 1,2 \rrbracket^2}$ are regular at infinity. Let us now prove that these asymptotics are consistent

with the one proposed for Ψ . Since Ψ is the solution to a companion-like system, we have $\Psi = \begin{pmatrix} \psi_1 & \psi_2 \\ \hbar \partial_\lambda \psi_1 & \hbar \partial_\lambda \psi_2 \end{pmatrix}$.

Hence, equation (2-38) is equivalent to

$$\begin{aligned} \Psi_{1,1}(\lambda) &= \exp \left(-\frac{1}{\hbar} \sum_{k=1}^{2r_\infty-2} \frac{t_{\infty,k}}{k} z^k - \frac{1}{2} \ln z + O(1) \right) \\ \Psi_{1,2}(\lambda) &= \exp \left(-\frac{1}{\hbar} \sum_{k=1}^{2r_\infty-2} (-1)^k \frac{t_{\infty,k}}{k} z^k - \frac{1}{2} \ln z + O(1) \right) \\ \Psi_{2,1}(\lambda) &= \left(-\frac{1}{2} \sum_{k=1}^{2r_\infty-2} t_{\infty,k} z^{k-2} - \frac{1}{4z^2} + o(z^{-3}) \right) \exp \left(-\frac{1}{\hbar} \sum_{k=1}^{2r_\infty-2} \frac{t_{\infty,k}}{k} z^k - \frac{1}{2} \ln z + O(1) \right), \\ \Psi_{2,2}(\lambda) &= \left(-\frac{1}{2} \sum_{k=1}^{2r_\infty-2} (-1)^k t_{\infty,k} z^{k-2} - \frac{1}{4z^2} + o(z^{-3}) \right) \exp \left(-\frac{1}{\hbar} \sum_{k=1}^{2r_\infty-2} (-1)^k \frac{t_{\infty,k}}{k} z^k - \frac{1}{2} \ln z + O(1) \right), \end{aligned} \quad (\text{A-4})$$

Thus, $\tilde{\Psi} = G_1 J \Psi$ is given by

$$\tilde{\Psi}(\lambda) = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} t_{\infty,2r_\infty-2} \lambda + g_0 + \frac{g(\lambda)}{\prod_{j=1}^g (\lambda - q_j)} & \frac{1}{\prod_{j=1}^g (\lambda - q_j)} \end{pmatrix} \begin{pmatrix} \Psi_{1,1}(\lambda) & \Psi_{1,2}(\lambda) \\ \Psi_{2,1}(\lambda) & \Psi_{2,2}(\lambda) \end{pmatrix} \quad (\text{A-5})$$

and behaves like

$$\begin{aligned} \tilde{\Psi}_{1,1}(\lambda) &= \exp \left(-\frac{1}{\hbar} \sum_{k=1}^{2r_\infty-2} \frac{t_{\infty,k}}{k} z^k - \frac{1}{2} \ln z + O(1) \right) \\ \tilde{\Psi}_{1,2}(\lambda) &= \exp \left(-\frac{1}{\hbar} \sum_{k=1}^{2r_\infty-2} (-1)^k \frac{t_{\infty,k}}{k} z^k - \frac{1}{2} \ln z + O(1) \right) \end{aligned}$$

$$\begin{aligned}
\tilde{\Psi}_{2,1}(\lambda) &= \exp\left(-\frac{1}{\hbar} \sum_{k=1}^{2r_\infty-2} \frac{t_{\infty,k}}{k} z^k - \frac{1}{2} \ln z + O(1)\right) \left[\frac{1}{2} t_{\infty,2r_\infty-2} z^2 + g_0 \right. \\
&\quad \left. + \left(-\frac{1}{2} t_{\infty,2r_\infty-2} z^{2r_\infty-4} - \frac{1}{2} t_{\infty,2r_\infty-3} z^{2r_\infty-5} + O(z^{2r_\infty-6})\right) \left(z^{-2r_\infty+6} + \sum_{j=1}^g q_j z^{2r_\infty+4} + O(z^{2r_\infty+2}) \right) \right] \\
&= \left(-\frac{1}{2} t_{\infty,2r_\infty-3} z + O(1)\right) \exp\left(-\frac{1}{\hbar} \sum_{k=1}^{2r_\infty-2} \frac{t_{\infty,k}}{k} z^k - \frac{1}{2} \ln z + O(1)\right) \\
\tilde{\Psi}_{2,2}(\lambda) &= \exp\left(-\frac{1}{\hbar} \sum_{k=1}^{2r_\infty-2} (-1)^k \frac{t_{\infty,k}}{k} z^k - \frac{1}{2} \ln z + O(1)\right) \left[\frac{1}{2} t_{\infty,2r_\infty-2} z^2 + g_0 \right. \\
&\quad \left. + \left(-\frac{1}{2} t_{\infty,2r_\infty-2} z^{2r_\infty-4} + \frac{1}{2} t_{\infty,2r_\infty-3} z^{2r_\infty-5} + O(z^{2r_\infty-6})\right) \left(z^{-2r_\infty+6} + \sum_{j=1}^g q_j z^{2r_\infty+4} + O(z^{2r_\infty+2}) \right) \right] \\
&= \left(\frac{1}{2} t_{\infty,2r_\infty-3} z + O(1)\right) \exp\left(-\frac{1}{\hbar} \sum_{k=1}^{2r_\infty-2} (-1)^k \frac{t_{\infty,k}}{k} z^k - \frac{1}{2} \ln z + O(1)\right) \tag{A-6}
\end{aligned}$$

in accordance with (A-3). Moreover, asymptotics (A-6) of $\tilde{\Psi}$ implies by direct computations that $(\hbar \partial_\lambda \tilde{\Psi}) \tilde{\Psi}^{-1} = \tilde{L}$ may be set in the form given by (2-13).

B Proof of Proposition 2.4

From Proposition 2.3, the local asymptotics of the wave functions ψ_1 and ψ_2 are

$$\begin{aligned}
\psi_1(\lambda) &\stackrel{\lambda \rightarrow \infty}{=} \exp\left(-\frac{1}{\hbar} \sum_{k=1}^{2r_\infty-2} \frac{t_{\infty,k}}{k} \lambda^{\frac{k}{2}} - \frac{1}{4} \ln \lambda + O(1)\right) \\
\psi_2(\lambda) &\stackrel{\lambda \rightarrow \infty}{=} \exp\left(-\frac{1}{\hbar} \sum_{k=1}^{2r_\infty-2} (-1)^k \frac{t_{\infty,k}}{k} \lambda^{\frac{k}{2}} - \frac{1}{4} \ln \lambda + O(1)\right) \tag{B-1}
\end{aligned}$$

In particular, the Wronskian $W(\lambda) = \psi_1(\lambda) \hbar \partial_\lambda \psi_2(\lambda) - \psi_2(\lambda) \hbar \partial_\lambda \psi_1(\lambda)$ is given by Definition 2.3:

$$W(\lambda) = W_0 \prod_{j=1}^g (\lambda - q_j) \exp\left(\frac{1}{\hbar} \int_0^\lambda \tilde{P}_1(s) ds\right) \tag{B-2}$$

The standard relation between $\text{Tr } L(\lambda)$ and the logarithmic derivative of the Wronskian provides the expected result of $L_{2,2}(\lambda, \hbar)$. As an intermediate step we define $Y_i(\lambda) = \frac{\hbar}{\psi_i(\lambda)} \partial_\lambda \psi_i(\lambda)$. Then

$$L_{2,1}(\lambda) = -Y_1(\lambda) Y_2(\lambda) - \hbar \frac{Y_2(\lambda) \partial_\lambda Y_1(\lambda) - Y_1(\lambda) \partial_\lambda Y_2(\lambda)}{Y_2(\lambda) - Y_1(\lambda)} \tag{B-3}$$

One may thus study the asymptotic behavior of $L_{2,1}$ at $\lambda \rightarrow \infty$. We have

$$\begin{aligned}
Y_1(\lambda) &\stackrel{\lambda \rightarrow \infty}{=} -\frac{1}{2} \sum_{k=1}^{2r_\infty-2} t_{\infty,k} \lambda^{\frac{k}{2}-1} - \frac{\hbar}{4\lambda} + O\left(\lambda^{-\frac{3}{2}}\right) \\
Y_2(\lambda) &\stackrel{\lambda \rightarrow \infty}{=} -\frac{1}{2} \sum_{k=1}^{2r_\infty-2} (-1)^k t_{\infty,k} \lambda^{\frac{k}{2}-1} - \frac{\hbar}{4\lambda} + O\left(\lambda^{-\frac{3}{2}}\right) \tag{B-4}
\end{aligned}$$

so that

$$-\hbar \frac{Y_2(\lambda) \partial_\lambda Y_1(\lambda) - Y_1(\lambda) \partial_\lambda Y_2(\lambda)}{Y_2(\lambda) - Y_1(\lambda)} \stackrel{\lambda \rightarrow \infty}{=} \frac{\hbar}{4} t_{\infty,2r_\infty-2} \lambda^{r_\infty-3} + O\left(\lambda^{r_\infty-4}\right) \tag{B-5}$$

Hence we obtain:

$$L_{2,1}(\lambda) \stackrel{\lambda \rightarrow \infty}{=} -\frac{1}{4} \sum_{i=1}^{2r_\infty-2} \sum_{j=1}^{2r_\infty-2} (-1)^j t_{\infty,i} t_{\infty,j} \lambda^{\frac{i+j}{2}-2} - \frac{\hbar}{2} t_{\infty,2r_\infty-2} \lambda^{r_\infty-3} + O(\lambda^{r_\infty-4}) \quad (\text{B-6})$$

Note that terms with odd values of $i + j$ cancel by symmetry, we end up with

$$\begin{aligned} L_{2,1}(\lambda) &\stackrel{\lambda \rightarrow \infty}{=} -\frac{1}{4} \sum_{\substack{(i,j) \in \llbracket 1, 2r_\infty-2 \rrbracket^2 \\ i+j \text{ even}}} (-1)^j t_{\infty,i} t_{\infty,j} \lambda^{\frac{i+j}{2}-2} + O(\lambda^{r_\infty-4}) \\ &\stackrel{\lambda \rightarrow \infty}{=} -\frac{1}{4} \sum_{k=r_\infty-2}^{2r_\infty-2} \sum_{j=2k-2r_\infty+6}^{2r_\infty-2} (-1)^j t_{\infty,j} t_{\infty,2k-j+4} \lambda^k \\ &\quad -\frac{1}{4} \sum_{j=1}^{2r_\infty-3} (-1)^j t_{\infty,j} t_{\infty,2r_\infty-j-2} \lambda^{r_\infty-3} + O(\lambda^{r_\infty-4}) \\ &= -\tilde{P}_2(\lambda) + O(\lambda^{r_\infty-4}) \end{aligned} \quad (\text{B-7})$$

Since $L_{2,1}(\lambda, \hbar)$ is a rational function of λ with only poles at infinity, we get that it is a polynomial in λ and the previous asymptotics provide its leading coefficients.

C Proof of Proposition 4.1

Let us first observe that the entry $[A_\alpha(\lambda)]_{1,2}$ is given by

$$[A_\alpha(\lambda)]_{1,2} = \frac{W_\alpha(\lambda)}{W(\lambda)} = \frac{Z_{\alpha,2}(\lambda) - Z_{\alpha,1}(\lambda)}{Y_2(\lambda) - Y_1(\lambda)} \quad (\text{C-1})$$

where we have defined

$$\begin{aligned} Z_{\alpha,i}(\lambda) &= \frac{\mathcal{L}_\alpha[\psi_i(\lambda)]}{\psi_i(\lambda)}, \quad \forall i \in \llbracket 1, 2 \rrbracket \\ W_\alpha(\lambda) &= \mathcal{L}_\alpha[\psi_2(\lambda)]\psi_1(\lambda) - \mathcal{L}_\alpha[\psi_1(\lambda)]\psi_2(\lambda) \end{aligned} \quad (\text{C-2})$$

and from Proposition 2.3 we have:

$$\begin{aligned} Z_{\alpha,1}(\lambda) &= -\sum_{k=1}^{2r_\infty-2} \frac{\alpha_{\infty,k}}{k} \lambda^{\frac{k}{2}} + O(1) \\ Z_{\alpha,2}(\lambda) &= -\sum_{k=1}^{2r_\infty-2} (-1)^k \frac{\alpha_{\infty,k}}{k} \lambda^{\frac{k}{2}} + O(1) \end{aligned} \quad (\text{C-3})$$

so that

$$Z_{\alpha,2}(\lambda) - Z_{\alpha,1}(\lambda) = 2 \sum_{j=1}^{r_\infty-1} \frac{\alpha_{\infty,2j-1}}{2j-1} \lambda^{j-\frac{1}{2}} + O(1) \quad (\text{C-4})$$

Thus, since

$$[A_\alpha(\lambda)]_{1,2} (Y_2(\lambda) - Y_1(\lambda)) = Z_{\alpha,2}(\lambda) - Z_{\alpha,1}(\lambda) \quad (\text{C-5})$$

using (B-4)

$$Y_2(\lambda) - Y_1(\lambda) = \sum_{j=1}^{r_\infty-1} t_{\infty,2j-1} \lambda^{j-\frac{3}{2}} + O(\lambda^{-\frac{3}{2}}) \quad (\text{C-6})$$

we obtain the form of the entry

$$A_{\alpha}(\lambda, \hbar)_{1,2} = \sum_{j=-1}^{r_{\infty}-3} \nu_{\infty,j}^{(\alpha)} \lambda^{-j} + O(\lambda^{-(r_{\infty}-2)}) \quad (\text{C-7})$$

where the coefficients $\left(\nu_{\infty,k}^{(\alpha)}\right)_{-1 \leq k \leq r_{\infty}-3}$ are recursively determined by

$$\left(\sum_{j=-1}^{r_{\infty}-3} \nu_{\infty,j}^{(\alpha)} \lambda^{-j} + O(\lambda^{-(r_{\infty}+2)})\right) \left(\sum_{k=1}^{r_{\infty}-1} t_{\infty,2k-1} \lambda^{k-1} + O(\lambda^{-1})\right) = 2 \sum_{k=1}^{r_{\infty}-1} \frac{\alpha_{\infty,2k-1}}{2k-1} \lambda^k + O(1). \quad (\text{C-8})$$

These relations may be rewritten in a $(r_{\infty}-1) \times (r_{\infty}-1)$ lower triangular Toeplitz matrix form:

$$M_{\infty} \begin{pmatrix} \nu_{\infty,-1}^{(\alpha)} \\ \vdots \\ \nu_{\infty,r_{\infty}-3}^{(\alpha)} \end{pmatrix} = \begin{pmatrix} \frac{2\alpha_{\infty,2r_{\infty}-3}}{(2r_{\infty}-3)} \\ \frac{2\alpha_{\infty,2r_{\infty}-5}}{(2r_{\infty}-5)} \\ \vdots \\ \frac{2\alpha_{\infty,1}}{1} \end{pmatrix} \text{ with } M_{\infty} = \begin{pmatrix} t_{\infty,2r_{\infty}-3} & 0 & \dots & & 0 \\ t_{\infty,2r_{\infty}-5} & t_{\infty,2r_{\infty}-3} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ t_{\infty,3} & & \ddots & \ddots & 0 \\ t_{\infty,1} & t_{\infty,3} & \dots & & t_{\infty,2r_{\infty}-3} \end{pmatrix} \quad (\text{C-9})$$

D Proof of Proposition 4.3

A straightforward computation shows that

$$[A_{\alpha}(\lambda)]_{1,1} = \frac{Z_{\alpha,1}(\lambda)Y_2(\lambda) - Z_{\alpha,2}(\lambda)Y_1(\lambda)}{Y_2(\lambda) - Y_1(\lambda)} \quad (\text{D-1})$$

which we rewrite as

$$[A_{\alpha}(\lambda)]_{1,1} (Y_2(\lambda) - Y_1(\lambda)) = Z_{\alpha,1}(\lambda)Y_2(\lambda) - Z_{\alpha,2}(\lambda)Y_1(\lambda) \quad (\text{D-2})$$

We proceed using (B-4) and (C-3) that give

$$\begin{aligned} Z_{\alpha,1}(\lambda)Y_2(\lambda) - Z_{\alpha,2}(\lambda)Y_1(\lambda) &= \frac{1}{2} \sum_{j=1}^{2r_{\infty}-2} \sum_{i=1}^{2r_{\infty}-2} \frac{\alpha_{\infty,j}}{j} t_{\infty,i} ((-1)^i - (-1)^j) \lambda^{\frac{i+j}{2}-1} + O(\lambda^{r_{\infty}-2}) \\ &= \sum_{s=1}^{r_{\infty}-1} \sum_{m=1}^{r_{\infty}-1} \left(\frac{\alpha_{\infty,2s-1}}{2s-1} t_{\infty,2m} - \frac{\alpha_{\infty,2s}}{2s} t_{\infty,2m-1} \right) \lambda^{m+s-\frac{3}{2}} + O(\lambda^{r_{\infty}-2}) \end{aligned} \quad (\text{D-3})$$

We make use (C-6) also in order to obtain

$$A_{\alpha}(\lambda, \hbar)_{1,1} = \sum_{j=1}^{r_{\infty}-1} c_{\infty,j}^{(\alpha)} \lambda^j + O(1) \quad (\text{D-4})$$

where the coefficients $\left(c_{\infty,k}^{(\alpha)}\right)_{1 \leq k \leq r_{\infty}-1}$ are recursively determined by

$$\left(\sum_{j=1}^{r_{\infty}-1} c_{\infty,j}^{(\alpha)} \lambda^j + O(1)\right) \left(\sum_{k=1}^{r_{\infty}-1} t_{\infty,2k-1} \lambda^{k-\frac{3}{2}} + O(\lambda^{-\frac{3}{2}})\right) = \sum_{s=1}^{r_{\infty}-1} \sum_{m=1}^{r_{\infty}-1} \left(\frac{\alpha_{\infty,2s-1}}{2s-1} t_{\infty,2m} - \frac{\alpha_{\infty,2s}}{2s} t_{\infty,2m-1} \right) \lambda^{m+s-\frac{3}{2}} + O(\lambda^{r_{\infty}-2}) \quad (\text{D-5})$$

These relations may be rewritten in a $(r_\infty - 1) \times (r_\infty - 1)$ lower triangular Toeplitz matrix form:

$$M_\infty \begin{pmatrix} c_{\infty, r_\infty - 1}^{(\alpha)} \\ \vdots \\ c_{\infty, k}^{(\alpha)} \\ \vdots \\ c_{\infty, 1}^{(\alpha)} \end{pmatrix} = \begin{pmatrix} \frac{\alpha_{\infty, 2r_\infty - 3}}{2r_\infty - 3} t_{\infty, 2r_\infty - 2} - \frac{\alpha_{\infty, 2r_\infty - 2}}{2r_\infty - 2} t_{\infty, 2r_\infty - 3} & & & \\ & \ddots & & \\ & & \sum_{m=k}^{r_\infty - 1} \left(\frac{\alpha_{\infty, 2k + 2r_\infty - 2m - 3}}{2k + 2r_\infty - 2m - 3} t_{\infty, 2m} - \frac{\alpha_{2k + 2r_\infty - 2m - 2}}{2k + 2r_\infty - 2m - 2} t_{\infty, 2m - 1} \right) & \\ & & & \ddots \\ & & & \sum_{m=1}^{r_\infty - 1} \left(\frac{\alpha_{\infty, 2r_\infty - 2m - 1}}{2r_\infty - 2m - 1} t_{\infty, 2m} - \frac{\alpha_{2r_\infty - 2m}}{2r_\infty - 2m} t_{\infty, 2m - 1} \right) \end{pmatrix} \quad (\text{D-6})$$

Finally, the coefficients $(\rho_j^{(\alpha)})_{1 \leq j \leq g}$ are obtained by looking at order $(\lambda - q_j)^{-3}$ of $\mathcal{L}_\alpha[L_{2,1}(\lambda)]$.

E Proof of Theorem 5.1

This appendix is devoted for the proof of Theorem 5.1.

E.1 Preliminary results

We start with the following lemma:

Lemma E.1. *For all $j \in \llbracket 1, g \rrbracket$:*

$$\sum_{k=0}^{r_\infty - 4} \sum_{i=1}^g H_{\infty, k} q_i^k \partial_{q_j} \mu_i^{(\alpha)} = -\mu_j^{(\alpha)} \sum_{k=0}^{r_\infty - 4} k H_{\infty, k} q_j^{k-1} \quad (\text{E-1})$$

Proof. The proof follows from the expression relating the coefficients $(\nu_{p,k}^{(\alpha)})$ and $(\mu_{p,k}^{(\alpha)})$ given by (4-11). Taking the derivative relatively to q_j and using the fact that $(\nu_{\infty, k}^{(\alpha)})_{-1 \leq k \leq r_\infty - 3}$ are independent of q_j gives:

$$\forall k \in \llbracket 0, r_\infty - 4 \rrbracket : \sum_{i=1}^g (\partial_{q_j} \mu_i^{(\alpha)}) q_i^k = -k \mu_j^{(\alpha)} q_j^{k-1} \quad (\text{E-2})$$

Thus

$$\sum_{k=0}^{r_\infty - 4} \sum_{i=1}^g H_{\infty, k} q_i^k \partial_{q_j} \mu_i^{(\alpha)} = -\mu_j^{(\alpha)} \left(\sum_{k=0}^{r_\infty - 4} k H_{\infty, k} q_j^{k-1} \right) \quad (\text{E-3})$$

so that the lemma is proved. \square

We may now provide an alternative expression for $\mathcal{L}_\alpha[p_j]$:

Proposition E.1. *Let $j \in \llbracket 1, g \rrbracket$, we have an alternative expression for $\mathcal{L}_\alpha[p_j]$:*

$$\begin{aligned} \mathcal{L}_\alpha[p_j] &= \hbar \sum_{i \neq j} \frac{(\mu_i^{(\alpha)} + \mu_j^{(\alpha)})(p_i - p_j)}{(q_j - q_i)^2} + \frac{\hbar}{2} \sum_{\substack{(r,s) \in \llbracket 1, g \rrbracket^2 \\ r \neq s}} \frac{(p_s - p_r)(\partial_{q_j} \mu_r^{(\alpha)} + \partial_{q_j} \mu_s^{(\alpha)})}{q_s - q_r} \\ &\quad - \mu_j^{(\alpha)} \left(\tilde{P}'_2(q_j) - p_j \tilde{P}'_1(q_j) \right) - \sum_{r=1}^g (\partial_{q_j} \mu_r^{(\alpha)}) \left(\tilde{P}_2(q_r) + p_r^2 - \tilde{P}_1(q_r) p_r \right) \\ &\quad + \hbar \nu_{\infty, -1}^{(\alpha)} p_j + \hbar \sum_{k=1}^{r_\infty - 1} k c_{\infty, k}^{(\alpha)} q_j^{k-1} \end{aligned} \quad (\text{E-4})$$

Proof. Using Lemma E.1, the expression (5-5) for $\mathcal{L}[p_j]$ becomes:

$$\begin{aligned} \mathcal{L}_\alpha[p_j] &= \hbar \sum_{i \neq j} \frac{(\mu_i^{(\alpha)} + \mu_j^{(\alpha)})(p_i - p_j)}{(q_j - q_i)^2} + \mu_j^{(\alpha)} \left(p_j \tilde{P}'_1(q_j) - \tilde{P}'_2(q_j) \right) + \hbar \nu_{\infty, -1}^{(\alpha)} p_j \\ &\quad + \hbar \sum_{k=1}^{r_\infty - 1} k c_{\infty, k}^{(\alpha)} q_j^{k-1} - \sum_{i=1}^g (\partial_{q_j} \mu_i^{(\alpha)}) \sum_{k=0}^{r_\infty - 4} H_{\infty, k} q_i^k \end{aligned} \quad (\text{E-5})$$

We now use (5-3) to get

$$\begin{aligned} \mathcal{L}_\alpha[p_j] &= \hbar \sum_{i \neq j} \frac{(\mu_i^{(\alpha)} + \mu_j^{(\alpha)})(p_i - p_j)}{(q_j - q_i)^2} + \mu_j^{(\alpha)} \left(p_j \tilde{P}'_1(q_j) - \tilde{P}'_2(q_j) \right) \\ &\quad + \hbar \nu_{\infty, -1}^{(\alpha)} p_j + \hbar \sum_{k=1}^{r_\infty - 1} k c_{\infty, k}^{(\alpha)} q_j^{k-1} \\ &\quad - \sum_{i=1}^g (\partial_{q_j} \mu_i^{(\alpha)}) \left[p_i^2 - \tilde{P}_1(q_i) p_i + \tilde{P}_2(q_i) + \hbar \sum_{r \neq i} \frac{p_r - p_i}{q_i - q_r} \right] \\ &= \hbar \sum_{i \neq j} \frac{(\mu_i^{(\alpha)} + \mu_j^{(\alpha)})(p_i - p_j)}{(q_j - q_i)^2} + \mu_j^{(\alpha)} \left(p_j \tilde{P}'_1(q_j) - \tilde{P}'_2(q_j) \right) \\ &\quad + \hbar \nu_{\infty, -1}^{(\alpha)} p_j + \hbar \sum_{k=1}^{r_\infty - 1} k c_{\infty, k}^{(\alpha)} q_j^{k-1} - \sum_{i=1}^g (\partial_{q_j} \mu_i^{(\alpha)}) \left(p_i^2 - \tilde{P}_1(q_i) p_i + \tilde{P}_2(q_i) \right) \\ &\quad + \hbar \sum_{i=1}^g (\partial_{q_j} \mu_i^{(\alpha)}) \sum_{r \neq i} \frac{p_r - p_i}{q_r - q_i} \end{aligned} \quad (\text{E-6})$$

The last sums may be split into a symmetric and anti-symmetric part: $\partial_{q_j} \mu_i^{(\alpha)} = \frac{1}{2}(\partial_{q_j} \mu_i^{(\alpha)} - \partial_{q_j} \mu_i^{(\alpha)}) + \frac{1}{2}(\partial_{q_j} \mu_i^{(\alpha)} + \partial_{q_j} \mu_i^{(\alpha)})$. The term involving $\partial_{q_j} \mu_i^{(\alpha)} - \partial_{q_j} \mu_i^{(\alpha)}$ is trivially zero because the sum is anti-symmetric so that we end up with

$$\begin{aligned} \mathcal{L}_\alpha[p_j] &= \hbar \sum_{i \neq j} \frac{(\mu_i^{(\alpha)} + \mu_j^{(\alpha)})(p_i - p_j)}{(q_j - q_i)^2} - \mu_j^{(\alpha)} \left(\tilde{P}'_2(q_j) - p_j \tilde{P}'_1(q_j) \right) \\ &\quad + \hbar \nu_{\infty, -1}^{(\alpha)} p_j + \hbar \sum_{k=1}^{r_\infty - 1} k c_{\infty, k}^{(\alpha)} q_j^{k-1} - \sum_{i=1}^g (\partial_{q_j} \mu_i^{(\alpha)}) \left(p_i^2 - \tilde{P}_1(q_i) p_i + \tilde{P}_2(q_i) \right) \\ &\quad + \frac{\hbar}{2} \sum_{i=1}^g \sum_{r \neq i} \frac{(p_r - p_i)(\partial_{q_j} \mu_i^{(\alpha)} + \partial_{q_j} \mu_r^{(\alpha)})}{q_r - q_i} \end{aligned} \quad (\text{E-7})$$

proving Proposition E.1. □

E.2 Proof of the Theorem 5.1

We may now proceed to the proof of Theorem 5.1. We recall that the Hamiltonian is given by:

$$\text{Ham}^{(\alpha)}(\mathbf{q}, \mathbf{p}) = -\frac{\hbar}{2} \sum_{\substack{(i,j) \in \llbracket 1, g \rrbracket^2 \\ i \neq j}} \frac{(\mu_i^{(\alpha)} + \mu_j^{(\alpha)})(p_i - p_j)}{q_i - q_j} - \hbar \sum_{j=1}^g (\nu_{\infty, 0}^{(\alpha)} p_j + \nu_{\infty, -1}^{(\alpha)} q_j p_j)$$

$$+ \sum_{j=1}^g \mu_j^{(\alpha)} \left(p_j^2 - \tilde{P}_1(q_j)p_j + \tilde{P}_2(q_j) \right) - \hbar \sum_{j=1}^g \sum_{k=0}^{r_\infty-1} c_{\infty,k}^{(\alpha)} q_j^k \quad (\text{E-8})$$

A straightforward computation from (5-6) and from the fact that the $(\nu_{\infty,k}^{(\alpha)})_{-1 \leq k \leq r_\infty-3}$ and $(c_{\infty,k}^{(\alpha)})_{1 \leq k \leq r_\infty-1}$ are independent of q_j provides

$$\begin{aligned} -\frac{\partial \text{Ham}^{(\alpha)}(\mathbf{q}, \mathbf{p})}{\partial q_j} &= \hbar \sum_{\substack{i \in \llbracket 1, g \rrbracket \\ i \neq j}} \frac{(\mu_i^{(\alpha)} + \mu_j^{(\alpha)})(p_i - p_j)}{(q_i - q_j)^2} + \frac{\hbar}{2} \sum_{\substack{(r,s) \in \llbracket 1, g \rrbracket^2 \\ r \neq s}} \frac{(\partial_{q_j} \mu_r^{(\alpha)} + \partial_{q_j} \mu_s^{(\alpha)})(p_r - p_s)}{q_r - q_s} \\ &\quad + \hbar \nu_{\infty,-1}^{(\alpha)} p_j - \sum_{i=1}^g \partial_{q_j} (\mu_i^{(\alpha)}) \left(p_i^2 - \tilde{P}_1(q_i)p_i + \tilde{P}_2(q_i) \right) \\ &\quad - \mu_j^{(\alpha)} \left(\tilde{P}_2'(q_j) - p_j \tilde{P}_1'(q_j) \right) + \hbar \sum_{k=1}^{r_\infty-1} k c_{\infty,k}^{(\alpha)} q_j^{k-1} \\ &\stackrel{\text{Prop. (E.1)}}{=} \mathcal{L}_\alpha[p_j] \end{aligned} \quad (\text{E-9})$$

Similarly a direct computation using the fact that $(\nu_{\infty,k}^{(\alpha)})_{-1 \leq k \leq r_\infty-3}$ and $(c_{\infty,k}^{(\alpha)})_{1 \leq k \leq r_\infty-1}$ and $(\mu_i^{(\alpha)})_{1 \leq i \leq g}$ are independent of p_j gives:

$$\frac{\partial \text{Ham}^{(\alpha)}(\mathbf{q}, \mathbf{p})}{\partial p_j} = -\hbar \sum_{\substack{i \in \llbracket 1, g \rrbracket \\ i \neq j}} \frac{\mu_i^{(\alpha)} + \mu_j^{(\alpha)}}{q_j - q_i} - \hbar \nu_{\infty,0}^{(\alpha)} - \hbar \nu_{\infty,-1}^{(\alpha)} q_j + \mu_j^{(\alpha)} \left(2p_j - \tilde{P}_1(q_j) \right) \quad (\text{E-10})$$

which is exactly $\mathcal{L}_\alpha[q_j]$ given by (5-1).

The last step is to verify that from Propositions 4.2 and 5.1:

$$\begin{aligned} &\sum_{j=1}^g \mu_j^{(\alpha)} \left(p_j^2 - \tilde{P}_1(q_j)p_j + \tilde{P}_2(q_j) + \hbar \sum_{i \neq j} \frac{p_i - p_j}{q_j - q_i} \right) \\ &= \left(\mu_1^{(\alpha)}, \dots, \mu_g^{(\alpha)} \right) \begin{pmatrix} p_1^2 - \tilde{P}_1(q_1)p_1 + \tilde{P}_2(q_1) + \hbar \sum_{i \neq 1} \frac{p_i - p_1}{q_1 - q_i} \\ \vdots \\ p_g^2 - \tilde{P}_1(q_g)p_g + \tilde{P}_2(q_g) + \hbar \sum_{i \neq g} \frac{p_i - p_g}{q_g - q_i} \end{pmatrix} \\ &= \left(\mu_1^{(\alpha)}, \dots, \mu_g^{(\alpha)} \right) (V_\infty)^t \begin{pmatrix} H_{\infty,0} \\ \vdots \\ H_{\infty,r_\infty-4} \end{pmatrix} = \left(\nu_{\infty,1}^{(\alpha)} \quad \dots \quad \nu_{\infty,r_\infty-3}^{(\alpha)} \right) \begin{pmatrix} H_{\infty,0} \\ \vdots \\ H_{\infty,r_\infty-4} \end{pmatrix} \\ &= \sum_{k=0}^{r_\infty-4} \nu_{\infty,k+1}^{(\alpha)} H_{\infty,k} \end{aligned} \quad (\text{E-11})$$

so that (E-8) becomes

$$\text{Ham}^{(\alpha)}(\mathbf{q}, \mathbf{p}) = \sum_{k=0}^{r_\infty-4} \nu_{\infty,k+1}^{(\alpha)} H_{\infty,k} - \hbar \sum_{j=1}^g \sum_{k=0}^{r_\infty-1} c_{\infty,k}^{(\alpha)} q_j^k - \hbar \nu_{\infty,0}^{(\alpha)} \sum_{j=1}^g p_j - \hbar \nu_{\infty,-1}^{(\alpha)} \sum_{j=1}^g q_j p_j \quad (\text{E-12})$$

F Proofs of the identities involving elementary symmetric polynomials

Let us prove Lemma 6.1. By definition we have

$$\prod_{j=1}^g (\lambda - x_j) = \sum_{i=0}^g (-1)^i e_i(\{x_1, \dots, x_g\}) \lambda^{g-i} \quad (\text{F-1})$$

Taking the derivative relatively to x_m provides:

$$\prod_{j \neq m} (\lambda - x_j) = \sum_{i=0}^g (-1)^{i-1} \frac{\partial e_i(\{x_1, \dots, x_g\})}{\partial x_m} \lambda^{g-i} \quad (\text{F-2})$$

Thus, $(-1)^{i-1} \frac{\partial e_i(\{x_1, \dots, x_g\})}{\partial x_m}$ is the coefficient of order λ^{g-i} of $\prod_{j \neq m} (\lambda - x_j)$. However, the last quantity is also given by

$$\begin{aligned} \prod_{j \neq m} (\lambda - x_j) &= \frac{\prod_{j=1}^g (\lambda - x_j)}{\lambda - x_m} = \left(\sum_{r=0}^g (-1)^r e_r(\{x_1, \dots, x_g\}) \lambda^{g-r} \right) \left(\sum_{s=0}^{\infty} x_m^s \lambda^{-s-1} \right) \\ &= \sum_{r=0}^g \sum_{s=0}^{\infty} (-1)^r e_r(\{x_1, \dots, x_g\}) x_m^s \lambda^{g-r-s-1} \end{aligned} \quad (\text{F-3})$$

Identifying the coefficient of order λ^{g-i} provides

$$(-1)^{i-1} \frac{\partial e_i(\{x_1, \dots, x_g\})}{\partial x_m} = \sum_{r=0}^{i-1} (-1)^r e_r(\{x_1, \dots, x_g\}) x_m^{i-1-r} \quad (\text{F-4})$$

proving the lemma.

Let us now prove Proposition 6.1. From, Lemma 6.1, we get that both polynomials take the same value at x_m (the value is $\frac{\partial e_i(\{x_1, \dots, x_g\})}{\partial x_m}$) for any $m \in \llbracket 1, g \rrbracket$. Since both sides are obviously polynomials of order at most $g-1$, we immediately get that they are equal.

Finally, let us prove Corollary 6.1. By definition

$$\begin{aligned} Q(\lambda) &= - \sum_{i=1}^g p_i \prod_{j \neq i} \frac{\lambda - q_j}{q_j - q_i} = - \sum_{i=1}^g P_i \left(\sum_{k=1}^g \frac{\partial e_i(\{q_1, \dots, q_g\})}{\partial q_k} \prod_{j \neq k} \frac{\lambda - q_j}{q_k - q_j} \right) \\ &\stackrel{\text{Prop 6.1}}{=} - \sum_{i=1}^g P_i \sum_{j=0}^{i-1} (-1)^j Q_{i-j-1} \lambda^j \\ &= \sum_{j=0}^{g-1} (-1)^{j-1} \left(\sum_{i=j+1}^g P_i Q_{i-j-1} \right) \lambda^j \end{aligned} \quad (\text{F-5})$$

ending the proof.

Let us now prove (6-11). We have for all $j \in \llbracket 1, n \rrbracket$:

$$\sum_{i=0}^{n-1} (-1)^{n-1-i} e_{n-1-i}(\{x_1, \dots, x_n\} \setminus \{x_j\}) \lambda^i = \prod_{k \neq i} (\lambda - q_k) = \frac{\prod_{k=1}^n (\lambda - x_k)}{(\lambda - x_j)}$$

$$\begin{aligned}
&= \left(\sum_{m=0}^g (-1)^{n-m} e_{n-m}(\{x_1, \dots, x_n\}) \lambda^m \right) \left(\sum_{p=0}^{\infty} x_j^p \lambda^{-1-p} \right) \\
&= \sum_{m=0}^g \sum_{p=0}^{\infty} (-1)^{n-m} e_{n-m}(\{x_1, \dots, x_n\}) x_j^p \lambda^{m-1-p} \\
&\stackrel{i=m-1-p}{=} \sum_{m=0}^g \sum_{i=-\infty}^{m-1} (-1)^{n-m} e_{n-m}(\{x_1, \dots, x_n\}) x_j^{m-1-i} \lambda^i \\
&= \sum_{i=-\infty}^{g-1} \sum_{m=i+1}^g (-1)^{n-m} e_{n-m}(\{x_1, \dots, x_n\}) x_j^{m-1-i} \lambda^i
\end{aligned} \tag{F-6}$$

Identifying the coefficient λ^i for $i \in \mathbb{Z}$ leads to

$$\begin{aligned}
(-1)^{n-1-i} e_{n-1-i}(\{x_1, \dots, x_n\} \setminus \{x_j\}) &= \sum_{m=i+1}^g (-1)^{n-m} e_{n-m}(\{x_1, \dots, x_n\}) x_j^{m-1-i}, \forall i \in \llbracket 0, n-1 \rrbracket \\
0 &= \sum_{m=i+1}^g (-1)^{n-m} e_{n-m}(\{x_1, \dots, x_n\}) x_j^{m-1-i}, \forall i < 0
\end{aligned} \tag{F-7}$$

which is equivalent to

$$\begin{aligned}
e_{n-i}(\{x_1, \dots, x_n\} \setminus \{x_j\}) &= \sum_{m=i}^g (-1)^{i-m} e_{n-m}(\{x_1, \dots, x_n\}) x_j^{m-i}, \forall i \in \llbracket 1, n \rrbracket \\
0 &= \sum_{m=i}^g (-1)^{n-m} e_{n-m}(\{x_1, \dots, x_n\}) x_j^{m-i}, \forall i \leq 0
\end{aligned} \tag{F-8}$$

Let us now prove Lemma 6.2. We have the trivial identity for large λ :

$$\begin{aligned}
\sum_{M=0}^{\infty} x_i^M \lambda^{-M-1} &= \frac{1}{\lambda - x_i} = \frac{\prod_{j=1}^n (\lambda - x_j)}{(\lambda - x_i) \prod_{j=1}^n (\lambda - x_j)} \\
&= \sum_{m=0}^n \sum_{r=0}^{\infty} \sum_{j=1}^{\infty} (-1)^{n-m} e_{n-m}(\{x_1, \dots, x_n\}) h_r(\{x_1, \dots, x_n\}) x_i^{j-1} \lambda^{m-n-r-j}
\end{aligned} \tag{F-9}$$

Identifying the coefficient of order λ^{-M-1} gives

$$x_i^M = \sum_{m=0}^n \sum_{j=1}^{\infty} (-1)^{n-m} e_{n-m}(\{x_1, \dots, x_n\}) h_{m-n+M-j+1}(\{x_1, \dots, x_n\}) x_i^{j-1} \tag{F-10}$$

but only terms with $m - n + M - j + 1 \geq 0$, i.e. $m \geq \text{Max}(j, j + n - 1 - M)$ contribute. In particular, since $j \leq m \leq n$, we also have $j \leq n$ ending the proof.

Let us now prove Proposition 6.2. Let $M \geq 0$, the system

$$\begin{pmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ 1 & x_2 & \dots & x_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \dots & x_n^{n-1} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix} = \begin{pmatrix} x_1^M \\ x_2^M \\ \vdots \\ x_n^M \end{pmatrix} \tag{F-11}$$

is equivalent to say, by inverting the Vandermonde matrix, that for all $i \in \llbracket 1, n \rrbracket$:

$$C_i = \sum_{j=1}^n \frac{(-1)^{n-i} e_{n-i}(\{x_1, \dots, x_n\} \setminus \{x_j\})}{\prod_{m \neq j} (x_j - x_m)} x_j^M \quad (\text{F-12})$$

The system is also equivalent to, for all $i \in \llbracket 1, n \rrbracket$:

$$\sum_{j=1}^n x_i^{j-1} C_j = x_i^M \quad (\text{F-13})$$

Lemma 6.2 implies that the last identity is equivalent to, for all $j \in \llbracket 1, n \rrbracket$

$$C_j = \sum_{m=\text{Max}(j, j+n-1-M)}^n (-1)^{n-m} e_{n-m}(\{x_1, \dots, x_n\}) h_{M+m-j-n+1}(\{x_1, \dots, x_n\}) \quad (\text{F-14})$$

Identifying (F-12) and (F-13) finally proves Proposition 6.2.

G Proof of Lemma 6.3

Let $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ be the canonical $2g$ symplectic matrix and define

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{with} \quad \begin{cases} A_{i,j} &= \frac{\partial Q_i}{\partial \tilde{Q}_j} \\ B_{i,j} &= \frac{\partial Q_i}{\partial \tilde{P}_j} = 0 \\ C_{i,j} &= \frac{\partial P_i}{\partial \tilde{Q}_j} = \sum_{r=1}^g \tilde{P}_r \frac{\partial^2 f_r(\alpha^{-1}Q_1 + \beta, \dots, \alpha^{-1}Q_g + \beta)}{\partial \tilde{Q}_j \partial Q_i} - \alpha^{-1} h'(\alpha^{-1}Q_i + \beta) \frac{\partial Q_i}{\partial \tilde{Q}_j} \\ D_{i,j} &= \frac{\partial P_i}{\partial \tilde{P}_j} = \frac{\partial f_j(\alpha^{-1}Q_1 + \beta, \dots, \alpha^{-1}Q_g + \beta)}{\partial Q_i} = \frac{\partial \tilde{Q}_j}{\partial Q_i} \end{cases} \quad (\text{G-1})$$

The change of coordinates is symplectic if and only if $M^t J M = J$ which is equivalent to prove that $A^t D = I$ and $(A^t C)$ is a symmetric matrix. We have for all $(i, j) \in \llbracket 1, g \rrbracket^2$:

$$(A^t D)_{i,j} = \sum_{k=1}^g A_{k,i} D_{k,j} = \sum_{k=1}^g \frac{\partial Q_k}{\partial \tilde{Q}_i} \frac{\partial \tilde{Q}_j}{\partial Q_k} = \frac{\partial \tilde{Q}_j}{\partial \tilde{Q}_i} = \delta_{i,j} \quad (\text{G-2})$$

and

$$\begin{aligned} (A^t C)_{i,j} &= \sum_{k=1}^g A_{k,i} C_{k,j} = \sum_{r=1}^g \tilde{P}_r \sum_{k=1}^g \frac{\partial Q_k}{\partial \tilde{Q}_i} \frac{\partial^2 f_r(\alpha^{-1}Q_1 + \beta, \dots, \alpha^{-1}Q_g + \beta)}{\partial \tilde{Q}_j \partial Q_k} \\ &\quad - \alpha^{-1} \sum_{k=1}^g \frac{\partial Q_k}{\partial \tilde{Q}_i} \frac{\partial Q_k}{\partial \tilde{Q}_j} h'(\alpha^{-1}Q_k + \beta) \\ &= \sum_{r=1}^g \tilde{P}_r \frac{\partial^2 f_r(\alpha^{-1}Q_1 + \beta, \dots, \alpha^{-1}Q_g + \beta)}{\partial \tilde{Q}_i \partial \tilde{Q}_j} - \alpha^{-1} \sum_{k=1}^g \frac{\partial Q_k}{\partial \tilde{Q}_i} \frac{\partial Q_k}{\partial \tilde{Q}_j} h'(\alpha^{-1}Q_k + \beta) \\ (A^t C)_{j,i} &= \sum_{k=1}^g A_{k,j} C_{k,i} = \sum_{r=1}^g \tilde{P}_r \sum_{k=1}^g \frac{\partial Q_k}{\partial \tilde{Q}_j} \frac{\partial^2 f_r(\alpha^{-1}Q_1 + \beta, \dots, \alpha^{-1}Q_g + \beta)}{\partial \tilde{Q}_i \partial Q_k} \\ &\quad - \alpha^{-1} \sum_{k=1}^g \frac{\partial Q_k}{\partial \tilde{Q}_j} \frac{\partial Q_k}{\partial \tilde{Q}_i} h'(\alpha^{-1}Q_k + \beta) \end{aligned}$$

$$= \sum_{r=1}^g \tilde{P}_r \frac{\partial^2 f_r(\alpha^{-1}Q_1 + \beta, \dots, \alpha^{-1}Q_g + \beta)}{\partial \tilde{Q}_j \partial \tilde{Q}_i} - \alpha^{-1} \sum_{k=1}^g \frac{\partial Q_k}{\partial \tilde{Q}_j} \frac{\partial Q_k}{\partial \tilde{Q}_i} h'(\alpha^{-1}Q_k + \beta) \quad (\text{G-3})$$

so that $(A^t B)_{i,j} = (A^t B)_{j,i}$ proving the lemma.

H Proof of Theorem 6.1

The proof is based on the computation of each term appearing in the Hamiltonian. Let us compute for $i \in \llbracket 1, g \rrbracket$:

$$\begin{aligned}
A_i &:= \hbar \sum_{j=1}^g \sum_{r \neq j} \frac{(-1)^{g-i} e_{g-i}(\{q_1, \dots, q_g\} \setminus \{q_j\}) p_r - p_j}{\prod_{a \neq j} (q_j - q_a)} \frac{p_r - p_j}{q_j - q_r} \\
&= -\hbar \sum_{j=1}^g \sum_{r \neq j} \frac{(-1)^{g-i} e_{g-i}(\{q_1, \dots, q_g\} \setminus \{q_j\}) p_r - p_j}{\prod_{a \neq j} (q_j - q_a)} \frac{p_r - p_j}{q_r - q_j} \\
(6.1) \quad &\stackrel{=}{=} -\hbar \sum_{j=1}^g \sum_{r \neq j} \frac{(-1)^{g-i} e_{g-i}(\{q_1, \dots, q_g\} \setminus \{q_j\})}{\prod_{a \neq j} (q_j - q_a)} \sum_{k=1}^g P_k \sum_{m=0}^{k-1} (-1)^m e_{k-1-m}(\{q_1, \dots, q_g\}) \frac{q_r^m - q_j^m}{q_r - q_j} \\
&= -\hbar \sum_{j=1}^g \sum_{r \neq j} \frac{(-1)^{g-i} e_{g-i}(\{q_1, \dots, q_g\} \setminus \{q_j\})}{\prod_{a \neq j} (q_j - q_a)} \sum_{k=1}^g P_k \sum_{m=0}^{k-1} (-1)^m e_{k-1-m}(\{q_1, \dots, q_g\}) \sum_{s=0}^{m-1} q_r^s q_j^{m-1-s} \\
&= -\hbar \sum_{j=1}^g \sum_{r=1}^g \frac{(-1)^{g-i} e_{g-i}(\{q_1, \dots, q_g\} \setminus \{q_j\})}{\prod_{a \neq j} (q_j - q_a)} \sum_{k=1}^g P_k \sum_{m=0}^{k-1} (-1)^m e_{k-1-m}(\{q_1, \dots, q_g\}) \sum_{s=0}^{m-1} q_r^s q_j^{m-1-s} \\
&\quad + \hbar \sum_{j=1}^g \frac{(-1)^{g-i} e_{g-i}(\{q_1, \dots, q_g\} \setminus \{q_j\})}{\prod_{a \neq j} (q_j - q_a)} \sum_{k=1}^g P_k \sum_{m=0}^{k-1} (-1)^m e_{k-1-m}(\{q_1, \dots, q_g\}) \sum_{s=0}^{m-1} q_j^{m-1} \\
&= -\hbar \sum_{k=1}^g \sum_{m=0}^{k-1} \sum_{s=0}^{m-1} (-1)^m P_k e_{k-1-m}(\{q_1, \dots, q_g\}) \sum_{r=1}^g q_r^s \sum_{j=1}^g \frac{(-1)^{g-i} e_{g-i}(\{q_1, \dots, q_g\} \setminus \{q_j\})}{\prod_{a \neq j} (q_j - q_a)} q_j^{m-1-s} \\
&\quad + \hbar \sum_{k=1}^g \sum_{m=0}^{k-1} \sum_{s=0}^{m-1} (-1)^m P_k e_{k-1-m}(\{q_1, \dots, q_g\}) \sum_{j=1}^g \frac{(-1)^{g-i} e_{g-i}(\{q_1, \dots, q_g\} \setminus \{q_j\})}{\prod_{a \neq j} (q_j - q_a)} q_j^{m-1} \\
(6.2) \quad &\stackrel{=}{=} -\hbar \sum_{k=1}^g \sum_{m=0}^{k-1} \sum_{s=0}^{m-1} (-1)^m P_k e_{k-1-m}(\{q_1, \dots, q_g\}) \sum_{r=1}^g q_r^s \delta_{i, m-s} \\
&\quad + \hbar \sum_{k=1}^g \sum_{m=0}^{k-1} \sum_{s=0}^{m-1} (-1)^m P_k e_{k-1-m}(\{q_1, \dots, q_g\}) \delta_{i, m} \\
&= -\hbar \sum_{k=1}^g \sum_{m=0}^{k-1} \sum_{s=0}^{m-1} (-1)^m P_k e_{k-1-m}(\{q_1, \dots, q_g\}) S_s(\{q_1, \dots, q_g\}) \delta_{s, m-i} \\
&\quad + \hbar \sum_{k=1}^g \sum_{m=0}^{k-1} (-1)^m m P_k e_{k-1-m}(\{q_1, \dots, q_g\}) \delta_{i, m} \\
&= -\hbar \sum_{k=i+1}^g \sum_{m=i}^{k-1} (-1)^m P_k e_{k-1-m}(\{q_1, \dots, q_g\}) S_{m-i}(\{q_1, \dots, q_g\})
\end{aligned}$$

$$\begin{aligned}
& + \hbar \sum_{k=1}^g \delta_{k-1 \geq i} (-1)^i P_k e_{k-1-i}(\{q_1, \dots, q_g\}) \\
= & -\hbar \sum_{k=i+1}^g \sum_{m=i}^{k-1} (-1)^m P_k e_{k-1-m}(\{q_1, \dots, q_g\}) S_{m-i}(\{q_1, \dots, q_g\}) \\
& + \hbar \sum_{k=i+1}^g (-1)^i P_k e_{k-1-i}(\{q_1, \dots, q_g\}) \\
= & -\hbar \sum_{k=i+1}^g \sum_{m=i+1}^{k-1} (-1)^m P_k e_{k-1-m}(\{q_1, \dots, q_g\}) S_{m-i}(\{q_1, \dots, q_g\}) \\
& - \hbar \sum_{k=i+1}^g (-1)^i P_k e_{k-1-i}(\{q_1, \dots, q_g\}) \\
& + \hbar \sum_{k=i+1}^g (-1)^i P_k e_{k-1-i}(\{q_1, \dots, q_g\}) \\
= & -\hbar \sum_{k=i+1}^g \sum_{m=i+1}^{k-1} (-1)^m P_k Q_{k-1-m} S_{m-i}(\{q_1, \dots, q_g\}) - \hbar \sum_{k=i+1}^g (-1)^i (g-i) P_k Q_{k-1-i} \quad (\text{H-1})
\end{aligned}$$

$$\begin{aligned}
B_i & := - \sum_{j=1}^g \frac{(-1)^{g-i} e_{g-i}(\{q_1, \dots, q_g\} \setminus \{q_j\})}{\prod_{a \neq j} (q_j - q_a)} \tilde{P}_1(q_j) p_j \\
& = \sum_{j=1}^g \sum_{s=0}^{g+1} \sum_{k=1}^g P_k \sum_{r=0}^{k-1} (-1)^r e_{k-1-r}(\{q_1, \dots, q_g\}) \frac{(-1)^{g-i} e_{g-i}(\{q_1, \dots, q_g\} \setminus \{q_j\})}{\prod_{a \neq j} (q_j - q_a)} t_{\infty, 2s+2} q_j^{s+r} \\
& = \sum_{k=1}^g \sum_{r=0}^{k-1} \sum_{s=0}^{g+1} (-1)^r P_k e_{k-1-r}(\{q_1, \dots, q_g\}) t_{\infty, 2s+2} \sum_{j=1}^g \frac{(-1)^{g-i} e_{g-i}(\{q_1, \dots, q_g\} \setminus \{q_j\})}{\prod_{a \neq j} (q_j - q_a)} q_j^{s+r} \\
& = \sum_{k=1}^g \sum_{r=0}^{k-1} \sum_{s=0}^{g-1-r} (-1)^r P_k e_{k-1-r}(\{q_1, \dots, q_g\}) t_{\infty, 2s+2} \sum_{j=1}^g \frac{(-1)^{g-i} e_{g-i}(\{q_1, \dots, q_g\} \setminus \{q_j\})}{\prod_{a \neq j} (q_j - q_a)} q_j^{s+r} \\
& \quad + \sum_{k=1}^g \sum_{r=0}^{k-1} \sum_{s=g-r}^{g+1} (-1)^r P_k e_{k-1-r}(\{q_1, \dots, q_g\}) t_{\infty, 2s+2} \sum_{j=1}^g \frac{(-1)^{g-i} e_{g-i}(\{q_1, \dots, q_g\} \setminus \{q_j\})}{\prod_{a \neq j} (q_j - q_a)} q_j^{s+r} \\
& \stackrel{(6.2)}{=} \sum_{k=1}^g \sum_{r=0}^{k-1} \sum_{s=0}^{g-1-r} (-1)^r P_k Q_{k-1-r}(\{q_1, \dots, q_g\}) t_{\infty, 2s+2} \delta_{s+r+1, i} \\
& \quad + \sum_{k=1}^g \sum_{r=0}^{k-1} \sum_{s=g-r}^{g+1} (-1)^r P_k Q_{k-1-r} t_{\infty, 2s+2} \sum_{m=i}^g (-1)^{g-m} Q_{g-m} h_{r+s+m-i-g+1}(\{q_1, \dots, q_g\}) \\
& = \sum_{k=1}^g \sum_{r=0}^{\text{Min}(k-1, i-1)} (-1)^r t_{\infty, 2i-2r} P_k Q_{k-1-r} \\
& \quad + \sum_{k=1}^g \sum_{r=0}^{k-1} \sum_{s=g-r}^{g+1} \sum_{m=i}^g (-1)^{g+r-m} t_{\infty, 2s+2} P_k Q_{k-1-r} Q_{g-m} h_{r+s+m-i-g+1}(\{q_1, \dots, q_g\}) \quad (\text{H-2})
\end{aligned}$$

$$\begin{aligned}
C_i &:= \sum_{j=1}^g \frac{(-1)^{g-i} e_{g-i}(\{q_1, \dots, q_g\} \setminus \{q_j\})}{\prod_{a \neq j} (q_j - q_a)} \tilde{P}_2(q_j) \\
&= \sum_{j=1}^g \sum_{r=g}^{2r_\infty-4} \tilde{P}_{\infty,r}^{(2)} \frac{(-1)^{g-i} e_{g-i}(\{q_1, \dots, q_g\} \setminus \{q_j\})}{\prod_{a \neq j} (q_j - q_a)} q_j^r \\
&\stackrel{(6.2)}{=} \sum_{r=g}^{2r_\infty-4} \sum_{m=i}^g (-1)^{g-m} \tilde{P}_{\infty,r}^{(2)} e_{g-m}(\{q_1, \dots, q_g\}) h_{r+m-i-g+1}(\{q_1, \dots, q_g\}) \\
&= \sum_{r=g}^{2r_\infty-4} \sum_{m=i}^g (-1)^{g-m} \tilde{P}_{\infty,r}^{(2)} Q_{g-m} h_{r+m-i-g+1}(\{q_1, \dots, q_g\}) \tag{H-3}
\end{aligned}$$

$$\begin{aligned}
E_i &:= \sum_{j=1}^g \frac{(-1)^{g-i} e_{g-i}(\{q_1, \dots, q_g\} \setminus \{q_j\})}{\prod_{a \neq j} (q_j - q_a)} p_j^2 \\
&= \sum_{j=1}^g \sum_{k_1=1}^g \sum_{k_2=1}^g P_{k_1} P_{k_2} \sum_{r_1=0}^{k_1-1} \sum_{r_2=0}^{k_2-1} (-1)^{r_1+r_2} Q_{k_1-1-r_1} Q_{k_2-1-r_2} \frac{(-1)^{g-i} e_{g-i}(\{q_1, \dots, q_g\} \setminus \{q_j\})}{\prod_{a \neq j} (q_j - q_a)} q_j^{r_1+r_2} \\
&= \sum_{k_1=1}^g \sum_{k_2=1}^g P_{k_1} P_{k_2} \sum_{r_1=0}^{k_1-1} \sum_{r_2=0}^{k_2-1} (-1)^{r_1+r_2} Q_{k_1-1-r_1} Q_{k_2-1-r_2} \sum_{j=1}^g \frac{(-1)^{g-i} e_{g-i}(\{q_1, \dots, q_g\} \setminus \{q_j\})}{\prod_{a \neq j} (q_j - q_a)} q_j^{r_1+r_2} \delta_{r_1+r_2 \leq g-1} \\
&\quad + \sum_{k_1=1}^g \sum_{k_2=1}^g P_{k_1} P_{k_2} \sum_{r_1=0}^{k_1-1} \sum_{r_2=0}^{k_2-1} (-1)^{r_1+r_2} Q_{k_1-1-r_1} Q_{k_2-1-r_2} \sum_{j=1}^g \frac{(-1)^{g-i} e_{g-i}(\{q_1, \dots, q_g\} \setminus \{q_j\})}{\prod_{a \neq j} (q_j - q_a)} q_j^{r_1+r_2} \delta_{r_1+r_2 \geq g} \\
&\stackrel{(6.2)}{=} \sum_{k_1=1}^g \sum_{k_2=1}^g P_{k_1} P_{k_2} \sum_{r_1=0}^{k_1-1} \sum_{r_2=0}^{k_2-1} (-1)^{r_1+r_2} Q_{k_1-1-r_1} Q_{k_2-1-r_2} \delta_{i, r_1+r_2+1} \delta_{r_1+r_2 \leq g-1} \\
&\quad + \sum_{k_1=1}^g \sum_{k_2=1}^g P_{k_1} P_{k_2} \sum_{r_1=0}^{k_1-1} \sum_{r_2=0}^{k_2-1} (-1)^{r_1+r_2} Q_{k_1-1-r_1} Q_{k_2-1-r_2} \delta_{r_1+r_2 \geq g} \sum_{m=i}^g (-1)^{g-m} Q_{g-m} h_{r_1+r_2+m-i-g+1}(\{q_1, \dots, q_g\}) \\
&= (-1)^{i-1} \sum_{k_1=1}^g \sum_{k_2=1}^g P_{k_1} P_{k_2} \sum_{r_1=\text{Max}(0, i-k_2)}^{\text{Min}(k_1-1, i-1)} Q_{k_1-1-r_1} Q_{k_2-i+r_1} \\
&\quad + \sum_{k_1=1}^g \sum_{k_2=1}^g P_{k_1} P_{k_2} \sum_{\substack{0 \leq r_1 \leq k_1-1 \\ 0 \leq r_2 \leq k_2-1 \\ r_1+r_2 \geq g}} (-1)^{r_1+r_2} Q_{k_1-1-r_1} Q_{k_2-1-r_2} \sum_{m=i}^g (-1)^{g-m} Q_{g-m} h_{r_1+r_2+m-i-g+1}(\{q_1, \dots, q_g\}) \tag{H-4}
\end{aligned}$$

We also have:

$$\begin{aligned}
-\hbar \sum_{i=1}^g p_i &= -\hbar \sum_{i=1}^g \sum_{m=1}^g P_m \frac{\partial e_m(\{q_1, \dots, q_g\})}{\partial q_i} \\
&\stackrel{\text{Lemma (6.1)}}{=} -\hbar \sum_{m=1}^g \sum_{i=1}^g \sum_{j=0}^{m-1} (-1)^j P_m e_{m-1-j}(\{q_1, \dots, q_g\}) q_i^j \\
&= -\hbar \sum_{m=1}^g \sum_{j=0}^{m-1} (-1)^j P_m e_{m-1-j}(\{q_1, \dots, q_g\}) \sum_{i=1}^g q_i^j \\
&= -\hbar \sum_{m=1}^g P_m \sum_{j=0}^{m-1} (-1)^j e_{m-1-j}(\{q_1, \dots, q_g\}) S_j(\{q_1, \dots, q_g\})
\end{aligned}$$

$$\begin{aligned}
& \stackrel{k=m-1}{=} -\hbar \sum_{k=0}^{g-1} P_{k+1} \sum_{j=0}^k (-1)^j e_{k-j}(\{q_1, \dots, q_g\}) S_j(\{q_1, \dots, q_g\}) \\
& \stackrel{(6-7)}{=} -\hbar \sum_{k=0}^{g-1} (g-k) Q_k P_{k+1}
\end{aligned} \tag{H-5}$$

and

$$\begin{aligned}
-\hbar \sum_{i=1}^g q_i P_i &= -\hbar \sum_{i=1}^g \sum_{m=1}^g q_i P_m \frac{\partial e_m(\{q_1, \dots, q_g\})}{\partial q_i} \\
& \stackrel{\text{Lemma (6.1)}}{=} -\hbar \sum_{m=1}^g \sum_{i=1}^g \sum_{j=0}^{m-1} (-1)^j P_m e_{m-1-j}(\{q_1, \dots, q_g\}) q_i^{j+1} \\
&= -\hbar \sum_{m=1}^g \sum_{j=0}^{m-1} (-1)^j P_m e_{m-1-j}(\{q_1, \dots, q_g\}) \sum_{i=1}^g q_i^{j+1} \\
&= -\hbar \sum_{m=1}^g P_m \sum_{j=0}^{m-1} (-1)^j e_{m-1-j}(\{q_1, \dots, q_g\}) S_{j+1}(\{q_1, \dots, q_g\}) \\
& \stackrel{k=m-1}{=} -\hbar \sum_{k=0}^{g-1} P_{k+1} \sum_{j=0}^k (-1)^j e_{k-j}(\{q_1, \dots, q_g\}) S_{j+1}(\{q_1, \dots, q_g\}) \\
& \stackrel{i=j+1}{=} -\hbar \sum_{k=0}^{g-1} P_{k+1} \sum_{i=1}^{k+1} (-1)^{i-1} e_{k+1-i}(\{q_1, \dots, q_g\}) S_i(\{q_1, \dots, q_g\}) \\
& \stackrel{(H-9)}{=} -\hbar \sum_{k=0}^{g-1} (k+1) P_{k+1} e_{k+1}(\{q_1, \dots, q_g\}) \\
&= -\hbar \sum_{k=0}^{g-1} (k+1) P_{k+1} Q_{k+1} \\
&= -\hbar \sum_{k=1}^g k P_k Q_k
\end{aligned} \tag{H-6}$$

where we have used a modified version of (6-7). Indeed, (6-7) implies that for any $k \in \llbracket 0, g-1 \rrbracket$:

$$\sum_{i=0}^{k+1} (-1)^i e_{k+1-i}(\{q_1, \dots, q_g\}) S_i(\{q_1, \dots, q_g\}) = (n-k-1) e_{k+1}(\{q_1, \dots, q_g\}) \tag{H-7}$$

Isolating the term $i=0$ and reminding that $S_0(\{q_1, \dots, q_g\}) = g$ gives:

$$\begin{aligned}
-(g-k-1) e_{k+1}(\{q_1, \dots, q_g\}) &= \sum_{i=0}^{k+1} (-1)^{i-1} e_{k+1-i}(\{q_1, \dots, q_g\}) S_i(\{q_1, \dots, q_g\}) \\
&= \sum_{i=1}^{k+1} (-1)^{i-1} e_{k+1-i}(\{q_1, \dots, q_g\}) S_i(\{q_1, \dots, q_g\}) - g e_{k+1}(\{q_1, \dots, q_g\})
\end{aligned} \tag{H-8}$$

i.e.

$$\sum_{i=1}^{k+1} (-1)^{i-1} e_{k+1-i}(\{q_1, \dots, q_g\}) S_i(\{q_1, \dots, q_g\}) = (k+1) e_{k+1}(\{q_1, \dots, q_g\}) \tag{H-9}$$

I Proof of Proposition 6.4

From Proposition 2.2 we have

$$[\tilde{A}_\alpha(\lambda)]_{1,2} = [A_\alpha(\lambda)]_{1,2} \left(\prod_{i=1}^g (\lambda - q_i) \right) \quad (\text{I-1})$$

Let us recall from (6-4) that:

$$[A_\alpha(\lambda)]_{1,2} = \nu_{\infty,-1}^{(\alpha)} \lambda + \nu_{\infty,0}^{(\alpha)} + \sum_{m=1}^g \nu_{\infty,m}^{(\alpha)} \lambda^{-m} + O(\lambda^{-g-1}) \quad (\text{I-2})$$

so that we get from (F-14):

$$\begin{aligned} [\tilde{A}_\alpha(\lambda)]_{1,2} &= \left(\sum_{m=-1}^g \nu_{\infty,m}^{(\alpha)} \lambda^{-m} + O(\lambda^{-m-1}) \right) \left(\sum_{r=0}^g (-1)^{g-r} Q_{g-r} \lambda^r \right) \\ &= \sum_{m=-1}^g \sum_{r=0}^g (-1)^{g-r} \nu_{\infty,m}^{(\alpha)} Q_{g-r} \lambda^{r-m} + O(\lambda^{-1}) \\ &= \sum_{m=-1}^g \sum_{r=\text{Max}(m,0)}^g (-1)^{g-r} \nu_{\infty,m}^{(\alpha)} Q_{g-r} \lambda^{r-m} + O(\lambda^{-1}) \\ &= \sum_{j=0}^{g+1} \left(\sum_{m=\text{Max}(-1,-j)}^{g-j} (-1)^{g-j-m} \nu_{\infty,m}^{(\alpha)} Q_{g-j-m} \right) \lambda^j + O(\lambda^{-1}) \end{aligned} \quad (\text{I-3})$$

Since we know that $[\tilde{A}_\alpha(\lambda)]_{1,2}$ is a polynomial in λ , we end up with

$$[\tilde{A}_\alpha(\lambda)]_{1,2} = \sum_{j=0}^{g+1} \left(\sum_{m=\text{Max}(-1,-j)}^{g-j} (-1)^{g-j-m} \nu_{\infty,m}^{(\alpha)} Q_{g-j-m} \right) \lambda^j \quad (\text{I-4})$$

Let us now compute $[\tilde{A}_\alpha(\lambda)]_{1,1}$. We have from Proposition 2.2:

$$[\tilde{A}_\alpha(\lambda)]_{1,1} = [A_\alpha(\lambda)]_{1,1} - \left(Q(\lambda) + \left(\frac{1}{2} t_{\infty,2r_\infty-2} \lambda + g_0 \right) \prod_{j=1}^g (\lambda - q_j) \right) [A_\alpha(\lambda)]_{1,2} \quad (\text{I-5})$$

Since we know that $[\tilde{A}_\alpha(\lambda)]_{1,1}$ is a polynomial in λ and using (4-18), we get

$$\begin{aligned} [\tilde{A}_\alpha(\lambda)]_{1,1} &= \sum_{i=0}^{r_\infty-1} c_{\infty,i}^{(\alpha)} \lambda^i - \left(Q(\lambda) + \left(\frac{1}{2} t_{\infty,2r_\infty-2} \lambda + g_0 \right) \sum_{s=0}^g (-1)^{g-s} Q_{g-s} \lambda^s \right) [A_\alpha(\lambda)]_{1,2} \\ &\stackrel{\text{Cor. 6.1}}{=} \sum_{i=0}^{r_\infty-1} c_{\infty,i}^{(\alpha)} \lambda^i \\ &\quad - \left(\sum_{j=0}^{g-1} (-1)^{j-1} \left(\sum_{r=j+1}^g P_r Q_{r-j-1} \right) \lambda^j \right) \left(\sum_{m=-1}^g \nu_{\infty,m}^{(\alpha)} \lambda^{-m} + O(\lambda^{-g-1}) \right) \\ &\quad - \left(\left(\frac{1}{2} t_{\infty,2r_\infty-2} \lambda + g_0 \right) \sum_{s=0}^g (-1)^{g-s} Q_{g-s} \lambda^s \right) \left(\sum_{m=-1}^g \nu_{\infty,m}^{(\alpha)} \lambda^{-m} + O(\lambda^{-g-1}) \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^{r_\infty-1} c_{\infty,i}^{(\alpha)} \lambda^i - \sum_{m=-1}^g \sum_{j=0}^{g-1} (-1)^{j-1} \nu_{\infty,m}^{(\alpha)} \left(\sum_{r=j+1}^g P_r Q_{r-j-1} \right) \lambda^{j-m} \\
&\quad - \left(\frac{1}{2} t_{\infty,2r_\infty-2} \lambda + g_0 \right) \sum_{m=-1}^g \sum_{s=0}^g (-1)^{g-s} Q_{g-s} \nu_{\infty,m}^{(\alpha)} \lambda^{s-m} + O(\lambda^{-1}) \\
\stackrel{i=j-m}{=} &\sum_{i=0}^{r_\infty-1} c_{\infty,i}^{(\alpha)} \lambda^i - \sum_{m=-1}^g \sum_{i=m}^{\text{Min}(g-1-m, g-1)} (-1)^{i+m-1} \nu_{\infty,m}^{(\alpha)} \left(\sum_{r=i+m+1}^g P_r Q_{r-i-m-1} \right) \lambda^i \\
&\quad - \left(\frac{1}{2} t_{\infty,2r_\infty-2} \lambda + g_0 \right) \sum_{i=0}^{g+1} \sum_{m=\text{Max}(-1, -i)}^{g-i} (-1)^{g-i-m} Q_{g-i-m} \nu_{\infty,m}^{(\alpha)} \lambda^i \\
&= \sum_{i=0}^{r_\infty-1} c_{\infty,i}^{(\alpha)} \lambda^i - \sum_{i=0}^g \sum_{m=\text{Max}(-1, -i)}^{g-1-i} (-1)^{i+m-1} \nu_{\infty,m}^{(\alpha)} \left(\sum_{r=i+m+1}^g P_r Q_{r-i-m-1} \right) \lambda^i \\
&\quad - \left(\frac{1}{2} t_{\infty,2r_\infty-2} \lambda + g_0 \right) \sum_{i=0}^{g+1} \sum_{m=\text{Max}(-1, -i)}^{g-i} (-1)^{g-i-m} Q_{g-i-m} \nu_{\infty,m}^{(\alpha)} \lambda^i
\end{aligned} \tag{I-6}$$

Let us now consider $[\tilde{A}_\alpha(\lambda)]_{2,2}$. The compatibility equation reads

$$\mathcal{L}_\alpha[\tilde{L}] = \hbar \partial_\lambda \tilde{A}_\alpha + [\tilde{A}_\alpha, \tilde{L}] \tag{I-7}$$

Taking the trace on both sides leads to

$$\mathcal{L}_\alpha[\text{Tr}(\tilde{L})] = \mathcal{L}_\alpha[\tilde{P}_1(\lambda)] = \hbar \partial_\lambda \text{Tr}(\tilde{A}_\alpha) \tag{I-8}$$

Since

$$\mathcal{L}_\alpha[\tilde{P}_1(\lambda)] = - \sum_{s=0}^{r_\infty-2} \mathcal{L}_\alpha[t_{\infty,2s+2}] \lambda^s = -\hbar \sum_{s=0}^{r_\infty-2} \alpha_{\infty,2s+2} \lambda^s \tag{I-9}$$

we get

$$[\tilde{A}_\alpha(\lambda)]_{2,2} = -[\tilde{A}_\alpha(\lambda)]_{1,1} - \sum_{s=0}^{r_\infty-2} \frac{1}{s+1} \alpha_{\infty,2s+2} \lambda^{s+1} + \tilde{c}_0 = -[\tilde{A}_\alpha(\lambda)]_{1,1} - \sum_{s=1}^{r_\infty-1} \frac{1}{s} \alpha_{\infty,2s} \lambda^s + \tilde{c}_0 \tag{I-10}$$

Let us now compute \tilde{c}_0 , we observe from (4-5) that

$$\text{Tr} A_\alpha(\lambda) = 2[A_\alpha(\lambda)]_{1,1} + \hbar \partial_\lambda [A_\alpha(\lambda)]_{1,2} + [A_\alpha(\lambda)]_{1,2} L(\lambda)_{2,2} \tag{I-11}$$

Moreover, from the gauge transformation we get:

$$\begin{aligned}
\text{Tr} \tilde{A}_\alpha(\lambda) &= \text{Tr} A_\alpha(\lambda) + \hbar \left(\prod_{j=1}^g (\lambda - q_j) \right) \mathcal{L}_\alpha \left[\frac{1}{\prod_{j=1}^g (\lambda - q_j)} \right] \\
&= \text{Tr} A_\alpha(\lambda) - \hbar \mathcal{L}_\alpha \left[\sum_{j=1}^g \log(\lambda - q_j) \right] \\
&= \text{Tr} A_\alpha(\lambda) + \hbar \sum_{j=1}^g \frac{\mathcal{L}_\alpha[q_j]}{\lambda - q_j}
\end{aligned} \tag{I-12}$$

But since $\tilde{A}_\alpha(\lambda)$ is a polynomial in λ , we may discard the last term and thus we end up with

$$\text{Tr} \tilde{A}_\alpha(\lambda) = \text{Tr} A_\alpha(\lambda) = 2[A_\alpha(\lambda)]_{1,1} + \hbar \partial_\lambda [A_\alpha(\lambda)]_{1,2} + [A_\alpha(\lambda)]_{1,2} L(\lambda)_{2,2} \tag{I-13}$$

The coefficient \tilde{c}_0 is thus the coefficients of λ^0 in the r.h.s.

$$\tilde{c}_0 = 2c_{\infty,0}^{(\alpha)} + \hbar\nu_{\infty,-1}^{(\alpha)} + \hbar g\nu_{\infty,-1}^{(\alpha)} \sum_{j=0}^{r_{\infty}-2} \tilde{P}_{\infty,j}^{(1)} \nu_{\infty,j}^{(\alpha)} = 2c_{\infty,0}^{(\alpha)} + \hbar(g+1)\nu_{\infty,-1}^{(\alpha)} - \sum_{j=0}^{r_{\infty}-2} t_{\infty,2j+2} \nu_{\infty,j}^{(\alpha)} \quad (\text{I-14})$$

where $\nu_{\infty,r_{\infty}-2}^{(\alpha)} = \sum_{j=1}^g \mu_j^{(\alpha)} q_j^{r_{\infty}-3}$. In the end

$$[\tilde{A}_{\alpha}(\lambda)]_{2,2} = -[\tilde{A}_{\alpha}(\lambda)]_{1,1} - \sum_{s=1}^{r_{\infty}-1} \frac{1}{s} \alpha_{\infty,2s} \lambda^s + 2c_{\infty,0}^{(\alpha)} + \hbar(g+1)\nu_{\infty,-1}^{(\alpha)} - \sum_{j=0}^{r_{\infty}-2} t_{\infty,2j+2} \nu_{\infty,j}^{(\alpha)} \quad (\text{I-15})$$

Let us now turn to $[\tilde{A}_{\alpha}(\lambda)]_{2,1}$. From the gauge transformation, we have denoting $P(\lambda) = \prod_{j=1}^g (\lambda - q_j)$:

$$\begin{aligned} [\tilde{A}_{\alpha}(\lambda)]_{2,1} &= ([A_{\alpha}(\lambda)]_{1,1} - [A_{\alpha}(\lambda)]_{2,2}) \left(\frac{Q(\lambda)}{P(\lambda)} + \frac{1}{2} t_{\infty,2r_{\infty}-2} \lambda + g_0 \right) + \frac{[A_{\alpha}(\lambda)]_{2,1}}{P(\lambda)} \\ &\quad - [A_{\alpha}(\lambda)]_{1,2} Q(\lambda) \left(\frac{Q(\lambda)}{P(\lambda)} + t_{\infty,2r_{\infty}-2} \lambda + 2g_0 \right) - \left(\frac{1}{2} t_{\infty,2r_{\infty}-2} \lambda + g_0 \right)^2 P(\lambda) [A_{\alpha}(\lambda)]_{1,2} \\ &\quad + \hbar \left(\mathcal{L}_{\alpha} \left[\frac{Q(\lambda) + \left(\frac{1}{2} t_{\infty,2r_{\infty}-2} \lambda + g_0 \right) P(\lambda)}{P(\lambda)} \right] - \left(Q(\lambda) + \left(\frac{1}{2} t_{\infty,2r_{\infty}-2} \lambda + g_0 \right) P(\lambda) \right) \mathcal{L}_{\alpha} \left[\frac{1}{P(\lambda)} \right] \right) \\ &= ([A_{\alpha}(\lambda)]_{1,1} - [A_{\alpha}(\lambda)]_{2,2}) \left(\frac{Q(\lambda)}{P(\lambda)} + \frac{1}{2} t_{\infty,2r_{\infty}-2} \lambda + g_0 \right) + \frac{[A_{\alpha}(\lambda)]_{2,1}}{P(\lambda)} \\ &\quad - [A_{\alpha}(\lambda)]_{1,2} Q(\lambda) \left(\frac{Q(\lambda)}{P(\lambda)} + t_{\infty,2r_{\infty}-2} \lambda + 2g_0 \right) - \left(\frac{1}{2} t_{\infty,2r_{\infty}-2} \lambda + g_0 \right)^2 P(\lambda) [A_{\alpha}(\lambda)]_{1,2} \\ &\quad + \hbar \frac{\mathcal{L}_{\alpha} [Q(\lambda) + \left(\frac{1}{2} t_{\infty,2r_{\infty}-2} \lambda + g_0 \right) P(\lambda)]}{P(\lambda)} \end{aligned} \quad (\text{I-16})$$

The last quantity gives $\frac{\hbar}{2} \alpha_{\infty,2r_{\infty}-2} + \mathcal{L}_{\alpha}[g_0] - \frac{1}{2} t_{\infty,2r_{\infty}-2} \sum_{j=1}^g \mathcal{L}_{\alpha}[q_j] + O(\lambda^{-1})$ which is equal to

$\frac{\hbar}{2} \alpha_{\infty,2r_{\infty}-2} + \frac{\hbar}{2} \alpha_{\infty,2r_{\infty}-4} + \frac{\hbar}{2} \alpha_{\infty,2r_{\infty}-2} \sum_{j=1}^g q_j$. Since we know that $[\tilde{A}_{\alpha}(\lambda)]_{2,1}$ is a polynomial in λ , we get:

$$\begin{aligned} [\tilde{A}_{\alpha}(\lambda)]_{2,1} &= ([A_{\alpha}(\lambda)]_{1,1} - [A_{\alpha}(\lambda)]_{2,2}) \left(\frac{Q(\lambda)}{P(\lambda)} + \frac{1}{2} t_{\infty,2r_{\infty}-2} \lambda + g_0 \right) + \frac{[A_{\alpha}(\lambda)]_{2,1}}{P(\lambda)} \\ &\quad - [A_{\alpha}(\lambda)]_{1,2} Q(\lambda) \left(\frac{Q(\lambda)}{P(\lambda)} + t_{\infty,2r_{\infty}-2} \lambda + 2g_0 \right) - \left(\frac{1}{2} t_{\infty,2r_{\infty}-2} \lambda + g_0 \right)^2 P(\lambda) [A_{\alpha}(\lambda)]_{1,2} \\ &\quad + \frac{\hbar}{2} \alpha_{\infty,2r_{\infty}-2} + \frac{\hbar}{2} \alpha_{\infty,2r_{\infty}-4} + \frac{\hbar}{2} \alpha_{\infty,2r_{\infty}-2} \sum_{j=1}^g q_j \end{aligned} \quad (\text{I-17})$$

Replacing $[A_{\alpha}(\lambda)]_{1,1}$ and $[A_{\alpha}(\lambda)]_{2,1}$ using (4-5) gives:

$$\begin{aligned} [\tilde{A}_{\alpha}(\lambda)]_{2,1} &= - \left(\hbar \partial_{\lambda} [A_{\alpha}(\lambda)]_{1,2} + [A_{\alpha}(\lambda)]_{1,2} L_{2,2}(\lambda) \right) \left(\frac{Q(\lambda)}{P(\lambda)} + \frac{1}{2} t_{\infty,2r_{\infty}-2} \lambda + g_0 \right) \\ &\quad + \frac{\hbar \partial_{\lambda} [A_{\alpha}(\lambda)]_{1,1} + [A_{\alpha}(\lambda)]_{1,1} L_{2,1}(\lambda)}{P(\lambda)} \\ &\quad - [A_{\alpha}(\lambda)]_{1,2} Q(\lambda) \left(\frac{Q(\lambda)}{P(\lambda)} + t_{\infty,2r_{\infty}-2} \lambda + 2g_0 \right) - \left(\frac{1}{2} t_{\infty,2r_{\infty}-2} \lambda + g_0 \right)^2 P(\lambda) [A_{\alpha}(\lambda)]_{1,2} \end{aligned}$$

$$\begin{aligned}
& +\frac{\hbar}{2}\alpha_{\infty,2r_{\infty}-2} + \frac{\hbar}{2}\alpha_{\infty,2r_{\infty}-4} + \frac{\hbar}{2}\alpha_{\infty,2r_{\infty}-2}Q_1 \\
= & -\left(\hbar\partial_{\lambda}[A_{\alpha}(\lambda)]_{1,2} + [A_{\alpha}(\lambda)]_{1,2}\left(\tilde{P}_1(\lambda) + \sum_{j=1}^g \frac{\hbar}{\lambda-q_j}\right)\right)\left(\frac{Q(\lambda)}{P(\lambda)} + \frac{1}{2}t_{\infty,2r_{\infty}-2}\lambda + g_0\right) \\
& + \frac{\hbar\partial_{\lambda}[A_{\alpha}(\lambda)]_{1,1} + [A_{\alpha}(\lambda)]_{1,2}L_{2,1}(\lambda)}{P(\lambda)} \\
& - [A_{\alpha}(\lambda)]_{1,2}Q(\lambda)\left(\frac{Q(\lambda)}{P(\lambda)} + t_{\infty,2r_{\infty}-2}\lambda + 2g_0\right) - \left(\frac{1}{2}t_{\infty,2r_{\infty}-2}\lambda + g_0\right)^2 P(\lambda)[A_{\alpha}(\lambda)]_{1,2} \\
& + \frac{\hbar}{2}\alpha_{\infty,2r_{\infty}-2} + \frac{\hbar}{2}\alpha_{\infty,2r_{\infty}-4} + \frac{\hbar}{2}\alpha_{\infty,2r_{\infty}-2}Q_1
\end{aligned} \tag{I-18}$$

Note that the polynomial part of $-\left(\hbar\partial_{\lambda}[A_{\alpha}(\lambda)]_{1,2} + [A_{\alpha}(\lambda)]_{1,2}\left(\sum_{j=1}^g \frac{\hbar}{\lambda-q_j}\right)\right)\left(\frac{Q(\lambda)}{P(\lambda)} + \frac{1}{2}t_{\infty,2r_{\infty}-2}\lambda + g_0\right)$ is given by (From corollary 6.1, we have $Q(\lambda) = (-1)^g P_g \lambda^{g-1} + O(\lambda^{g-2})$)

$$\begin{aligned}
& -\frac{\hbar}{2}\nu_{\infty,-1}^{(\alpha)}t_{\infty,2r_{\infty}-2}\lambda^2 - \hbar\nu_{\infty,-1}^{(\alpha)}g_0\lambda - \hbar\nu_{\infty,-1}^{(\alpha)}(-1)^g P_g \\
& -\frac{\hbar}{2}gt_{\infty,2r_{\infty}-2}\nu_{\infty,-1}^{(\alpha)}\lambda - \hbar gg_0\nu_{\infty,-1}^{(\alpha)} + \frac{\hbar}{2}t_{\infty,2r_{\infty}-2}\nu_{\infty,-1}^{(\alpha)}Q_1 - \frac{\hbar}{2}gt_{\infty,2r_{\infty}-2}\nu_{\infty,0}^{(\alpha)} \tag{I-19}
\end{aligned}$$

Keeping only the polynomial part of (I-18) leads to

$$\begin{aligned}
[\tilde{A}_{\alpha}(\lambda)]_{2,1} = & -\frac{\hbar}{2}\nu_{\infty,-1}^{(\alpha)}t_{\infty,2r_{\infty}-2}\lambda^2 - \hbar\nu_{\infty,-1}^{(\alpha)}g_0\lambda - \hbar\nu_{\infty,-1}^{(\alpha)}(-1)^g P_g \\
& -\frac{\hbar}{2}gt_{\infty,2r_{\infty}-2}\nu_{\infty,-1}^{(\alpha)}\lambda - \hbar gg_0\nu_{\infty,-1}^{(\alpha)} + \frac{\hbar}{2}t_{\infty,2r_{\infty}-2}\nu_{\infty,-1}^{(\alpha)}Q_1 - \frac{\hbar}{2}gt_{\infty,2r_{\infty}-2}\nu_{\infty,0}^{(\alpha)} \\
& - [A_{\alpha}(\lambda)]_{1,2}\tilde{P}_1(\lambda)\left(\frac{Q(\lambda)}{P(\lambda)} + \frac{1}{2}t_{\infty,2r_{\infty}-2}\lambda + g_0\right) \\
& + \hbar(r_{\infty}-1)c_{\infty,r_{\infty}-1}^{(\alpha)}\lambda + \hbar(r_{\infty}-1)c_{\infty,r_{\infty}-1}^{(\alpha)}Q_1 + \hbar(r_{\infty}-2)c_{\infty,r_{\infty}-2}^{(\alpha)} + \nu_{\infty,-1}^{(\alpha)}H_{\infty,r_{\infty}-4} - \frac{\tilde{P}_2(\lambda)[A_{\alpha}(\lambda)]_{1,2}}{P(\lambda)} \\
& - [A_{\alpha}(\lambda)]_{1,2}Q(\lambda)\left(\frac{Q(\lambda)}{P(\lambda)} + t_{\infty,2r_{\infty}-2}\lambda + 2g_0\right) - \left(\frac{1}{2}t_{\infty,2r_{\infty}-2}\lambda + g_0\right)^2 P(\lambda)[A_{\alpha}(\lambda)]_{1,2} \\
& + \frac{\hbar}{2}\alpha_{\infty,2r_{\infty}-2} + \frac{\hbar}{2}\alpha_{\infty,2r_{\infty}-4} + \frac{\hbar}{2}\alpha_{\infty,2r_{\infty}-2}Q_1 \\
= & -\frac{\hbar}{2}\nu_{\infty,-1}^{(\alpha)}t_{\infty,2r_{\infty}-2}\lambda^2 - \hbar\nu_{\infty,-1}^{(\alpha)}g_0\lambda - \hbar\nu_{\infty,-1}^{(\alpha)}(-1)^g P_g \\
& -\frac{\hbar}{2}gt_{\infty,2r_{\infty}-2}\nu_{\infty,-1}^{(\alpha)}\lambda - \hbar gg_0\nu_{\infty,-1}^{(\alpha)} + \frac{\hbar}{2}t_{\infty,2r_{\infty}-2}\nu_{\infty,-1}^{(\alpha)}Q_1 - \frac{\hbar}{2}gt_{\infty,2r_{\infty}-2}\nu_{\infty,0}^{(\alpha)} \\
& + \hbar(r_{\infty}-1)c_{\infty,r_{\infty}-1}^{(\alpha)}\lambda + \hbar(r_{\infty}-1)c_{\infty,r_{\infty}-1}^{(\alpha)}Q_1 + \hbar(r_{\infty}-2)c_{\infty,r_{\infty}-2}^{(\alpha)} + \nu_{\infty,-1}^{(\alpha)}H_{\infty,r_{\infty}-4} \\
& + \frac{\hbar}{2}\alpha_{\infty,2r_{\infty}-2} + \frac{\hbar}{2}\alpha_{\infty,2r_{\infty}-4} + \frac{\hbar}{2}\alpha_{\infty,2r_{\infty}-2}Q_1 \\
& + [A_{\alpha}(\lambda)]_{1,2}\left[\frac{-Q(\lambda)\tilde{P}_1(\lambda) - \tilde{P}_2(\lambda) - Q(\lambda)^2}{P(\lambda)} - \left(\frac{1}{2}t_{\infty,2r_{\infty}-2}\lambda + g_0\right)\tilde{P}_1(\lambda)\right. \\
& \left. - (t_{\infty,2r_{\infty}-2}\lambda + 2g_0)Q(\lambda) - \left(\frac{1}{2}t_{\infty,2r_{\infty}-2}\lambda + g_0\right)^2 P(\lambda)\right] \\
= & -\frac{\hbar}{2}\nu_{\infty,-1}^{(\alpha)}t_{\infty,2r_{\infty}-2}\lambda^2 - \hbar\nu_{\infty,-1}^{(\alpha)}g_0\lambda - \hbar\nu_{\infty,-1}^{(\alpha)}(-1)^g P_g \\
& -\frac{\hbar}{2}gt_{\infty,2r_{\infty}-2}\nu_{\infty,-1}^{(\alpha)}\lambda - \hbar gg_0\nu_{\infty,-1}^{(\alpha)} + \frac{\hbar}{2}t_{\infty,2r_{\infty}-2}\nu_{\infty,-1}^{(\alpha)}Q_1 - \frac{\hbar}{2}gt_{\infty,2r_{\infty}-2}\nu_{\infty,0}^{(\alpha)} \\
& + \hbar(r_{\infty}-1)c_{\infty,r_{\infty}-1}^{(\alpha)}\lambda + \hbar(r_{\infty}-1)c_{\infty,r_{\infty}-1}^{(\alpha)}Q_1 + \hbar(r_{\infty}-2)c_{\infty,r_{\infty}-2}^{(\alpha)} + \nu_{\infty,-1}^{(\alpha)}H_{\infty,r_{\infty}-4} \\
& + \frac{\hbar}{2}\alpha_{\infty,2r_{\infty}-2} + \frac{\hbar}{2}\alpha_{\infty,2r_{\infty}-4} + \frac{\hbar}{2}\alpha_{\infty,2r_{\infty}-2}Q_1 \\
& - \sum_{i=0}^g \sum_{j=\text{Max}(i-1,0)}^{g-1} \sum_{s=g+i-j-1}^{g+1} \sum_{r=j+1}^g \sum_{m=1}^{s+j-g-i} (-1)^j t_{\infty,2s+2}\nu_{\infty,m}^{(\alpha)} h_{s+j-m-g-i} P_r Q_{r-j-1} \lambda^i
\end{aligned}$$

$$\begin{aligned}
& - \sum_{i=0}^{r_\infty} \sum_{j=\text{Max}(g, g+i-1)}^{2r_\infty-4} \sum_{m=-1}^{j-g-i} \nu_{\infty, m}^{(\alpha)} h_{j-g-m-i} \tilde{P}_{\infty, j}^{(2)} \lambda^i \\
& - \sum_{i=0}^g \sum_{j_1=0}^{g-1} \sum_{j_2=0}^{g-1} \sum_{m=-1}^{j_1+j_2-g-i} (-1)^{j_1+j_2} \nu_{\infty, m}^{(\alpha)} h_{j_1+j_2-g-m-i} \sum_{r_1=j_1+1}^g \sum_{r_2=j_2+1}^g P_{r_1} P_{r_2} Q_{r_1-j_1-1} Q_{r_2-j_2-1} \lambda^i \\
& + \left(\frac{1}{2} t_{\infty, 2r_\infty-2} \lambda + g_0 \right) \sum_{i=0}^{r_\infty-1} \sum_{s=\text{Max}(i-1, 0)}^{r_\infty-2} t_{\infty, 2s+2} \nu_{\infty, s-i}^{(\alpha)} \lambda^i \\
& - (t_{\infty, 2r_\infty-2} \lambda + 2g_0) \sum_{i=0}^g \sum_{j=\text{Max}(i-1, 0)}^{g-1} \sum_{r=j+1}^g (-1)^{j-1} \nu_{\infty, j-i}^{(\alpha)} P_r Q_{r-j-1} \lambda^i \\
& - \left(\frac{1}{2} t_{\infty, 2r_\infty-2} \lambda + g_0 \right) \sum_{i=0}^{2g+1} \sum_{j=\text{Max}(i-1, 0)}^g (-1)^{g-j} Q_{g-j} \nu_{\infty, j-i}^{(\alpha)} \lambda^i \tag{I-20}
\end{aligned}$$

where we have $\nu_{\infty, r_\infty-2}^{(\alpha)} = \sum_{j=1}^g \mu_j^{(\alpha)} q_j^g$.

J Proof of Proposition 7.1

Let $k \in \llbracket 1, r_\infty - 1 \rrbracket$ and consider \mathbf{w}_k . It is obvious from (4-8) (whose r.h.s. only implies odd entries of α) that $\nu_j^{(\mathbf{w}_k)} = 0$ for all $j \in \llbracket -1, r_\infty - 3 \rrbracket$. Consequently, (4-11) implies that $\mu_j^{(\mathbf{w}_k)} = 0$ for all $j \in \llbracket 1, g \rrbracket$. For $j \in \llbracket 1, r_\infty - 1 \rrbracket$, the j^{th} line of the r.h.s. of (4-20) is given by $\sum_{m=r_\infty-j}^{r_\infty-1} \left(\frac{\alpha_{\infty, 4r_\infty-2j-2m-3}}{4r_\infty-2j-2m-3} t_{\infty, 2m} - \frac{\alpha_{\infty, 4r_\infty-2j-2m-2}}{4r_\infty-2j-2m-2} t_{\infty, 2m-1} \right)$. For $\alpha = \mathbf{w}_k$ it reduces only to $-\frac{1}{2k} t_{\infty, 4r_\infty-3-2j-2k} \delta_{j \geq r_\infty-k}$. Thus we get:

$$\begin{pmatrix} t_{\infty, 2r_\infty-3} & 0 & \dots & 0 \\ t_{\infty, 2r_\infty-5} & t_{\infty, 2r_\infty-3} & 0 & \ddots \\ \vdots & \ddots & \ddots & \vdots \\ t_{\infty, 3} & & \ddots & 0 \\ t_{\infty, 1} & t_{\infty, 3} & \dots & t_{\infty, 2r_\infty-3} \end{pmatrix} \begin{pmatrix} c_{\infty, r_\infty-1}^{(\mathbf{w}_k)} \\ c_{\infty, r_\infty-2}^{(\mathbf{w}_k)} \\ \vdots \\ c_{\infty, 1}^{(\mathbf{w}_k)} \end{pmatrix} = -\frac{1}{2k} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ t_{\infty, 2r_\infty-3} \\ \vdots \\ t_{\infty, 2r_\infty-2k-1} \end{pmatrix} \tag{J-1}$$

We recognize that the r.h.s. is $-\frac{1}{2k}$ times the $(r_\infty - k)^{\text{th}}$ column of the matrix on the l.h.s. so that we immediately get $c_{\infty, j}^{(\mathbf{w}_k)} = -\frac{1}{2k} \delta_{j, k}$ for all $j \in \llbracket 1, r_\infty - 1 \rrbracket$.

Let us now take $k \in \llbracket -1, r_\infty - 3 \rrbracket$ and consider \mathbf{u}_k . The $(k+2)^{\text{th}}$ column of M_∞ is given by $\sum_{i=k+2}^{r_\infty-1} t_{\infty, 2r_\infty-2i+2k+1} \mathbf{e}_i$. Similarly, the r.h.s. of (4-8) only implies odd indexes of \mathbf{u}_k and it is given by

$$\begin{aligned}
\text{RHS} &= \sum_{i=1}^{r_\infty-1} \frac{2\alpha_{2r_\infty-1-2i}^{(\mathbf{u}_k)}}{2r_\infty-1-2i} \mathbf{e}_i \stackrel{m=r_\infty-i}{=} \sum_{m=1}^{r_\infty-1} \frac{2\alpha_{2m-1}^{(\mathbf{u}_k)}}{2m-1} \mathbf{e}_{r_\infty-m} \\
&= \sum_{m=1}^{r_\infty-k-2} t_{\infty, 2m+1+2k} \mathbf{e}_{r_\infty-m} \stackrel{i=r_\infty-m}{=} \sum_{i=k+2}^{r_\infty-1} t_{\infty, 2r_\infty-2i+1+2k} \mathbf{e}_i \tag{J-2}
\end{aligned}$$

We recognize the $(k+2)^{\text{th}}$ column of M_∞ . Hence, we conclude that $(\nu_{-1}^{(\mathbf{u}_k)}, \dots, \nu_{r_\infty-3}^{(\mathbf{u}_k)})^t = \mathbf{e}_{k+2}$, i.e. $\nu_j^{(\mathbf{u}_k)} = \delta_{j, k}$ for all $j \in \llbracket -1, r_\infty - 3 \rrbracket$.

Let us now take $i \in \llbracket 1, r_\infty - 1 \rrbracket$ and compute the i^{th} line of the r.h.s. of (4-20). It is given by:

$$\begin{aligned}
\text{RHS}_i &= \sum_{m=r_\infty-i}^{r_\infty-1} \frac{\alpha_{\infty,4r_\infty-2i-2m-3}^{(\mathbf{u}_k)}}{4r_\infty-2i-2m-3} t_{\infty,2m} - \sum_{m=r_\infty-i}^{r_\infty-1} \frac{\alpha_{4r_\infty-2i-2m-2}^{(\mathbf{u}_k)}}{4r_\infty-2i-2m-2} t_{\infty,2m-1} \\
&\stackrel{s=2r_\infty-i-m-1}{=} \sum_{s=r_\infty-i}^{r_\infty-1} \frac{\alpha_{\infty,2s-1}^{(\mathbf{u}_k)}}{2s-1} t_{\infty,4r_\infty-2i-2s-2} - \sum_{s=r_\infty-i}^{r_\infty-1} \frac{\alpha_{2s}^{(\mathbf{u}_k)}}{2s} t_{\infty,4r_\infty-2i-2s-3} \\
&= \frac{1}{2} \sum_{s=r_\infty-i}^{r_\infty-k-2} t_{\infty,2s+2k+1} t_{\infty,4r_\infty-2i-2s-2} - \frac{1}{2} \sum_{s=r_\infty-i}^{r_\infty-k-2} t_{\infty,2s+2k+2} t_{\infty,4r_\infty-2i-2s-3} \\
&\stackrel{m=2r_\infty-2-i-s-k}{=} \frac{1}{2} \sum_{m=r_\infty-i}^{r_\infty-k-2} t_{\infty,4r_\infty-2i-2m-3} t_{\infty,2m+2k+2} - \frac{1}{2} \sum_{s=r_\infty-i}^{r_\infty-k-2} t_{\infty,2s+2k+2} t_{\infty,4r_\infty-2i-2s-3} \\
&= 0 \tag{J-3}
\end{aligned}$$

Thus, the r.h.s. of (4-20) is null for $\alpha = \mathbf{u}_k$ so that since M_∞ is invertible, we get $c_{\infty,j}^{(\mathbf{u}_k)} = 0$ for all $j \in \llbracket 1, r_\infty - 1 \rrbracket$.

K Proof of Theorem 7.1

Let $j \in \llbracket 1, g \rrbracket$ and $k \in \llbracket 1, r_\infty - 1 \rrbracket$. From Proposition 7.1 we have $\nu_i^{(\mathbf{w}_k)} = 0$ for all $i \in \llbracket -1, r_\infty - 3 \rrbracket$. Consequently, from (4-11), $\mu_i^{(\mathbf{w}_k)} = 0$ for all $i \in \llbracket 1, g \rrbracket$. Therefore $\mathcal{L}_{\mathbf{w}_k}[q_j] = 0$ from Theorem 5.1. Similarly, we also have from Proposition 7.1 that $c_{\infty,i}^{(\mathbf{w}_k)} = -\frac{1}{2k} \delta_{i,k}$ for all $i \in \llbracket 1, r_\infty - 1 \rrbracket$ so that Theorem 5.1 provides

$$\mathcal{L}_{\mathbf{w}_k}[p_j] = \hbar \sum_{i=1}^{r_\infty-1} i c_{\infty,i}^{(\mathbf{w}_k)} q_j^{i-1} = -\frac{\hbar}{2} q_j^{k-1} \tag{K-1}$$

Let us now consider $\mathcal{L}_{\mathbf{u}_{-1}}$. From Proposition 7.1 we have $\nu_i^{(\mathbf{u}_{-1})} = \delta_{i,-1}$ for all $i \in \llbracket -1, r_\infty - 2 \rrbracket$. Consequently, from (4-11), $\mu_i^{(\mathbf{u}_{-1})} = 0$ for all $i \in \llbracket 1, g \rrbracket$. From Theorem 5.1, we get that

$$\mathcal{L}_{\mathbf{u}_{-1}}[q_j] = -\hbar \nu_{\infty,-1}^{(\mathbf{u}_{-1})} q_j = -\hbar q_j \tag{K-2}$$

Similarly, since from Proposition 7.1 we have $c_{\infty,i}^{(\mathbf{u}_{-1})} = 0$ for all $i \in \llbracket 1, r_\infty - 1 \rrbracket$, Theorem 5.1 provides:

$$\mathcal{L}_{\mathbf{u}_{-1}}[p_j] = \hbar \nu_{\infty,-1}^{(\mathbf{u}_{-1})} p_j = \hbar p_j \tag{K-3}$$

Let us now consider $\mathcal{L}_{\mathbf{u}_0}$. From Proposition 7.1 we have $\nu_i^{(\mathbf{u}_0)} = \delta_{i,0}$ for all $i \in \llbracket -1, r_\infty - 2 \rrbracket$. Consequently, from (4-11), $\mu_i^{(\mathbf{u}_0)} = 0$ for all $i \in \llbracket 1, g \rrbracket$. From Theorem 5.1, we get that

$$\mathcal{L}_{\mathbf{u}_{-1}}[q_j] = -\hbar \nu_{\infty,0}^{(\mathbf{u}_0)} = -\hbar \tag{K-4}$$

Similarly, since from Proposition 7.1 we have $c_{\infty,i}^{(\mathbf{u}_0)} = 0$ for all $i \in \llbracket 1, r_\infty - 1 \rrbracket$, Theorem 5.1 provides:

$$\mathcal{L}_{\mathbf{u}_0}[p_j] = 0 \tag{K-5}$$

L Proof of Proposition 7.2

Since

$$t_{\infty, 2r_{\infty}-3} = 2T_2^{\frac{2r_{\infty}-3}{2}}, \quad t_{\infty, 2r_{\infty}-5} = (2r_{\infty}-5)T_1 T_2^{\frac{2r_{\infty}-5}{2}} \quad (\text{L-1})$$

we first observe that we may rewrite for all $k \in \llbracket 1, g \rrbracket$:

$$\begin{aligned} \tau_k &= \sum_{i=0}^{k-1} \frac{(-1)^i \left(\prod_{s=1}^i (2r_{\infty} - 2k + 2s - 7) \right) \left(\frac{1}{2} t_{\infty, 2r_{\infty}-5} \right)^i \left(\frac{1}{2} t_{\infty, 2r_{\infty}-3} \right)^{-\frac{(2r_{\infty}-3)i+2r_{\infty}-5-2k}{2r_{\infty}-3}}}{i!(2r_{\infty}-5)^i} \frac{1}{2} t_{\infty, 2r_{\infty}-5-2k+2i} \\ &\quad + \frac{(-1)^k \left(\prod_{s=1}^k (2r_{\infty} - 2k + 2s - 7) \right) \left(\frac{1}{2} t_{\infty, 2r_{\infty}-5} \right)^{k+1} \left(\frac{1}{2} t_{\infty, 2r_{\infty}-3} \right)^{-\frac{(k+1)(2r_{\infty}-5)}{2r_{\infty}-3}}}{(k+1)(k-1)!(2r_{\infty}-5)^k} \\ &= \sum_{i=0}^{k-1} \frac{(-1)^i \left(\prod_{s=1}^i (2r_{\infty} - 2k + 2s - 7) \right) T_1^i T_2^{-\frac{2r_{\infty}-5+2i-2k}{2}}}{2^i i!} \frac{1}{2} t_{\infty, 2r_{\infty}-5-2k+2i} \\ &\quad + \frac{(-1)^k \left(\prod_{s=1}^{k+1} (2r_{\infty} - 2k + 2s - 7) \right) T_1^{k+1}}{2^{k+1}(k+1)(k-1)!} \\ &\stackrel{j=k-i}{=} \sum_{j=1}^k \frac{(-1)^{k-j} \left(\prod_{s=1}^{k-j} (2r_{\infty} - 2k + 2s - 7) \right) T_1^{k-j} T_2^{-\frac{2r_{\infty}-5-2j}{2}}}{2^{k-j}(k-j)!} \frac{1}{2} t_{\infty, 2r_{\infty}-5-2j} \\ &\quad + \frac{(-1)^k \left(\prod_{s=1}^{k+1} (2r_{\infty} - 2k + 2s - 7) \right) T_1^{k+1}}{2^{k+1}(k+1)(k-1)!} \end{aligned} \quad (\text{L-2})$$

We now insert the ansatz

$$t_{\infty, 2r_{\infty}-5-2j} = 2T_2^{\frac{2r_{\infty}-5-2j}{2}} \left(\sum_{p=1}^j \alpha_p^{(j)} T_1^{j-p} \tau_p + T_1^{j+1} \alpha_0^{(j)} \right), \quad \forall j \in \llbracket 1, r_{\infty}-3 \rrbracket \quad (\text{L-3})$$

which gives:

$$\begin{aligned} \tau_k &= \sum_{j=1}^k \sum_{p=1}^j \frac{(-1)^{k-j} \left(\prod_{s=1}^{k-j} (2r_{\infty} - 2k + 2s - 7) \right) T_1^{k-p} \alpha_p^{(j)} \tau_p}{2^{k-j}(k-j)!} \\ &\quad + \sum_{j=1}^k \frac{(-1)^{k-j} \left(\prod_{s=1}^{k-j} (2r_{\infty} - 2k + 2s - 7) \right) T_1^{k+1} \alpha_0^{(j)}}{2^{k-j}(k-j)!} \\ &\quad + \frac{(-1)^k \left(\prod_{s=0}^k (2r_{\infty} - 2k + 2s - 7) \right) T_1^{k+1}}{2^{k+1}(k+1)(k-1)!} \\ &\stackrel{m=k+1-s}{=} \sum_{p=1}^k \sum_{j=p}^k \frac{(-1)^{k-j} \left(\prod_{m=j+1}^k (2r_{\infty} - 2m - 5) \right) T_1^{k-p} \alpha_p^{(j)} \tau_p}{2^{k-j}(k-j)!} \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^k \frac{(-1)^{k-j} \left(\prod_{m=j+1}^k (2r_\infty - 2m - 5) \right) T_1^{k+1} \alpha_0^{(j)}}{2^{k-j} (k-j)!} \\
& + \frac{(-1)^k \left(\prod_{m=0}^k (2r_\infty - 2m - 5) \right) T_1^{k+1}}{2^{k+1} (k+1)(k-1)!} \\
& \stackrel{r=j-p}{=} \sum_{p=1}^k \sum_{r=0}^{k-p} \frac{(-1)^{k-r-p} \left(\prod_{m=r+p+1}^k (2r_\infty - 2m - 5) \right) T_1^{k-p} \alpha_p^{(r+p)} \tau_p}{2^{k-r-p} (k-r-p)!} \\
& + \sum_{j=1}^k \frac{(-1)^{k-j} \left(\prod_{m=j+1}^k (2r_\infty - 2m - 5) \right) T_1^{k+1} \alpha_0^{(j)}}{2^{k-j} (k-j)!} \\
& + \frac{(-1)^k \left(\prod_{m=0}^k (2r_\infty - 2m - 5) \right) T_1^{k+1}}{2^{k+1} (k+1)(k-1)!}
\end{aligned} \tag{L-4}$$

Let us now take

$$\forall j \in \llbracket 1, r_\infty - 3 \rrbracket : \alpha_0^{(j)} = \frac{\prod_{m=0}^j (2r_\infty - 2m - 5)}{2^{j+1} (j+1)!} \tag{L-5}$$

and prove that for all $k \in \llbracket 1, r_\infty - 3 \rrbracket$:

$$D_k \stackrel{\text{def}}{=} \sum_{j=1}^k \frac{(-1)^{k-j} \left(\prod_{m=j+1}^k (2r_\infty - 2m - 5) \right) T_1^{k+1} \alpha_0^{(j)}}{2^{k-j} (k-j)!} + \frac{(-1)^k \left(\prod_{m=0}^k (2r_\infty - 2m - 5) \right) T_1^{k+1}}{2^{k+1} (k+1)(k-1)!} = 0 \tag{L-6}$$

Indeed we have:

$$\begin{aligned}
D_k &= \sum_{j=1}^k \frac{(-1)^{k-j} \left(\prod_{m=0}^k (2r_\infty - 2m - 5) \right) T_1^{k+1}}{(j+1)! 2^{k+1} (k-j)!} + \frac{(-1)^k \left(\prod_{m=0}^k (2r_\infty - 2m - 5) \right) T_1^{k+1}}{2^{k+1} (k+1)(k-1)!} \\
& \stackrel{i=j+1}{=} \sum_{i=2}^k \frac{(-1)^{k-i+1} \left(\prod_{m=0}^k (2r_\infty - 2m - 5) \right) T_1^{k+1}}{i! 2^{k+1} (k-i+1)!} + \frac{(-1)^k \left(\prod_{m=0}^k (2r_\infty - 2m - 5) \right) T_1^{k+1}}{2^{k+1} (k+1)(k-1)!} \\
&= \frac{(-1)^{k+1}}{(k+1)!} \left(\prod_{m=0}^k (2r_\infty - 2m - 5) \right) T_1^{k+1} \sum_{i=2}^k \frac{(-1)^i (k+1)!}{i! 2^{k+1} (k+1-i)!} \\
& \quad + \frac{(-1)^k \left(\prod_{m=0}^k (2r_\infty - 2m - 5) \right) T_1^{k+1}}{2^{k+1} (k+1)(k-1)!} \\
&= \frac{(-1)^{k+1}}{2^{k+1} (k+1)!} \left(\prod_{m=0}^k (2r_\infty - 2m - 5) \right) T_1^{k+1} (0 - 1 - (-1)(k+1))
\end{aligned}$$

$$\begin{aligned}
& + \frac{(-1)^k k \left(\prod_{m=0}^k (2r_\infty - 2m - 5) \right) T_1^{k+1}}{2^{k+1} (k+1)!} \\
= & 0
\end{aligned} \tag{L-7}$$

Thus, under the choice (L-5), equation (L-4) simplifies into

$$\tau_k = \sum_{p=1}^k \sum_{r=0}^{k-p} \frac{(-1)^{k-r-p} \left(\prod_{m=r+p+1}^k (2r_\infty - 2m - 5) \right) T_1^{k-p} \alpha_p^{(r+p)} \tau_p}{2^{k-r-p} (k-r-p)!} \tag{L-8}$$

We now rewrite:

$$\begin{aligned}
\prod_{m=r+p+1}^k (2r_\infty - 2m - 5) &= \frac{\prod_{m=r+p+1}^{r_\infty-3} (2r_\infty - 2m - 5)}{\prod_{m=k+1}^{r_\infty-3} (2r_\infty - 2m - 5)} \frac{\prod_{m=p+1}^{r_\infty-3} (2r_\infty - 2m - 5)}{\prod_{m=p+1}^{r_\infty-3} (2r_\infty - 2m - 5)} \\
&= \frac{1}{\prod_{m=p+1}^{r+p} (2r_\infty - 2m - 5)} \frac{\prod_{m=p+1}^{r_\infty-3} (2r_\infty - 2m - 5)}{\prod_{m=k+1}^{r_\infty-3} (2r_\infty - 2m - 5)}
\end{aligned} \tag{L-9}$$

and we take

$$\forall (p, j) \in \llbracket 1, j \rrbracket \times \llbracket 1, r_\infty - 3 \rrbracket : \alpha_p^{(j)} = \frac{\prod_{m=p+1}^j (2r_\infty - 2m - 5)}{2^{j-p} (j-p)!} \tag{L-10}$$

so that $\alpha_p^{(r+p)} = \frac{2^{-r} \prod_{m=p+1}^{r+p} (2r_\infty - 2m - 5)}{r!}$. Thus, the sum from $r = 0$ to $k - p$ in (L-8) reduces to $\sum_{r=0}^{k-p} \frac{(-1)^{-r}}{r! (k-r-p)!} = (k-p)! (1-1)^{k-p} = \delta_{k-p=0}$. We obtain:

$$\sum_{p=1}^k \sum_{r=0}^{k-p} \frac{(-1)^{k-r-p} \left(\prod_{m=r+p+1}^k (2r_\infty - 2m - 5) \right) T_1^{k-p} \alpha_p^{(r+p)} \tau_p}{(k-r-p)!} = \tau_k \tag{L-11}$$

Thus, we conclude that for all $j \in \llbracket 1, r_\infty - 3 \rrbracket$:

$$t_{\infty, 2r_\infty - 5 - 2j} = 2T_2^{\frac{2r_\infty - 5 - 2j}{2}} \left(\sum_{p=1}^j \frac{\prod_{m=p+1}^j (2r_\infty - 2m - 5)}{2^{j-p} (j-p)!} T_1^{j-p} \tau_p + T_1^{j+1} \frac{\prod_{m=0}^j (2r_\infty - 2m - 5)}{2^{j+1} (j+1)!} \right) \tag{L-12}$$

In other words, taking $k = r_\infty - 2 - j$ we get for all $k \in \llbracket 1, r_\infty - 3 \rrbracket$

$$t_{\infty, 2k-1} = 2T_2^{\frac{2k-1}{2}} \left(\sum_{p=1}^{r_\infty - k - 2} \frac{\prod_{m=p+1}^{r_\infty - k - 2} (2r_\infty - 2m - 5)}{2^{r_\infty - k - p - 2} (r_\infty - k - p - 2)!} T_1^{r_\infty - k - p - 2} \tau_p + T_1^{r_\infty - 1 - k} \frac{\prod_{m=0}^{r_\infty - k - 2} (2r_\infty - 2m - 5)}{2^{r_\infty - 1 - k} (r_\infty - 1 - k)!} \right) \tag{L-13}$$

M Proof of Proposition 7.5

Let $j \in \llbracket 1, g \rrbracket$. Let $k \in \llbracket 1, r_\infty - 1 \rrbracket$ and consider $\mathcal{L}_{\mathbf{w}_k}$. Since $\mathcal{L}_{\mathbf{w}_k}[T_1] = \mathcal{L}_{\mathbf{w}_k}[T_2] = \mathcal{L}_{\mathbf{w}_k}[q_j] = 0$ from Proposition 7.4, we immediately get $\mathcal{L}_{\mathbf{w}_k}[\tilde{q}_j] = 0$. Moreover from Proposition 3-3, we observe that:

$$\mathcal{L}_{\mathbf{w}_k}[\tilde{P}_1(\lambda)] = - \sum_{i=0}^{r_\infty-2} \mathcal{L}_{\mathbf{w}_k}[t_{\infty,2i+2}]\lambda^i = -\hbar\lambda^{k-1} \quad (\text{M-1})$$

so that Theorem 7.1 provides for all $j \in \llbracket 1, g \rrbracket$:

$$\mathcal{L}_{\mathbf{w}_k}[\tilde{p}_j] = T_2^{-1} \left(\mathcal{L}_{\mathbf{w}_k}[p_j] - \frac{1}{2} \mathcal{L}_{\mathbf{w}_k}[\tilde{P}_1](q_j) \right) = T_2^{-1} \left(-\frac{\hbar}{2} q_j^{k-1} + \frac{1}{2} \hbar q_j^{k-1} \right) = 0 \quad (\text{M-2})$$

Let us now consider $\mathcal{L}_{\mathbf{u}_{-1}}$. From Proposition 7.4, we have $\mathcal{L}_{\mathbf{u}_{-1}}[T_2] = \hbar T_2$ and $\mathcal{L}_{\mathbf{u}_{-1}}[T_1] = 0$. We also have from Theorem 7.1 $\mathcal{L}_{\mathbf{u}_{-1}}[q_j] = -\hbar q_j$ and $\mathcal{L}_{\mathbf{u}_{-1}}[p_j] = \hbar p_j$. Finally we observe that

$$\begin{aligned} \mathcal{L}_{\mathbf{u}_{-1}}[\tilde{P}_1(\lambda)] &= -\hbar \sum_{i=0}^{r_\infty-2} \mathcal{L}_{\mathbf{u}_{-1}}[t_{\infty,2i+2}]\lambda^i = -\frac{\hbar}{2} \sum_{i=0}^{r_\infty-2} \sum_{r=1}^{2r_\infty-2} r t_{\infty,r} \partial_{t_{\infty,r}}[t_{\infty,2i+2}]\lambda^i \\ &= -\frac{\hbar}{2} \sum_{i=0}^{r_\infty-2} (2i+2) t_{\infty,2i+2} \lambda^i \end{aligned} \quad (\text{M-3})$$

Thus, we get for all $j \in \llbracket 1, g \rrbracket$:

$$\begin{aligned} \mathcal{L}_{\mathbf{u}_{-1}}[\tilde{q}_j] &= \mathcal{L}_{\mathbf{u}_{-1}}[T_2]q_j + T_2 \mathcal{L}_{\mathbf{u}_{-1}}[q_j] + \mathcal{L}_{\mathbf{u}_{-1}}[T_1] = \hbar T_2 q_j + T_2(-\hbar q_j) = 0 \\ \mathcal{L}_{\mathbf{u}_{-1}}[\tilde{p}_j] &= -\frac{\mathcal{L}_{\mathbf{u}_{-1}}[T_2]}{T_2^2} \left(p_j - \frac{1}{2} \tilde{P}_1(q_j) \right) + T_2^{-1} \left(\mathcal{L}_{\mathbf{u}_{-1}}[p_j] - \frac{1}{2} \mathcal{L}_{\mathbf{u}_{-1}}[\tilde{P}_1](q_j) - \frac{1}{2} \mathcal{L}_{\mathbf{u}_{-1}}[q_j] \tilde{P}'_1(q_j) \right) \\ &= -\hbar T_2^{-1} \left(p_j - \frac{1}{2} \tilde{P}_1(q_j) \right) \\ &\quad + \hbar T_2^{-1} \left(p_j + \frac{1}{4} \sum_{i=0}^{r_\infty-2} (2i+2) t_{\infty,2i+2} q_j^i - \frac{1}{2} q_j \sum_{i=1}^{r_\infty-2} i t_{\infty,2i+2} q_j^{i-1} \right) \\ &= \hbar T_2^{-1} \left(-\frac{1}{2} \sum_{i=0}^{r_\infty-2} t_{\infty,2i+2} q_j^i + \frac{1}{4} \sum_{i=0}^{r_\infty-2} (2i+2) t_{\infty,2i+2} q_j^i - \frac{1}{2} \sum_{i=1}^{r_\infty-2} i t_{\infty,2i+2} q_j^i \right) \\ &= 0 \end{aligned} \quad (\text{M-4})$$

Let us now consider $\mathcal{L}_{\mathbf{u}_0}$. From Proposition 7.4, we have $\mathcal{L}_{\mathbf{u}_0}[T_2] = 0$ and $\mathcal{L}_{\mathbf{u}_0}[T_1] = \hbar T_2$. We also have from Theorem 7.1 $\mathcal{L}_{\mathbf{u}_0}[q_j] = -\hbar$ and $\mathcal{L}_{\mathbf{u}_0}[p_j] = 0$. Finally we observe that

$$\begin{aligned} \mathcal{L}_{\mathbf{u}_0}[\tilde{P}_1(\lambda)] &= -\hbar \sum_{i=0}^{r_\infty-2} \mathcal{L}_{\mathbf{u}_0}[t_{\infty,2i+2}]\lambda^i \\ &= -\frac{\hbar}{2} \sum_{i=0}^{r_\infty-2} \sum_{r=1}^{2r_\infty-4} r t_{\infty,r+2} \partial_{t_{\infty,r}}[t_{\infty,2i+2}]\lambda^i \\ &= -\frac{\hbar}{2} \sum_{i=0}^{r_\infty-3} (2i+2) t_{\infty,2i+4} \lambda^i \stackrel{i=s-1}{=} -\hbar \sum_{s=1}^{r_\infty-2} s t_{\infty,2s+2} \lambda^{s-1} \end{aligned} \quad (\text{M-5})$$

Thus, we get for all $j \in \llbracket 1, g \rrbracket$:

$$\mathcal{L}_{\mathbf{u}_0}[\tilde{q}_j] = \mathcal{L}_{\mathbf{u}_0}[T_2]q_j + T_2 \mathcal{L}_{\mathbf{u}_0}[q_j] + \mathcal{L}_{\mathbf{u}_0}[T_1] = -\hbar T_2 + \hbar T_2 = 0$$

$$\begin{aligned}
\mathcal{L}_{\mathbf{u}_0}[\check{p}_j] &= -\frac{\mathcal{L}_{\mathbf{u}_0}[T_2]}{T_2^2} \left(p_j - \frac{1}{2} \tilde{P}_1(q_j) \right) + T_2^{-1} \left(\mathcal{L}_{\mathbf{u}_0}[p_j] - \frac{1}{2} \mathcal{L}_{\mathbf{u}_0}[\tilde{P}_1](q_j) - \frac{1}{2} \mathcal{L}_{\mathbf{u}_0}[q_j] \tilde{P}'_1(q_j) \right) \\
&= 0 + T_2^{-1} \left(0 + \frac{\hbar}{2} \sum_{s=1}^{r_\infty-2} s t_{\infty, 2s+2} q_j^{s-1} - \frac{\hbar}{2} \sum_{i=1}^{r_\infty-2} i t_{\infty, 2i+2} q_j^{i-1} \right) \\
&= 0
\end{aligned} \tag{M-6}$$

N Proof of Theorem 7.2

Let us first observe that a function $f(t_{\infty,1}, t_{\infty,2}, \dots, t_{\infty, 2r_\infty-3})$ solution of $\mathcal{L}_{\mathbf{w}_k}[f] = 0$ for all $k \in \llbracket 1, r_\infty - 1 \rrbracket$ is independent of $(t_{\infty,2}, \dots, t_{\infty, 2r_\infty-2})$. Hence, the function f may only depend on odd irregular times:

$$f(t_{\infty,1}, \dots, t_{\infty, 2r_\infty-2}) = g(t_{\infty,1}, t_{\infty,3}, \dots, t_{\infty, 2r_\infty-3}) \tag{N-1}$$

Let us now consider $\mathcal{L}_{\mathbf{u}_{-1}}[g] = 0$. It is equivalent to

$$0 = \mathcal{L}_{\mathbf{u}_{-1}}[g] = \frac{\hbar}{2} \sum_{m=1}^{r_\infty-1} (2m-1) t_{\infty, 2m-1} \partial_{t_{\infty, 2m-1}}[g] \tag{N-2}$$

whose solutions are arbitrary functions of

$$y_j = \frac{\frac{1}{2} t_{\infty, 2j-1}}{\left(\frac{1}{2} t_{\infty, 2r_\infty-3} \right)^{\frac{2j-1}{2r_\infty-3}}} \text{ with } j \in \llbracket 1, r_\infty - 2 \rrbracket \tag{N-3}$$

In other words:

$$0 = \mathcal{L}_{\mathbf{u}_{-1}}[g] \Leftrightarrow g(t_{\infty,1}, t_{\infty,3}, \dots, t_{\infty, 2r_\infty-3}) = h(y_1, \dots, y_{r_\infty-2}) \tag{N-4}$$

Let us now translate this result to $\mathcal{L}_{\mathbf{u}_0}[h] = 0$. We find

$$\begin{aligned}
0 &= \mathcal{L}_{\mathbf{u}_0}[h(y_1, \dots, y_{r_\infty-2})] = \frac{\hbar}{2} \sum_{m=1}^{r_\infty-2} (2m-1) t_{\infty, 2m+1} \partial_{t_{\infty, 2m+1}}[h(y_1, \dots, y_{r_\infty-2})] \\
&= \frac{\hbar}{2} \sum_{m=1}^{r_\infty-2} (2m-1) t_{\infty, 2m+1} \sum_{r=1}^{r_\infty-2} \frac{\partial y_r}{\partial t_{\infty, 2m+1}} \partial_{y_r} h(y_1, \dots, y_{r_\infty-2}) \\
&= \frac{\hbar}{2} (2r_\infty - 5) \frac{1}{2} t_{\infty, 2r_\infty-3} \left(\frac{1}{2} t_{\infty, 2r_\infty-3} \right)^{-\frac{2r_\infty-5}{2r_\infty-3}} \partial_{y_{r_\infty-2}} h(y_1, \dots, y_{r_\infty-2}) \\
&\quad + \frac{\hbar}{2} \sum_{m=1}^{r_\infty-3} (2m-1) \frac{1}{2} t_{\infty, 2m+1} \left(\frac{1}{2} t_{\infty, 2r_\infty-3} \right)^{-\frac{2m-1}{2r_\infty-3}} \partial_{y_m} h(y_1, \dots, y_{r_\infty-2}) \\
&= \frac{\hbar}{2} \left(\frac{1}{2} t_{\infty, 2r_\infty-3} \right)^{\frac{2}{2r_\infty-3}} \left[(2r_\infty - 5) \partial_{y_{r_\infty-2}} h(y_1, \dots, y_{r_\infty-2}) \right. \\
&\quad \left. + \sum_{m=1}^{r_\infty-3} (2m-1) \frac{1}{2} t_{\infty, 2m+1} \left(\frac{1}{2} t_{\infty, 2r_\infty-3} \right)^{-\frac{2m+1}{2r_\infty-3}} \partial_{y_m} h(y_1, \dots, y_{r_\infty-2}) \right] \\
&= \frac{\hbar}{2} \left(\frac{1}{2} t_{\infty, 2r_\infty-3} \right)^{\frac{2}{2r_\infty-3}} \left[(2r_\infty - 5) \partial_{y_{r_\infty-2}} h(y_1, \dots, y_{r_\infty-2}) \right. \\
&\quad \left. + \sum_{m=1}^{r_\infty-3} (2m-1) y_{m+1} \partial_{y_m} h(y_1, \dots, y_{r_\infty-2}) \right]
\end{aligned} \tag{N-5}$$

We proceed using the following lemma.

Lemma N.1. *The general solutions of the differential equation*

$$(2r_\infty - 5)\partial_{y_{r_\infty-2}}h(y_1, \dots, y_{r_\infty-2}) + \sum_{m=1}^{r_\infty-3} (2m-1)y_{m+1}\partial_{y_m}h(y_1, \dots, y_{r_\infty-2}) = 0 \quad (\text{N-6})$$

are arbitrary functions of

$$\begin{aligned} f_1(y_1, \dots, y_{r_\infty-2}) &= y_{r_\infty-3} - \frac{(2r_\infty - 7)}{2(2r_\infty - 5)}y_{r_\infty-2}^2 \\ &\vdots \\ f_k(y_1, \dots, y_{r_\infty-2}) &= y_{r_\infty-2-k} + \sum_{i=1}^{k-1} \frac{(-1)^i \left(\prod_{s=1}^i (2r_\infty - 2k + 2s - 7) \right) y_{r_\infty-2}^i y_{r_\infty-2-k+i}}{i!(2r_\infty - 5)^i} \\ &\quad + \frac{(-1)^k \left(\prod_{s=1}^{k-1} (2r_\infty - 2k + 2s - 7) \right) (2r_\infty - 7) y_{r_\infty-2}^{k+1}}{(k+1)(k-1)!(2r_\infty - 5)^k} \\ &= \sum_{i=0}^{k-1} \frac{(-1)^i \left(\prod_{s=1}^i (2r_\infty - 2k + 2s - 7) \right) y_{r_\infty-2}^i y_{r_\infty-2-k+i}}{i!(2r_\infty - 5)^i} \\ &\quad + \frac{(-1)^k \left(\prod_{s=1}^k (2r_\infty - 2k + 2s - 7) \right) y_{r_\infty-2}^{k+1}}{(k+1)(k-1)!(2r_\infty - 5)^k} \end{aligned} \quad (\text{N-7})$$

where $k \in \llbracket 1, r_\infty - 3 \rrbracket$.

Proof. Let $k \in \llbracket 1, r_\infty - 3 \rrbracket$. We have:

$$\begin{aligned} (2r_\infty - 5)\partial_{y_{r_\infty-2}}f_k &= \sum_{i=1}^{k-1} \frac{(-1)^i \left(\prod_{s=1}^i (2r_\infty - 2k + 2s - 7) \right) y_{r_\infty-2}^{i-1} y_{r_\infty-2-k+i}}{(i-1)!(2r_\infty - 5)^{i-1}} \\ &\quad + \frac{(-1)^k \left(\prod_{s=1}^{k-1} (2r_\infty - 2k + 2s - 7) \right) (2r_\infty - 7) y_{r_\infty-2}^k}{(k-1)!(2r_\infty - 5)^{k-1}} \\ &= \sum_{i=1}^k \frac{(-1)^i \left(\prod_{s=1}^i (2r_\infty - 2k + 2s - 7) \right) y_{r_\infty-2}^{i-1} y_{r_\infty-2-k+i}}{(i-1)!(2r_\infty - 5)^{i-1}} \end{aligned} \quad (\text{N-8})$$

Moreover, we have:

$$\begin{aligned} \sum_{m=1}^{r_\infty-3} (2m-1)y_{m+1}\partial_{y_m}f_k &= \sum_{i=0}^{k-1} \frac{(-1)^i (2r_\infty - 2k + 2i - 5) \left(\prod_{s=1}^i (2r_\infty - 2k + 2s - 7) \right) y_{r_\infty-2}^i y_{r_\infty-1-k+i}}{i!(2r_\infty - 5)^i} \\ &\quad - \sum_{\substack{j=i+1 \\ j=1}}^k \frac{(-1)^j (2r_\infty - 2k + 2j - 7) \left(\prod_{s=1}^{j-1} (2r_\infty - 2k + 2s - 7) \right) y_{r_\infty-2}^{j-1} y_{r_\infty-k+j-2}}{(j-1)!(2r_\infty - 5)^{j-1}} \end{aligned} \quad (\text{N-9})$$

since in the first equality only $m = r_\infty - 2 - k + i$ provides non-vanishing contributions. We now observe that (N-8) and (N-9) provides opposite contributions so that

$$(2r_\infty - 5)\partial_{y_{r_\infty-2}}f_k(y_1, \dots, y_{r_\infty-2}) + \sum_{m=1}^{r_\infty-3} (2m-1)y_{m+1}\partial_{y_m}f_k(y_1, \dots, y_{r_\infty-2}) = 0 \quad (\text{N-10})$$

□

Combining Lemma N.1 with (N-3), we obtain arbitrary functions of

$$\begin{aligned}
\tau_k = & \sum_{i=0}^{k-1} \frac{(-1)^i \left(\prod_{s=1}^i (2r_\infty - 2k + 2s - 7) \right) \left(\frac{1}{2} t_{\infty, 2r_\infty - 5} \right)^i \left(\frac{1}{2} t_{\infty, 2r_\infty - 3} \right)^{-\frac{(2r_\infty - 3)i + 2r_\infty - 5 - 2k}{2r_\infty - 3}} \frac{1}{2} t_{\infty, 2r_\infty - 5 - 2k + 2i}}{i! (2r_\infty - 5)^i} \\
& + \frac{(-1)^k \left(\prod_{s=1}^k (2r_\infty - 2k + 2s - 7) \right) \left(\frac{1}{2} t_{\infty, 2r_\infty - 5} \right)^{k+1} \left(\frac{1}{2} t_{\infty, 2r_\infty - 3} \right)^{-\frac{(k+1)(2r_\infty - 5)}{2r_\infty - 3}}}{(k+1)(k-1)! (2r_\infty - 5)^k}
\end{aligned} \tag{N-11}$$

with $k \in \llbracket 1, r_\infty - 3 \rrbracket$.

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