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► To cite this version:

Jie Jiang, Philippe Laurençot. Global bounded classical solutions to a parabolic-elliptic chemotaxis model with local sensing and asymptotically unbounded motility. *Bulletin of the London Mathematical Society*, 2023, 56 (2), pp.681–700. 10.1112/blms.12958 . hal-04018777

HAL Id: hal-04018777

<https://hal.science/hal-04018777>

Submitted on 8 Mar 2023

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GLOBAL BOUNDED CLASSICAL SOLUTIONS TO A PARABOLIC-ELLIPTIC CHEMOTAXIS MODEL WITH LOCAL SENSING AND ASYMPTOTICALLY UNBOUNDED MOTILITY

JIE JIANG AND PHILIPPE LAURENÇOT

ABSTRACT. Global existence and boundedness of classical solutions are shown for a parabolic-elliptic chemotaxis system with local sensing when the motility function is assumed to be unbounded at infinity. The cornerstone of the proof is the derivation of L^∞ -estimates on the second component of the system and is achieved by various comparison arguments.

1. INTRODUCTION

Let Ω be a smooth bounded domain of \mathbb{R}^N , $N \geq 1$, and consider the initial-boundary value problem

$$\partial_t u = \Delta(u\gamma(v)), \quad (t, x) \in (0, \infty) \times \Omega, \quad (1.1a)$$

$$0 = \Delta v - v + u, \quad (t, x) \in (0, \infty) \times \Omega, \quad (1.1b)$$

$$\nabla(u\gamma(v)) \cdot \mathbf{n} = \nabla v \cdot \mathbf{n} = 0, \quad (t, x) \in (0, \infty) \times \partial\Omega, \quad (1.1c)$$

$$u(0) = u^{in}, \quad x \in \Omega, \quad (1.1d)$$

where \mathbf{n} denotes the outward unit normal vector field to $\partial\Omega$.

The system (1.1) is a classical parabolic-elliptic simplification of the well-known fully parabolic system of partial differential equations named after Keller & Segel, which is originally proposed in their seminal work [20] to model the chemotaxis phenomenon due to a local sensing mechanism:

$$\partial_t u = \Delta(u\gamma(v)), \quad (t, x) \in (0, \infty) \times \Omega, \quad (1.2a)$$

$$\tau \partial_t v = \Delta v - v + u, \quad (t, x) \in (0, \infty) \times \Omega, \quad (1.2b)$$

$$\nabla(u\gamma(v)) \cdot \mathbf{n} = \nabla v \cdot \mathbf{n} = 0, \quad (t, x) \in (0, \infty) \times \partial\Omega, \quad (1.2c)$$

$$(u, \tau v)(0) = (u^{in}, \tau v^{in}), \quad x \in \Omega, \quad (1.2d)$$

Here, u and v represent the cell density and the chemical concentration, respectively. The cellular motility γ is a positive function on $(0, \infty)$ and its dependence on v accounts for the influence of the

Date: March 8, 2023.

2020 Mathematics Subject Classification. 35K51 - 35K55 - 35A01.

Key words and phrases. chemotaxis - global existence - boundedness - comparison.

chemical signal on the motion of cells which is better seen after expanding

$$\Delta(u\gamma(v)) = \operatorname{div}(\gamma(v)\nabla u + u\gamma'(v)\nabla v) :$$

the diffusion of cells in space is governed by the values of $\gamma(v)$, while the (chemotactic) bias on their motion induced by the signal is attractive when $\gamma'(v) < 0$ and repulsive when $\gamma'(v) > 0$. More recently, an extended model involving a third component n accounting for the nutrient level has been introduced in [23] and reads:

$$\partial_t u = \Delta(u\gamma(v)) + \theta u f(n), \quad (t, x) \in (0, \infty) \times \Omega, \quad (1.3a)$$

$$\tau \partial_t v = \Delta v - v + u, \quad (t, x) \in (0, \infty) \times \Omega, \quad (1.3b)$$

$$\partial_t n = \Delta n - \theta u f(n), \quad (t, x) \in (t, x) \in (0, \infty) \times \Omega, \quad (1.3c)$$

also supplemented with no-flux boundary conditions and initial conditions. In [23], the motility γ is assumed to be a positive non-increasing function, which reflects a repressive effect of the signal on cellular motility. Formation of spatially periodic patterns are shown by numerical simulations and experimental analysis in a growing bacteria population merely under this motility control. We note that setting $\theta = 0$ in (1.3) cancels the coupling between u and n and we then recover the system (1.2) for (u, v) .

The mathematical analysis of the Keller–Segel-type system (1.2) and its variants (such as (1.1) or (1.3)) has attracted a lot of interest in recent years. Besides the quasilinear structure of the cell's equation (1.2a) coupling in a nonlinear way the dynamics of the cells and the signal, the cells' equation (1.2a) features a (signal-dependent) degeneracy when γ vanishes, which is likely to occur at infinity ($\gamma(s) = s^{-k}$, $k > 0$, or $\gamma(s) = e^{-\chi s}$, $\chi > 0$, for instance) or at zero ($\gamma(s) = s^k$, $k > 0$, for instance). In that case, equation (1.2a) is a second-order degenerate quasilinear parabolic equation which, unlike the celebrated porous medium equation $\partial_t z - \Delta z^m = 0$, $m > 1$, does not have a variational structure in general. Nevertheless, energy methods turn out to be adequate in some special cases which we describe now. When $\gamma \in W^{1,\infty}(0, \infty)$ ranges in a compact subinterval of $(0, \infty)$, thereby excluding the aforementioned possible degeneracy, and Ω is convex, the existence of a global bounded solution to (1.2) is shown in space dimension $N = 2$ in [26], along with the existence of global weak solutions in higher space dimensions $N \geq 3$. When the motility γ is the specific algebraically decaying function $\gamma(s) = c_0 s^{-k}$ with $c_0 > 0$, it is proved in [29] that global bounded classical solutions whatever the value of $k > 0$, provided that c_0 is sufficiently small. The smallness condition on c_0 is subsequently relaxed in [1, 27] when $\tau = 0$ and $k < \frac{2}{N-2}$. When $\gamma(s) = 1/(c + s^k)$, $c \geq 0$, global weak solutions to (1.2) are constructed in [7] when $N \in \{1, 2, 3\}$ and $k \in (0, k_N)$ with $k_1 = \infty$, $k_2 = 2$, and $k_3 = 4/3$. When the motility function γ is a negative exponential, $\gamma(s) = e^{-\chi s}$ with $\chi > 0$, global existence of weak solutions is established in [6] in arbitrary space dimensions $N \geq 1$. Moreover, when $N = 2$, there is a threshold value of $\|u^{in}\|_1$, which is exactly the same as that of the minimal Keller–Segel system according to [19], separating two different dynamics of (1.2): global classical solutions to (1.2) exist when $\|u^{in}\|_1$ lies beyond the threshold value, while there are initial conditions with $\|u^{in}\|_1$ above the threshold value for which the corresponding classical solution to (1.2) becomes unbounded in finite or infinite time.

In a series of works [8–14, 16–18, 22, 24, 25, 28], a different approach is developed and proved to be efficient in generic cases. The main idea is to exploit the specific structure of (1.2a) to derive a quasilinear degenerate parabolic equation involving a non-local term for an auxiliary unknown function. A systematic argument based on comparison techniques, monotonicity tricks, iteration procedures as well as applications of abstract semi-group theory gives rise to advanced theories on well-posedness and qualitative behavior of the solutions.

Let us summarize the results obtained in the literature for the initial-boundary value problem (1.2) or its variant (1.3) (also supplemented with no-flux boundary conditions) by this alternative method. When $\tau > 0$, it is proved in [12, 14, 18] that there exists a unique global classical solution to (1.2) for a large class of non-negative initial conditions under a rather general assumption on γ :

$$\gamma \in C^3((0, \infty)), \quad \gamma > 0 \quad \text{on} \quad (0, \infty), \quad (1.4)$$

$$\gamma' \leq 0 \quad \text{on} \quad (0, \infty), \quad (1.5)$$

and

$$\limsup_{s \rightarrow \infty} \gamma(s) < 1/\tau. \quad (1.6)$$

The monotonicity (1.5) of γ is actually not needed, as shown recently in [28], where the existence of global classical solutions to (1.2) is established under the only assumptions (1.4) and (1.6). In the same vein, global weak solutions are constructed in [22] when γ satisfies (1.4) and $\sup_{s>0} \gamma(s) < 1/\tau$, and in [8] when γ satisfies (1.4) and decays at most algebraically at infinity. All these existence results share the property that no finite time blowup occurs in (1.2), a feature which significantly differs from the minimal Keller–Segel system or the logarithmic Keller–Segel system which both involve a linear cellular diffusion and are well-known to trigger finite time singularities.

Furthermore, the boundedness of global solutions to (1.2) is shown to be closely related to the decay property of γ at infinity. Specifically, for γ satisfying (1.4), (1.5), and vanishing at infinity

$$\lim_{s \rightarrow \infty} \gamma(s) = 0, \quad (1.7)$$

it is proved that exponential decay is critical in space dimension $N = 2$ in the following sense: there exist unbounded solutions to (1.2) when $\gamma(s) = e^{-\chi s}$ for some $\chi > 0$ and the L^1 -norm of u^{in} is sufficiently large [12, 19], whereas all solutions to (1.2) are bounded if γ decays slower than a negative exponential function; that is, γ satisfies

$$\liminf_{s \rightarrow \infty} e^{\chi s} \gamma(s) > 0$$

for all $\chi > 0$, typical examples including $\gamma(s) = s^{-k_1} \log^{-k_2}(1+s)$ with $(k_1, k_2) \in (0, \infty) \times [0, \infty)$, or $\gamma(s) = e^{-s^\alpha}$ with $0 < \alpha < 1$. When $N \geq 3$ and $\gamma(s) \sim s^{-k}$, boundedness of global solutions holds true for $k < N/(N-2)$ [13, 18], which in particular improves previous results for the specific case $\tau = 0$ and $\gamma(s) = s^{-k}$ by substantially enlarging the admissible range of k . Recently, the above boundedness results have been improved in [28] by removing the monotonicity requirement (1.5) and replacing the asymptotically vanishing property (1.7) by the asymptotically smallness assumption (1.6). Moreover, if γ simply satisfies (1.4) and is bounded by positive constants from above and below, so that (1.2a) is uniformly parabolic, existence of uniform-in-time bounded solution is also proved in [28] in any

bounded domain Ω in arbitrary space dimension $N \geq 1$, thus improving the corresponding one in [26] obtained in two-dimensional convex domains.

As for the initial-boundary value problem (1.1), which is recovered from (1.2) by setting formally $\tau = 0$, the existence and boundedness/unboundedness results available for (1.2) are still true under the same assumptions by setting $1/0 = \infty$ in (1.6). Indeed, global existence of classical solutions is proved in our previous work [17] for motility functions γ satisfying (1.4) and such that $\gamma \in L^\infty(s, \infty)$ for any $s > 0$. Note that, thanks to the continuity (1.4) of γ , the latter is obviously equivalent to

$$\limsup_{s \rightarrow \infty} \gamma(s) < \infty. \quad (1.8)$$

According to the above discussion, the studies of (1.1) and (1.2) performed up to now require the motility γ to be a bounded function on $[s, \infty)$ for any $s > 0$, as implied by, either (1.6), or the assumption that γ ranges in a compact subinterval of $(0, \infty)$. This raises the natural question of the dynamics of (1.1) and (1.2) when γ becomes unbounded near infinity, and the main purpose of the present contribution is a detailed analysis of the dynamics of (1.1) in that case.

We first show that the initial-boundary value problem (1.1) has a unique global bounded classical solution when γ is, either non-decreasing, or asymptotically unbounded.

Theorem 1.1. *Assume that γ satisfies (1.4), as well as, either $\gamma' \geq 0$ in $(0, \infty)$, or*

$$\limsup_{s \rightarrow \infty} \gamma(s) = \infty. \quad (1.9)$$

Suppose that $u^{in} \in W_+^{1,\infty}(\Omega)$ with $m = \|u^{in}\|_1 > 0$, where

$$W_+^{1,\infty}(\Omega) := \{z \in W^{1,\infty}(\Omega) : z \geq 0 \text{ in } \Omega\}$$

and we use the short notation $\|\cdot\|_p$ for the norm $\|\cdot\|_{L^p(\Omega)}$ with $p \in [1, \infty]$.

Then there is a unique non-negative global classical solution

$$u \in C([0, \infty) \times \bar{\Omega}) \cap C^{1,2}((0, \infty) \times \bar{\Omega}), \quad v \in C^{1,2}((0, \infty) \times \bar{\Omega}),$$

to (1.1) which satisfies the conservation of matter

$$\|u(t)\|_1 = m = \|u^{in}\|_1, \quad t \geq 0, \quad (1.10)$$

and is uniformly bounded; that is,

$$\sup_{t \in (0, \infty)} (\|u(t)\|_\infty + \|v(t)\|_\infty) < \infty. \quad (1.11)$$

Combining the outcome of Theorem 1.1 with [17, Theorem 1.1] provides the existence of a unique non-negative classical solution to (1.1) under the sole assumption (1.4) on γ , as stated now for future reference.

Corollary 1.2. *Assume that γ satisfies (1.4) and that $u^{in} \in W_+^{1,\infty}(\Omega)$ with $m = \|u^{in}\|_1 > 0$. Then there is a unique non-negative global classical solution to (1.1) which satisfies the conservation of matter (1.10).*

Another consequence of Theorem 1.1 is the boundedness of the non-negative global classical solution to (1.1) given by Corollary 1.2 in the chemorepulsive case $\gamma' \geq 0$. The dynamics is more complex in the chemoattractive case $\gamma' \leq 0$ as the solution to (1.1) may blow up in infinite time according to [10].

We next supplement Theorem 1.1 with explicit L^∞ -bounds on u or v when γ is endowed with additional properties. First, define the (unbounded) linear operator \mathcal{A} on $L^2(\Omega)$ by

$$\begin{aligned} D(\mathcal{A}) &:= \{z \in H^2(\Omega) : \nabla z \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}, \\ \mathcal{A}[z] &:= z - \Delta z, \quad z \in D(\mathcal{A}). \end{aligned} \tag{1.12}$$

Then $v = \mathcal{A}^{-1}[u]$ according to (1.1b) and (1.1c) and $v^{in} := \mathcal{A}^{-1}[u^{in}] \in W^{3,p}(\Omega)$ for any $1 \leq p < \infty$ by standard regularity theory of elliptic equations. In particular, v^{in} is bounded and positive in $\bar{\Omega}$, the latter following from Lemma 2.2 below while the former is ensured by Sobolev embeddings.

Now, we show that explicit upper and lower bounds on v are available under a local monotonicity assumption on γ , implying the existence of an invariant region for the second component v .

Theorem 1.3. *Assume that γ satisfies (1.4) and that $u^{in} \in W_+^{1,\infty}(\Omega)$ with $m = \|u^{in}\|_1 > 0$. Let (u, v) be the corresponding non-negative classical solution to (1.1) given by Corollary 1.2. If*

$$\gamma' \geq 0 \quad \text{on} \quad \left[\min_{\bar{\Omega}}\{v^{in}\}, \max_{\bar{\Omega}}\{v^{in}\} \right], \tag{1.13}$$

then the solution to (1.1) satisfies the uniform bound (1.11). More precisely,

$$\min_{\bar{\Omega}}\{v^{in}\} \leq v(t, x) \leq \max_{\bar{\Omega}}\{v^{in}\}, \quad (t, x) \in [0, \infty) \times \bar{\Omega}.$$

In the same vein, we have the following explicit upper bounds on u .

Theorem 1.4. *Assume that γ satisfies (1.4), along with (1.13), and that $u^{in} \in W_+^{1,\infty}(\Omega)$ with $m = \|u^{in}\|_1 > 0$. Let (u, v) be the corresponding non-negative global classical solution to (1.1) provided by Corollary 1.2. Then*

$$\|u(t)\|_\infty \leq \frac{\|u^{in}\gamma(v^{in})\|_\infty}{\gamma(\min_{\bar{\Omega}}\{v^{in}\})} \leq \frac{\|u^{in}\|_\infty \gamma(\max_{\bar{\Omega}}\{v^{in}\})}{\gamma(\min_{\bar{\Omega}}\{v^{in}\})}, \quad t \geq 0.$$

Assume in addition that $\gamma'' \leq 0$ in $[\min_{\bar{\Omega}}\{v^{in}\}, \max_{\bar{\Omega}}\{v^{in}\}]$. Then $\|u(t)\|_\infty \leq \|u^{in}\|_\infty$ for $t \geq 0$.

We finally deal with the large time behaviour of solutions to (1.1) when γ is non-decreasing and report that there is no pattern formation in that case, as intuitively expected in the chemorepulsive regime.

Theorem 1.5. *Assume that γ satisfies (1.4) and (1.13). Consider $u^{in} \in W_+^{1,\infty}(\Omega)$ with $m = \|u^{in}\|_1 > 0$ and denote the corresponding non-negative global classical solution to (1.1) provided by Corollary 1.2 by (u, v) . Then*

$$\lim_{t \rightarrow \infty} \left(\left\| u(t) - \frac{m}{|\Omega|} \right\|_\infty + \left\| v(t) - \frac{m}{|\Omega|} \right\|_{W^{1,\infty}} \right) = 0.$$

A similar stability result is proved in [1, Theorem 1.3] when $\gamma(s) = s^{-k}$ for some $k \in (0, 1]$. Their proof relies on the availability of a Liapunov functional for (1.1) when $\gamma(s) = s^{-k}$ for some $k \in (0, 1]$ which turns out to be also a Liapunov functional when γ is non-decreasing. The proof of Theorem 1.5 is thus the same as that of [1, Theorem 1.3], to which we refer.

The cornerstone of the proof of our existence and boundedness results is the derivation of an upper bound for the second component v , which is achieved by a new delicate comparison argument developed in the current contribution. Let us illustrate the difficulty to be faced and sketch the main idea in the simple case where γ satisfies (1.4) and $\gamma' \geq 0$ on $(0, \infty)$. To begin with, we derive from (1.1a)-(1.1b) the following key identity

$$\partial_t v - \gamma(v)\Delta v + v\gamma(v) = \mathcal{A}^{-1}[u\gamma(v)], \quad (t, x) \in (0, \infty) \times \Omega, \quad (1.14)$$

recalling that \mathcal{A} is defined in (1.12). This identity is uncovered in [10] and has since then been used efficiently to investigate the global existence and boundedness of classical solutions to (1.1) and its variants, see [8, 9, 11–14, 16–18, 22, 24, 25, 28], where an upper bound on γ is essentially used to control the non-local source term on the right-hand side of (1.14). Indeed, supposing that $\gamma(v)$ is bounded from above on $[0, T]$ by some possibly time-dependent upper bound $\gamma^*(T) > 0$, an application of the elliptic comparison principle yields an upper control of the non-local term by a term growing linearly as a function of v which reads

$$\mathcal{A}^{-1}[u\gamma(v)] \leq \gamma^*(T)\mathcal{A}^{-1}[u] = \gamma^*(T)v, \quad (t, x) \in [0, T] \times \bar{\Omega}.$$

We also refer to [17] for a more tricky derivation of a sublinear control of this term when γ is non-increasing on $(0, \infty)$. Then a systematic argument based on comparison techniques and Moser-Alikakos iterations is developed in [10, 11, 17, 28] to derive estimate for v in $L^\infty((0, T) \times \Omega)$.

In contrast, when γ is non-decreasing and unbounded at infinity, the non-local term features in principle a superlinear growth with respect to v . Indeed, since $\gamma(v) \leq \gamma(\|v\|_\infty)$, it follows that

$$\mathcal{A}^{-1}[u\gamma(v)] \leq \gamma(\|v\|_\infty)v, \quad (t, x) \in (0, \infty) \times \Omega,$$

and thus

$$\partial_t v - \gamma(v)\Delta v + v\gamma(v) \leq \gamma(\|v\|_\infty)v, \quad (t, x) \in (0, \infty) \times \Omega. \quad (1.15)$$

The previous methods fail in this case due to the superlinear dependence of the right-hand side on $\|v\|_\infty$.

To overcome this difficulty, we develop a novel approach to derive time-independent upper bounds on v solely relying on comparison techniques. Specifically, introducing the solution V to the ordinary differential equation

$$\begin{aligned} \frac{dV}{dt} + V\gamma(V) &= \gamma(\|v\|_\infty)V, & t \geq 0, \\ V(0) &= \|v^{in}\|_\infty, \end{aligned} \quad (1.16)$$

recalling that $v^{in} = \mathcal{A}^{-1}[u^{in}]$, an immediate consequence of (1.16) is that V is a supersolution to (1.15) and we deduce from the parabolic comparison principle that $v(t, x) \leq V(t)$ for $(t, x) \in [0, \infty) \times \bar{\Omega}$. In particular, $\|v(t)\|_\infty \leq V(t)$ for $t \geq 0$ and the monotonicity of γ , the non-negativity of V , and (1.16)

entail that $dV/dt = (\gamma(\|v\|_\infty) - \gamma(V))V \leq 0$. As a result, we have $\|v(t)\|_\infty \leq V(t) \leq \|v^{in}\|_\infty$ for $t \geq 0$, as stated in Theorem 1.3. The lower bound in Theorem 1.3 is derived with a similar argument.

In the general case where γ satisfies (1.4) but need not be monotone, the main idea is to split γ as a sum of its increasing and decreasing parts. A more delicate argument combining the above technique and the monotonicity trick developed in our previous work [17] is carried out to derive the time-independent upper bound for v ; see Section 3.2. Once the upper bound for v is obtained, we accomplish the proof according to Proposition 2.3.

The remainder of this paper is organized as follows. In Section 2, we recall the local existence result and establish a blowup criterion, which ensures that global existence of classical solutions to (1.1) is a direct, though far from straightforward, consequence of the boundedness of v . We then derive the uniform-in-time boundedness of v by the new comparison technique and hence prove Theorem 1.1 in Section 3. In addition, we discuss the continuous dependence of classical solutions of (1.1) on initial conditions. We next derive explicit L^∞ -estimates for v and u in Section 4 and Section 5, respectively, under additional conditions on γ . Finally, we study the stabilization of solutions to (1.1) and prove Theorem 1.5 in Section 6.

2. PRELIMINARIES

We first state the local well-posedness of (1.1), which can be proved as in [1, Lemma 3.1].

Proposition 2.1. *Assume that γ satisfies (1.4) and that $u^{in} \in W_+^{1,\infty}(\Omega)$ with $m = \|u^{in}\|_1 > 0$. Then the initial-boundary value problem (1.1) has a unique non-negative classical solution*

$$u \in C([0, T_{\max}) \times \bar{\Omega}) \cap C^{1,2}((0, T_{\max}) \times \bar{\Omega}), \quad v \in C^{1,2}((0, T_{\max}) \times \bar{\Omega}),$$

defined on a maximal time interval $[0, T_{\max})$ with $T_{\max} \in (0, \infty]$ which satisfies

$$\|u(t)\|_1 = m = \|u^{in}\|_1, \quad t \in [0, T_{\max}). \quad (2.1)$$

In addition, if $T_{\max} < \infty$, then

$$\lim_{t \rightarrow T_{\max}} \|u(t)\|_\infty = \infty. \quad (2.2)$$

We next recall the following lemma [15, Lemma 2.1].

Lemma 2.2. *There is $\omega_* > 0$ depending only on Ω such that, for any $f \in L^1(\Omega)$ satisfying*

$$f \geq 0 \quad \text{a.e. in } \Omega \quad \text{and} \quad \|f\|_1 = m,$$

there holds

$$\mathcal{A}^{-1}[f] \geq m\omega_* \quad \text{in } \Omega,$$

recalling that the elliptic operator \mathcal{A} is defined by (1.12).

We now fix γ satisfying (1.4) and $u^{in} \in W_+^{1,\infty}(\Omega)$ with $m = \|u^{in}\|_1 > 0$ and denote the corresponding classical solution to (1.1) defined on $[0, T_{\max})$ provided by Proposition 2.1 by (u, v) . We then infer from (1.1) that v solves

$$\partial_t v - \gamma(v)\Delta v + v\gamma(v) = \mathcal{A}^{-1}[u\gamma(v)], \quad (t, x) \in (0, T_{\max}) \times \Omega, \quad (2.3a)$$

$$\nabla v \cdot \mathbf{n} = 0, \quad (t, x) \in (0, T_{\max}) \times \partial\Omega, \quad (2.3b)$$

$$v(0) = v^{in}, \quad x \in \Omega, \quad (2.3c)$$

where

$$v^{in} = \mathcal{A}^{-1}[u^{in}]. \quad (2.4)$$

Observe that the properties of u^{in} , the elliptic comparison principle, Lemma 2.2, (2.1), and (2.4) imply that $v^{in} \in L^\infty(\Omega)$ with $\|v^{in}\|_\infty \leq \|u^{in}\|_\infty$ and

$$v(t, x) \geq v_* := m\omega_* > 0, \quad (t, x) \in [0, T_{\max}) \times \bar{\Omega}. \quad (2.5)$$

We conclude this section by showing that an L^∞ -estimate on v on $(0, T)$ for some $T > 0$ guarantees that $T_{\max} \geq T$.

Proposition 2.3. *Under the assumption of Proposition 2.1, if there is $T > 0$ such that*

$$\mathcal{V}(T) := \sup_{[0, T] \cap [0, T_{\max})} \{\|v(t)\|_\infty\} < \infty, \quad (2.6)$$

then $T_{\max} \geq T$ and

$$\mathcal{U}(T) := \sup_{[0, T]} \{\|u(t)\|_\infty\} < \infty. \quad (2.7)$$

In addition, if (2.6) holds true for all $T > 0$ and there is $\mathcal{V}_\infty > 0$ such that $\mathcal{V}(T) \leq \mathcal{V}_\infty$ for all $T > 0$, then there is $\mathcal{U}_\infty > 0$ such that $\mathcal{U}(T) \leq \mathcal{U}_\infty$ for all $T > 0$.

Proof. We outline the proof here since it relies on a well-established argument already described in previous works [17, 18, 28], see also [13, 14].

First, arguing as in [17, Lemma 3.2], we derive a Hölder estimate for v in $C^\alpha([0, T] \times \bar{\Omega})$, with a possible dependence upon T of both the estimate and the exponent $\alpha \in (0, 1)$. Let us mention that, as already noticed in [17], no monotonicity property of γ is needed here and the proof only requires positive upper and lower bounds on $\gamma(v)$.

The second step is the derivation of estimates for v in $L^\infty((0, T), W^{2,r}(\Omega))$ for $r > 1$, still possibly depending on T , as well as on r . The just established Hölder continuity of v enables us to regard $\gamma(v)\Delta$ as a generator of a parabolic evolution operator on $L^r(\Omega)$ for $r \in (1, \infty)$. Then estimates in $L^\infty((0, T), W^{2,r}(\Omega))$ of v follows from (2.3) and applications of the abstract theory for non-autonomous parabolic equations developed in [2–5]; see [17, Sect. 3.3 and Sect. 4.2] for detailed proofs.

Finally, we finish the proof of Proposition 2.3 by establish estimates for u in $L^\infty((0, T), L^r(\Omega))$, $r \in [1, \infty]$, by a bootstrap argument as done in [1]. \square

3. GLOBAL EXISTENCE AND BOUNDEDNESS

The purpose of this section is two-fold: we show the global existence and uniform-in-time boundedness of classical solutions to (1.1), as well as their continuous dependence on initial data.

According to Proposition 2.3, it is sufficient to prove that the L^∞ -norm of v is bounded on $[0, T_{\max})$ by a time-independent bound to obtain Theorem 1.1. In order to illustrate the approach we use and

avoid some technicalities, we first provide a proof under the additional assumption that γ is non-decreasing on $(0, \infty)$, see Section 3.1. The proof in the general case is then given in Section 3.2.

3.1. L^∞ -estimate on v : the non-decreasing case.

Lemma 3.1. *Assume γ satisfy (1.4) and $\gamma' \geq 0$ on $(0, \infty)$. Then*

$$\|v(t)\|_\infty \leq \|v^{in}\|_\infty, \quad t \in [0, T_{\max}). \quad (3.1)$$

Proof. Owing to (1.1b), the monotonicity of γ , and the non-negativity of u , there holds

$$u\gamma(v) \leq \gamma(\|v\|_\infty)\mathcal{A}[v] \quad \text{in } (0, T_{\max}) \times \Omega.$$

Consequently, the elliptic comparison implies that

$$\mathcal{A}^{-1}[u\gamma(v)] \leq \gamma(\|v\|_\infty)v \quad \text{in } (0, T_{\max}) \times \Omega,$$

and it readily follows from (2.3) that v satisfies

$$\partial_t v - \gamma(v)\Delta v + v\gamma(v) \leq \gamma(\|v\|_\infty)v, \quad (t, x) \in (0, T_{\max}) \times \Omega, \quad (3.2a)$$

$$\nabla v \cdot \mathbf{n} = 0, \quad (t, x) \in (0, T_{\max}) \times \partial\Omega, \quad (3.2b)$$

$$v(0) = v^{in}, \quad x \in \Omega. \quad (3.2c)$$

Introducing the solution V to the ordinary differential equation

$$\frac{dV}{dt} + V\gamma(V) = \gamma(\|v\|_\infty)V, \quad t \in (0, T_{\max}), \quad (3.3a)$$

$$V(0) = \|v^{in}\|_\infty, \quad (3.3b)$$

we infer from (3.2), (3.3), and the parabolic comparison principle that

$$v(t, x) \leq V(t), \quad (t, x) \in (0, T_{\max}) \times \bar{\Omega}. \quad (3.4)$$

In particular, for $t \in (0, T_{\max})$, there holds $\|v(t)\|_\infty \leq V(t)$ and we deduce from (3.3a), the non-negativity of V , and the monotonicity of γ that

$$\frac{dV}{dt}(t) + V(t)\gamma(V(t)) \leq \gamma(V(t))V(t), \quad t \in (0, T_{\max}).$$

Hence, $dV/dt \leq 0$ in $(0, T_{\max})$, from which (3.1) follows after using (3.3b) and (3.4). \square

3.2. L^∞ -estimate on v : the general case. When γ is not monotone but becomes unbounded near infinity, the previous argument no longer works and requires to be suitably adapted. To this end, we begin with the following auxiliary result.

Lemma 3.2. *Under the assumptions (1.4) and (1.9), there is $s_* \geq \|v^{in}\|_\infty$ such that*

$$\gamma(s_*) = \max_{s \in [v_*, s_*]} \{\gamma(s)\}, \quad (3.5)$$

recalling that the lower bound v_* on v is defined in (2.5).

We may obviously choose $s_* = \|v^{in}\|_\infty$ when γ is non-decreasing.

Proof. Let $j \geq 1$ be a positive integer and set

$$M_j := \max_{s \in [v_*, j \|v^{in}\|_\infty]} \{\gamma(s)\},$$

$$s_j := \sup \{s \in [v_*, j \|v^{in}\|_\infty] : \gamma(s) = M_j\},$$

so that

$$\gamma(s_j) = M_j = \max_{s \in [v_*, s_j]} \{\gamma(s)\}.$$

Then $(M_j)_{j \geq 1}$ and $(s_j)_{j \geq 1}$ are non-decreasing sequences of positive real numbers and the unbound-
edness (1.9) of γ at infinity guarantees that

$$\lim_{j \rightarrow \infty} M_j = \infty \quad \text{and} \quad \lim_{j \rightarrow \infty} s_j = \infty.$$

Consequently,

$$j_0 := \inf \{j \geq 1 : s_j \geq \|v^{in}\|_\infty\} < \infty.$$

Setting $s_* := s_{j_0}$, it is clear that $s_* \geq \|v^{in}\|_\infty$ with

$$\gamma(s_*) = \gamma(s_{j_0}) = M_{j_0} = \max_{s \in [v_*, s_{j_0}]} \{\gamma(s)\} = \max_{s \in [v_*, s_*]} \{\gamma(s)\},$$

and the proof is complete. \square

We next define

$$\gamma'_i(s) := \begin{cases} 0, & s \in [v_*, s_*], \\ (\gamma'(s))_+ = \max \{\gamma'(s), 0\}, & s \geq s_*, \end{cases} \quad (3.6)$$

and

$$\gamma'_d(s) := \begin{cases} 0, & s \in [v_*, s_*], \\ -(\gamma'(s))_- = \min \{\gamma'(s), 0\}, & s \geq s_*, \end{cases} \quad (3.7)$$

with $\gamma_i(s_*) = \gamma_d(s_*) = 0$ and notice that

$$\gamma_i \geq 0 \geq \gamma_d \quad \text{on} \quad [v_*, \infty) \quad (3.8)$$

and

$$\gamma(s) = \gamma(s_*) + \gamma_i(s) + \gamma_d(s), \quad s \in [s_*, \infty). \quad (3.9)$$

In addition,

$$\gamma_i(s) = \gamma_d(s) = 0, \quad s \in [v_*, s_*]. \quad (3.10)$$

We also set

$$\Gamma_d(s) := \int_{s_*}^s \gamma_d(\sigma) d\sigma, \quad s \in [v_*, \infty), \quad (3.11)$$

and deduce from (3.8) and (3.10) that

$$\begin{aligned} \Gamma_d(s) &= 0, & s \in [v_*, s_*], \\ 0 \leq \Gamma_d(s) &\leq (s - s_*)\gamma_d(s) \geq s\gamma_d(s), & s \geq s_*. \end{aligned} \quad (3.12)$$

With this notation, we are in a position to derive an upper bound on $u\gamma(v)$.

Lemma 3.3. *There holds*

$$u\gamma(v) \leq u[\gamma(s_*) + \gamma_i(\|v\|_\infty)] + \mathcal{A}[\Gamma_d(v)] \quad \text{in } (0, T_{\max}) \times \Omega.$$

Proof. Let $t \in (0, T_{\max})$. We first consider the case where $\|v(t)\|_\infty \geq s_*$. Then, for $x \in \Omega$, either $v(t, x) \geq s_*$ and it follows from (1.1b), (3.9), the monotonicity of γ_i and γ_d , and the non-negativity of u that

$$\begin{aligned} u(t, x)\gamma(v(t, x)) &= u(t, x)\gamma(s_*) + u(t, x)\gamma_i(v(t, x)) + u(t, x)\gamma_d(v(t, x)) \\ &\leq u(t, x)\gamma(s_*) + u(t, x)\gamma_i(\|v(t)\|_\infty) + \gamma_d(v(t, x))(v - \Delta v)(t, x) \\ &= u(t, x)\gamma(s_*) + u(t, x)\gamma_i(\|v(t)\|_\infty) - \operatorname{div}(\gamma_d(v)\nabla v)(t, x) \\ &\quad + v(t, x)\gamma_d(v(t, x)) + \gamma'_d(v(t, x))|\nabla v(t, x)|^2 \\ &\leq u(t, x)\gamma(s_*) + u(t, x)\gamma_i(\|v(t)\|_\infty) - \Delta\Gamma_d(v)(t, x) + v(t, x)\gamma_d(v(t, x)). \end{aligned}$$

We next use (3.12) to arrive at

$$u(t, x)\gamma(v(t, x)) \leq u(t, x)[\gamma(s_*) + \gamma_i(\|v(t)\|_\infty)] + \mathcal{A}[\Gamma_d(v)](t, x).$$

Or $v_* \leq v(t, x) < s_*$, and there is $r > 0$ such $v_* \leq v(t, y) < s_*$ for all $y \in B_r(x) \subset \Omega$ by the continuity of v and (2.5). Therefore, $\Gamma_d(v(t)) \equiv 0$ in $B_r(x)$ by (3.12) and $\mathcal{A}[\Gamma_d(v)](t, x) = 0$. Consequently, recalling (3.5),

$$u(t, x)\gamma(v(t, x)) \leq u(t, x)\gamma(s_*) \leq u(t, x)[\gamma(s_*) + \gamma_i(\|v(t)\|_\infty)] + \mathcal{A}[\Gamma_d(v)](t, x).$$

We next argue as above to conclude that, when $\|v(t)\|_\infty < s_*$, the inequality

$$u(t, x)\gamma(v(t, x)) \leq u(t, x)\gamma(s_*) \leq u(t, x)[\gamma(s_*) + \gamma_i(\|v(t)\|_\infty)] + \mathcal{A}[\Gamma_d(v)](t, x)$$

also holds true for all $x \in \Omega$. This completes the proof. \square

After this preparation, we are ready to establish an L^∞ -estimate on v .

Proposition 3.4. *Recalling that s_* is defined in Lemma 3.2, there holds*

$$\|v(t)\|_\infty \leq s_*, \quad t \in [0, T_{\max}]. \quad (3.13)$$

Proof. It readily follows from Lemma 3.3 and the elliptic comparison principle that

$$\mathcal{A}^{-1}[u\gamma(v)] \leq [\gamma(s_*) + \gamma_i(\|v\|_\infty)]v + \Gamma_d(v) \quad \text{in } (0, T_{\max}) \times \Omega.$$

Consequently, recalling (2.3), we realize that v satisfies

$$\partial_t v - \gamma(v)\Delta v + v\gamma(v) \leq [\gamma(s_*) + \gamma_i(\|v\|_\infty)]v + \Gamma_d(v), \quad (t, x) \in (0, T_{\max}) \times \Omega, \quad (3.14a)$$

$$\nabla v \cdot \mathbf{n} = 0, \quad (t, x) \in (0, T_{\max}) \times \partial\Omega, \quad (3.14b)$$

$$v(0) = v^{in}, \quad x \in \Omega. \quad (3.14c)$$

We next introduce the solution V to the following ordinary differential equation

$$\frac{dV}{dt} + V\gamma(V) = [\gamma(s_*) + \gamma_i(\|v(t)\|_\infty)]V + \Gamma_d(V), \quad t \in (0, T_{\max}), \quad (3.15a)$$

$$V(0) = s_*, \quad (3.15b)$$

noticing that the non-positivity (3.12) of Γ_d indeed guarantees that V is well-defined on $[0, T_{\max})$. Since $s_* \geq \|v^{in}\|_\infty$ by Lemma 3.2, we deduce from (3.14), (3.15), and the parabolic comparison principle that

$$\|v(t)\|_\infty \leq V(t), \quad t \in (0, T_{\max}). \quad (3.16)$$

Let $T \in (0, T_{\max})$. On the one hand, the continuity of v , the positivity of γ , and the non-positivity (3.12) of Γ_d imply that

$$V(t) \leq \mathcal{V}_T := s_* \exp \left\{ T \left[\gamma(s_*) + \sup_{\tau \in [0, T]} \{\gamma_i(\|v(\tau)\|_\infty)\} \right] \right\}, \quad t \in [0, T]. \quad (3.17)$$

On the other hand, owing to the monotonicity of γ_i , it follows from (3.15a) and (3.17) that

$$\frac{dV}{dt} + V\gamma(V) \leq [\gamma(s_*) + \gamma_i(V)]V + \Gamma_d(V), \quad t \in (0, T_{\max}). \quad (3.18)$$

Now, set $G(s) := \Gamma_d(s) - s\gamma_d(s)$ for $s \geq v_*$ and notice that $G(s) = 0$ for $s \in [v_*, s_*]$ by (3.10) and (3.12). We infer from (3.9) and (3.18) that

$$\begin{aligned} \frac{d(V - s_*)_+}{dt} &\leq [\gamma(s_*) + \gamma_i(V) - \gamma(V)]V \text{sign}_+(V - s_*) + \Gamma_d(V) \text{sign}_+(V - s_*) \\ &= [\gamma(s_*) + \gamma_i(V) - \gamma(s_*) - \gamma_i(V) - \gamma_d(V)]V \text{sign}_+(V - s_*) + \Gamma_d(V) \text{sign}_+(V - s_*) \\ &= (G(V) - G(s_*)) \text{sign}_+(V - s_*) \\ &= \frac{G(V) - G(s_*)}{V - s_*} (V - s_*)_+. \end{aligned}$$

Now, since $G'(s) = -s\gamma'_d(s) \geq 0$ for $s \geq s_*$, we deduce from (3.17) that

$$0 \leq \frac{G(V) - G(s_*)}{V - s_*} (V - s_*)_+ \leq (V - s_*)_+ \sup_{s \in [s_*, \mathcal{V}_T]} \{G'(s)\}, \quad t \in [0, T],$$

and we conclude that $V(t) \leq s_*$ for $t \in [0, T]$. As T is arbitrary in $(0, T_{\max})$, we have shown (3.13) in view of (3.16). \square

Proof of Theorem 1.1. Theorem 1.1 now readily follows from Proposition 3.4 and Proposition 2.3. \square

Proof of Corollary 1.2. Either γ satisfies (1.9) and Corollary 1.2 is an immediate consequence of Theorem 1.1. Or γ belongs to $L^\infty(v_*, \infty)$ and Corollary 1.2 follows at once from [17, Theorem 1.1]. \square

3.3. Continuous dependence. An interesting corollary of the previous analysis is the continuous dependence of the solutions to (1.1) with respect to the initial condition u^{in} , a property which is likely to follow as well from the proof of [1, Lemma 3.1]. This property is needed later in the proof of Theorem 1.3. We recall that, according to Corollary 1.2, the problem (1.1) is well-posed in a classical sense under the sole assumption (1.4) on γ .

Proposition 3.5. *Assume that γ satisfies (1.4) and consider a sequence $(u_j^{in})_{j \geq 1}$ in $W_+^{1,\infty}(\Omega)$ and $u^{in} \in W_+^{1,\infty}(\Omega)$ such that $\|u^{in}\|_1 > 0$ and*

$$\lim_{j \rightarrow \infty} \|u_j^{in} - u^{in}\|_\infty = 0. \quad (3.19)$$

For $j \geq 1$, let (u_j, v_j) be the classical solution to (1.1) with initial condition u_j^{in} and denote the classical solution to (1.1) with initial condition u^{in} by (u, v) , their existence being guaranteed by Corollary 1.2. Then, for any $T > 0$,

$$\lim_{j \rightarrow \infty} \sup_{t \in [0, T]} \{ \|(u_j - u)(t)\|_\infty + \|(v_j - v)(t)\|_\infty \} = 0. \quad (3.20)$$

Proof. Setting

$$v_j^{in} := \mathcal{A}^{-1}[u_j^{in}], \quad j \geq 1, \quad \text{and} \quad v^{in} := \mathcal{A}^{-1}[u^{in}],$$

it readily follows from (3.19) that we may assume without loss of generality that there are $m_0 > 0$ and $M_0 > 0$ such that

$$\|u_j^{in}\|_1 \geq m_0 \quad \text{and} \quad \|v_j^{in}\|_\infty \leq M_0, \quad j \geq 1. \quad (3.21)$$

We then argue as in Section 3.2 (with m_0 instead of m and M_0 instead of $\|v^{in}\|_\infty$) to establish that there is $s_0 \geq M_0$ such that

$$m_0 \omega_* \leq v_j(t, x) \leq s_0, \quad (t, x) \in [0, \infty) \times \bar{\Omega}, \quad j \geq 1, \quad (3.22)$$

the lower bound in (3.22) being a consequence of Lemma 2.2 and (3.21). Owing to (3.22), we next proceed as in [17, Proposition 4.6] to show that there is $\alpha \in (0, 1/2)$ such that

$$(v_j)_{j \geq 1} \quad \text{is bounded in} \quad BUC^{2\alpha}([0, \infty), C^{2\alpha}(\bar{\Omega})). \quad (3.23)$$

In addition, arguing as in [17, Proposition 4.7], we obtain that, for any $p \in (N, \infty)$, there is $C_0(p) > 0$ such that

$$\|v_j(t)\|_{W^{2,p}} \leq C_0(p), \quad t \geq 0, \quad j \geq 1. \quad (3.24)$$

Fix $p > 2N/(1-2\alpha)$. Since $C^{2\alpha}(\bar{\Omega})$ is continuously embedded in $L^p(\Omega)$ and interpolation and Sobolev inequalities guarantee that there are $C_1(p) > 0$ and $C_2(p) > 0$ such that

$$\|z\|_{C^{1+\alpha}} \leq C_1(p) \|z\|_{W^{3/2,p}} \leq C_2(p) \|z\|_{W^{2,p}}^{3/4} \|z\|_p^{1/4}, \quad z \in W^{2,p}(\Omega),$$

we infer from (3.24) and the above inequality that, for $j \geq 1$ and $(t, s) \in [0, \infty)^2$,

$$\begin{aligned} \|v_j(t) - v_j(s)\|_{C^{1+\alpha}} &\leq C_2(p) \left(\|v_j(t)\|_{W^{2,p}}^{3/4} + \|v_j(s)\|_{W^{2,p}}^{3/4} \right) \|v_j(t) - v_j(s)\|_p^{1/4} \\ &\leq 2C_0(p)^{3/4} C_2(p) |\Omega|^{1/4p} \|v_j(t) - v_j(s)\|_{C^{2\alpha}}^{1/4}. \end{aligned}$$

Hence, owing to (3.23), we conclude that

$$(v_j)_{j \geq 1} \text{ is bounded in } BUC^{\alpha/2}([0, \infty), C^{1+\alpha}(\bar{\Omega})). \quad (3.25a)$$

It then follows from (3.22), (3.25a), and parabolic regularity for linear equations [21, Chapter V, Theorem 1.1] applied to (1.1a) considered as a linear equation for u_j that there is $\alpha_1 \in (0, 1)$ such that

$$(u_j)_{j \geq 1} \text{ is bounded in } C^{\alpha_1/2, \alpha_1}([0, T] \times \bar{\Omega}) \text{ for any } T > 0. \quad (3.25b)$$

We now use elliptic regularity theory to deduce from (1.1b) and (3.25b) that

$$(v_j)_{j \geq 1} \text{ is bounded in } C^{\alpha_1/2, 2+\alpha_1}([0, T] \times \bar{\Omega}) \text{ for any } T > 0, \quad (3.25c)$$

which, together with parabolic regularity applied to (1.1a), see [21, Chapter IV], gives that

$$(u_j)_{j \geq 1} \text{ is bounded in } C^{(2+\alpha_1)/2, 2+\alpha_1}((0, T) \times \bar{\Omega}) \text{ for any } T > 0. \quad (3.25d)$$

We now infer from (3.25) and the Arzelà-Ascoli theorem that there are a subsequence of $(u_j, v_j)_{j \geq 1}$ (not relabeled) and

$$\tilde{u} \in C([0, \infty) \times \bar{\Omega}) \cap C^{1,2}((0, \infty) \times \bar{\Omega}), \quad \tilde{v} \in C([0, \infty), C^1(\bar{\Omega}))$$

such that

$$\lim_{j \rightarrow \infty} \sup_{t \in [0, T]} \{ \|(u_j - \tilde{u})(t)\|_{\infty} + \|(v_j - \tilde{v})(t)\|_{C^1} \} = 0 \text{ for any } T > 0. \quad (3.26)$$

Since (u_j, v_j) solves (1.1) with initial condition u_j^{in} , we combine (3.25) with the convergence (3.26) to conclude that (\tilde{u}, \tilde{v}) is a non-negative global classical solution to (1.1) with initial condition u^{in} ; that is, $(\tilde{u}, \tilde{v}) = (u, v)$ by the uniqueness of classical solutions to (1.1), see Proposition 2.1, so that it is the whole sequence $((u_j, v_j))_{j \geq 1}$ which actually converges to (u, v) and satisfies (3.26). We have thus completed the proof. \square

4. EXPLICIT UPPER AND LOWER BOUNDS ON v

The main building block of the proof of Theorem 1.3 is the following lemma, which is somewhat a generalisation of Lemma 3.1.

Lemma 4.1. *Assume that γ satisfies (1.4) and let $u^{in} \in W_+^{1,\infty}(\Omega)$ with $m = \|u^{in}\|_1 > 0$. We denote the corresponding global classical solution to (1.1) by (u, v) , see Corollary 1.2. Assume further that there are $0 < a < b$ such that*

$$\gamma' \geq 0 \text{ on } [a, b], \quad (4.1)$$

$$a < \min_{\bar{\Omega}} \{v^{in}\} \leq \max_{\bar{\Omega}} \{v^{in}\} < b, \quad (4.2)$$

where $v^{in} = \mathcal{A}^{-1}[u^{in}]$. Then

$$a \leq v(t, x) \leq b, \quad (t, x) \in [0, \infty) \times \bar{\Omega}.$$

Proof. For $t \geq 0$, we put

$$v_* \leq \mu(t) := \min_{x \in \bar{\Omega}} \{v(t, x)\} \leq M(t) := \max_{x \in \bar{\Omega}} \{v(t, x)\},$$

which are well-defined and positive according to elliptic regularity and Lemma 2.2. Owing to the continuity of v , it follows from (4.2) that

$$\tau := \inf \{t > 0 : M(t) > b \text{ or } \mu(t) < a\} \in (0, \infty].$$

In particular,

$$a \leq \mu(t) \leq M(t) \leq b, \quad t \in [0, \tau). \quad (4.3)$$

By (4.1), (4.3), and the non-negativity of u ,

$$u\gamma(\mu) \leq u\gamma(v) \leq u\gamma(M) \quad \text{in } [0, \tau) \times \bar{\Omega},$$

and the elliptic comparison principle and (1.1b) imply that

$$\gamma(\mu)v = \mathcal{A}^{-1}[\gamma(\mu)u] \leq \mathcal{A}^{-1}[\gamma(v)u] \leq \mathcal{A}^{-1}[\gamma(M)u] = \gamma(M)v \quad \text{in } [0, \tau) \times \bar{\Omega}. \quad (4.4)$$

It follows from (2.3) and (4.4) that v satisfies

$$\gamma(M)v \geq \partial_t v - \gamma(v)\Delta v + v\gamma(v) \geq \gamma(\mu)v, \quad (t, x) \in (0, \tau) \times \Omega, \quad (4.5a)$$

$$\nabla v \cdot \mathbf{n} = 0, \quad (t, x) \in (0, \tau) \times \partial\Omega, \quad (4.5b)$$

$$v(0) = v^{in}, \quad x \in \Omega. \quad (4.5c)$$

The parabolic comparison principle then entails that

$$V_u(t) \geq v(t, x) \geq V_l(t), \quad (t, x) \in [0, \tau) \times \bar{\Omega}, \quad (4.6)$$

where V_l and V_u are the solutions to the ordinary differential equations

$$\frac{dV_l}{dt} + V_l\gamma(V_l) = \gamma(\mu)V_l, \quad t \in (0, \tau), \quad (4.7a)$$

$$V_l(0) = \mu(0), \quad (4.7b)$$

and

$$\frac{dV_u}{dt} + V_u\gamma(V_u) = \gamma(M)V_u, \quad t \in (0, \tau), \quad (4.8a)$$

$$V_u(0) = M(0), \quad (4.8b)$$

respectively. Since $b > V_u(0) \geq V_l(0) > a$ by (4.2),

$$\tau_* := \inf \{t \in (0, \tau) : V_u(t) > b \text{ or } V_l(t) < a\} \in (0, \tau],$$

and we infer from (4.1), (4.3), and (4.6) that

$$\gamma(V_l(t)) \leq \gamma(\mu(t)) \leq \gamma(v(t, x)) \leq \gamma(M(t)) \leq \gamma(V_u(t)), \quad (t, x) \in [0, \tau_*) \times \bar{\Omega}. \quad (4.9)$$

Combining (4.6), (4.7), (4.8), and (4.9), we deduce that

$$\frac{dV_l}{dt}(t) \geq 0 \geq \frac{dV_u}{dt}(t), \quad t \in (0, \tau_*).$$

Hence, owing to (4.2)

$$V_l(t) \geq V_l(0) > a \quad \text{and} \quad V_u(t) \leq V_u(0) < b, \quad t \in [0, \tau_*),$$

so that $\tau_* = \tau$. Gathering (4.6) and the above inequalities, we conclude that

$$a < V_l(t) \leq v(t, x) \leq V_u(t) < b, \quad (t, x) \in [0, \tau) \times \bar{\Omega}.$$

Consequently, $\tau = \infty$ and the proof of Lemma 4.1 is complete. \square

Proof of Theorem 1.3. If $u^{in} \equiv m/|\Omega|$, then $(u, v) = (m/|\Omega|, m/|\Omega|)$ and Theorem 1.3 is obvious. Let us thus assume that u^{in} is not constant, so that v^{in} is also not constant. We then pick $\nu \in (\min_{\bar{\Omega}}\{v^{in}\}, \max_{\bar{\Omega}}\{v^{in}\})$ and define

$$u_j^{in} := \frac{j-1}{j}u^{in} + \frac{1}{j}\nu, \quad v_j^{in} := \mathcal{A}^{-1}[u_j^{in}] = \frac{j-1}{j}v^{in} + \frac{1}{j}\nu, \quad j \geq 1.$$

Clearly, for each $j \geq 1$,

$$\begin{aligned} \min_{\bar{\Omega}}\{v^{in}\} &< \frac{j-1}{j} \min_{\bar{\Omega}}\{v^{in}\} + \frac{1}{j}\nu = \min_{\bar{\Omega}}\{v_j^{in}\}, \\ \max_{\bar{\Omega}}\{v^{in}\} &> \frac{j-1}{j} \max_{\bar{\Omega}}\{v^{in}\} + \frac{1}{j}\nu = \max_{\bar{\Omega}}\{v_j^{in}\}, \end{aligned}$$

and we infer from Lemma 4.1 that the global classical solution (u_j, v_j) to (1.1) with initial condition u_j^{in} satisfies

$$\min_{\bar{\Omega}}\{v^{in}\} \leq v_j(t, x) \leq \max_{\bar{\Omega}}\{v^{in}\}, \quad (t, x) \in [0, \infty) \times \bar{\Omega}.$$

We then pass to the limit $j \rightarrow \infty$ in the above inequality with the help of Proposition 3.5 to complete the proof. \square

5. EXPLICIT L^∞ -ESTIMATES ON u

This section is devoted to the proof of Theorem 1.4 and we begin with the case of non-increasing and concave motility function γ . In this case, the evolution equation (2.3a) for v is not used and the proof relies on direct computations involving (1.1a) and (1.1b) and comparison arguments.

Proof of Theorem 1.4 (Part 1). Let us thus consider a function γ satisfying (1.4) and being non-decreasing and concave on $[\min_{\bar{\Omega}}\{v^{in}\}, \max_{\bar{\Omega}}\{v^{in}\}]$. By (1.1a) and (1.1b),

$$\begin{aligned} \partial_t u &= \operatorname{div}(\gamma(v)\nabla u + u\gamma'(v)\nabla v) \\ &= \gamma(v)\Delta u + 2\gamma'(v)\nabla v \cdot \nabla u + u\gamma''(v)|\nabla v|^2 + u\gamma'(v)\Delta v \\ &= \gamma(v)\Delta u + 2\gamma'(v)\nabla v \cdot \nabla u + u\gamma''(v)|\nabla v|^2 + u\gamma'(v)(v - u). \end{aligned}$$

Since $v = \mathcal{A}^{-1}[u]$ and $u \leq \|u\|_\infty$ in Ω , the elliptic comparison principle implies that

$$v(t, x) \leq \|u(t)\|_\infty, \quad (t, x) \in [0, \infty) \times \bar{\Omega}.$$

Consequently, recalling that v ranges in $[\min_{\bar{\Omega}}\{v^{in}\}, \max_{\bar{\Omega}}\{v^{in}\}]$ by Theorem 1.3, it follows from the monotonicity and concavity of γ and the non-negativity of u that

$$u\gamma''(v)|\nabla v|^2 + u\gamma'(v)(v - u) \leq u\gamma'(v)(\|u\|_\infty - u) \leq Yu(\|u\|_\infty - u)$$

with $Y := \gamma'(\min_{\bar{\Omega}}\{v^{in}\}) \geq 0$ and we conclude that

$$\partial_t u - \gamma(v)\Delta u - 2\gamma'(v)\nabla v \cdot \nabla u \leq Yu(\|u\|_\infty - u) \quad \text{in } (0, \infty) \times \Omega, \quad (5.1a)$$

$$\nabla u \cdot \mathbf{n} = 0 \quad \text{on } (0, \infty) \times \partial\Omega, \quad (5.1b)$$

$$u(0) = u^{in} \quad \text{in } \Omega. \quad (5.1c)$$

Introducing the solution $U \in C^1([0, \infty))$ to the ordinary differential equation

$$\frac{dU}{dt} = YU(\|u\|_\infty - U), \quad t > 0, \quad (5.2a)$$

$$U(0) = \|u^{in}\|_\infty, \quad (5.2b)$$

it readily follows from (5.1), (5.2), and the parabolic comparison principle that

$$u(t, x) \leq U(t), \quad (t, x) \in [0, \infty) \times \bar{\Omega}. \quad (5.3)$$

In particular, $\|u(t)\|_\infty \leq U(t)$ for all $t \geq 0$ by (5.3) and we infer from (5.2a) and the non-negativity of Y that $dU/dt \leq 0$ in $(0, \infty)$. Therefore, $U(t) \leq U(0)$ for $t \geq 0$ which, together with (5.2b) and (5.3), completes the proof. \square

We next turn to general non-decreasing motility functions γ and use the comparison principle applied to the parabolic equation satisfied by the auxiliary function $u\gamma(v)$.

Proof of Theorem 1.4 (Part 2). We assume here that γ satisfies (1.4) and (1.13). Introducing $\varphi := u\gamma(v)$, we infer from (1.1a) and (2.3a) that

$$\begin{aligned} \partial_t \varphi &= \gamma(v)\partial_t u + u\gamma'(v)\partial_t v = \gamma(v)\Delta \varphi + u\gamma'(v)(\mathcal{A}^{-1}[\varphi] - \varphi) \\ &= \gamma(v)\Delta \varphi + \frac{\gamma'}{\gamma}(v)\varphi(\mathcal{A}^{-1}[\varphi] - \varphi). \end{aligned}$$

Since

$$\mathcal{A}^{-1}[\varphi] \leq \|\varphi\|_\infty \quad \text{in } [0, \infty) \times \bar{\Omega}$$

by the elliptic comparison principle, it follows from the monotonicity of γ , the non-negativity of φ , and Theorem 1.3 that

$$\frac{\gamma'}{\gamma}(v)\varphi(\mathcal{A}^{-1}[\varphi] - \varphi) \leq \frac{\gamma'}{\gamma}(v)\varphi(\|\varphi\|_\infty - \varphi) \leq Y\|\gamma'(v)\|_\infty\varphi(\|\varphi\|_\infty - \varphi),$$

with $1/Y := \gamma(\min_{\bar{\Omega}}\{v^{in}\}) > 0$. Therefore, φ satisfies

$$\partial_t \varphi - \gamma(v)\Delta \varphi \leq Y\|\gamma'(v)\|_\infty\varphi(\|\varphi\|_\infty - \varphi) \quad \text{in } (0, \infty) \times \Omega, \quad (5.4a)$$

$$\nabla \varphi \cdot \mathbf{n} = 0 \quad \text{on } (0, \infty) \times \partial\Omega, \quad (5.4b)$$

$$\varphi(0) = u^{in}\gamma(v^{in}) \quad \text{in } \Omega. \quad (5.4c)$$

Let $\psi \in C^1([0, \infty))$ be the solution to the ordinary differential equation

$$\frac{d\psi}{dt} = Y\|\gamma'(v)\|_\infty\psi(\|\varphi\|_\infty - \psi), \quad t > 0, \quad (5.5a)$$

$$\psi(0) = \|u^{in}\gamma(v^{in})\|_\infty. \quad (5.5b)$$

Owing to (5.4), (5.5), and the parabolic comparison principle,

$$\varphi(t, x) \leq \psi(t), \quad (t, x) \in [0, \infty) \times \bar{\Omega}. \quad (5.6)$$

In particular, for $t > 0$, $\|\varphi(t)\|_\infty \leq \psi(t)$ and we deduce from (5.5a) that $d\psi/dt \leq 0$ on $(0, \infty)$. Combining the just established time monotonicity of ψ with (5.6) gives

$$\|\varphi(t)\|_\infty \leq \psi(t) \leq \psi(0) = \|u^{in}\gamma(v^{in})\|_\infty, \quad t \geq 0. \quad (5.7)$$

Now, on the one hand, the monotonicity of γ and Theorem 1.3 imply that

$$\gamma(\min_{\bar{\Omega}}\{v^{in}\})u(t, x) \leq \varphi(t, x) \leq \|\varphi(t)\|_\infty, \quad (t, x) \in [0, \infty) \times \bar{\Omega}. \quad (5.8)$$

On the other hand, the elliptic comparison principle and the definition $v^{in} = \mathcal{A}^{-1}[u^{in}]$ entail that $\|v^{in}\|_\infty \leq \|u^{in}\|_\infty$ and we use once more the monotonicity of γ to obtain

$$\|u^{in}\gamma(v^{in})\|_\infty \leq \|u^{in}\|_\infty\gamma(\max_{\bar{\Omega}}\{v^{in}\}). \quad (5.9)$$

Gathering (5.7), (5.8), and (5.9) completes the proof. \square

6. LARGE TIME BEHAVIOUR FOR NON-DECREASING γ

Proof of of Theorem 1.5. As already pointed out in the introduction, the monotonicity of γ implies the existence of a Liapunov function. Indeed, it readily follows from (1.1) that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\nabla v\|_2^2 + \|v\|_2^2) &= \int_{\Omega} v \partial_t (v - \Delta v) \, dx = \int_{\Omega} v \partial_t u \, dx \\ &= \int_{\Omega} u \gamma(v) \Delta v \, dx = \int_{\Omega} \gamma(v) (v - \Delta v) \Delta v \, dx \\ &= - \int_{\Omega} (v \gamma'(v) + \gamma(v)) |\nabla v|^2 \, dx - \int_{\Omega} \gamma(v) |\Delta v|^2 \, dx, \end{aligned}$$

and the monotonicity and positivity of γ , along with Theorem 1.3, guarantee that the right-hand side of the above identity is non-positive. We then proceed as in the proof of [1, Theorem 1.3] to complete that of Theorem 1.5. \square

ACKNOWLEDGMENTS

Jiang is supported by National Natural Science Foundation of China (NSFC) under grants No. 12271505 & No. 12071084, the Training Program of Interdisciplinary Cooperation of Innovation Academy for Precision Measurement Science and Technology, CAS (No. S21S3202), and by Knowledge Innovation Program of Wuhan-Basic Research (No. 2022010801010135).

REFERENCES

- [1] J. AHN AND C. YOON, *Global well-posedness and stability of constant equilibria in parabolic-elliptic chemotaxis systems without gradient sensing*, *Nonlinearity*, 32 (2019), pp. 1327–1351.
- [2] H. AMANN, *Dynamic theory of quasilinear parabolic systems. III: Global existence*, *Math. Z.*, 202 (1989), pp. 219–250.
- [3] ———, *Dynamic theory of quasilinear parabolic equations. II: Reaction-diffusion systems*, *Differ. Integral Equ.*, 3 (1990), pp. 13–75.
- [4] ———, *Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems*, in *Function spaces, differential operators and nonlinear analysis. Survey articles and communications of the international conference held in Friedrichsroda, Germany, September 20-26, 1992*, Stuttgart: B. G. Teubner Verlagsgesellschaft, 1993, pp. 9–126.
- [5] ———, *Linear and quasilinear parabolic problems. Vol. 1: Abstract linear theory*, vol. 89 of *Monogr. Math.*, Basel, Basel: Birkhäuser, 1995.
- [6] M. BURGER, PH. LAURENÇOT, AND A. TRESCASES, *Delayed blow-up for chemotaxis models with local sensing*, *J. Lond. Math. Soc. (2)*, 103 (2021), pp. 1596–1617.
- [7] L. DESVILLETES, Y.-J. KIM, A. TRESCASES, AND C. YOON, *A logarithmic chemotaxis model featuring global existence and aggregation*, *Nonlinear Anal. Real World Appl.*, 50 (2019), pp. 562–582.
- [8] L. DESVILLETES, PH. LAURENÇOT, A. TRESCASES, AND M. WINKLER, *Weak solutions to triangular cross diffusion systems modeling chemotaxis with local sensing*, *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, 226 (2023), p. 26. Id/No 113153.
- [9] K. FUJIE, *A short remark on comparison estimates in a chemotaxis system with local sensing*. to appear in *Discrete Contin. Dynam. Systems B*, 2023.
- [10] K. FUJIE AND J. JIANG, *Global existence for a kinetic model of pattern formation with density-suppressed motilities*, *J. Differ. Equations*, 269 (2020), pp. 5338–5378.
- [11] K. FUJIE AND J. JIANG, *Boundedness of classical solutions to a degenerate Keller–Segel type model with signal-dependent motilities*, *Acta Appl. Math.*, 176 (2021), p. 36. Id/No 3.
- [12] K. FUJIE AND J. JIANG, *Comparison methods for a Keller–Segel-type model of pattern formations with density-suppressed motilities*, *Calc. Var. Partial Differ. Equ.*, 60 (2021), p. 37. Id/No 92.
- [13] K. FUJIE AND T. SENBA, *Global boundedness of solutions to a parabolic-parabolic chemotaxis system with local sensing in higher dimensions*, *Nonlinearity*, 35 (2022), pp. 3777–3811.
- [14] ———, *Global existence and infinite time blow-up of classical solutions to chemotaxis systems of local sensing in higher dimensions*, *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, 222 (2022), p. 7. Id/No 112987.
- [15] K. FUJIE, M. WINKLER, AND T. YOKOTA, *Boundedness of solutions to parabolic-elliptic Keller–Segel systems with signal-dependent sensitivity*, *Math. Methods Appl. Sci.*, 38 (2015), pp. 1212–1224.
- [16] J. JIANG, *Boundedness and exponential stabilization in a parabolic-elliptic Keller–Segel model with signal-dependent motilities for local sensing chemotaxis*, *Acta Math. Sci., Ser. B, Engl. Ed.*, 42 (2022), pp. 825–846.
- [17] J. JIANG AND PH. LAURENÇOT, *Global existence and uniform boundedness in a chemotaxis model with signal-dependent motility*, *J. Differ. Equations*, 299 (2021), pp. 513–541.
- [18] J. JIANG, PH. LAURENÇOT, AND Y. ZHANG, *Global existence, uniform boundedness, and stabilization in a chemotaxis system with density-suppressed motility and nutrient consumption*, *Commun. Partial Differ. Equations*, 47 (2022), pp. 1024–1069.
- [19] H.-Y. JIN AND Z.-A. WANG, *Critical mass on the Keller–Segel system with signal-dependent motility*, *Proc. Amer. Math. Soc.*, 148 (2020), pp. 4855–4873.
- [20] E. F. KELLER AND L. A. SEGEL, *Model for chemotaxis*, *J. Theoret. Biol.*, 30 (1971), pp. 225–234.

- [21] O. A. LADYZHENSKAYA, V. A. SOLONNIKOV, AND N. N. URAL'TSEVA, *Linear and quasi-linear equations of parabolic type. Translated from the Russian by S. Smith*, vol. 23 of Transl. Math. Monogr., American Mathematical Society (AMS), Providence, RI, 1968.
- [22] H. LI AND J. JIANG, *Global existence of weak solutions to a signal-dependent Keller-Segel model for local sensing chemotaxis*, Nonlinear Anal., Real World Appl., 61 (2021), p. 14. Id/No 103338.
- [23] C. LIU, X. FU, L. LIU, X. REN, C. K. L. CHAU, S. LI, L. XIANG, H. ZENG, G. CHEN, L.-H. TANG, P. LENZ, X. CUI, W. HUANG, T. HWA, AND J.-D. HUANG, *Sequential establishment of stripe patterns in an expanding cell population*, Science, 334 (2011), pp. 238–241.
- [24] W. LYU AND Z.-A. WANG, *Global classical solutions for a class of reaction-diffusion system with density-suppressed motility*, Electron Res. Arch., 30 (2022), pp. 995–1015.
- [25] ———, *Logistic damping effect in chemotaxis models with density-suppressed motility*, Adv. Nonlinear Anal., 12 (2023), pp. 336–355.
- [26] Y.-S. TAO AND M. WINKLER, *Effects of signal-dependent motilities in a Keller-Segel-type reaction-diffusion system*, Math. Models Methods Appl. Sci., 27 (2017), pp. 1645–1683.
- [27] Z.-A. WANG, *On the parabolic-elliptic Keller-Segel system with signal-dependent motilities: a paradigm for global boundedness and steady states*, Math. Methods Appl. Sci., 44 (2021), pp. 10881–10898.
- [28] Y. XIAO AND J. JIANG, *Global existence and uniform boundedness in a fully parabolic Keller–Segel system with non-monotonic signal-dependent motility*, J. Differ. Equations, 354 (2023), pp. 403–429.
- [29] C. YOON AND Y.-J. KIM, *Global existence and aggregation in a Keller–Segel model with Fokker–Planck diffusion*, Acta Appl. Math., 149 (2017), pp. 101–123.

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