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### ► To cite this version:

K. Benali, B. Krour, M. Atif Benatta, K. Hafid, M Bachir Bouiadjra, et al.. Investigation of dynamic behavior of prestressed FRP plate intended for strengthening prestressed RC beam. *Engineering Structures*, 2023, 280, pp.115690. 10.1016/j.engstruct.2023.115690 . hal-04019368

**HAL Id: hal-04019368**

**<https://hal.science/hal-04019368>**

Submitted on 31 Mar 2023

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**Investigation of dynamic behavior of prestressed FRP plate intended for strengthening  
prestressed RC beam**

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**Abstract :**

Strengthening concrete beams by prestressed or not prestressed composite plates has become one of the widely used techniques in civil engineering with the aim of improving the durability of infrastructures, specifically against seismic solicitations. However, this type of reinforcement is quite new for prestressed concrete beams. Before using a composite plate for strengthening prestressed concrete beams, it is important to know its dynamic characteristics. In this paper, an analytical model based on the high-order theory and the laminate theory is proposed in order to well determine the free vibration characteristics of prestressed composite plate. The presence of prestressing tension effort in the composite plate, which was not taken into consideration in earlier studies for slender composite plates, is the main innovation of the current research. Parametric studies varying prestressing effort and fiber orientations are carried out in aim to find the best way to use the composite plate in the reinforcing of prestressed concrete beams. For comparative purposes, a finite element model is carried out in order to explore the behavior of the composite plate under several boundary conditions (B.Cs). As an important result, this research reveals that prestressing effort stiffens the composite plate and makes it a strong choice for reinforcing prestressed concrete beams by increasing natural frequencies and reducing deflexion.

**Keywords:** prestressed composite plates; laminate theory; high order theory, prestressed concrete beams, natural frequencies, finite element model.

## 1. Introduction

For any modern society, the road network represents the main vein of economy and development. Bridges are essential elements of the road network and are mainly built in prestressed concrete in order to overcome the most important obstacles. Due to their high cost, it is imperative to protect them and guarantee their sustainability utilizing various methods, including bonding an externally laminated composite plate.

In fact, the use of laminated plates is very popular in several fields, such as space, automotive, and the reinforcement of civil engineering structures. This wide use is due to several advantages like lightness, mechanical and chemical resistance, reduced maintenance, and freedom of shape. Like any civil engineering structure, prestressed concrete beams require reinforcement for rehabilitation purposes or to increase flexural strength and shear capacity, especially under dynamic load cases.

Reinforcement of RC beams by external bonding of composite plates is a very popular technique in the field of reinforcement. Indeed, many scientists produced several papers dealing with various fields of FRP (fiber reinforced polymer) composite material use.

Indeed, Muhammad Aslam et al. [1] give a thorough literature review that highlights the flexural performance of reinforced RC beams made of prestressed FRP. Among other findings, it has been highlighted that CFRPs are more likely to exhibit the benefits of other FRP types and that prestressed CFRPs increase the structure's flexural strength and ductility. An experimental study has been carried out by Jingsi Huo et al. [2] for a reinforced concrete beam strengthened by CFRP composite material under static and impact loads. The study showed that the arrangement and amount of CFRP as well as the impact energy have a crucial role in the dynamic behavior of the studied RC beams. The fatigue behavior has also been investigated experimentally by Huang. H et al. [3] who tested posttensioned prestressed CFRP sheets strengthening RC beams. The developed technique presented a good improvement in the fatigue

behavior of the system. Additionally, Wang. G. et al. [4] conducted an experimental investigation of the bending and fatigue behavior of RC beams bonded to prestressed FRP composite plates with two different anchorage systems. The results showed that the load-carrying capacities were improved up to 30%, and the stiffness was also improved, especially under fatigue loading.

The anchorage system plays a significant role in the strengthening and repair of existing structures in prestressed carbon fiber reinforced polymer (CFRP) plates. That is why Mohee, F. M. and A. Al-Mayah [5] developed a numerical design method for a friction-based mechanical anchor. Also, Li. C and G. Xian [6] conducted a novel experimental study by proposing a new wedge-bond anchor in aim to improve the bearing capacity of the anchorage system and reduce the interfacial stresses through the adhesive layer at the anchorage zone.

These studies lead us to explore the effect of different B.Cs. on the dynamic behavior of prestressed composite plates.

A prestressed FRP reinforcement-enhanced RC beam is fully analyzed in an experimental program by Justas Slaitas et al. [7] under the influence of an external load. The effectiveness of RC beams reinforced with prestressed basalt fiber-reinforced polymer (BFRP) laminates was parametrically investigated by Xinquan Chang et al [8], who came to the conclusion that strengthening with prestressed BFRP laminates significantly increases the cracking load of RC beams. These studies made it clear how crucial it is to use composite plates to reinforce RC beams under static, dynamic, and long-term behavior.

In a theoretical point of view, several studies have been developed. Y.M. Ghugal and R. P. Simpi [9] group them together, citing the different theories of isotropic plates from the well-known Classical Plate Theory (CPT) developed by G.R. Kirchhoff [10], which is based on the hypothesis that the deflection caused by shear deformation can be neglected compared to the deflection generated by the curvature of the plate. Also, First-order Shear Deformation Theory

(FSDT) which presumes that the transverse shear strain is constant across the thickness of the plate, and the Higher-order Shear Deformation Theory (HSDT), which is based on the power series expansion for displacements with respect to the thickness coordinate. More refined theories have been proposed in aim to give more accuracy in the determination of the FRP plate behavior. Indeed, T. Houari et al. [11] proposed a new quasi-3D HSDT bending plate theory, and A. Bouhadra et al. [12] dealt with the effect of thickness stretching.

Furthermore, Lazreg Hadji and Fabrice Bernard [13] developed a new simple higher order shear and normal deformation theory for bending and free vibration analysis of functionally graded material (FGM) beams resting on an elastic foundation, considering the effect of the porosity. This study constitutes the basis for our paper insofar as the elastic foundation may represent the adhesive layer usually used for bonding composite plates to a concrete beam. Other studies on the vibration of composite plates have been carried out in order to propose more accurate analytical models predicting the dynamic behavior of advanced materials.

Emrah Madenci[14] The purpose of his study is to ascertain how the pultruded GFRP responds to vibration, buckling stresses, and flexure responses while taking fiber and layer combinations into account. To achieve this goal, Comprehensive experimental, numerical, and analytical studies have been carried out. Emrah Madenci et al. [15] have also investigated the flexure performances of pultruded glass FRP (P-GFRP) composite beams.

P. Malekzadeh and A. R. Vosoughi [16] studied the large amplitude free vibration of thin and moderately thick angle-ply laminated rectangular plates using classical plate theory and the first order shear deformation plate theory, and solved the governing equation with the differential quadrature method (DQM) method. E. Asadi and S. J. Fariborz [17] studied the effect of various B.Cs. using higher-order shear deformation theory to obtain the governing equations of composite plates under dynamic excitation. A non-polynomial zigzag theory is recently proposed by Aniket Chanda and Rosalin Sahoo [18] to study the free vibration and transient

responses of multilayered laminated composites and sandwich plates subjected to various time-dependent loads and different forms of blast loads. A. Rahmani et al. [19] dealt with the vibrational behavior of anti-symmetric laminated composite plates resting on visco-elastic foundations and taking into account the temperature effect. Several B.Cs are also treated.

In a numerical point of view, P. Dey et al. [20] developed a finite element for the free vibration analysis of composite plates representing high orders of displacement with few degrees of freedom. The elements procure satisfaction in terms of accuracy and cost of computation.

All these studies have been carried out under static and dynamic loadings. However, few studies have been done on the vibration behavior of the laminated plates in the presence of the initial stress. In fact, for the reinforcement of bridge beams, which are relatively longer, the prestressing of the reinforcement plate is indispensable, so the effect of the reinforcement on the vibration of the structural element will be significant. Hence, studying the dynamic behavior of the reinforcement plate in free vibration becomes very important. There are few papers in the literature investigating this subject, C.M. Wang et al. [21] presented in their article a variety of methods for the derivation of the governing equations and B.Cs. for prestressed beams in free vibration using Eringen's nonlocal elasticity theory and Timoshenko beam theory. Brunelle, E.J., and Robertson [22], Yamaki and Chiba [23], Chen and al. [24], all investigated the nonlinear vibration analysis of Functionally Graded Plates under arbitrary initial stress conditions. The effect of prestress force on the vibration characteristics has been carried-out by Orłowska A. et al. [25] for the eccentrically prestressed glass fiber reinforced polymer composite beams. The study is based on a theoretical background, a numerical model, and an experimental program involving varying the prestress level. The most important result is that the natural frequencies of the first bending mode increase and the two subsequent bending frequencies decrease due to the prestressing force. Jalala Chafik et al. [26] in this study, the author used a specific machine to perform tensile tests on carbon/epoxy, basalt/epoxy,

glass/epoxy, and hybrid/epoxy composites to determine the ultimate strength and stress-strain relationship.

It has been well demonstrated that the external bonding of a composite plate to an RC beam causes a high level of stress at the interface. In fact, utilizing laminate theory, A. Tounsi and S. Benyoucef [27] created an analytical technique to forecast the distributions of interfacial stresses in concrete beams reinforced by composite plates. Using analytical and numerical methods, Krour et al. [28] investigated the bearing capacity of an RC beam while accounting for interfacial stresses and fiber orientations. This research served as the impetus for our decision to investigate how the fiber orientation parameter affects the dynamic behavior of FRP plates because it may be the difference between strong interfacial pressures and debonding failure.

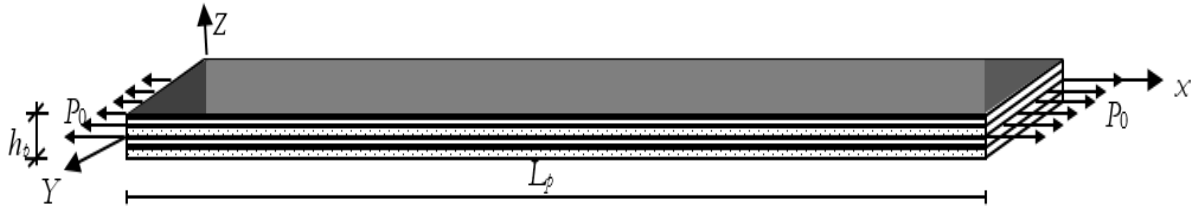
The external bonding of a prestressed FRP laminated composite plate, which creates additional pre-stressing, is the fundamental method of reinforcing prestressed concrete bridge girders. This paper will focus on the study of the vibratory behavior of a laminated composite plate using a refined displacement field and the laminate theory under prestressing force.

The key originality of the present work in comparison to other studies is the merging of the two approaches and the inclusion of the prestressing force for a composite plate designed to support a prestressed concrete beam. In addition, unlike the other studies, more than one FRP are tested. Additionally, different B.Cs. are taken into account in order to well represent the anchorage system. Furthermore, the elastic foundations are considered in order to predict vibratory behavior in the presence of an adhesive layer. A finite element model is carried out for comparison purposes with the present analytical model. A non-dimensional comparison is also performed to situate our model in relation to the existing literature.

## **2. Theory and formulations:**



The studied plate is a composite laminate composed of  $N$  perfectly bonded layers, each of which is composed of an isotropic polymer reinforced by fibers as shown in the figure (Fig. 1).



**Fig. 1.** Coordinate system and prestressed laminated plate

## 2.1. Constitutive relations

In this study, the fibers are distributed throughout the volume of the layer. The calculation of the physical properties of the layer may be obtained using the following relationship, as described by V.V. Vasiliev, E. Morozov [29]:

$$\begin{aligned}
 E_{11} &= V_f \cdot E_{11f} + V_m \cdot E_m \\
 \frac{1}{E_{22}} &= \left( \frac{V_m \cdot E_{11f} + V_f \cdot E_m}{E_{11f} \cdot E_m} \right) \\
 G_{12} &= \left( \frac{G_m \cdot G_{12f}}{V_f \cdot G_m + V_m \cdot G_{12f}} \right)
 \end{aligned} \tag{1}$$

Where,  $E_{11f}$  and  $G_{12f}$  are the Young's modulus and shear modulus, respectively, of the fiber, ' $E_m$ ' and ' $G_m$ ' are the engineering properties of the polymer matrix, ' $V_f$ ' and ' $V_m$ ' are the volume fractions of the fiber and the polymer matrix, respectively. Given the volume fractions ' $V_i$ ', the density and the Poisson's ratio can be calculated as follows, using the same rules:

$$\begin{aligned}
 \rho &= V_f \cdot \rho_f + V_m \cdot \rho_m \\
 \nu_{12} &= V_f \cdot \nu_f + V_m \cdot \nu_m
 \end{aligned} \quad \text{With} \quad V_f + V_m = 1 \tag{2}$$

The subscripts ( $f, m$ ) represent fiber and matrix respectively.

## 2.2. Kinematics

Due to the geometry of the studied cases, a laminated composite beam is investigated. The components of the displacement field along  $x$  and  $z$  directions of any point within the beam are based on the refined high-order beam theory, which is under the following assumptions:

- The in plane displacement  $u$  in the  $x$  direction is composed of the extension component, the bending component, and the shear component.
- The transverse displacement  $w$  consists of the bending component  $w_b$  and the shear component  $w_s$ , which are functions of the coordinate  $x$  only.
- The in-plane displacement in the  $y$  direction is neglected because the dimension in the  $x$  direction is much more important in comparison with the dimension in the  $y$  direction.

Therefore, the displacement field can be expressed as:

$$\begin{cases} u(x, z, t) = u_0(x, t) - z \cdot w_{b,x}(x, t) - \psi(z)w_{s,x}(x, t) \\ w(x, z, t) = w_b(x, t) + w_s(x, t) \end{cases} \quad (3)$$

Where, “ $u_0$ ” represents the beam’s mid-plane displacements along the  $x$  direction, ‘ $w_b$ ’ and ‘ $w_s$ ’ represents the bending and shear components of transverse displacement in  $z$  direction, respectively. Besides,  $\psi(z)$  is a shape function that covers the shear strain and stress towards thickness via a particular profile.

In this research, the Third Order Shear Deformation Theory (TSDT), developed by Reissner ([30], [31]), Krishna Murty [32], Lewinski ([33],[34]), and generalized by Reddy [35], is implemented. The shear stress distribution is represented by:

$$\psi(z) = -\frac{1}{4}z - \frac{5}{3}\left(\frac{z^3}{h^2}\right) \quad (4)$$

It should be noted that, unlike the first-order shear deformation theory, this theory does not require shear correction factors. The total strain vector associated with the displacement field equation (3) is expressed as:

$$\begin{cases} \varepsilon_{xx}(x, z, t) = u_{,x}(x, z, t) \\ \varepsilon_{xx}(x, z, t) = u_{0,x}(x, t) - z \cdot w_{b,xx}(x, t) - \psi(z) \cdot w_{s,xx}(x, t) \\ \gamma_{xz}(x, z, t) = u_{,z}(x, z, t) + w_{,x}(x, z, t) \\ \gamma_{xz}(x, z, t) = (1 - \psi_{,x}(z)) w_{s,x}(x, t) \\ \gamma_{xz}(x, z, t) = \xi(z) w_{s,x}(x, t) \end{cases} \quad (5.a)$$

With:

$$\xi(z) = (1 - \psi_{,x}(z)) \quad (5.b)$$

### 2.3. Equations of motion

Herein, Hamilton's principle is used to get the equations of motion of the stratified beam resting on an elastic foundation. The principle can be expressed as follows:

$$\int_{t_1}^{t_2} (\delta U + \delta W - \delta K) dt = 0 \quad (6)$$

where ' $\delta U$ ', ' $\delta W$ ' and ' $\delta K$ ' are the strain energy variation, the variation of the work of the external load, and the variation of the kinetic energy, respectively. Since the stratified beam is resting on an elastic foundation, the variation in the strain energy can be stated as follows:

$$\delta U = \delta U_b + \delta U_f \quad (7)$$

Knowing that ' $\delta U_b$ ' and ' $\delta U_f$ ' are the variations of strain energy of the stratified beam and foundation, respectively, the strain energy of the beam is calculated as:

$$\begin{aligned} \delta U_b &= \iint_{A L} (\sigma_{xx} \delta \varepsilon_{xx} + \tau_{xz} \delta \gamma_{xz}) dx dA \\ \delta U_b &= b \int_L (N_x \delta u_{0,x} - M_x \delta w_{b,xx} - P_x \delta w_{s,xx} + Q_x \delta w_{s,x}) dx \end{aligned} \quad (8)$$

Where: 
$$[N_x, M_x, P_x, Q_x] = \sum_{n=1}^N \int_{h_n}^{h_{n+1}} \sigma_{xx} [1, z, \psi(z), \xi(z)] dz \quad (9)$$

The virtual potential energy of the elastic foundation is expressed as follows:

$$\delta U_f = \int_L (k_w w_0 \delta w_0 + k_s w_{0,x} \delta w_{0,x}) dx \quad (10)$$

With: ‘ $k_w$ ’ and ‘ $k_s$ ’ are the Winkler and shearing layer spring constants, respectively. The plate is subjected to an axial compressive load; the variation of the potential energy of the prestress loads is as C.M. Wang et al. [21] expressed:

$$\delta V = \int_0^{L_p} \sigma_0 A w_{0,x} \delta w_{0,x} dx \quad (11)$$

where  $\sigma_0$  is the initial tensile elastic stress by assuming a positive value for tension.

The virtual kinetic energy is taking the following form:

$$\delta K = \int_A \int_L \rho [\dot{u} \delta \dot{u} + \dot{w}_0 \delta \dot{w}_0] dA dx \quad (12.a)$$

from where:

$$\delta K = b \int_L \left\{ \begin{aligned} &I_0 (\dot{u}_0 \delta \dot{u}_0 + \dot{w}_0 \delta \dot{w}_0) - I_1 (\dot{u}_0 \delta \dot{w}_{b,x} + \dot{w}_{b,x} \delta \dot{u}_0) + I_2 (\dot{w}_{b,x} \delta \dot{w}_{b,x}) \\ &- J_1 (\dot{u}_0 \delta \dot{w}_{s,x} + \dot{w}_{s,x} \delta \dot{u}_0) + J_2 (\dot{w}_{s,x} \delta \dot{w}_{b,x} + \dot{w}_{b,x} \delta \dot{w}_{s,x}) + K_1 (\dot{w}_{s,x} \delta \dot{w}_{s,x}) \end{aligned} \right\} dx \quad (12.b)$$

where:

$$(I_0, I_1, I_2) = \int_{-h/2}^{h/2} \rho (1, z, z^2) dz \quad (12.c)$$

$$(J_1, J_2, K_1) = \int_{-h/2}^{h/2} \rho (f(z), z, f(z), [f(z)]^2) dz$$

By substituting the expressions of energies  $\delta U$ ,  $\delta W$  and  $\delta K$  eq. (7)-(11)-(12) in the eq. (6) we obtain the following governing equations:

$$\begin{aligned} \delta u_0 : N_{x,x} &= I_0 \ddot{u}_0 - I_1 \ddot{w}_{b,x} - J_1 \ddot{w}_{s,x} \\ \delta w_b : M_{x,xx} + N_{0x} (w_{b,xx} + w_{s,xx}) + k_w (w_s + w_b) + k_s (w_{b,x} + w_{s,x}) &= I_0 (\ddot{w}_b + \ddot{w}_s) + I_1 \ddot{u}_{0,x} \\ &- I_2 \ddot{w}_{b,xx} - J_2 \ddot{w}_{s,xx} \\ \delta w_s : P_{x,xx} + Q_{x,x} + N_{0x} (w_{b,xx} + w_{s,xx}) + k_w (w_s + w_b) + k_s (w_{b,x} + w_{s,x}) &= I_0 (\ddot{w}_b + \ddot{w}_s) + J_1 \ddot{u}_{0,x} \\ &- J_2 \ddot{w}_{b,xx} - K_1 \ddot{w}_{s,xx} \end{aligned} \quad (13)$$

To obtain the equations of motion in terms of displacements ( $u_0(x,t)$ ;  $w_0(x,t)$ ) by considering linear elasticity behavior defined by the generalized Hooke's law and using a stiffness matrix connecting the strain field to the stress field as it's shown below:

$$\{\sigma\} = [C] \cdot \{\varepsilon\} \quad (14.a)$$

From where:

$$\begin{aligned} \sigma_{.xx} &= \bar{Q}_{11} \varepsilon_{.xx} \\ \sigma_{.xz} &= \bar{Q}_{55} \gamma_{.xz} \end{aligned} \quad (14.b)$$

The laminated beam is produced by many layers; the direction of the fibers may change from one layer to another. To study the behavior of these laminates, it is necessary to transform each layer from the local axis to the global axis of the entire plate. Reporting the elastic behavior of each layer to the global system ( $x,y,z$ ) is gotten from the following relationships, improved by J.R. Vinson and R.L. Sierakowski [36]:

$$\begin{aligned} \bar{Q}_{11} &= (Q_{11} \cos^4 \theta + Q_{22} \sin^4 \theta + 2(Q_{12} + Q_{66}) \sin^2 \theta \cos^2 \theta) \\ \bar{Q}_{55} &= (G_{13} \cos^2 \theta + G_{23} \sin^2 \theta) \end{aligned} \quad (15)$$

Where  $\theta$  is the angle between the local x-axis and the global x-axis of each layer.

The stiffness coefficients'  $Q_{ij}$  ', the components of the '[C]' matrix can be calculated using the material properties from equations (1) and (2) ,and they are expressed as follows:

$$Q_{11} = \frac{E_{11}}{1 - \nu_{12} \cdot \nu_{21}} \quad Q_{12} = \frac{\nu_{12} E_{22}}{1 - \nu_{12} \cdot \nu_{21}} \quad Q_{22} = \frac{E_{22}}{1 - \nu_{12} \cdot \nu_{21}} \quad Q_{66} = G_{12} \quad (16)$$

Substituting equations (8), (13) and (14) into equation (12) the equations of motions become as follows:

$$\begin{aligned}
\delta u_0 &: A_1 u_{0,xx} - B_1 w_{b,xxx} - B_s w_{s,xxx} = I_0 \ddot{u}_0 - I_1 \ddot{w}_{b,x} - J_1 \ddot{w}_{s,x} \\
\delta w_b &: B u_{0,xxx} - D_1 w_{b,xxxx} - D_s w_{s,xxxx} + N_0 (w_{b,xx} + w_{s,xx}) = I_0 (\ddot{w}_b + \ddot{w}_s) + I_1 \ddot{u}_{0,x} \\
&\quad - I_2 \ddot{w}_{b,xx} - J_2 \ddot{w}_{s,xx} \\
\delta w_s &: B_s u_{0,xxx} - D_s w_{b,xxxx} - H_s w_{s,xxxx} + A_s w_{s,xx} + N_0 (w_{b,xx} + w_{s,xx}) = I_0 (\ddot{w}_b + \ddot{w}_s) + J_1 \ddot{u}_{0,x} \\
&\quad - J_2 \ddot{w}_{b,xx} - K_1 \ddot{w}_{s,xx}
\end{aligned}
\tag{17}$$

Where:

$$\begin{aligned}
[A_1, B_1, D_1] &= b \sum_{n=1}^N \int_{h_n}^{h_{n+1}} \bar{Q}_{11\theta_i} [1, z, z^2] dz \\
[B_s, D_s] &= b \sum_{n=1}^N \int_{h_n}^{h_{n+1}} \bar{Q}_{11\theta_i} [\psi(z), z\psi(z)] dz \\
[H_s] &= b \sum_{n=1}^N \int_{h_n}^{h_{n+1}} \bar{Q}_{11\theta_i} [\psi(z)^2] dz \\
[A_{sij}] &= b \sum_{n=1}^N \int_{h_n}^{h_{n+1}} \bar{Q}_{55\theta_i} [\xi(z)^2] dz
\end{aligned}
\tag{18}$$

### 3. Analytical solution:

For a very close solution of the equilibrium equations and in order to evaluate the vibratory behavior of the beam, the shape functions, which are also functions of the B.Cs. are proposed to satisfy the embedding and simple support at the edges, which are simplified as follows:

For simply support edge condition:  $w_b(0, L) = w_s(0, L) = M_x = 0$  (19)

For clamped edge condition:  $u_0(0, L) = w_b(0, L) = w_s(0, L) = 0$  (20)

The Navier's solution procedure is employed in the present study to derive the characteristics of the embedded and the simply supported plate at both ends and resting on an elastic foundation. Indeed, the use of an elastic foundation may represent the stiffness of the adhesive layer usually employed in the reinforcement of a prestressed concrete beam. The used B.Cs. may represent the anchorage system used for prestressing the composite plate at its end. Thus,

the solution form for ‘ $u(x, t)$ ’, ‘ $w_b(x, t)$ ’ and ‘ $w_s(x, t)$ ’ that satisfies those conditions takes the following form:

$$\begin{aligned} u_0(x, t) &= \sum_{n=1}^{\infty} U_n \cdot \frac{\partial X(x)}{\partial x} \cdot e^{I \cdot \omega \cdot t} \\ w_b(x, t) &= \sum_{n=1}^{\infty} W_{bn} \cdot X(x) \cdot e^{I \cdot \omega \cdot t} \\ w_s(x, t) &= \sum_{n=1}^{\infty} W_{sn} \cdot X(x) \cdot e^{I \cdot \omega \cdot t} \end{aligned} \quad (21)$$

Where: ‘ $U_n$ ’, ‘ $W_{bn}$ ’ and ‘ $W_{sn}$ ’ are arbitrary parameters to be determined. Analytically, the spatial displacement function for both B.Cs. are:

Simply supported simply supported plate strip (S-S):

$$X(x) = \sin(\lambda_n \cdot x) \quad \text{with} \quad \lambda_n = \frac{n}{L} \pi \quad (22)$$

Clamped Clamped plate strip (C-C):

$$X(x) = \sinh(\lambda_n \cdot x) - \sin(\lambda_n \cdot x) + \alpha_n \cdot (\cosh(\lambda_n \cdot x) - \cos(\lambda_n \cdot x)) \quad (23)$$

$$\text{With: } \alpha_n = \frac{\sin(\lambda_n \cdot L) + \sinh(\lambda_n \cdot L)}{\cos(\lambda_n \cdot L) + \cosh(\lambda_n \cdot L)} \quad \text{and} \quad \lambda_n = \frac{2 \cdot n + 1}{2 \cdot L} \pi \quad (24)$$

Substituting eq. (21) into the governing equations (17), we get next closed-form solution of vibration problem of the FRP:

$$[[K] - \omega^2 [M]] = 0 \quad (25)$$

Where:

$$[K] = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ & k_{22} & k_{23} \\ sym & & k_{33} \end{bmatrix}; \quad [M] = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ & m_{22} & m_{23} \\ sym & & m_{33} \end{bmatrix} \quad (26)$$

The stiffness and masse matrices are given as bellows:

$$\left\{ \begin{array}{l} k_{11} = -(A + N_0)\alpha^2 \\ k_{12} = B\alpha^3 \\ k_{13} = B_s\alpha^3 \\ k_{22} = -(D_1 + N_0)\alpha^2 \\ k_{23} = -D_s\alpha^4 - N_0\alpha^2 \\ k_{33} = H_s - (A_s + N_0)\alpha^2 \end{array} \right. ; \left\{ \begin{array}{l} m_{11} = -I_0 \\ m_{12} = I_1\alpha \\ m_{13} = J_1\alpha \\ m_{22} = -I_2\alpha^2 - I_0 \\ m_{23} = -J_2\alpha^2 - I_0 \\ m_{33} = -K_2\alpha^2 - I_0 \end{array} \right. \quad (27)$$

To get the eigenfrequencies of the prestressed plate, we fix  $N_0$  the prestress force since it's involved in the stiffness matrix and the determinant of the equation (25) is set equal to zero.

#### 4. Results and discussions:

In this study, the reinforcing fibers of the plate were varied for the same polymer matrix. We have chosen three fibers that are of different natures and have different characteristics, as shown in table 1:

**Table. 1.** Fibers and matrix characteristics

Proprieties		$E_1$ (GPa)	$E_2$ (GPa)	$\rho$ (kg/m <sup>3</sup> )	$\nu_f$
Fibers type	Carbon	390	12	1800	0.35
	Glass R	86	86	2500	0.25
	Aramid	130	130	1450	0.4
Polymer matrix		3.1	3.1	1200	0.3

Fiber and matrix volume fractions are determined to be 54% and 46%, respectively, for each layer of the plate. The geometric properties of the composite laminated plate projected for strengthening a bridge girder are summarized in Table 2.

**Table. 2.** Geometric properties

	Length (L) m	Width (b) m	Thickness (h) mm
FRP Plate	20	0.3	5

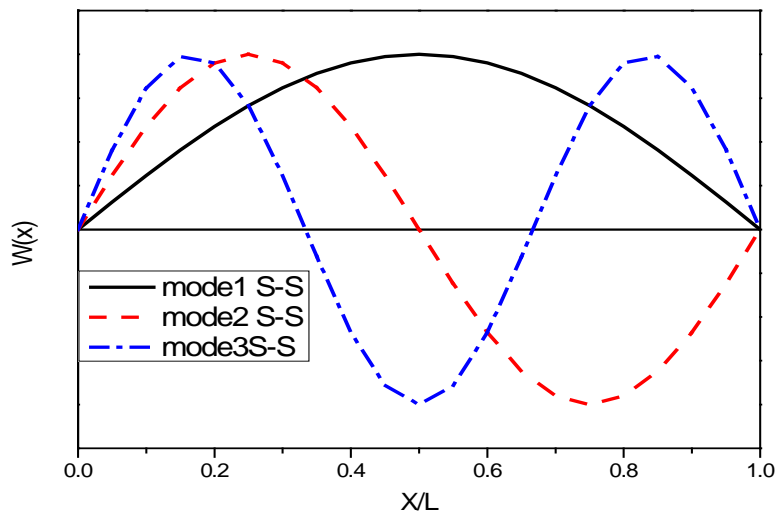


#### 4.1. Comparison study

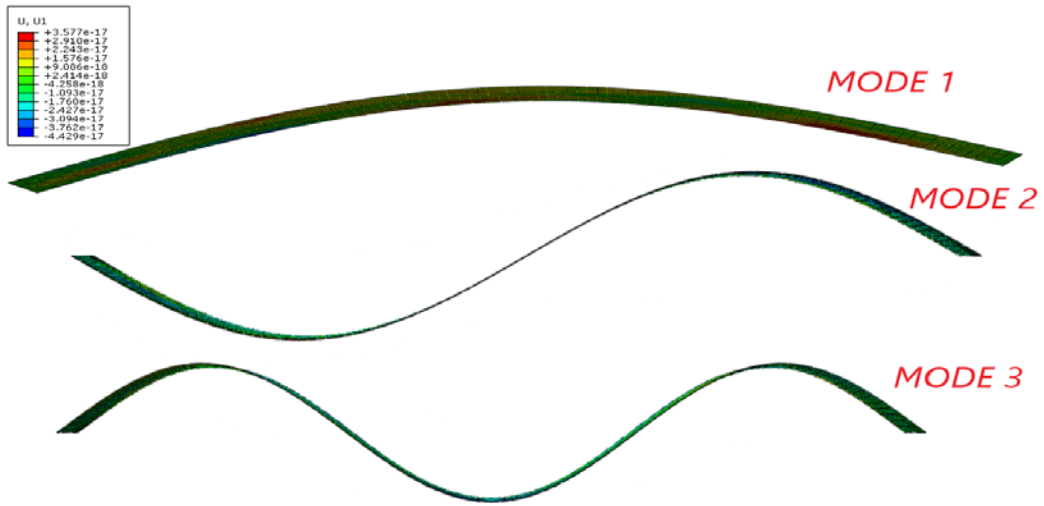
This section contrasts the suggested analytical model with models from the literature and a finite element model produced with the general-purpose package Abaqus [37].

The finite element shell S8-R is used to simulate the plate. This element has eight nodes, using reduced integration, with a mesh size of 5 mm and is a doubly curved thick shell.

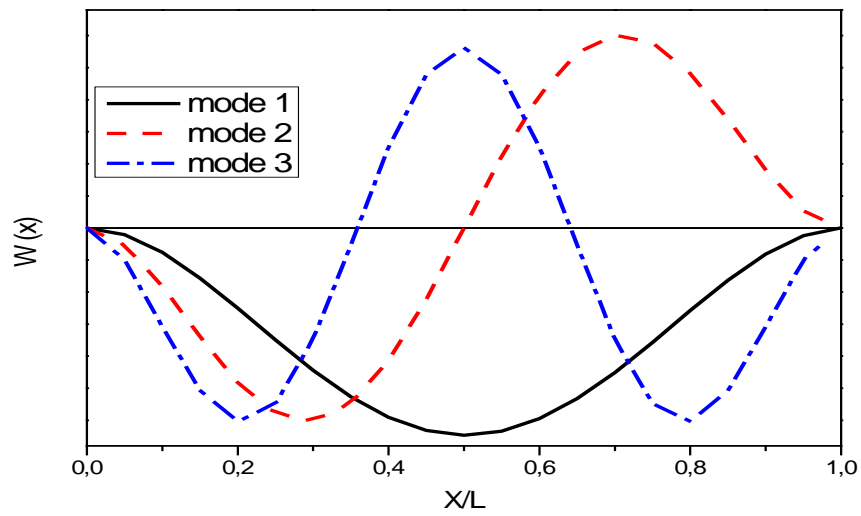
Figure 2 shows a comparison of the shapes of the three first modes for a  $0^\circ$  stratified CFRP plate strip between the analytical and finite element model. A plate with the dimensions (250x25x5) mm was the subject of the experiment, which was conducted under various boundary conditions. It is evident that there is good agreement between the two models in terms of mode shapes.



(a) Analytic three first modes shape of the simply supported Plate



(b) Numeric three first modes shape of the simply supported Plate tripe



(c) Analytic three first modes shape of the embedded Plate



(d) Numeric three first modes shape of the embedded Plate stripe

**Fig.2.** First fundamental three modes analytical and numerical under various B.Cs

Table.3 shows the Eigen frequencies for the first three fundamental modes and under different boundary conditions. According to these results, we can conclude that the proposed model which is a closed-form solution seems in a substantial matching with the FEM results.

**Table. 3.** The comparison of natural frequencies (Hz) of the CFRP

Boundary conditions	Modes	Frequency Hz		Error (%)
		FEM	Present	
S-S	Mode1	2.6780	2.6936	0.57 %
	Mode2	10.7120	10.7728	0.56 %
	Mode3	24.1020	24.2332	0.54 %
C-C	Mode1	6.0806	6.0615	0.31 %
	Mode2	16.7540	16.8277	0.51 %
	Mode3	32.8330	33.3202	0.55 %

The proposed model and those of Reddy et al. [38] and Drici et al. [39] are compared for a non-dimensional problem, and the results are in good match, as shown in Table 4. These results show the influence of the geometric dimensions on the non-dimensional natural frequencies, which rise with an increase in the ratio  $L/h$ . and

**Table. 4.** The comparison of non-dimensional natural frequencies  $\varpi = \omega.L^2 \sqrt{(\rho.h / E.I)}$  of an isotropic beam ( $L=10, E=30 \times 10^6, \nu=0.3, \rho=1$ )

L/h	Reddy 2007	Drici et al	Present
10	9.7454	9.7077	9.7075
20	9.8381	/	9.8281
100	9.8683	/	9.8679

#### 4.2. Effect of the prestress on the natural frequencies

In this part, the results of free vibration of a stratified composite plate stripe with four unidirectional layers of carbon fibers, Glass Fibers and Aramid Fibers (**Table.1**), under prestress forces for both B.Cs (C-C, S-S) are presented. Table. 5 shows the influence of the variation of the prestress force, under different modes of vibration and for different B.Cs, on the periods and frequencies of a CFRP plate. The case of no prestress applied to the composite plates is presented in the same table.

**Table. 5.** Effect of prestress on the CFRP plate eigenvalues

Boundary conditions	$P_0^*$ (kN)	Mode 01		Mode 02		Mode 03	
		Periods (s)	Frequency ( $s^{-1}$ )	Periods (s)	Frequency ( $s^{-1}$ )	Periods (s)	Frequency ( $s^{-1}$ )
S-S	0	14.8268	0.0674	3.7067	0.2698	1.6475	0.6070
	50	0.4945	2.0223	0.2468	4.0513	0.1641	6.0937
	100	0.3498	2.8591	0.1747	5.7230	0.1163	8.5964
	150	0.2856	3.5013	0.1427	7.0066	0.0951	10.5196
	200	0.2474	4.0428	0.1236	8.0890	0.0824	12.1420
C-C	0	6.5897	0.1518	2.3723	0.4215	1.2104	0.8262
	50	0.4438	2.2535	0.2280	4.3865	0.1550	6.4514
	100	0.3141	3.1833	0.1616	6.1891	0.1101	9.0858
	150	0.2566	3.8972	0.1320	7.5742	0.0900	11.1126
	200	0.2223	4.4992	0.1144	8.7426	0.0780	12.8231

Where  $P_0$  is the prestress force

In tables 6 and 7, the results of free vibration of the unidirectional, four-layered composite plate of Glass-R fibers and Aramid fibers, respectively, under different modes of vibration and for different B.Cs are presented. The increase in frequencies is also directly proportional to the increase in prestress force.

**Table 6.** Effect of prestress on the GFRP plate eigenvalues

Boundary conditions	$P_0^*$ (kN)	Mode 01		Mode 02		Mode 03	
		Periods (s)	Frequency ( $s^{-1}$ )	Periods (s)	Frequency ( $s^{-1}$ )	Periods (s)	Frequency ( $s^{-1}$ )
S-S	0	34.8103	0.0287	8.7026	0.1149	3.8678	0.2585
	50	0.5535	1.8068	0.2766	3.6150	0.1843	5.4259
	100	0.3914	2.5550	0.1957	5.1111	0.1304	7.6690
	150	0.3196	3.1292	0.1598	6.2592	0.1065	9.3908
	200	0.2768	3.6133	0.1384	7.2272	0.0922	10.8425
C-C	0	15.4712	0.0646	5.5697	0.1795	2.8416	0.3519
	50	0.4973	2.0107	0.2560	3.9068	0.1745	5.7298
	100	0.3518	2.8428	0.1811	5.5222	0.1235	8.0953
	150	0.2872	3.4814	0.1479	6.7620	0.1009	9.9117
	200	0.2488	4.0198	0.1281	7.8074	0.0874	11.4432

**Table 7.** Effect of prestress on the Aramid FRP plate eigenvalues

Boundary conditions	$P_0^*$ (kN)	Mode 01		Mode 02		Mode 03	
		Periods (s)	Frequency ( $s^{-1}$ )	Periods (s)	Frequency ( $s^{-1}$ )	Periods (s)	Frequency ( $s^{-1}$ )
S-S	0	23.7592	0.0421	5.9398	0.1684	2.6399	0.3788
	50	0.4625	2.1621	0.2311	4.3267	0.1539	6.4961
	100	0.3271	3.0574	0.1635	6.1165	0.1089	9.1791
	150	0.2671	3.7444	0.1335	7.4902	0.0890	11.2388
	200	0.2313	4.3236	0.1156	8.6484	0.0771	12.9757
C-C	0	10.5597	0.0947	3.8015	0.2631	1.9395	0.5156
	50	0.4155	2.4066	0.2138	4.6772	0.1457	6.8623
	100	0.2939	3.4021	0.1513	6.6094	0.1032	9.6910
	150	0.2400	4.1661	0.1236	8.0927	0.0843	11.8635
	200	0.2079	4.8103	0.1070	9.3434	0.0730	13.6956

It can be seen clearly that the increase in Eigen frequencies of the composite plate is directly proportional to the increase in the prestress force, as shown in figure 5 for  $k_w=0$ .

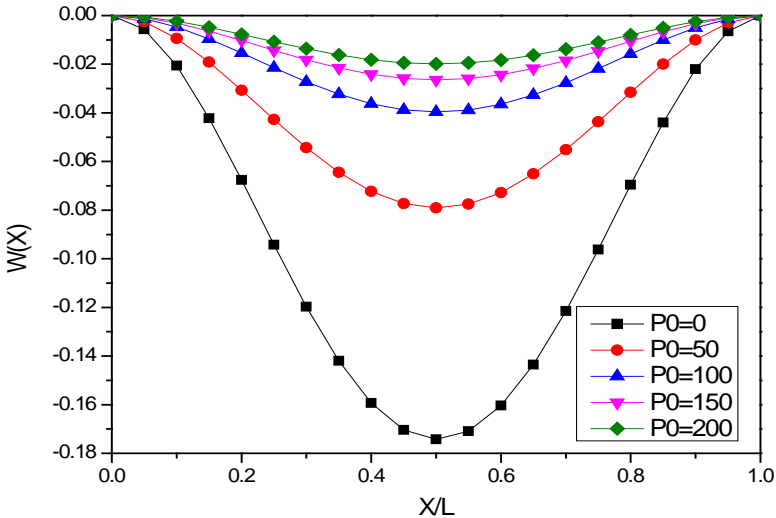
It can also be noted that the frequencies increase by 126%, 56%, and 36% when switching from simple support to embedding for the 1<sup>st</sup>, 2<sup>nd</sup>, and 3<sup>rd</sup> modes, respectively, depending on the prestressing.

When prestressing force is present, the increase in the first mode is approximately 8%, 11%, and approximately 5%, respectively, with a minor drop in this rate when prestressing force is increased. This justifies a convergence of the frequency value for both cases of B.Cs.

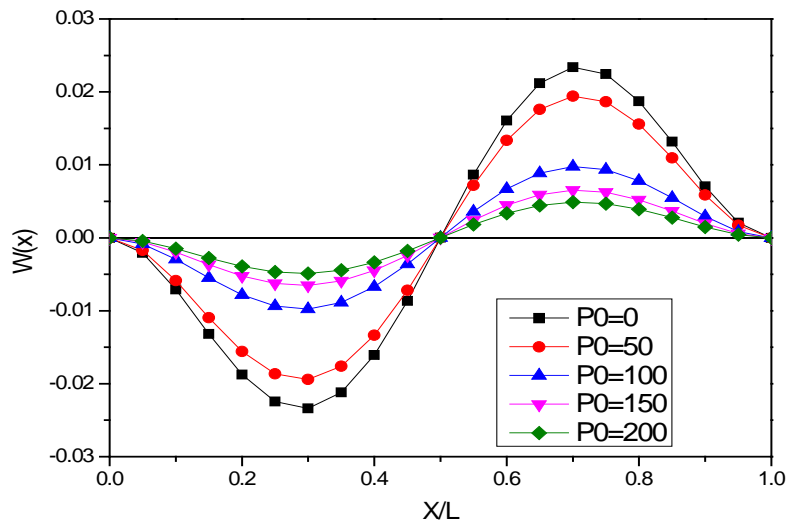
When comparing the behavior of CFRP, GFRP, and Aramid FRP plates, it can be seen that these plates (CFRP, GFRP, and Aramid FRP) behave differently without prestress and seem to have almost the same behavior in the presence of the prestress force.

### 4.3. Effect of the prestress on the amplitude

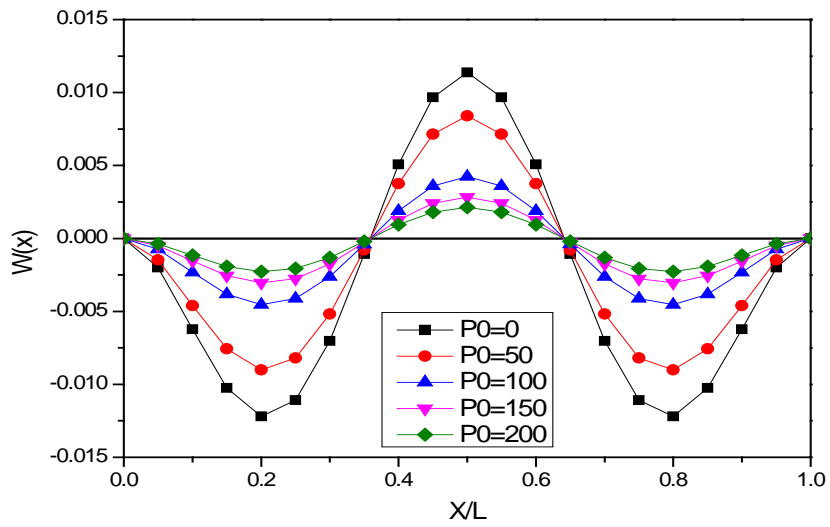
Figure 3 presents the effect of the prestress force on the amplitude of the composite plate in different modes of vibration. These results are found by applying a unit load along the plate under different values of prestress. We notice from these results that the amplitudes of the plate are inversely proportional to the prestress force. It can be noted that the results are very close in terms of values, as mentioned in the previous paragraph; therefore, the results for the CFRP plate embedded at its ends are presented.



(a) First mode amplitude



(b) Second mode amplitude

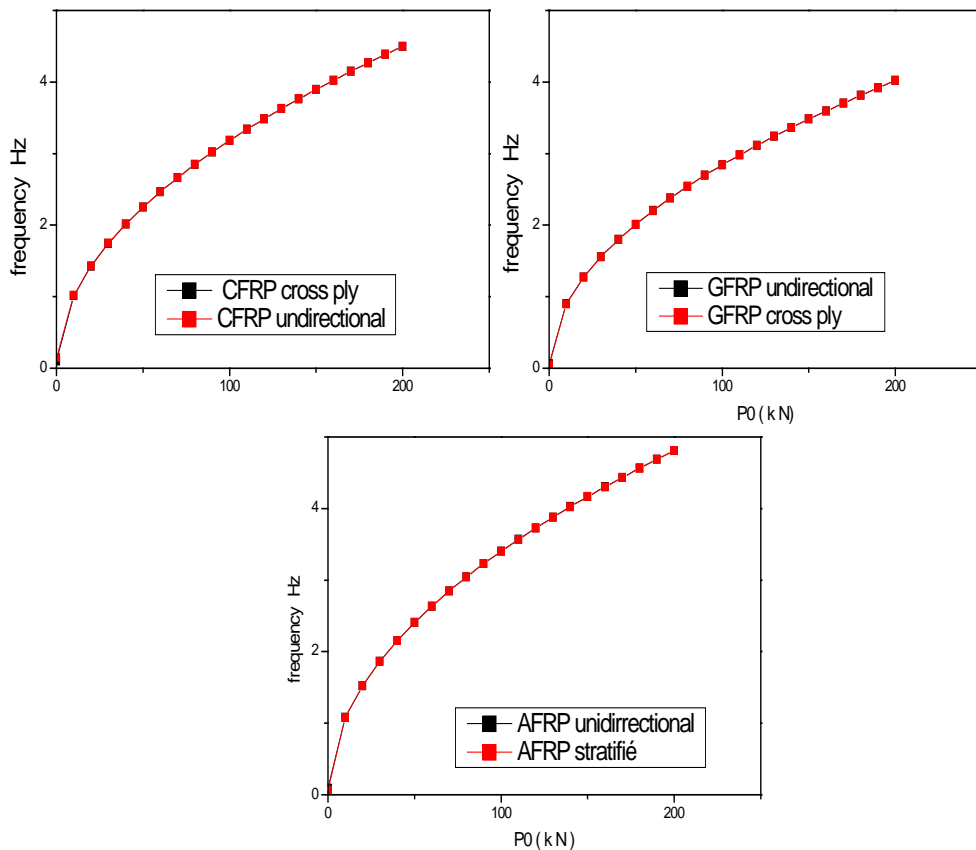


(c) Third mode amplitude

**Fig.3.** Amplitude of CFRP at “  $t= T/2$  ” under various prestress values

#### 4.4 Effect of the fiber orientation on the natural pulsation

The figure.4 presents the evolution of the Eigen frequencies according to the variation of the prestress applied to the laminated Four-layered unidirectional  $[0^\circ]$  composite plate and to the laminated Four-layered plate oriented as  $[0^\circ/90^\circ/0^\circ/90^\circ]$  with respect to its longitudinal axis. It can be clearly seen that the frequencies of the two studied cases are very close to each other. This results is interesting since it is well demonstrated by Krour et al [28] that when layers are oriented as  $[0^\circ/90^\circ/0^\circ/90^\circ]$  gives a weak level of interfacial stresses at the adhesive layer and procure a good reinforcement. We can also notice that the results are so similar even if we change the fiber type.



**Fig.4.** Variation of the frequencies of the CFRP in function of the variation of prestress values



#### 4.5 Effect of plates theories

In this part, the effect of the variation of the function shape  $\psi(z)$  is investigated. For simplicity we take the classical plate theory to be compared to the used theory, the displacement field is as described by Euler–Bernoulli beam theory [38]:

$$\begin{cases} u(x, z, t) = u_0(x, t) - z \cdot w_{0,x}(x, t) \\ w(x, z, t) = w_0(x, t) \end{cases} \quad (28)$$

By following the same work path, we got the results listed in **Table.8**, which presents a comparison of the periods “ $T$ ” and frequencies “ $f$ ” of the C-C composite prestressed plate for the two theories, the TSDT developed by Reddy [31] and the Euler–Bernoulli beam theory. The obtained results are perfectly identical. Consequentially, due to its simplicity, it is practical to use the classical plate theory for studying such slender plates.

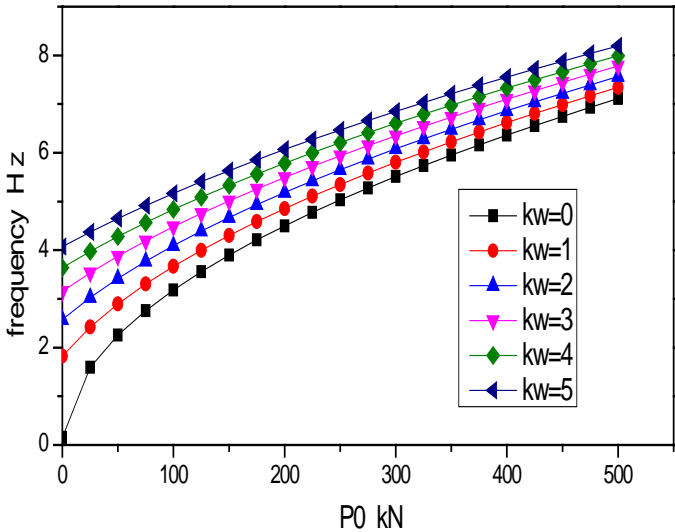
**Table. 8.** Frequencies of C-C CFRP plate under TSDT and CBT theories

$P_0$ (kN)	$T$ (s)		$f$ (s <sup>-1</sup> )	
	CPT	TSDT	CPT	TSDT
0	6.5897	6.5897	0.1518	0.1518
50	0.4438	0.4438	2.2535	2.2535
100	0.3141	0.3141	3.1833	3.1833
150	0.2566	0.2566	3.8972	3.8972
200	0.2223	0.2223	4.4992	4.4992

#### 4.6. Effect of elastic foundation

As described in the governing equations, the elastic foundation can be part of the solution, representing, in the case of the reinforcement of a prestressed concrete beam, the stiffness of the adhesive layer. Thus, the behavior of the composite plate reposing on an elastic foundation is investigated, with or without the presence of prestressing force. Figure.4 shows the variation

of the natural frequencies of the simply supported plate in relation to the variation of the prestress force under various values of the transverse elastic foundation stiffness. Depending on the pre-stressing, it can be seen that the Winkler parameter representing transverse stiffness increases the free vibration frequencies considerably. This increase becomes less and less important with the increase of the prestressing force, which expresses the convergence of the results for the highest prestressing forces in the figure 5. However, the Pasternak parameter representing shear stiffness weakly decreases the natural period comparing to the Winkler parameter. In presence of the prestress force, the role played by the elastic foundation is less important. It is essentially due to the prestress force effect which is preponderant.



**Fig.5.** variation of the proper pulsation of the CFRP in function of the variation of prestress for different values of the elastic foundation stiffness

**5. Conclusion**

In the present paper, a study of the free vibration of a prestressed composite plate was performed using a refined theory. In order to investigate the dynamic performance of the plate, the effect of prestress force on the Eigen frequencies, boundary conditions, fiber orientation, and the presence of the elastic foundation, is investigated. To validate this model, a comparative study with other studies was carried out, as well as numerical modeling using finite element coding software (ABAQUS). From the above discussions, it can be concluded that:

- The applied prestress force significantly increases the frequency of the plate, which results a decrease in the deflection of the plate. Consequently, the bending stiffness of the composite plate is well improved, more so considering that even a change in boundary conditions has no discernible effect on the results.
- The bending stiffness of the fibers and the application of the prestressing force both affect the plate's Eigen frequency. It is clear that these plates respond differently in the absence of prestress and appear to behave nearly identically when prestress is applied.
- The prestress force also eliminates the effect of the variation in fiber orientation.
- The importance of the elastic foundation's involvement diminishes as the prestressing force rises since this increase becomes less and less significant. It is essentially due to the prestress force effect, which is preponderant.

Based on the above conclusions, this technique of prestressing the composite plate may be a good solution for strengthening structural elements such as prestressed concrete beams by minimizing their deflection, stiffening the plate, and ensuring additional external pre-stress.

As perspectives, future research will examine the impact of using this prestressed laminated plate on a prestressed concrete beam's dynamic behavior under forced and free vibration as well as potential enhancements.

## **Acknowledgements**

The present work was financially supported by the Algerian ministry of higher education and scientific research. A gratefully acknowledgement is addressed to the Thematic Agency for Research in Science and Technology (ATRST) of Algeria.

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