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Disjoint odd circuits in a bridgeless cubic graph can be quelled by a single perfect matching

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Abstract

Let G be a bridgeless cubic graph. The Berge–Fulkerson Conjecture (1970s) states that G admits a list of six perfect matchings such that each edge of G belongs to exactly two of these perfect matchings. If answered in the affirmative, two other recent conjectures would also be true: the Fan–Raspaud Conjecture (1994), which states that G admits three perfect matchings such that every edge of G belongs to at most two of them; and a conjecture by Mazzuocolo (2013), which states that G admits two perfect matchings whose deletion yields a bipartite subgraph of G . It can be shown that given an arbitrary perfect matching of G , it is not always possible to extend it to a list of three or six perfect matchings satisfying the statements of the Fan–Raspaud and the Berge–Fulkerson conjectures, respectively. In this paper, we show that given any 1^+ -factor F (a spanning subgraph of G such that its vertices have degree at least 1) and an arbitrary edge e of G , there always exists a perfect matching M of G containing e such that $G \setminus (F \cup M)$ is bipartite. Our result implies Mazzuocolo’s conjecture, but not only. It also implies that given any collection of disjoint odd circuits in G , there exists a perfect

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matching of G containing at least one edge of each circuit in this collection.

Keywords: factor, perfect matching, cubic graph, snark, S_4 -Conjecture, Fan–Raspaud Conjecture, Berge–Fulkerson Conjecture
2000 MSC: 05C15, 05C70

1. Introduction

The behaviour of perfect matchings in cubic graphs is amongst the most well-studied themes in graph theory. One of the first classical results was made by Petersen in 1891 [18], who showed that every bridgeless cubic graph admits a perfect matching. These graphs not only do admit a perfect matching, but in 2011, one of the most prominent conjectures about perfect matchings in bridgeless cubic graphs was completely solved by Esperet *et al.* [1]. The conjecture, proposed by Lovász and Plummer in the 1970s, stated that the number of perfect matchings in a bridgeless cubic graph grows exponentially with its order (see [9]). Another conjecture which has baffled mathematicians for more than 50 years is the following.

Conjecture 1.1 (Fulkerson, 1971 [3]). *Every bridgeless cubic graph G admits six perfect matchings such that each edge in G is contained in exactly two of these six perfect matchings.*

Such a list of six perfect matchings shall be referred to as a *Fulkerson cover*. Although initially stated by Fulkerson, Conjecture 1.1 is also attributed to Berge and has been widely referred to as the Berge–Fulkerson Conjecture. In hindsight, such a name was more than appropriate as Fulkerson’s conjecture was actually shown to be equivalent to the following seemingly weaker conjecture made by Berge (see [13] for more details).

Conjecture 1.2 (Berge, unpublished). *Every bridgeless cubic graph G admits five perfect matchings such that every edge in G is contained in at least one of these five perfect matchings.*

In the sequel, in order to avoid confusion, the Berge–Fulkerson Conjecture shall refer exclusively to the statement by Fulkerson, which, despite being quite simple and uncomplicated, remains widely open. As the years went by, in an attempt to advance in this direction, weaker assertions started to be considered. Such an example is a conjecture by Fan and Raspaud.

29 **Conjecture 1.3** (Fan–Raspaud, 1994 [2]). *Every bridgeless cubic graph ad-*
30 *mits three perfect matchings M_1, M_2, M_3 such that $M_1 \cap M_2 \cap M_3 = \emptyset$.*

31 Three perfect matchings satisfying this property shall be referred to as
32 an *FR-triple*. As one can readily deduce, any three perfect matchings from
33 a Fulkerson cover form an FR-triple. Although, as stated by the authors
34 in [2], this conjecture is weaker than the Berge–Fulkerson Conjecture; once
35 again, it remains unresolved. The best result in this direction is a result
36 by the second author and Škoviera [11] who show that the Fan–Raspaud
37 Conjecture is true for bridgeless cubic graphs having oddness 2. The *oddness*
38 of a bridgeless cubic graph G is the least (even) number of odd circuits in a
39 2-factor amongst all possible 2-factors of G . Seemingly stronger formulations
40 of the Fan–Raspaud Conjecture are discussed in [15, 17, 20], and shown to
41 be equivalent to the original formulation.

42 A *Fano colouring* of a cubic graph G is a colouring of the edges of G with
43 the points of the Fano plane such that the colours given to the three edges
44 incident to a vertex of G form a line of the Fano plane, where the latter
45 consists of seven points and seven lines (see Figure 1(a)). A cubic graph
46 admits a Fano colouring if and only if it is bridgeless (see [4]). In [10], the
47 second author and Škoviera deal with the question of how many points and
48 lines of the Fano plane are sufficient to colour all bridgeless cubic graphs.
49 It is shown that for a Fano colouring of a snark (a non-3-edge-colourable
50 bridgeless cubic graph), all seven points and at least four lines are needed.
51 In fact, it is proven that six, and conjectured that four, lines of the Fano
52 plane colour every bridgeless cubic graph. For $i \in \{4, 5\}$, the statement that
53 for every bridgeless cubic graph there exists a Fano colouring with at most
54 i lines is denoted as the \mathcal{F}_i -Conjecture. It is shown that the \mathcal{F}_4 -Conjecture
55 is equivalent to the Fan–Raspaud Conjecture, and that the \mathcal{F}_5 -Conjecture is
56 equivalent to saying that every bridgeless cubic graph G admits two perfect
57 matchings whose intersection does not contain any odd edge-cut of G . In
58 2013, Mazzuocolo proposed the following conjecture, which is equivalent to
59 the \mathcal{F}_5 -Conjecture restricted to only those odd edge-cuts that separate odd
60 circuits.

61 **Conjecture 1.4** (Mazzuocolo, 2013 [12]). *For any bridgeless cubic graph*
62 *G , there exist two perfect matchings such that the deletion of their union*
63 *leaves a bipartite subgraph of G .*

64 Let H and G be graphs. An *H -colouring* of G is a map $f : E(G) \rightarrow E(H)$
65 such that for any vertex $u \in V(G)$ there exists a vertex $v \in V(H)$ with

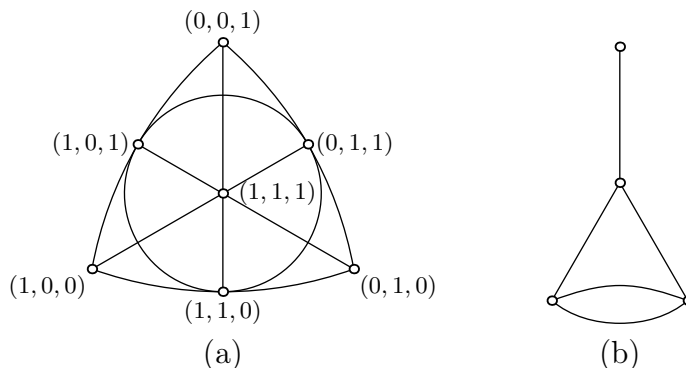


Figure 1: (a) The Fano plane (the configuration where points are non-zero elements of \mathbb{Z}_2^3 and lines are triples that sum up to zero); (b) The graph S_4 .

66 $f(\partial_G(u)) = \partial_H(v)$, where $\partial_G(u)$ and $\partial_H(v)$ respectively denote the sets of
 67 edges in G and H incident to the vertices u and v . Conjecture 1.4 was later
 68 referred to as the S_4 -Conjecture by Mazzuocolo and the last author, since
 69 it is equivalent to colouring the edges of G with the edges of the graph S_4
 70 shown in Figure 1(b) (see [14, 15, 20] for more details). Consequently, as
 71 in the last two cited documents, given a bridgeless cubic graph G , a pair of
 72 perfect matchings whose deletion yields a bipartite subgraph of G shall be
 73 referred to as an S_4 -pair.

74 In 2013, Mkrtchyan [16] showed that if the Petersen Colouring Conjecture
 75 by Jaeger [5] is true, the Petersen graph is the unique connected bridgeless
 76 cubic graph which can colour all bridgeless cubic graphs (colouring is done in
 77 the same manner as H -colourings defined above). In [14] this was extended
 78 and it was shown that if one was to remove all degree conditions on the graph
 79 H by which we are colouring, any bridgeless cubic graph can be coloured by
 80 the Petersen graph, and by a unique other graph: S_4 (see [14] for more details
 81 about the uniqueness of S_4). Related results can be found in [6, 7].

82 Clearly, by the result in [11], any bridgeless cubic graph having oddness 2
 83 admits an S_4 -pair. The authors of [15, 20] extend this result to bridgeless cu-
 84 bic graphs having oddness 4. The authors also show that if the S_4 -Conjecture
 85 is true, then there exist cubic graphs admitting a bridge for which an S_4 -pair
 86 exists – this is not possible when considering the Berge–Fulkerson Conjec-
 87 ture and the weaker conjectures mentioned above, because if a cubic graph
 88 admitting a perfect matching contains a bridge, then, every perfect matching
 89 of this graph must intersect the bridge (itself an odd edge-cut).

90 **Theorem 1.5** ([15, 20]). *Let G be a connected cubic graph having k bridges,*
 91 *all of which lie on a single path, for some positive integer k . If the S_4 -*
 92 *Conjecture is true, then G admits an S_4 -pair.*

93 *1.1. Important definitions and notation*

94 Graphs considered in this paper may contain parallel edges, but they
 95 cannot contain loops, unless otherwise stated. We speak about a *simple*
 96 graph if parallel edges are not allowed.

97 Let G be a graph and (V_1, V_2) be a partition of its vertex set, that is,
 98 $V_1 \cup V_2 = V(G)$ and $V_1 \cap V_2 = \emptyset$. Then, by $E(V_1, V_2)$ we denote the set
 99 of edges having one endvertex in V_1 and one in V_2 ; we call such a set an
 100 *edge-cut*. An edge which itself is an edge-cut of size one is a *bridge*.

101 An edge-cut $X = E(V_1, V_2)$ is called *cyclic* if both graphs $G[V_1]$ and
 102 $G[V_2]$, obtained from G after deleting X , contain a *circuit* (a 2-regular con-
 103 nected subgraph). The *cyclic edge-connectivity* of a graph G is defined as
 104 the smallest size of a cyclic edge-cut in G if G admits one; it is defined as
 105 $|E(G)| - |V(G)| + 1$ otherwise. For cubic graphs, the latter only concerns
 106 three graphs: K_4 , $K_{3,3}$, and the graph consisting of two vertices joined by
 107 three parallel edges, whose cyclic edge-connectivity is thus 3, 4, and 2, re-
 108 spectively.

109 Let G be a bridgeless cubic graph. A 1^+ -*factor* of G is the edge set
 110 of a spanning subgraph of G such that its vertices have degree 1, 2 or 3.
 111 In particular, a *perfect matching* and a *2-factor* of G are 1^+ -factors whose
 112 vertices have exactly degree 1 and 2, respectively. A *parity subgraph* J of a
 113 graph G is a subgraph H such that the parity of the degree of every vertex is
 114 the same in H as is in G . For cubic graphs, a parity subgraph is a 1^+ -factor
 115 with no vertices of degree 2. We remark that in [8], parity subgraphs (also
 116 known as *joins* in the literature) have been already studied with respect to
 117 problems discussed above.

118 **Definition 1.6.** A 1^+ -factor F and a perfect matching M of G form a
 119 *quelling pair* if $G \setminus (F \cup M)$ is bipartite.

120 **2. Disjoint odd circuits in bridgeless cubic graphs**

121 In [15, 20], a stronger problem than the S_4 -Conjecture is proposed.

122 **Problem 2.1** ([15, 20]). Establish whether any perfect matching of a bridge-
 123 less cubic graph can be extended to an S_4 -pair.

124 It is known that not every perfect matching can be extended to an FR-
125 triple and consequently, neither to a Fulkerson cover. In fact, consider the
126 Petersen graph P and expand each of its vertices into a triangle (see Figure
127 2). Let the resulting graph on 30 vertices be denoted by P' , and let N be the
128 set of edges in P' corresponding to $E(P)$. The set N is a perfect matching
129 of P' , and it is not difficult to see that it cannot be extended to an FR-triple
130 or to a Fulkerson cover. However, as can be seen in Figure 2, N (shown as
131 dashed lines) can in fact be extended to an S_4 -pair of P' (with the edges of
132 the second perfect matching shown as dotted lines).

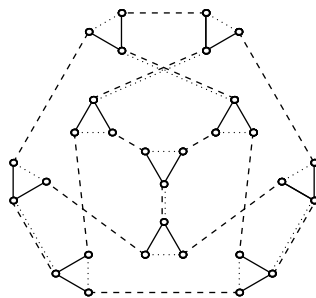


Figure 2: An example of an S_4 -pair in the graph P' obtained from the Petersen graph P by replacing each vertex by a triangle.

133 Moreover, saying that Problem 2.1 is true for every bridgeless cubic graph
134 is equivalent to the following statement: for every bridgeless cubic graph G ,
135 given any collection \mathcal{O} of disjoint odd circuits of G , there exists a perfect
136 matching M such that $M \cap E(C) \neq \emptyset$, for every $C \in \mathcal{O}$ (see [15, 20]). One
137 implication is clearly obvious. Thus, assume that every perfect matching
138 of any bridgeless cubic graph can be extended to an S_4 -pair, and consider
139 a collection of disjoint odd circuits in a bridgeless cubic graph G . Expand
140 every vertex not covered by the circuits in the collection to a triangle, and
141 let the resulting bridgeless cubic graph be denoted by G' . The initial odd
142 circuits and all the new expanded triangles give a 2-factor R of G' . Let
143 $N' = E(G') \setminus R$. By our assumption, there exists a perfect matching M' such
144 that $G' \setminus (N' \cup M')$ is bipartite, implying that M' intersects all the odd circuits
145 in R , including all the new expanded triangles. Let M'_Δ be the set of edges
146 belonging simultaneously to M' and the new expanded triangles. One can
147 immediately see that the set of the edges of G corresponding to $M' \setminus M'_\Delta$ is a
148 perfect matching of G intersecting all the odd circuits in the initial collection
149 of odd circuits in G , proving the equivalence of the two statements.

150 *2.1. Main result*

151 We now proceed to proving our main result, and consequently solve Prob-
 152 lem 2.1 and the S_4 -Conjecture.

153 **Theorem 2.2.** *Let G be a bridgeless cubic graph. Let F be a 1^+ -factor of G
 154 and let $e \in E(G)$. Then, there exists a perfect matching M of G such that
 155 $e \in M$, and F and M are a quelling pair.*

156 *Proof.* Let G be a minimum counterexample to the above statement. Let
 157 $e \in E(G)$ be an edge of G such that there exists a 1^+ -factor which cannot be
 158 extended to a quelling pair by a perfect matching containing e . Amongst all
 159 such 1^+ -factors of G , we can choose an inclusion-wise maximal one, denoted
 160 by F . By the choice of F , we may assume that $e \in F$ and that $G \setminus F$ contains
 161 only (disjoint) odd circuits, that is, F is a join. Let the set of components of
 162 $G \setminus F$ be denoted by \mathcal{O} .

163 If G is 3-edge-colourable, then its edge set decomposes into three perfect
 164 matchings. It suffices to take M to be the one containing e . Then $G \setminus M$ is
 165 a union of even circuits, and so F and M are a quelling pair by definition.
 166 Therefore, G is not 3-edge-colourable, and hence, it is not bipartite. In
 167 particular, G is not the unique bipartite cubic graph on two vertices.

168 It is easy to see that we may assume that G is connected.

169 **Claim 1.** The graph G does not have any 2-edge-cuts.

170 *Proof of Claim 1.* Suppose that G admits a 2-edge-cut $E(V', V'')$ with
 171 $E(V', V'') = \{f_1, f_2\} =: X$, where $f_1 = v'_1 v''_1$ and $f_2 = v'_2 v''_2$ for some
 172 $v'_1, v'_2 \in V'$ and $v''_1, v''_2 \in V''$. Since G is bridgeless, $v'_1 \neq v'_2$ and $v''_1 \neq v''_2$.
 173 See Figure 3 for an illustration.

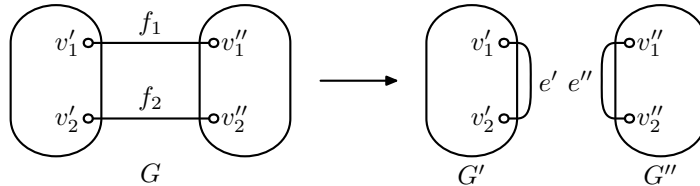


Figure 3: The graphs G' and G'' when G admits a 2-edge-cut $\{f_1, f_2\}$.

175 By the choice of F , we have that either X is contained in the edge set of
 176 some odd circuit C_X in \mathcal{O} , or $X \subset F$. Let G' and G'' be the two graphs on

177 the vertex sets V' and V'' obtained from G after deleting X and adding the
 178 edges $e' = v'_1v'_2$ and $e'' = v''_1v''_2$, respectively. Let

$$F' = \begin{cases} F \cap E(G') & \text{if } F \cap X = \emptyset, \\ (F \cap E(G')) \cup \{e'\} & \text{otherwise;} \end{cases}$$

179 and let

$$F'' = \begin{cases} F \cap E(G'') & \text{if } F \cap X = \emptyset, \\ (F \cap E(G'')) \cup \{e''\} & \text{otherwise.} \end{cases}$$

180 Clearly, F' (F'') is a 1^+ -factor of G' (G'' , respectively).

181 Without loss of generality, we may assume that at least one of the end-
 182 vertices of e corresponds to a vertex of G' . First, consider the case when
 183 $e \in X$. Observe that by our choice of F , $X \subset F$. By minimality of G , there
 184 exists a perfect matching M' of G' (M'' of G'') containing e' (e'') such that
 185 F' and M' (F'' and M'') are a quelling pair of G' (of G'' , respectively). Con-
 186 sequently, $M = M' \cup M'' \cup X \setminus \{e', e''\}$ is a perfect matching of G containing
 187 e . Moreover, since $X \subset M$, F and M are a quelling pair of G , contradicting
 188 our initial assumption.

189 Therefore, e corresponds to an edge in $G' \setminus e'$. For simplicity, we shall
 190 refer to the edge in G' corresponding to e , with the same name, that is, by
 191 e . By minimality of G , there exists a perfect matching M' of G' containing
 192 e such that F' and M' are a quelling pair of G' .

193 If $e' \in M'$, then let M'' be a perfect matching of G'' containing e'' such
 194 that F'' and M'' are a quelling pair of G'' . Such a perfect matching M'' exists
 195 by minimality of G . Once again, $M = M' \cup M'' \cup X \setminus \{e', e''\}$ is a perfect
 196 matching of G containing e , and since $X \subset M$, F and M are a quelling pair
 197 of G . Thus, $e' \notin M'$.

198 In this case, let e''' be an edge adjacent to e'' in G'' . By minimality of G ,
 199 there exists a perfect matching M'' of G'' containing e''' such that F'' and M''
 200 are a quelling pair of G'' . Since $e'' \notin M''$, $M = M' \cup M''$ is a perfect matching
 201 of G containing e . It remains to prove that F and M are a quelling pair of G .
 202 Every odd circuit $C \neq C_X$ in \mathcal{O} corresponds to an odd circuit either in $G' \setminus F'$
 203 or in $G'' \setminus F''$. The odd circuit C_X (if it exists), corresponds to two circuits
 204 C'_X and C''_X in $G' \setminus F'$ and $G'' \setminus F''$, respectively, of different parity. Therefore,
 205 at least one of them (the odd one) is hit, that is, intersected, by at least one
 206 edge of M' or M'' in G' or G'' , respectively, and thus, C_X is hit by at least one
 207 edge of M in G . Indeed, F and M are a quelling pair of G , a contradiction. ■

208

209 In particular, Claim 1 implies that G does not have any parallel edges,
 210 that is, G is simple.

211

212 **Claim 2.** The graph G does not have any cyclic 3-edge-cuts.

213 *Proof of Claim 2.* Suppose that G admits a cyclic 3-edge-cut $E(V', V'')$ with
 214 $E(V', V'') = \{f_1, f_2, f_3\} =: X$, where each $f_i = v'_i v''_i$, for some $v'_1, v'_2, v'_3 \in V'$
 215 and $v''_1, v''_2, v''_3 \in V''$. Since G has no 2-edge-cuts, the vertices $v'_1, v'_2, v'_3, v''_1, v''_2, v''_3$
 216 are all distinct.

217 By the choice of F , the parity of $|F \cap X|$ is odd, that is, either $|F \cap X| =$
 218 3 , meaning there is no odd circuit in \mathcal{O} intersecting X , or $|F \cap X| = 1$,
 219 meaning the cut X is intersected by a unique circuit C_X in \mathcal{O} . Without loss
 220 of generality, we shall assume that when $|F \cap X| = 1$, $F \cap X = f_1$.

221 Let G' and G'' be the two graphs obtained from G after deleting X and
 222 joining the vertices v'_i to a new vertex v' , and the vertices v''_i to a new vertex
 223 v'' . For each $i \in \{1, 2, 3\}$, let $e'_i = v'_i v'$ and $e''_i = v''_i v''$.

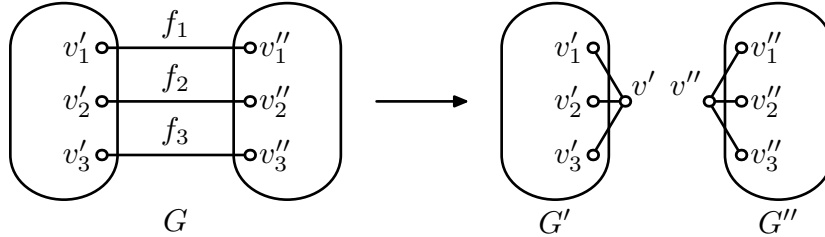


Figure 4: The graphs G' and G'' when G admits a cyclic 3-edge-cut $\{f_1, f_2, f_3\}$.

224 Let

$$F' = \begin{cases} (F \cap E(G')) \cup \{e'_1\} & \text{if } |F \cap X| = 1, \\ (F \cap E(G')) \cup \{e'_1, e'_2, e'_3\} & \text{otherwise.} \end{cases}$$

225 Similarly, let

$$F'' = \begin{cases} (F \cap E(G'')) \cup \{e''_1\} & \text{if } |F \cap X| = 1, \\ (F \cap E(G'')) \cup \{e''_1, e''_2, e''_3\} & \text{otherwise.} \end{cases}$$

226 It is not hard to see that F' (F'') is a 1^+ -factor of G' (of G'' , respectively).
 227 Every odd circuit $C \neq C_X$ in \mathcal{O} corresponds to an odd circuit either in $G' \setminus F'$
 228 or in $G'' \setminus F''$. The odd circuit C_X (if it exists) corresponds to two circuits
 229 C'_X and C''_X in $G' \setminus F'$ and $G'' \setminus F''$, respectively, having different parity.

230 Without loss of generality, we may assume that at least one of the end-
 231 vertices of e corresponds to a vertex in V' . We consider two cases, depending
 232 on the size of $|F \cap X|$.

233 **Case A.** First, consider the case when $|F \cap X| = 3$. When $e \in X$, say $e =$
 234 f_1 , then, by minimality of G , there exists a perfect matching M' of G' (M''
 235 of G'') containing e'_1 (e''_1), such that F' and M' (F'' and M'') form a quelling
 236 pair of G' (of G'' , respectively). Consequently, $M = M' \cup M'' \cup \{f_1\} \setminus \{e'_1, e''_1\}$
 237 is a perfect matching of G containing $e = f_1$, and moreover, F and M are a
 238 quelling pair of G .

239 It remains to consider the case when $e \notin X$, and so the endvertices of e
 240 both correspond to vertices in G' . Once again, for simplicity, we shall refer
 241 to this edge as e . Let M' be a perfect matching of G' containing e such that
 242 F' and M' are a quelling pair of G' . Without loss of generality, assume that
 243 $e'_1 \in M'$. Let M'' be a perfect matching of G'' containing e''_1 such that F'' and
 244 M'' are a quelling pair of G'' . Let $M = M' \cup M'' \cup \{f_1\} \setminus \{e'_1, e''_1\}$. This is a
 245 perfect matching of G containing e , and as before, F and M form a quelling
 246 pair of G , a contradiction.

247 **Case B.** Suppose that $|F \cap X| = 1$. When $e \in X$, say $e = f_i$, then,
 248 by minimality of G , there exists a perfect matching M' of G' (M'' of G'')
 249 containing e'_i (e''_i), such that F' and M' (F'' and M'') form a quelling pair of
 250 G' (of G'' , respectively). Consequently, $M = M' \cup M'' \cup \{f_i\} \setminus \{e'_i, e''_i\}$ is a
 251 perfect matching of G containing $e = f_i$. To observe that F and M form a
 252 quelling pair of G , it suffices to note that the only odd circuit in \mathcal{O} possibly
 253 not intersected by M is C_X . However, this can only happen if $i = 1$, and, if
 254 this is the case, then either C'_X or C''_X is odd, and so it is hit by an edge of
 255 M' or M'' , not incident to v' or v'' . However, this means that C_X is also hit
 256 by the corresponding edge of M in G , a contradiction.

257 It remains to consider the case when $e \notin X$, and so the endvertices of e
 258 both correspond to vertices in G' . Once again, for simplicity, we shall refer
 259 to this edge as e . Let M' be a perfect matching of G' containing e such that
 260 F' and M' are a quelling pair of G' . We have $e'_i \in M'$ for some $i \in \{1, 2, 3\}$.
 261 Let M'' be a perfect matching of G'' containing e''_i such that F'' and M'' are
 262 a quelling pair of G'' . Let $M = M' \cup M'' \cup \{f_i\} \setminus \{e'_i, e''_i\}$. This is a perfect
 263 matching of G containing e . As before, F and M form a quelling pair of G ,
 264 unless $i = 1$ and no edge of G' or G'' corresponding to an edge of C_X is hit
 265 by M' or M'' , which is impossible since either C'_X or C''_X is an odd circuit. ■

266

267 From this point on we may assume that G has no cyclic 3-edge-cuts.

268 Therefore, it is cyclically 4-edge-connected, unless $G = K_4$, but for this
 269 particular graph it is easy to see that it is not a counterexample, since for
 270 every edge e , the complement of (the only) perfect matching containing e is
 271 an even circuit.

272 We now consider the edges at distance 2 from e (distance measured as
 273 the distance of the corresponding vertices in the line graph of G).

274 **Claim 3.** Let f be an edge at distance 2 from e . Then, $f \in F$.

275 *Proof of Claim 3.* We will use a procedure that transforms a cubic graph
 276 G into a cubic graph G' smaller than G , such that every perfect matching
 277 of G' containing a certain edge can be extended into a perfect matching of
 278 G containing the corresponding edge. This operation was already used by
 279 Voorhoeve [19] to study perfect matchings in bipartite cubic graphs and it is
 280 a key ingredient for counting perfect matchings in general in [1].

281 Let $f = uv$, let the neighbours of u distinct from v be α and γ , and let
 282 the neighbours of v distinct from u be β and δ . In particular, since G is
 283 cyclically 4-edge-connected, these four vertices are all distinct. Without loss
 284 of generality, we may assume that α is an endvertex of e .

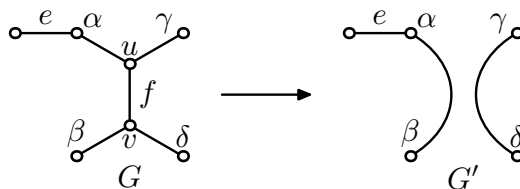


Figure 5: The vertices $\alpha, \beta, \gamma, \delta$ and an $(\alpha\beta : \gamma\delta)_{uv}$ -reduction.

285 As shown in Figure 5, we obtain a smaller graph (possibly containing par-
 286 allel edges) by deleting the endvertices of f (together with all edges incident
 287 to them) and adding the edges $\alpha\beta$ and $\gamma\delta$. Let this resulting graph be G' .
 288 We shall say that G' is obtained after an $(\alpha\beta : \gamma\delta)_{uv}$ -reduction. It is well-
 289 known that when applying this operation, the cyclic edge-connectivity of a
 290 cubic graph can drop by at most 2. Since G is cyclically 4-edge-connected,
 291 G' is bridgeless.

292 Let the edge in G' corresponding to e , and the vertices in G' correspond-
 293 ing to $\alpha, \beta, \gamma, \delta$ be denoted by the same name. We recall that any perfect
 294 matching of G' which contains e can be extended to a perfect matching of
 295 G containing the edge e (see also Figure 6). In fact, let M' be a perfect
 296 matching of G' containing e . This is extended to a perfect matching M of G
 297 containing e as follows:

$$M = \begin{cases} M' \cup \{u\gamma, v\delta\} \setminus \{\gamma\delta\} & \text{if } \gamma\delta \in M', \\ M' \cup \{f\} & \text{otherwise.} \end{cases}$$

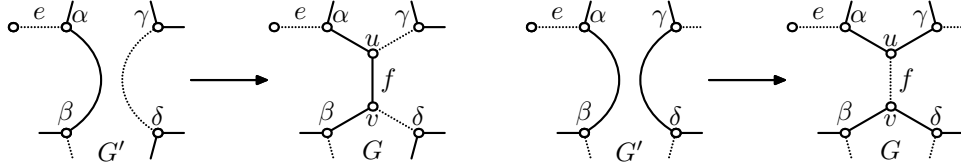


Figure 6: Extending a perfect matching of G' containing e to a perfect matching of G containing e . Dotted lines represent edges in M or M' .

298 Suppose that for some edge f at distance 2 from e we have $f \notin F$. By the
 299 choice of F , f is in some odd circuit C_f in \mathcal{O} . This means that exactly one
 300 of $u\alpha$ and $u\gamma$, and exactly one of $v\beta$ and $v\delta$ belong to C_f . Without loss of
 301 generality, we may assume that $u\alpha \in F$ if and only if $v\beta \in F$ (otherwise, we
 302 rename β and δ). Let G' be the graph obtained from G after an $(\alpha\beta : \gamma\delta)_{uv}$ -
 303 reduction. Let F' be the 1^+ -factor of G' as follows:

$$F' = \begin{cases} F \cup \{\alpha\beta\} \setminus \{u\alpha, v\beta\} & \text{if } \{u\alpha, v\beta\} \subset F, \\ F \cup \{\gamma\delta\} \setminus \{u\gamma, v\delta\} & \text{otherwise.} \end{cases}$$

304 This is portrayed in Figure 7.



Figure 7: If $f \notin F$, then we apply induction on G' – the graph obtained from G after an $(\alpha\beta : \gamma\delta)_{uv}$ -reduction. Dashed lines represent edges in F' or F .

305 Observe that every odd circuit $C \neq C_f$ remains the same in G' ; and
 306 that C_f is transformed into an odd circuit C'_f in G' , shorter than C_f by two
 307 edges. Since G' is bridgeless and its order is strictly less than G , it is not a
 308 counterexample. Let M' be a perfect matching of G' containing e such that
 309 F' and M' are a quelling pair. We extend this perfect matching to a perfect
 310 matching M of G containing e as described above (see Figure 6), and claim
 311 that F and M are a quelling pair. Every odd circuit $C' \neq C'_f$ in $G' \setminus F'$ is hit

312 by an edge of M' in G' , and so the corresponding circuit C is hit by the corre-
 313 sponding edge of M in G . The odd circuit C'_f is hit by an edge M' in G' , and
 314 so the corresponding circuit C_f is hit by the corresponding edge in G , unless
 315 the hitting edge is $\alpha\beta$ (or $\gamma\delta$), but then C_f is hit by at least one edge (in
 316 fact, two edges) on the path from α to β (or from γ to δ , respectively) which
 317 contains u and v . Hence, F and M are a quelling pair of G , a contradiction. ■

318

319 From this point on we may assume that for every edge f at distance 2
 320 from e we have $f \in F$. As a consequence, by the maximality of F , we have
 321 that all the edges at distance at most 1 from e are in F , otherwise there
 322 would be a component in \mathcal{O} which is not a circuit, a contradiction. Once
 323 again, let us consider an edge $f = uv$ at distance 2 from e . Let α, β, γ and
 324 δ be as defined in Claim 3. The edge $u\gamma$ is at distance 2 from e , and so
 325 $u\gamma \in F$, implying that $u\alpha \in F$ as well. Furthermore, it can be easily seen
 326 that $v\beta \in F$ if and only if $v\delta \in F$.

327 Consider first the case when $\{v\beta, v\delta\} \subset F$. Then, in the graph G' ,
 328 obtained by an $(\alpha\beta : \gamma\delta)_{uv}$ -reduction, the 1^+ -factor $F' = F \cup \{\alpha\beta, \gamma\delta\} \setminus$
 329 $\{u\alpha, v\beta, uv, u\gamma, v\delta\}$ can be extended to a quelling pair of G' by a perfect
 330 matching M' containing e . It is easy to see that the perfect matching M (of
 331 G) containing e obtained as an extension of M' forms a quelling pair with F
 332 in G , a contradiction. This implies that $v\beta$ and $v\delta$ do not belong to F .

333 When $\{v\beta, v\delta\} \cap F = \emptyset$, we cannot use the reduction portrayed in Figure
 334 5 as we do not have a guarantee that we can obtain a perfect matching M
 335 intersecting the odd circuit in \mathcal{O} containing the edges $v\beta$ and $v\delta$. Since G is
 336 cyclically 4-edge-connected, this latter odd circuit is of length at least 5. Let
 337 δ, v, β, y, z be consecutive and distinct vertices on this odd circuit (see Figure
 338 8). Moreover, let w and x be the vertices in G such that $\{w\beta, xy\} \subset F$. We
 339 proceed by applying an $(\alpha\beta : \gamma\delta)_{uv}$ -reduction followed by an $(\alpha x : wz)_{\beta y}$ -
 340 reduction as portrayed in Figure 8.

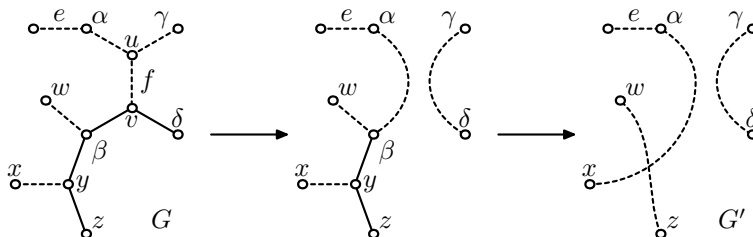


Figure 8: An $(\alpha\beta : \gamma\delta)_{uv}$ -reduction followed by an $(\alpha x : wz)_{\beta y}$ -reduction.

341 Let the resulting graph after these two reductions be denoted by G' . We
 342 remark that G' may have parallel edges and even loops. Let $F' = F \cap$
 343 $E(G - \{u, v, \beta, y\}) \cup \{\alpha x, wz, \gamma\delta\}$. Since G' is obtained by applying twice
 344 the reduction at an edge at distance 2 from e , a perfect matching of G'
 345 containing e can always be extended to a perfect matching of G containing
 346 e (recall that βy cannot be adjacent to e , otherwise $\beta y \in F$). Moreover, any
 347 such matching contains either the edge βy or the edge yz , and so it hits the
 348 odd circuit in \mathcal{O} containing the edges $v\beta$ and $v\delta$. Therefore, as long as G'
 349 is bridgeless, by minimality of G , there exists a perfect matching M' of G'
 350 containing e such that F' and M' are a quelling pair of G' . This extends to a
 351 perfect matching M of G containing e such that F and M are a quelling pair
 352 of G . This contradicts our initial assumption that G is a counterexample.
 353 Consequently, G' must admit some bridge g . Let $\Omega_1 = \{\alpha, \gamma, \delta, w, x, z\}$ and
 354 let $\Omega_2 = \{u, v, \beta, y\}$.

355 In order to prove that G' is bridgeless, we prove that it is 2-edge-connected
 356 (which is a stronger property). Before considering an edge-cut of size one, we
 357 prove that the graph G' is connected. For, suppose that G' is disconnected.
 358 Then G' admits at least two components. Label the vertices of a first compo-
 359 nent by A , and the remaining vertices by B . This shall result in a labelling
 360 of the vertices of G' such that each edge in G' is monochromatic (an edge in
 361 G' is said to be monochromatic if its endvertices have the same label). For
 362 the three edges αx , wz , and $\gamma\delta$, either their endpoints receive all the same
 363 label, or two of them receive one label and the remaining four receive the
 364 other. We first extend this labelling of $V(G')$ to a partial labelling of $V(G)$
 365 by giving to the vertices in $V(G) - \Omega_2$ the same label they had in G' . Next,
 366 in both cases, we label all the vertices in Ω_2 with the label appearing four
 367 or six times in Ω_1 , thus obtaining a labelling of G such that there are at
 368 most two edges which are not monochromatic. However, this means that G
 369 is either disconnected or contains a 2-edge-cut, a contradiction.

370 Therefore, G' must be connected. We label the vertices of $G' \setminus \{g\}$ with
 371 labels A and B depending in which connected component of $G' \setminus \{g\}$ they
 372 belong to. Consequently, G' has exactly one edge which is not monochro-
 373 matic: the bridge g . We consider different cases depending on the number
 374 of vertices in Ω_1 labelled A in G' , and show that, in each case, a bridge in
 375 G' would imply that G is not cyclically 4-edge-connected. Without loss of
 376 generality, we shall assume that the number of vertices in Ω_1 labelled A is at
 377 least the number of vertices in Ω_1 labelled B . We consider four cases.

378 (B0) All the vertices in Ω_1 are labelled A in G' .

379 As above, we extend this labelling to a partial labelling of G , and then
380 give label A to all the vertices in Ω_2 . However, this means that G has
381 exactly one edge, corresponding to g , which is not monochromatic, a
382 contradiction, since G does not admit a bridge.

383 (B1) Exactly 5 vertices in Ω_1 are labelled A in G' .

384 This means that one of the edges in $\{\alpha x, wz, \gamma\delta\}$ is the bridge g . Once
385 again, we extend this labelling to a partial labelling of G , and then
386 give label A to all the vertices in Ω_2 . However, this means that G has
387 an edge which has exactly one endvertex in Ω_1 labelled B and exactly
388 one endvertex in Ω_2 labelled A , implying that G admits a bridge, a
389 contradiction once again.

390 (B2) Exactly 4 vertices in Ω_1 are labelled A in G' .

391 Since G' has exactly one edge which is not monochromatic, the edges
392 in $\{\alpha x, wz, \gamma\delta\}$ are all monochromatic. As in the previous cases, we
393 extend this labelling to a partial labelling of $V(G)$, and then give label
394 A to all the vertices in Ω_2 . However, this means that G has exactly two
395 edges each having exactly one endvertex in Ω_1 labelled B and exactly
396 one endvertex in Ω_2 labelled A . These two edges together with the
397 edge in G corresponding to the bridge g form a 3-edge-cut in G . Since
398 G has no cyclic 3-edge-cuts, this cut must be trivial – it separates a
399 singleton from the rest of the graph. Therefore, one of the edges αx ,
400 wz , or $\gamma\delta$ is a loop in G' , and so $\alpha = x$, $w = z$, or $\gamma = \delta$ in G . If $w = z$
401 or $\gamma = \delta$, then the vertices w, β, y , or γ, u, v would induce a triangle in
402 G , which is impossible since G is cyclically 4-edge-connected. If $\alpha = x$,
403 then the edges βy and yz (which are not in F) would be at distance at
404 most 2 from e , a contradiction once again.

405 (B3) Exactly 3 vertices in Ω_1 are labelled A in G' .

406 Since G' has exactly one edge which is not monochromatic, there is
407 exactly one edge in $\{\alpha x, wz, \gamma\delta\}$ which is not monochromatic. As in
408 the preceding cases, we extend this labelling to a partial labelling of
409 $V(G)$, and then give label A to all the vertices in Ω_2 . However, this
410 means that G has exactly three edges each having exactly one endvertex
411 in Ω_1 labelled B and exactly one endvertex in Ω_2 labelled A , and so

412 these three edges separate a singleton from the rest of the graph. In
413 particular, it means that either $\alpha = x$, $w = z$ or $\gamma = \delta$, which is
414 impossible as we have seen in the previous case.

415 Thus, G' is bridgeless, a contradiction to our assumption. \square

416 Here are some consequences of our main result.

417 **Corollary 2.3.** *Let G be a bridgeless cubic graph. For every parity subgraph*
418 *J of G there exists a perfect matching M such that J and M are a quelling*
419 *pair.*

420 **Corollary 2.4.** *Let G be a bridgeless cubic graph and let \mathcal{O} be a collection*
421 *of disjoint odd circuits of G . Then, there exists a perfect matching M such*
422 *that $M \cap E(C) \neq \emptyset$, for every $C \in \mathcal{O}$.*

423 **Corollary 2.5.** *Every bridgeless cubic graph admits an S_4 -pair.*

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