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A tailored Chain Ladder methodology for inflation analysis in reserving triangles

Sébastien FARKAS (Allianz en France) and Amélie ROUX (Allianz en France)

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Abstract

This report deals with the estimation of inflation coefficients obtained with the Taylor method. In this report, we will first present the challenge behind the search for an efficient estimate of inflation. Then we will show that to obtain these coefficients it is enough to make a Chain Ladder project in calendar view.

Key words: Inflation, Chain Ladder, Taylor.

1 Introduction

If inflation is a recurrent media subject, evoked between financial and economic analysts, in the stock exchange, in many magazines and financial reports, but also in many economic models, the sanitary and economic crisis caused by the Covid-19 epidemic as well as the current geopolitical context has put inflation, more than ever, at the heart of political and societal debates.

Concretely and simply, inflation is the loss of purchasing power of money. It is often accompanied by economic policies, as recently with the inflation premium for the start of the school year 2022, implemented by the government of Jean Castex.

While inflation is often a concern for households, it plays a very ambiguous role in finance; some fear it and others welcome it. This is particularly the case in assurance. Moreover, when it comes to assets, inflation has a positive impact: the insurer will be able to invest heavily in debt securities, which will have an optimal return. However, this inflation will have a negative impact on liabilities and may result in under-provisioning if measured incorrectly. In fact, insurers have in their portfolio sinistres will be paid later, so this can entrain a period of uncertainty more longer and they will have an important inflationist effect on payments and those on the cost of provisions. In fact, insurers hold claims in their portfolio that will be settled over several years, which will lead to a longer period of uncertainty and will have a consequent inflationary effect on payments and therefore on the cost of reserves. In other words, insurers must have reserves for both their short and long lines of business in order to meet their future liabilities, the reverse production cycle. Therefore, inflation is an important part of the long-tail risk. Due to their nature, the efficient estimation of inflation over the long run becomes difficult and this can impact the technical reserves and thus the insurers' profits more widely. As a result, the welfare of the insurance portfolio depends to a large extent on inflation. Thus, controlling inflation is necessary for the prosperity of insurance companies and a skill to be acquired by actuaries. Whether or not they restate historical inflation in their triangles, its knowledge is fundamental.

So how do we handle inflation in our data and which model is appropriate for a good inflation estimate? To follow up on this, the Allianz group proposed to use the Taylor method to obtain the inflation coefficients, a method that is not widely used and that seems at first sight quite complex. While trying to understand this method, we found a direct link between this method and the widely used Chain Ladder method. This is the main issue of our approach and of the work that will be presented in this report.

First, we will give some reminders about the Chain Ladder method as well as about different constructions of triangles that we will use in this demonstration. Then, we will present the Taylor method, its usefulness, and its construction, then we will expose our results. Finally, we will propose some generalizations and deepening that we can use thanks to these results.

2 Reminders: Chain Ladder method and construction of triangles

In actuarial science, development triangles are the basis of many models. Development triangles are used to calculate reserves and ultimate expenses, which are crucial data in actuarial science. Indeed, the stakes of a correct prediction of the evolution of the cost

of the claims are necessary for an effective control of the risks. The data in these triangles can be of different kinds: payments, expenses, number of claims, etc.

Here is an example of a classic triangle, where the data are the number of claims and are represented by the coefficients $a_{i,j}$, $i, j \in [0, n]$ represent, respectively, the year of occurrence and the year of development of claims. In other words, the coefficient $a_{i,j}$ represents the number of claims that occurred in year i and were paid in year $i+j$.

Occurrence\Development	0	1	...	j	...	n-1	n
0	$a_{0,0}$	$a_{0,1}$...	$a_{0,j}$...	$a_{0,n-1}$	$a_{0,n}$
1	$a_{1,0}$	$a_{1,1}$...	$a_{1,j}$...	$a_{1,n-1}$	
⋮	⋮	⋮		⋮	⋮		
$n-j$	$a_{n-j,0}$	$a_{n-j,1}$...	$a_{n-j,j}$			
⋮	⋮	⋮					
$n-1$	$a_{n-1,0}$	$a_{n-1,1}$					
n	$a_{n,0}$						

The Chain-Ladder method is a deterministic method of estimation, it is a method of calculating ultimate expenses based on development triangles where the data is accumulated. Furthermore, it is the most widely used method for estimating reserves. It is based on the idea that previous claims develop in a similar way to previous ones and independently of their year of occurrence. The calculation of these reserves, which are the elements located in the lower triangle of the rectangle above, are calculated from development coefficients from the development years (columns), in fact, they are ratios of sums.

To illustrate this method, let us fix $n=5$ and keep the same notations:

Occurrence\Development	0	1	2	3	4	5
0	$a_{0,0}$	$a_{0,1}$	$a_{0,2}$	$a_{0,3}$	$a_{0,4}$	$a_{0,5}$
1	$a_{1,0}$	$a_{1,1}$	$a_{1,2}$	$a_{1,3}$	$a_{1,4}$...
2	$a_{2,0}$	$a_{2,1}$	$a_{2,2}$	$a_{2,3}$
3	$a_{3,0}$	$a_{3,1}$	$a_{3,2}$
4	$a_{4,0}$	$a_{4,1}$
5	$a_{5,0}$

The coefficients of development, allowing to calculate the provisionals, (the empty cells of the rectangle) are:

$$C_j = \frac{\sum_{k=0}^{n-j} b_{k,j}}{\sum_{k=j}^n b_{k,j-1}} \quad \forall j \in [0, n]$$

Thus, for $j = 1$ we have: $C_1 = \frac{\sum_{k=0}^4 a_{k,1}}{\sum_{k=1}^5 a_{k,0}}$ and $a_{0,1} * C_1 = a_{5,1}$

Hence

Occurrence\Development	0	1	2	3	4	5
0	$a_{0,0}$	$a_{0,1}$	$a_{0,2}$	$a_{0,3}$	$a_{0,4}$	$a_{0,5}$
1	$a_{1,0}$	$a_{1,1}$	$a_{1,2}$	$a_{1,3}$	$a_{1,4}$...
2	$a_{2,0}$	$a_{2,1}$	$a_{2,2}$	$a_{2,3}$
3	$a_{3,0}$	$a_{3,1}$	$a_{3,2}$
4	$a_{4,0}$	$a_{4,1}$
5	$a_{5,0}$	$a_{5,1}$

By determining all the coefficients C_j , we can complete the lower triangle of the table, corresponding to the provisions:

Survenance\Développement	0	1	2	3	4	5
0	$a_{0,0}$	$a_{0,1}$	$a_{0,2}$	$a_{0,3}$	$a_{0,4}$	$a_{0,5}$
1	$a_{1,0}$	$a_{1,1}$	$a_{1,2}$	$a_{1,3}$	$a_{1,4}$	$a_{1,5}$
2	$a_{2,0}$	$a_{2,1}$	$a_{2,2}$	$a_{2,3}$	$a_{2,4}$	$a_{2,5}$
3	$a_{3,0}$	$a_{3,1}$	$a_{3,2}$	$a_{3,1}$	$a_{3,4}$	$a_{3,5}$
4	$a_{4,0}$	$a_{4,1}$	$a_{4,2}$	$a_{4,3}$	$a_{4,4}$	$a_{4,5}$
5	$a_{5,0}$	$a_{5,1}$	$a_{5,2}$	$a_{5,3}$	$a_{5,4}$	$a_{5,5}$

We have seen the construction of the triangles in relation to their years of occurrence, but other projections are also possible, such as the calendar view. In fact, it is a question of projecting the data according to the year of development instead of the year of occurrence of the claims, in concrete terms, it is a different arrangement of the diagonals: "they will be flattened".

So by projecting in calendar view this triangle:

Occurence\Development	0	1	2	3	4	5
0	$a_{0,0}$	$a_{0,1}$	$a_{0,2}$	$a_{0,3}$	$a_{0,4}$	$a_{0,5}$
1	$a_{1,0}$	$a_{1,1}$	$a_{1,2}$	$a_{1,3}$	$a_{1,4}$...
2	$a_{2,0}$	$a_{2,1}$	$a_{2,2}$	$a_{2,3}$
3	$a_{3,0}$	$a_{3,1}$	$a_{3,2}$
4	$a_{4,0}$	$a_{4,1}$
5	$a_{5,0}$

We obtain:

Occurence\Development	0	1	2	3	4	5
0	$a_{0,0}$
1	$a_{1,0}$	$a_{0,1}$
2	$a_{2,0}$	$a_{1,1}$	$a_{0,2}$
3	$a_{3,0}$	$a_{2,1}$	$a_{1,2}$	$a_{0,3}$
4	$a_{4,0}$	$a_{3,1}$	$a_{2,2}$	$a_{1,3}$	$a_{0,4}$...
5	$a_{5,0}$	$a_{4,1}$	$a_{3,2}$	$a_{2,3}$	$a_{1,4}$	$a_{0,5}$

3 Taylor method

Inflation is a risk to which every insurer is subject, and its control is necessary. Moreover, modeling inflation can be tricky and finding an efficient method is a major issue, taken very seriously in insurance companies. Indeed, a correct estimation of inflation allows to minimize the risk of under-provisioning. For this purpose, the Allianz Group has proposed to use the Verbeek Taylor method, which is much less known in the usual non-life insurance methods than the Chain Ladder method. However, unlike the Chain Ladder method, it allows to determine the inflation factors: $\lambda_j \forall j \in [0, n]$. Indeed, the Chain Ladder method does not take into account the management of historical inflation in the future, and this method only works if inflation is constant.

The Taylor model is an iterative calculation method.

Occurence\Development	0	1	...	j	...	n-1	n
0	$r_0\lambda_0$	$r_1\lambda_1$...	$r_j\lambda_j$...	$r_{n-1}\lambda_{n-1}$	$r_n\lambda_n$
1	$r_0\lambda_1$	$r_1\lambda_2$...	$r_j\lambda_{j+1}$...	$r_{n-1}\lambda_n$	
...		
$n-j$	$r_0\lambda_{n-j}$	$r_1\lambda_{n-j+1}$...	$r_j\lambda_n$			
...					
$n-1$	$r_0\lambda_{n-1}$	$r_1\lambda_n$					
n	$r_0\lambda_n$						

where:

- v_j is the sum of $n-j$ observable elements of column j such that $v_j = \sum_{k=0}^{n-j} a_{k,j}$
- d_j is the sum of $j+1$ elements of the diagonal j such that $d_j = \sum_{k=0}^j a_{k,j-k}$
- r_j represents the payment candency such that $r_j = \frac{v_j}{\lambda_j + \lambda_{j+1} + \dots + \lambda_n}$ such that $\sum_{j=0}^n r_j = 1$
- λ_j is an inflation factor such that $\lambda_j = \frac{d_j}{1 - r_n - r_{n-1} - \dots - r_{h+1}}$ and $\lambda_n = d_n$

We notice the iterative character of this method: to know λ_3 , it is necessary to have determined λ_4 and λ_5 , si $n = 5$

4 Results

As seen in the previous section, the Taylor method is much more complex to understand than the Chain Ladder method. It is by trying to understand it in order to master it, that we found a direct link between the two.

This proposition is really good, because on the first hand, we have a method for estimate inflation and on the other hand, it's a method that we are comfortable, we have trust on it because we use a lot.

This proposal is really satisfactory, since on the one hand, we have a method for estimating inflation and on the other hand, it is a method with which we are comfortable and confident, since we already use it very often.

In fact, in order to find the λ_j , inflation coefficients, we just have to do a Chain Ladder in calendar view on the triangle of average costs. Thus, we will assume that the $a_{i,j}$ coefficients represent average costs.

Let's illustrate this, with $n=5$:

Occurence\Development	0	1	2	3	4	5
0	$a_{0,0}$	$a_{0,1}$	$a_{0,2}$	$a_{0,3}$	$a_{0,4}$	$a_{0,5}$
1	$a_{1,0}$	$a_{1,1}$	$a_{1,2}$	$a_{1,3}$	$a_{1,4}$...
2	$a_{2,0}$	$a_{2,1}$	$a_{2,2}$	$a_{2,3}$
3	$a_{3,0}$	$a_{3,1}$	$a_{3,2}$
4	$a_{4,0}$	$a_{4,1}$
5	$a_{5,0}$

We projet on calendar view:

Occurence\Development	0	1	2	3	4	5
0	$a_{0,0}$
1	$a_{1,0}$	$a_{0,1}$
2	$a_{2,0}$	$a_{1,1}$	$a_{0,2}$
3	$a_{3,0}$	$a_{2,1}$	$a_{1,2}$	$a_{0,3}$
4	$a_{4,0}$	$a_{3,1}$	$a_{2,2}$	$a_{1,3}$	$a_{0,4}$...
5	$a_{5,0}$	$a_{4,1}$	$a_{3,2}$	$a_{2,3}$	$a_{1,4}$	$a_{0,5}$

we accumulate the data:

Occurence\Development	0	1	2	3	4	5
0	$b_{0,0}$
1	$b_{1,0}$	$b_{1,1}$
2	$b_{2,0}$	$b_{2,1}$	$b_{2,2}$
3	$b_{3,0}$	$b_{3,1}$	$b_{3,2}$	$b_{3,3}$
4	$b_{4,0}$	$b_{4,1}$	$b_{4,2}$	$b_{4,3}$	$b_{4,4}$...
5	$b_{5,0}$	$b_{5,1}$	$b_{5,2}$	$b_{5,3}$	$b_{5,4}$	$b_{5,5}$

where $b_{i,j} = \sum_{k=i-j}^i a_{k,i-k} \forall i, j \in N$ tel que $i - j \geq 0$

Let us note, from now on c_j the coefficients of passage (development) of Chain Ladder in calendar view: $c_j = \frac{\sum_{k=j}^n b_{k,j}}{\sum_{k=j}^n b_{k,j-1}}$, which make it possible to complete the upper triangle.

Hence:

Occurence\Development	0	1	2	3	4	5
0	$b_{0,0}$	$b_{0,1}$	$b_{0,2}$	$b_{0,3}$	$b_{0,4}$	$b_{0,5}$
1	$b_{1,0}$	$b_{1,1}$	$b_{1,2}$	$b_{1,3}$	$b_{1,4}$	$b_{1,5}$
2	$b_{2,0}$	$b_{2,1}$	$b_{2,2}$	$b_{2,3}$	$b_{2,4}$	$b_{2,5}$
3	$b_{3,0}$	$b_{3,1}$	$b_{3,2}$	$b_{3,3}$	$b_{4,3}$	$b_{5,3}$
4	$b_{4,0}$	$b_{4,1}$	$b_{4,2}$	$b_{4,3}$	$b_{4,4}$	$b_{4,5}$
5	$b_{5,0}$	$b_{5,1}$	$b_{5,2}$	$b_{5,3}$	$b_{5,4}$	$b_{5,5}$

We have thus determined the inflation coefficients λ_j $j \in [0, 5]$ where $\lambda_5 = b_{5,5}$, $\lambda_4 = b_{4,5}$, $\lambda_3 = b_{3,5}$, $\lambda_2 = b_{2,5}$, $\lambda_1 = b_{1,5}$ et $\lambda_0 = b_{0,5}$.

Once the approach is illustrated, let's move on to the proposition:

$$\forall h < n \quad \lambda_h = b_{h,h} \times \prod_{k=h+1}^n c_k \text{ and si } h = n \quad \lambda_n = b_{n,n}$$

You can find the mathematical demonstration in the appendix.

We have also implemented this method, to illustrate its efficiency. Here are some box-plots illustrating the estimation of inflation with this new method:

Illustration of performances according to portfolio size

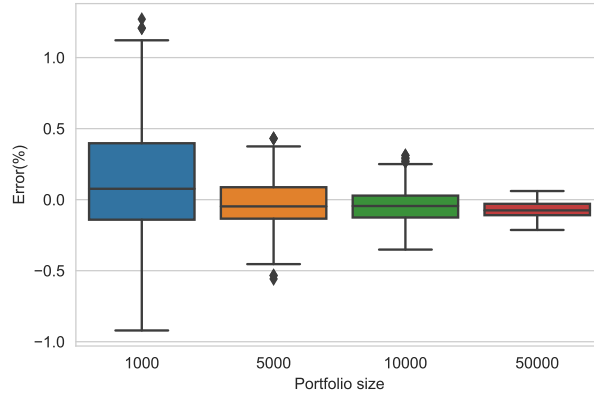
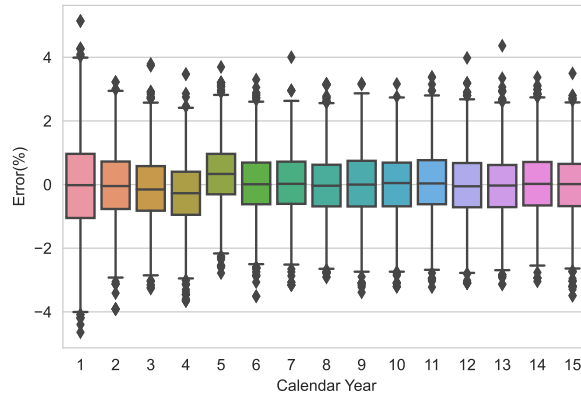


Illustration of performances according to calendar year



We can thus see that for each year of occurrence ($n=15$), the error is almost zero, and it is even more so when the size of the portfolio increases. This confirms the relevance of our proposal.

5 Generalization and deepening

5.1 Obtaining the triangle of average costs

As mentioned before, this demonstration is based on a triangle of average costs. Therefore we wanted to focus on the obtaining of this triangle, more precisely on the way to divide the triangle of costs (payments), in order to improve the performance of the model.

Here are the 3 methods we are interested in:

Method 1: Method 1 is the usual method used in actuarial science, which consists in starting from the triangle of the evolution of the number of claims and making a chain ladder in order to obtain the vector of the ultimate number of claims for each year of occurrence. Then divide the triangle of the evolution of the costs by this vector, and finally project the triangle obtained in calendar view then cumulate it.

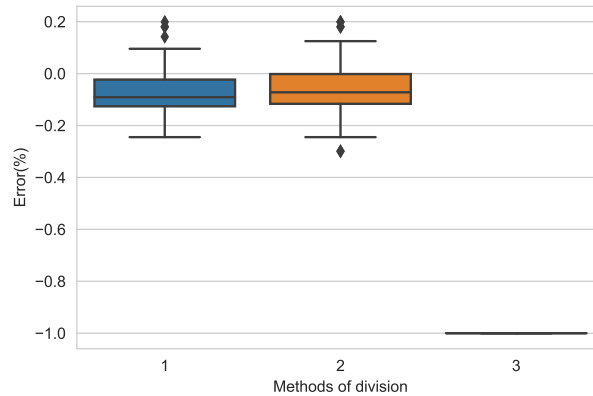
Method 2: Method 2 is a method for which we will use a simulated data: the number of claims which comes from the sum of the columns for each line of the rectangle of the number of claims that we simulated. Then we divide the triangle of the evolution of the costs by this ultimate number of claims simulated, and finally, we project the obtained triangle in calendar view and cumulate it.

Method 3: For method 3, we first cumulate the triangle of the evolution of the number of claims and the triangle of the evolution of the costs, then we divide the two, and project the obtained triangle in calendar view. For this method, we will not cumulate the data since the coefficients have already been cumulated in the previous step.

At the end of these 3 methods, we can directly make a Chain-Ladder, and thus complete the upper triangle and recover in the last column of the rectangle the Taylor coefficients, which are the inflation coefficients !

Comparison of the 3 methods:

Illustration of performances according to methods of division



Thus, we can see that method 2 seems to estimate inflation the best, this seems consistent since it is the method that has the most information. The method 1 also estimates the inflation very well, it is moreover less extensive than the method 1. The method 3 as for it has an estimation error of 1 %.

5.2 Generalization of Chain Ladder

Since it has been shown that the inflation estimate search is really just a Chain Ladder projected into a calendar view of the average cost triangle, we wondered if we could apply the same generalizations that we are used to doing with the Chain Ladder method. For this purpose, we implemented 3 methods calculating in a different way the Chain Ladder development coefficients.

Exclusion method ratio 0: The method 0, is the calculation of the coefficients of passage of classic way.

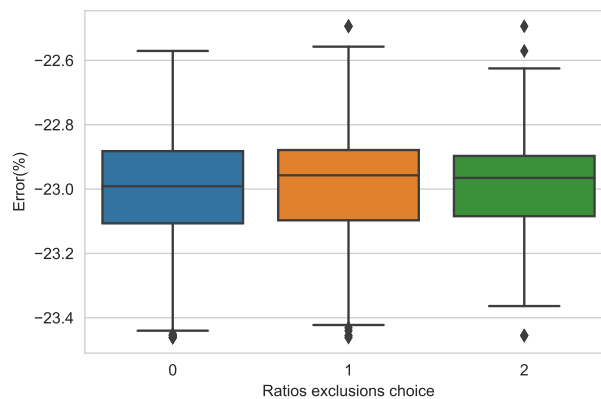
Exclusion ratio method 1: The method 1 is the calculation of the coefficients of passage with exclusion of the min and the max on the triangle of the ratio.

Ratio exclusion method 2: Method 2 is the calculation of the pass coefficients with selection of the last 5 diagonals and exclusion of the min and max on the ratio triangle.

For this, we deliberately simulated a triangle with volatile data, hence the gross error.

Comparison of the 3 methods:

Illustration of performances according to ratios exclusions choice with method 0



Thus, we note that methods 2 and 3 that exclude data provide a better estimate of inflation. To this end, we can consider using these Chain Ladder generalization methods with the Taylor method.

6 Conclusion

As we have discussed and explained throughout this report, inflation estimation is a very important topic for insurers. The Taylor Verbeek method allows for the estimation of inflation, however, it is a little known method and quite complicated to understand and interpret. The Chain Ladder method, on the other hand, is a basic method of calculating reserves in non-life insurance with fairly simple reasoning. The fact that the Taylor method has been shown to be a Chain Ladder projected into a calendar view should simplify and help the inflation estimation calculations.

7 Appendix: Demonstration

7.1 References

The purpose of this demonstration is to show that the Taylor method is nothing more than a Chain Ladder in calendar view.

We will need to introduce the following terms and formulas, already seen earlier in the report:

The $a_{i,j}$ represent the elements of the average cost triangle. The $b_{i,j}$, represent the elements of the average cost triangle in calendar view. Assuming it is the same triangle, the two expressions are related so that:

$$b_{i,j} = \sum_{k=i-j}^i a_{k,i-k} \quad \forall i, j \in N \text{ tel que } i - j \geq 0$$

$$b_{i,j} = b_{i,i} \times \prod_{k=i+1}^j c_k \quad \forall i, j \in N \text{ tel que } i - j \leq 0.$$

NOTA BENE: When one uses the formula above and that $i - j = 0$ one then has the product for k going $j+1$ to j of c_k , by default one considers that this quantity is equal to 1.

The following expressions and formulas are relative to the Taylor method, and therefore relative to the triangle of average costs:

- v_j is the sum of the $n - j$ observable elements of column j such that $v_j = \sum_{k=0}^{n-j} a_{k,j}$
- d_j is the sum of the $j + 1$ elements of diagonal j such that $d_j = \sum_{k=0}^j a_{k,j-k}$
- r_j represents the payment rate such that $r_j = \frac{v_j}{\lambda_j + \lambda_{j+1} + \dots + \lambda_n}$
- λ_j is an inflation factor such that $\lambda_j = \frac{d_j}{1 - r_n - r_{n-1} - \dots - r_{h+1}}$

And finally, the term c_j is the Chain Ladder crossing coefficient in the average cost triangle in calendar view and is defined such that $c_j = \frac{\sum_{k=j}^n b_{k,j}}{\sum_{k=j}^n b_{k,j-1}}$

Thus we will show that:

$$\forall h < n \quad \lambda_h = b_{h,h} \times \prod_{k=h+1}^n c_k \quad \text{and for } h=n \quad \lambda_h = b_{n,n}$$

7.2 Demonstration λ_h

We know that:

$$\lambda_h = \frac{d_h}{1 - r_n - r_{n-1} - \dots - r_{h+1}} = \frac{b_{h,h}}{1 - r_n - r_{n-1} - \dots - r_{h+1}}$$

and we want to show that

$$\lambda_h = b_{h,h} \times \prod_{k=h+1}^n c_k$$

so we want to show that

Proposal 1:

$$\frac{1}{1 - r_n - r_{n-1} - \dots - r_{h+1}} = \prod_{k=h+1}^n c_k \quad (1)$$

To show this equality(1) we will make a recurrence proof on h. We suppose (1) to be true for the rank $n, n-1, \dots, h+1$ and we will show it for the rank h .

For that it is enough to show the following 4 propositions:

Proposal 2:

Show (1) is equivalent at show that:

$$1 - r_{h+1} \times \frac{b_{n,n} \times \prod_{k=h+2}^{n-1} c_k}{b_{n,n-1}} = \frac{1}{c_{h+1}} \quad (2)$$

Proposal 3:

$$1 - r_{h+1} \frac{b_{n,n} \times \prod_{k=h+2}^{n-1} c_k}{b_{n,n-1}} = \frac{(b_{n,n-1} + \sum_{i=h+1}^{n-1} b_{i,i} \prod_{k=i+1}^{n-1} c_k) \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} - \sum_{i=0}^{n-(h+1)} a_{i,h+1} \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{(b_{n,n-1} + \sum_{i=h+1}^{n-1} b_{i,i} \prod_{k=i+1}^{n-1} c_k) \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1}} \quad (3)$$

Proposal 4:

$$(b_{n,n-1} + \sum_{i=h+1}^{n-1} b_{i,i} \prod_{k=i+1}^{n-1} c_k) \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} = b_{n,n-1} \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} + \sum_{j=h+1}^{n-1} b_{j,j} \left[\left(\prod_{i=j+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) \left(\prod_{i=h+2}^j \sum_{k=i}^n b_{k,i-1} \right) \right] \quad (4)$$

Proposal 5:

$$b_{n,n-1} \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} + \sum_{j=h+1}^{n-1} b_{j,j} \left[\left(\prod_{i=j+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) \left(\prod_{i=h+2}^j \sum_{k=i}^n b_{k,i-1} \right) \right] = \prod_{i=h+1}^{n-1} \sum_{k=i}^n b_{k,i} \quad (5)$$

Indeed, if we assume the above equalities to be true, then we have :

According to (3), we have:

$$1 - r_{h+1} \frac{b_{n,n} \times \prod_{k=h+2}^{n-1} c_k}{b_{n,n-1}} = \frac{(b_{n,n-1} + \sum_{i=h+1}^{n-1} b_{i,i} \prod_{k=i+1}^{n-1} c_k) \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} - \sum_{i=0}^{n-(h+1)} a_{i,h+1} \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{(b_{n,n-1} + \sum_{i=h+1}^{n-1} b_{i,i} \prod_{k=i+1}^{n-1} c_k) \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1}}$$

According to (4), we have:

$$= \frac{b_{n,n-1} \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} + \sum_{j=h+1}^{n-1} b_{j,j} \left[\left(\prod_{i=j+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) \left(\prod_{i=h+2}^j \sum_{k=i}^n b_{k,i-1} \right) \right] - \sum_{i=0}^{n-(h+1)} a_{i,h+1} \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{b_{n,n-1} \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} + \sum_{j=h+1}^{n-1} b_{j,j} \left[\left(\prod_{i=j+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) \left(\prod_{i=h+2}^j \sum_{k=i}^n b_{k,i-1} \right) \right]}$$

And after (5), we have:

$$\begin{aligned} &= \frac{\prod_{i=h+1}^{n-1} \sum_{k=i}^n b_{k,i} - \sum_{i=0}^{n-(h+1)} a_{i,h+1} \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{\prod_{i=h+1}^{n-1} \sum_{k=i}^n b_{k,i}} \\ &= \frac{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i} \times \sum_{k=h+1}^n b_{k,h+1} - \sum_{i=0}^{n-(h+1)} a_{i,h+1} \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{\prod_{i=h+1}^{n-1} \sum_{k=i}^n b_{k,i}} \\ &= \frac{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i} \times \left(\sum_{k=h+1}^n b_{k,h+1} - \sum_{i=0}^{n-(h+1)} a_{i,h+1} \right)}{\prod_{i=h+1}^{n-1} \sum_{k=i}^n b_{k,i}} \\ &= \frac{\sum_{k=h+1}^n b_{k,h}}{\sum_{k=h+1}^n b_{k,h+1}} \\ &= \frac{1}{c_{h+1}} \end{aligned}$$

Now according to Proposition 2, showing (1) is equivalent to showing (2), and we have just shown (2), hence (1).

Let us now prove (2), (3), (4) and (5)

Proof of (2):

First, by the recurrence hypothesis, we know the expression of λ_{h+1} , so we know that :

$$\frac{1}{1 - r_n - r_{n-1} - \dots - r_{h+2}} = \prod_{k=h+2}^{n-1} c_k \times c_n$$

So we know the expression of:

$$1 - r_n - r_{n-1} - \dots - r_{h+2} = \frac{1}{c_n \times \prod_{k=h+2}^{n-1} c_k}$$

so,

$$\begin{aligned} 1 - r_n - r_{n-1} - \dots - r_{h+2} - r_{h+1} &= (1 - r_n - r_{n-1} - \dots - r_{h+2}) - r_{h+1} \\ &= \frac{1}{c_n \times \prod_{k=h+2}^{n-1} c_k} - r_{h+1} \\ &= \frac{b_{n,n-1}}{b_{n,n}} \times \frac{1}{\prod_{k=h+2}^{n-1} c_k} \left(1 - r_{h+1} \times \frac{b_{n,n} \times \prod_{k=h+2}^{n-1} c_k}{b_{n,n-1}} \right) \end{aligned}$$

so it is enough to show that :

$$1 - r_{h+1} \times \frac{b_{n,n} \times \prod_{k=h+2}^{n-1} c_k}{b_{n,n-1}} = \frac{1}{c_{h+1}}$$

Demonstration of (3):

$$\begin{aligned}
1 - r_{h+1} \frac{b_{n,n} \times \prod_{k=h+2}^{n-1} c_k}{b_{n,n-1}} &= 1 - r_{h+1} \times \frac{b_{n,n}}{b_{n,n-1}} \times \frac{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1}} \\
&= 1 - \frac{v_{h+1}}{\lambda_n + \lambda_{n-1} + \dots + \lambda_{h+1}} \times \frac{b_{n,n}}{b_{n,n-1}} \times \frac{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1}}
\end{aligned}$$

So using the recurrence hypothesis on the λ_k pour $k \in [h+1, n]$

$$\begin{aligned}
&= 1 - \frac{\sum_{i=0}^{n-(h+1)} a_{i,h+1}}{b_{n,n} + \sum_{i=h+1}^{n-1} \lambda_i} \times \frac{b_{n,n}}{b_{n,n-1}} \times \frac{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1}} \\
&= 1 - \frac{\sum_{i=0}^{n-(h+1)} a_{i,h+1}}{b_{n,n} + \sum_{i=h+1}^{n-1} b_{i,i} \prod_{k=i+1}^{n-1} c_k \times c_n} \times \frac{b_{n,n}}{b_{n,n-1}} \times \frac{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1}} \\
&= 1 - \frac{\sum_{i=0}^{n-(h+1)} a_{i,h+1}}{b_{n,n} + (\sum_{i=h+1}^{n-1} b_{i,i} \prod_{k=i+1}^{n-1} c_k) \times \frac{b_{n,n}}{b_{n,n-1}}} \times \frac{b_{n,n}}{b_{n,n-1}} \times \frac{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1}} \\
&= 1 - \frac{\sum_{i=0}^{n-(h+1)} a_{i,h+1}}{b_{n,n} (1 + (\sum_{i=h+1}^{n-1} b_{i,i} \prod_{k=i+1}^{n-1} c_k) \frac{1}{b_{n,n-1}})} \times \frac{b_{n,n}}{b_{n,n-1}} \times \frac{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1}} \\
&= 1 - \frac{\sum_{i=0}^{n-(h+1)} a_{i,h+1}}{b_{n,n-1} + \sum_{i=h+1}^{n-1} b_{i,i} \prod_{k=i+1}^{n-1} c_k} \times \frac{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1}} \\
&= \frac{(b_{n,n-1} + \sum_{i=h+1}^{n-1} b_{i,i} \prod_{k=i+1}^{n-1} c_k) \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} - \sum_{i=0}^{n-(h+1)} a_{i,h+1} \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{(b_{n,n-1} + \sum_{i=h+1}^{n-1} b_{i,i} \prod_{k=i+1}^{n-1} c_k) \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1}}
\end{aligned}$$

Demonstration of (4):

First, by rewriting (3) in terms of the expression of c_k , on a:

$$\begin{aligned}
&b_{n,n-1} + \sum_{i=h+1}^{n-1} b_{i,i} \prod_{k=i+1}^{n-1} c_k = \\
&b_{n,n-1} + b_{n-1,n-1} + b_{n-2,n-2} \frac{\sum_{k=n-1}^n b_{k,n-1}}{\sum_{k=n-1}^n b_{k,n-2}} + \dots + b_{h+2,h+2} \frac{\prod_{i=h+3}^{n-1} \sum_{k=i}^n b_{k,i}}{\prod_{i=h+3}^{n-1} \sum_{k=i}^n b_{k,i-1}} + b_{h+1,h+1} \frac{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1}}
\end{aligned}$$

By putting in the same denominator

$$= \frac{(b_{n,n-1} + b_{n-1,n-1}) \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} + b_{n-2,n-2} \sum_{k=n-1}^n b_{k,n-1} \times \prod_{i=h+2}^{n-2} \sum_{k=i}^n b_{k,i-1} + \dots + b_{h+1,h+1} \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i}}{\prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1}}$$

So

$$\begin{aligned}
&(b_{n,n-1} + \sum_{i=h+1}^{n-1} b_{i,i} \prod_{k=i+1}^{n-1} c_k) \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} \\
&= (b_{n,n-1} + b_{n-1,n-1}) \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} + b_{n-2,n-2} \sum_{k=n-1}^n b_{k,n-1} \times \prod_{i=h+2}^{n-2} \sum_{k=i}^n b_{k,i-1} + \dots + b_{h+1,h+1} \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i} \\
&= (b_{n,n-1} + b_{n-1,n-1}) \prod_{i=n}^{n-1} \sum_{k=i}^n b_{k,i} \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} + b_{n-2,n-2} \prod_{i=n-1}^{n-1} \sum_{k=i}^n b_{k,i} \times \prod_{i=h+2}^{n-2} \sum_{k=i}^n b_{k,i-1} + \dots + b_{h+1,h+1} \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i} \times \prod_{i=h+2}^{h+1} \sum_{k=i}^n b_{k,i-1}
\end{aligned}$$

$$= b_{n,n-1} \times \prod_{i=n}^{n-1} \sum_{k=i}^n b_{k,i} \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} + \sum_{j=h+1}^{n-1} b_{j,j} \times \left[\left(\prod_{i=j+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) \times \left(\prod_{i=h+2}^j \sum_{k=i}^n b_{k,i-1} \right) \right]$$

that is to say

$$= b_{n,n-1} \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} + \sum_{j=h+1}^{n-1} b_{j,j} \times \left[\left(\prod_{i=j+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) \times \left(\prod_{i=h+2}^j \sum_{k=i}^n b_{k,i-1} \right) \right]$$

Demonstration of (5):

We want to show that:

$$b_{n,n-1} \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} + \sum_{j=h+1}^{n-1} b_{j,j} \left[\left(\prod_{i=j+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) \left(\prod_{i=h+2}^j \sum_{k=i}^n b_{k,i-1} \right) \right] = \prod_{i=h+1}^{n-1} \sum_{k=i}^n b_{k,i}$$

Show this equality by recurrence

Let us assume the property is true for rank $h+1$ and show that it is still true for rank h .

Thus we want to show that :

$$b_{n,n-1} \times \prod_{i=h+1}^{n-1} \sum_{k=i}^n b_{k,i-1} + \sum_{j=h}^{n-1} b_{j,j} \left[\left(\prod_{i=j+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) \left(\prod_{i=h+1}^j \sum_{k=i}^n b_{k,i-1} \right) \right] = \prod_{i=h}^{n-1} \sum_{k=i}^n b_{k,i}$$

To do this, let's show the recurrence hypothesis in the above expression:

$$\begin{aligned} &= b_{n,n-1} \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} \times \sum_{k=h+1}^n b_{k,h} + \sum_{j=h+1}^{n-1} b_{j,j} \left[\left(\prod_{i=j+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) \left(\prod_{i=h+2}^j \sum_{k=i}^n b_{k,i-1} \right) \left(\sum_{k=h+1}^n b_{k,h} \right) \right] \\ &\quad + b_{h,h} \left(\prod_{i=h+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) \left(\prod_{i=h+1}^h \sum_{k=i}^n b_{k,i-1} \right) \\ &= \sum_{k=h+1}^n b_{k,h} \left[b_{n,n-1} \times \prod_{i=h+2}^{n-1} \sum_{k=i}^n b_{k,i-1} + \sum_{j=h+1}^{n-1} b_{j,j} \left(\prod_{i=j+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) \left(\prod_{i=h+2}^j \sum_{k=i}^n b_{k,i-1} \right) \right] \\ &\quad + b_{h,h} \left(\prod_{i=h+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) \left(\prod_{i=h+1}^h \sum_{k=i}^n b_{k,i-1} \right) \\ &= \sum_{k=h+1}^n b_{k,h} \left(\prod_{i=h+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) + b_{h,h} \left(\prod_{i=h+1}^{n-1} \sum_{k=i}^n b_{k,i} \right) \\ &= \prod_{i=h+1}^{n-1} \sum_{k=i}^n b_{k,i} \times \left(\sum_{k=h}^n b_{k,h} \right) \\ &= \prod_{i=h}^{n-1} \sum_{k=i}^n b_{k,i} \end{aligned}$$

hence (1).

FIN

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