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Inflation in reserving triangles :
Taylor and Verbeek techniques are equivalent to
a Chain Ladder on a calendar triangle.

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Abstract

In this paper, we study the estimation of past inflation based on reserving triangles of average claims costs. More precisely, we focus on methodologies introduced by H. G. Verbeek in 1972 and by G.C. Taylor in 1977 that allows to estimate such an historical inflation index based on average claims costs triangle. Those methods rely on recursive formula that challenges the understanding of underlying assumptions and that can complexify the interpretation of the results. We alternatively show the equivalence of those techniques with a Chain Ladder on a specifically rearranged triangle. We therefore highlights the spirit of that technics that firstly compute the ultimate average claims costs by calendar year and secondly deduce the inflation index based on calendar evolutions. Moreover, this allow to rationalize the underlying assumptions of Taylor and Verbeek methodology, using the mathematical conceptualization of Chain Ladder by Mack in 1993. We illustrate those findings on simulated data.

Key words: Reserving; Inflation; Taylor; Verbeek; Chain Ladder.

Short title: Claims inflation : Taylor and Verbeek techniques are equivalent to Chain Ladder.

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1 Introduction

Inflation (resp. deflation) quantify the rate of decrease (resp. increase) of the value of money. At global economic scale, inflation is one of the indicators closely monitored because significant values can challenge people trust in the money system. For instance this is officially the only indicator on which the objective of the European Central Bank is evaluated. Indeed, from 2003 and until 2021, the European Central Bank had to ensure that the mid-term inflation was close but lower to 2% per year. As from 2021, the objective is more symmetric because the European Central Bank has ensure that the mid-term inflation is close to 2% per year. On a financial point of view, the inflation also impact the real interest rates. For insurers the real interest rates are important because the rentability of the assets depends on interest rates and the evaluation of the liabilities depends on the inflation of claims costs.

The insurance business and especially insurance liabilities are also very sensitive to inflation. Thus, monitoring inflation is necessary for relevant level of pricing and reserving. Usually, claims reserves are computed by Chain Ladder methodology, explained by Mack (1993). This methodology allows to compute future developments of claims costs based on the past developments that may have been perturbed historical inflation. Hence the past inflation is implicitly projected in the future, in the sense that there is no parametric assumption on past and future inflation. However, this implicit projection may be challenging for several reasons : we do not control this implicit inflation by a parameter and the underlying assumptions of Chain Ladder may suffer.

Also because the future may not be similar to the past, it can therefore be interested to retreat the claims triangle using an inflation index and then to compute future cash flows based on a parametric assumption. We can use external inflation index but also an experience index based on average claims costs triangles. This technique, introduced by Taylor (1977) and Verbeek (1972) relies on a recursive formula. Several methodologies proposed to extend this methodologies such as Wüthrich (2010), Gigante et al. (2019) and Beard (1977). In this paper, we complete those analysis by showing the equivalence between Taylor and Verbeek technique with an application of the well-known Chain Ladder technique on a calendar triangle. This finding legitimate the applications of all extensions of Chain Ladder in order to estimate claims past inflation on insurance portfolios.

2 Methodology introduced by Taylor and Verbeek

Inflation is a risk to which every insurer is subject, and its control is necessary. Moreover, modeling inflation can be tricky and finding an efficient method is a major issue, taken very seriously in insurance companies. Indeed, a correct estimation of inflation allows to minimize the risk of under-provisioning. For this purpose, the Allianz Group has proposed to use the Verbeek Taylor method, which is much less known in the usual non-life insurance methods than the Chain Ladder method. However, unlike the Chain Ladder method, it allows to determine the inflation factors: $\lambda_j \forall j \in [0, n]$. Indeed, the Chain Ladder method does not take into account the management of historical inflation in the future, and this method only works if inflation is constant.

The Taylor model is an iterative calculation method.

Occurrence\Development	0	1	...	j	...	n-1	n
0	$r_0\lambda_0$	$r_1\lambda_1$...	$r_j\lambda_j$...	$r_{n-1}\lambda_{n-1}$	$r_n\lambda_n$
1	$r_0\lambda_1$	$r_1\lambda_2$...	$r_j\lambda_{j+1}$...	$r_{n-1}\lambda_n$	
⋮	⋮	⋮		⋮	⋮		
$n-j$	$r_0\lambda_{n-j}$	$r_1\lambda_{n-j+1}$...	$r_j\lambda_n$			
⋮	⋮	⋮					
$n-1$	$r_0\lambda_{n-1}$	$r_1\lambda_n$					
n	$r_0\lambda_n$						

where:

- v_j is the sum of $n-j$ observable elements of column j such that $v_j = \sum_{k=0}^{n-j} a_{k,j}$
- d_j is the sum of $j+1$ elements of the diagonal j such that $d_j = \sum_{k=0}^j a_{k,j-k}$
- r_j represents the payment candency such that $r_j = \frac{v_j}{\lambda_j + \lambda_{j+1} + \dots + \lambda_n}$ such that $\sum_{j=0}^n r_j = 1$

- λ_j is an inflation factor such that $\lambda_j = \frac{d_j}{1-r_n-r_{n-1}-\dots-r_{h+1}}$ and $\lambda_n = d_n$

We notice the iterative character of this method. For instance for $n = 5$, to be able to know λ_3 , it is necessary to have previously computed λ_4 and λ_5 .

3 Equivalence with a Chain Ladder on a calendar triangle

As seen in the previous section, the Taylor method is much more complex to understand than the Chain Ladder method. It is by trying to understand it in order to master it, that we found a direct link between the two.

This proposition is really good, because on the first hand, we have a method for estimate inflation and on the other hand, it's a method that we are comfortable, we have trust on it because we use a lot.

This proposal is really satisfactory, since on the one hand, we have a method for estimating inflation and on the other hand, it is a method with which we are comfortable and confident, since we already use it very often.

In fact, in order to find the λ_j , inflation coefficients, we just have to do a Chain Ladder in calendar view on the triangle of average costs. Thus, we will assume that the $a_{i,j}$ coefficients represent average costs.

Let's illustrate this, with $n=5$:

Occurrence \ Development	0	1	2	3	4	5
0	$a_{0,0}$	$a_{0,1}$	$a_{0,2}$	$a_{0,3}$	$a_{0,4}$	$a_{0,5}$
1	$a_{1,0}$	$a_{1,1}$	$a_{1,2}$	$a_{1,3}$	$a_{1,4}$...
2	$a_{2,0}$	$a_{2,1}$	$a_{2,2}$	$a_{2,3}$
3	$a_{3,0}$	$a_{3,1}$	$a_{3,2}$
4	$a_{4,0}$	$a_{4,1}$
5	$a_{5,0}$

We projet on calendar view:

Occurrence\Development	0	1	2	3	4	5
0	$a_{0,0}$
1	$a_{1,0}$	$a_{0,1}$
2	$a_{2,0}$	$a_{1,1}$	$a_{0,2}$
3	$a_{3,0}$	$a_{2,1}$	$a_{1,2}$	$a_{0,3}$
4	$a_{4,0}$	$a_{3,1}$	$a_{2,2}$	$a_{1,3}$	$a_{0,4}$...
5	$a_{5,0}$	$a_{4,1}$	$a_{3,2}$	$a_{2,3}$	$a_{1,4}$	$a_{0,5}$

we accumulate the data:

Occurrence\Development	0	1	2	3	4	5
0	$b_{0,0}$
1	$b_{1,0}$	$b_{1,1}$
2	$b_{2,0}$	$b_{2,1}$	$b_{2,2}$
3	$b_{3,0}$	$b_{3,1}$	$b_{3,2}$	$b_{3,3}$
4	$b_{4,0}$	$b_{4,1}$	$b_{4,2}$	$b_{4,3}$	$b_{4,4}$...
5	$b_{5,0}$	$b_{5,1}$	$b_{5,2}$	$b_{5,3}$	$b_{5,4}$	$b_{5,5}$

where $b_{i,j} = \sum_{k=i-j}^i a_{k,i-k} \forall i, j \in N$ tel que $i - j \geq 0$

Let us note, from now on c_j the coefficients of passage (development) of Chain Ladder in calendar view: $c_j = \frac{\sum_{k=j}^n b_{k,j}}{\sum_{k=j}^n b_{k,j-1}}$, which make it possible to complete the upper triangle.

Hence:

Occurrence\Development	0	1	2	3	4	5
0	$b_{0,0}$	$b_{0,1}$	$b_{0,2}$	$b_{0,3}$	$b_{0,4}$	$b_{0,5}$
1	$b_{1,0}$	$b_{1,1}$	$b_{1,2}$	$b_{1,3}$	$b_{1,4}$	$b_{1,5}$
2	$b_{2,0}$	$b_{2,1}$	$b_{2,2}$	$b_{2,3}$	$b_{2,4}$	$b_{2,5}$
3	$b_{3,0}$	$b_{3,1}$	$b_{3,2}$	$b_{3,3}$	$b_{4,3}$	$b_{5,3}$
4	$b_{4,0}$	$b_{4,1}$	$b_{4,2}$	$b_{4,3}$	$b_{4,4}$	$b_{4,5}$
5	$b_{5,0}$	$b_{5,1}$	$b_{5,2}$	$b_{5,3}$	$b_{5,4}$	$b_{5,5}$

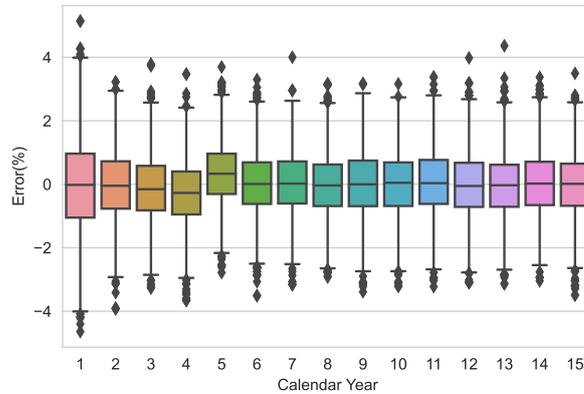
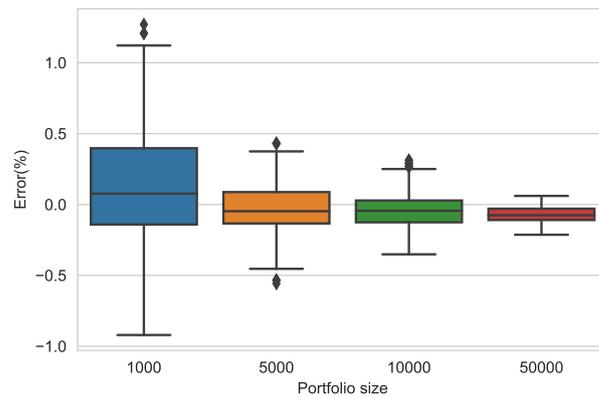
We have thus determined the inflation coefficients λ_j $j \in [0, 5]$ where $\lambda_5 = b_{5,5}$, $\lambda_4 = b_{4,5}$, $\lambda_3 = b_{3,5}$, $\lambda_2 = b_{2,5}$, $\lambda_1 = b_{1,5}$ et $\lambda_0 = b_{0,5}$.

Once the approach is illustrated, let's move on to the proposition:

$$\forall h < n \quad \lambda_h = b_{h,h} \times \prod_{k=h+1}^n c_k \text{ and si } h = n \quad \lambda_n = b_{n,n}$$

You can find the mathematical demonstration in the appendix.

We have also implemented this method, to illustrate its efficiency. Here are some box-plots illustrating the estimation of inflation with this new method:



We can thus see that for each year of occurrence ($n=15$), the error is almost zero, and it is even more so when the size of the portfolio increases. This confirms the relevance of our proposal.

4 Simulation study

4.1 Impact of average cost triangle computation

As mentioned before, this demonstration is based on a triangle of average costs. Therefore we wanted to focus on the obtaining of this triangle, more precisely on the way to divide the triangle of costs (payments), in order to improve the performance of the model.

Here are the 3 methods we are interested in:

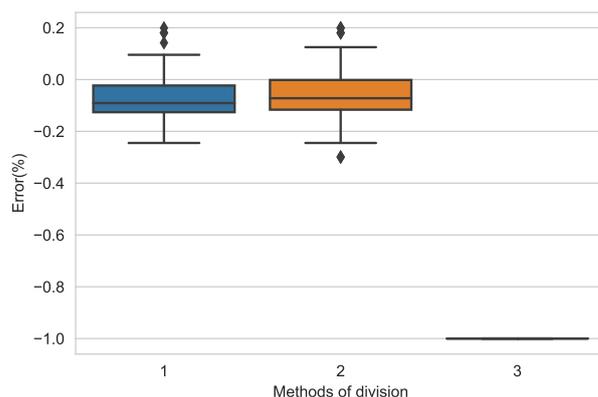
Method 1: Method 1 is the usual method used in actuarial science, which consists in starting from the triangle of the evolution of the number of claims and making a chain ladder in order to obtain the vector of the ultimate number of claims for each year of occurrence. Then divide the triangle of the evolution of the costs by this vector, and finally project the triangle obtained in calendar view then cumulate it.

Method 2: Method 2 is a method for which we will use a simulated data: the number of claims which comes from the sum of the columns for each line of the rectangle of the number of claims that we simulated. Then we divide the triangle of the evolution of the costs by this ultimate number of claims simulated, and finally, we project the obtained triangle in calendar view and cumulate it.

Method 3: For method 3, we first cumulate the triangle of the evolution of the number of claims and the triangle of the evolution of the costs, then we divide the two, and project the obtained triangle in calendar view. For this method, we will not cumulate the data since the coefficients have already been cumulated in the previous step.

At the end of these 3 methods, we can directly make a Chain-Ladder, and thus complete the upper triangle and recover in the last column of the rectangle the Taylor coefficients, which are the inflation coefficients !

Comparison of the 3 methods:



Thus, we can see that method 2 seems to estimate inflation the best, this seems consistent since it is the method that has the most information. The method 1 also estimates the inflation very well, it is moreover less extensive than the method 1. The method 3 as for it has an estimation error of 1 %.

4.2 Impact of Chain Ladder ratio selections

Since it has been shown that the inflation estimate search is really just a Chain Ladder projected into a calendar view of the average cost triangle, we wondered if we could apply the same generalizations that we are used to doing with the Chain Ladder method. For this purpose, we implemented 3 methods calculating in a different way the Chain Ladder development coefficients.

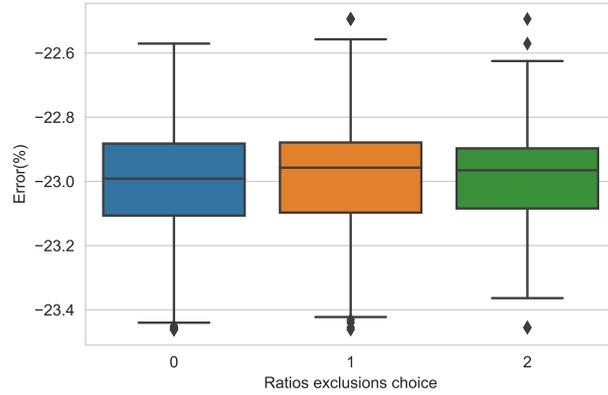
Exclusion method ratio 0: The method 0, is the calculation of the coefficients of passage of classic way.

Exclusion ratio method 1: The method 1 is the calculation of the coefficients of passage with exclusion of the min and the max on the triangle of the ratio.

Ratio exclusion method 2: Method 2 is the calculation of the pass coefficients with selection of the last 5 diagonals and exclusion of the min and max on the ratio triangle.

For this, we deliberately simulated a triangle with volatile data, hence the gross error.

Comparison of the 3 methods:



Thus, we note that methods 2 and 3 that exclude data provide a better estimate of inflation. To this end, we can consider using these Chain Ladder generalization methods with the Taylor method.

5 Conclusion

As we have discussed and explained throughout this report, inflation estimation is a very important topic for insurers. The Taylor Verbeek method allows for the estimation of inflation, however, it is a little known method and quite complicated to understand and interpret. The Chain Ladder method, on the other hand, is a basic method of calculating reserves in non-life insurance with fairly simple reasoning. The fact that the Taylor method has been shown to be a Chain Ladder projected into a calendar view should simplify and help the inflation estimation calculations.

6 Appendix

We know that:

$$\lambda_h = \frac{d_h}{1 - r_n - r_{n-1} - \dots - r_{h+1}} = \frac{b_{h,h}}{1 - r_n - r_{n-1} - \dots - r_{h+1}}$$

and we want to show that

$$\lambda_h = b_{h,h} \times \prod_{k=h+1}^n c_k$$

so we want to show that

Proposal 1:

$$\boxed{\frac{1}{1 - r_n - r_{n-1} - \dots - r_{h+1}} = \prod_{k=h+1}^n c_k}$$

To show this equality (1) we will make a recurrence proof on h . We suppose (1) to be true for the rank $n, n-1, \dots, h+1$ and we will show it for the rank h .

For that it is enough to show the following 4 propositions:

Proposal 2:

Show (1) is equivalent at show that:

$$\boxed{1 - r_{h+1} \times \frac{b_{n,n} \times \prod_{k=h+2}^{n-1} c_k}{b_{n,n-1}} = \frac{1}{c_{h+1}}}$$

Proposal 3:

$$\boxed{1 - r_{h+1} \frac{b_{n,n} \times \prod_{k=h+2}^{n-1} c_k}{b_{n,n-1}} = \frac{(b_{n,n-1} + \sum_{i=h+1}^{n-1} b_{i,i} \prod_{k=i+1}^{n-1} c_k) \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} - \sum_{i=0}^{n-(h+1)} a_{i,h+1} \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{(b_{n,n-1} + \sum_{i=h+1}^{n-1} b_{i,i} \prod_{k=i+1}^{n-1} c_k) \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1}}}$$

Proposal 4:

$$\boxed{(b_{n,n-1} + \sum_{i=h+1}^{n-1} b_{i,i} \prod_{k=i+1}^{n-1} c_k) \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} = b_{n,n-1} \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} + \sum_{j=h+1}^{n-1} b_{j,j} \left[\left(\prod_{i=j+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) \left(\prod_{i=h+2}^j \sum_{k=i}^n b_{k,i-1} \right) \right]}$$

Proposal 5:

$$\boxed{b_{n,n-1} \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} + \sum_{j=h+1}^{n-1} b_{j,j} \left[\left(\prod_{i=j+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) \left(\prod_{i=h+2}^j \sum_{k=i}^n b_{k,i-1} \right) \right] = \prod_{i=h+1}^{n-1} \sum_{k=i}^n b_{k,i}}$$

Indeed, if we assume the above equalities to be true, then we have :

According to (3), we have:

$$1 - r_{h+1} \frac{b_{n,n} \times \prod_{k=h+2}^{n-1} c_k}{b_{n,n-1}} = \frac{(b_{n,n-1} + \sum_{i=h+1}^{n-1} b_{i,i} \prod_{k=i+1}^{n-1} c_k) \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} - \sum_{i=0}^{n-(h+1)} a_{i,h+1} \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{(b_{n,n-1} + \sum_{i=h+1}^{n-1} b_{i,i} \prod_{k=i+1}^{n-1} c_k) \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1}}$$

According to (4), we have:

$$= \frac{b_{n,n-1} \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} + \sum_{j=h+1}^{n-1} b_{j,j} \left[\left(\prod_{i=j+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) \left(\prod_{i=h+2}^j \sum_{k=i}^n b_{k,i-1} \right) \right] - \sum_{i=0}^{n-(h+1)} a_{i,h+1} \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{b_{n,n-1} \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} + \sum_{j=h+1}^{n-1} b_{j,j} \left[\left(\prod_{i=j+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) \left(\prod_{i=h+2}^j \sum_{k=i}^n b_{k,i-1} \right) \right]}$$

And after (5), we have:

$$\begin{aligned} &= \frac{\prod_{i=h+1}^{n-1} \sum_{k=i}^n b_{k,i} - \sum_{i=0}^{n-(h+1)} a_{i,h+1} \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{\prod_{i=h+1}^{n-1} \sum_{k=i}^n b_{k,i}} \\ &= \frac{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i} \times \sum_{k=h+1}^n b_{k,h+1} - \sum_{i=0}^{n-(h+1)} a_{i,h+1} \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{\prod_{i=h+1}^{n-1} \sum_{k=i}^n b_{k,i}} \\ &= \frac{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i} \times \left(\sum_{k=h+1}^n b_{k,h+1} - \sum_{i=0}^{n-(h+1)} a_{i,h+1} \right)}{\prod_{i=h+1}^{n-1} \sum_{k=i}^n b_{k,i}} \\ &= \frac{\sum_{k=h+1}^n b_{k,h}}{\sum_{k=h+1}^n b_{k,h+1}} \\ &= \frac{1}{c_{h+1}} \end{aligned}$$

Now according to Proposition 2, showing (1) is equivalent to showing (2), and we have just shown (2), hence (1).

Let us now prove (2), (3), (4) and (5)

Proof of (2):

First, by the recurrence hypothesis, we know the expression of λ_{h+1} , so we know that

:

$$\frac{1}{1 - r_n - r_{n-1} - \dots - r_{h+2}} = \prod_{k=h+2}^{n-1} c_k \times c_n$$

So we know the expression of:

$$1 - r_n - r_{n-1} - \dots - r_{h+2} = \frac{1}{c_n \times \prod_{k=h+2}^{n-1} c_k}$$

so,

$$\begin{aligned} 1 - r_n - r_{n-1} - \dots - r_{h+2} - r_{h+1} &= (1 - r_n - r_{n-1} - \dots - r_{h+2}) - r_{h+1} \\ &= \frac{1}{c_n \times \prod_{k=h+2}^{n-1} c_k} - r_{h+1} \end{aligned}$$

$$= \frac{b_{n,n-1}}{b_{n,n}} \times \frac{1}{\prod_{k=h+2}^{n-1} c_k} \left(1 - r_{h+1} \times \frac{b_{n,n} \times \prod_{k=h+2}^{n-1} c_k}{b_{n,n-1}} \right)$$

so it is enough to show that :

$$1 - r_{h+1} \times \frac{b_{n,n} \times \prod_{k=h+2}^{n-1} c_k}{b_{n,n-1}} = \frac{1}{c_{h+1}}$$

Demonstration of (3):

$$\begin{aligned} 1 - r_{h+1} \frac{b_{n,n} \times \prod_{k=h+2}^{n-1} c_k}{b_{n,n-1}} &= 1 - r_{h+1} \times \frac{b_{n,n}}{b_{n,n-1}} \times \frac{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1}} \\ &= 1 - \frac{v_{h+1}}{\lambda_n + \lambda_{n-1} + \dots + \lambda_{h+1}} \times \frac{b_{n,n}}{b_{n,n-1}} \times \frac{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1}} \end{aligned}$$

So using the recurrence hypothesis on the λ_k pour $k \in [h+1, n]$

$$\begin{aligned} &= 1 - \frac{\sum_{i=0}^{n-(h+1)} a_{i,h+1}}{b_{n,n} + \sum_{i=h+1}^{n-1} \lambda_i} \times \frac{b_{n,n}}{b_{n,n-1}} \times \frac{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1}} \\ &= 1 - \frac{\sum_{i=0}^{n-(h+1)} a_{i,h+1}}{b_{n,n} + \sum_{i=h+1}^{n-1} b_{i,i} \prod_{k=i+1}^{n-1} c_k \times c_n} \times \frac{b_{n,n}}{b_{n,n-1}} \times \frac{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1}} \\ &= 1 - \frac{\sum_{i=0}^{n-(h+1)} a_{i,h+1}}{b_{n,n} + (\sum_{i=h+1}^{n-1} b_{i,i} \prod_{k=i+1}^{n-1} c_k) \times \frac{b_{n,n}}{b_{n,n-1}}} \times \frac{b_{n,n}}{b_{n,n-1}} \times \frac{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1}} \\ &= 1 - \frac{\sum_{i=0}^{n-(h+1)} a_{i,h+1}}{b_{n,n} (1 + (\sum_{i=h+1}^{n-1} b_{i,i} \prod_{k=i+1}^{n-1} c_k) \frac{1}{b_{n,n-1}})} \times \frac{b_{n,n}}{b_{n,n-1}} \times \frac{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1}} \\ &= 1 - \frac{\sum_{i=0}^{n-(h+1)} a_{i,h+1}}{b_{n,n-1} + \sum_{i=h+1}^{n-1} b_{i,i} \prod_{k=i+1}^{n-1} c_k} \times \frac{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1}} \end{aligned}$$

$$= \frac{(b_{n,n-1} + \sum_{i=h+1}^{n-1} b_{i,i} \prod_{k=i+1}^{n-1} c_k) \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} - \sum_{i=0}^{n-(h+1)} a_{i,h+1} \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{(b_{n,n-1} + \sum_{i=h+1}^{n-1} b_{i,i} \prod_{k=i+1}^{n-1} c_k) \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1}}$$

Demonstration of (4):

First, by rewriting (3) in terms of the expression of c_k , on a:

$$b_{n,n-1} + \sum_{i=h+1}^{n-1} b_{i,i} \prod_{k=i+1}^{n-1} c_k =$$

$$b_{n,n-1} + b_{n-1,n-1} + b_{n-2,n-2} \frac{\sum_{k=n-1}^n b_{k,n-1}}{\sum_{k=n-1}^n b_{k,n-2}} + \dots + b_{h+2,h+2} \frac{\prod_{i=h+3}^{n-1} \sum_{k=i}^n b_{k,i}}{\prod_{i=h+3}^{n-1} \sum_{k=i}^n b_{k,i-1}} + b_{h+1,h+1} \frac{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1}}$$

By putting in the same denominator

$$= \frac{(b_{n,n-1} + b_{n-1,n-1}) \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} + b_{n-2,n-2} \sum_{k=n-1}^n b_{k,n-1} \times \prod_{i=h+2}^{n-2} \sum_{k=i}^n b_{k,i-1} + \dots + b_{h+1,h+1} \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1}}$$

So

$$(b_{n,n-1} + \sum_{i=h+1}^{n-1} b_{i,i} \prod_{k=i+1}^{n-1} c_k) \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1}$$

$$= (b_{n,n-1} + b_{n-1,n-1}) \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} + b_{n-2,n-2} \sum_{k=n-1}^n b_{k,n-1} \times \prod_{i=h+2}^{n-2} \sum_{k=i}^n b_{k,i-1} + \dots + b_{h+1,h+1} \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}$$

$$= (b_{n,n-1} + b_{n-1,n-1}) \prod_{i=n}^{n-1} \sum_{k=i}^n b_{k,i} \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} + b_{n-2,n-2} \prod_{i=n-1}^{n-1} \sum_{k=i}^n b_{k,i} \times \prod_{i=h+2}^{n-2} \sum_{k=i}^n b_{k,i-1} + \dots + b_{h+1,h+1} \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i} \times \prod_{i=h+2}^{h+1} \sum_{k=i}^n b_{k,i-1}$$

$$= b_{n,n-1} \times \prod_{i=n}^{n-1} \sum_{k=i}^n b_{k,i} \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} + \sum_{j=h+1}^{n-1} b_{j,j} \times \left[\left(\prod_{i=j+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) \times \left(\prod_{i=h+2}^j \sum_{k=i}^n b_{k,i-1} \right) \right]$$

that is to say

$$= b_{n,n-1} \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} + \sum_{j=h+1}^{n-1} b_{j,j} \times \left[\left(\prod_{i=j+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) \times \left(\prod_{i=h+2}^j \sum_{k=i}^n b_{k,i-1} \right) \right]$$

Demonstration of (5):

We want to show that:

$$b_{n,n-1} \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} + \sum_{j=h+1}^{n-1} b_{j,j} \left[\left(\prod_{i=j+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) \left(\prod_{i=h+2}^j \sum_{k=i}^n b_{k,i-1} \right) \right] = \prod_{i=h+1}^{n-1} \sum_{k=i}^n b_{k,i}$$

Show this equality by recurrence

Let us assume the property is true for rank $h+1$ and show that it is still true for rank h .

Thus we want to show that :

$$b_{n,n-1} \times \prod_{i=h+1}^{n-1} \sum_{k=i}^n b_{k,i-1} + \sum_{j=h}^{n-1} b_{j,j} \left[\left(\prod_{i=j+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) \left(\prod_{i=h+1}^j \sum_{k=i}^n b_{k,i-1} \right) \right] = \prod_{i=h}^{n-1} \sum_{k=i}^n b_{k,i}$$

To do this, let's show the recurrence hypothesis in the above expression:

$$\begin{aligned} &= b_{n,n-1} \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} \times \sum_{k=h+1}^n b_{k,h} + \sum_{j=h+1}^{n-1} b_{j,j} \left[\left(\prod_{i=j+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) \left(\prod_{i=h+2}^j \sum_{k=i}^n b_{k,i-1} \right) \left(\sum_{k=h+1}^n b_{k,h} \right) \right] \\ &\quad + b_{h,h} \left(\prod_{i=h+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) \left(\prod_{i=h+1}^h \sum_{k=i}^n b_{k,i-1} \right) \\ &= \sum_{k=h+1}^n b_{k,h} \left[b_{n,n-1} \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} + \sum_{j=h+1}^{n-1} b_{j,j} \left(\prod_{i=j+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) \left(\prod_{i=h+2}^j \sum_{k=i}^n b_{k,i-1} \right) \right] \\ &\quad + b_{h,h} \left(\prod_{i=h+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) \left(\prod_{i=h+1}^h \sum_{k=i}^n b_{k,i-1} \right) \\ &= \sum_{k=h+1}^n b_{k,h} \left(\prod_{i=h+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) + b_{h,h} \left(\prod_{i=h+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) \\ &= \prod_{i=h+1}^{n-1} \sum_{k=i}^n b_{k,i} \times \left(\sum_{k=h}^n b_{k,h} \right) \end{aligned}$$

$$= \prod_{i=h}^{n-1} \sum_{k=i}^n b_{k,i}$$

hence we obtain the result.

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