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Sharing values for multi-choice games: an axiomatic approach

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Abstract

A Sharing value for transferable utility games distributes the Harsanyi dividend of each coalition among the players in the coalition's support. Such distribution is done according to a certain sharing system that determines the Sharing value. In this paper, we extend Sharing values to multi-choice games. Multi-choice games are a generalization of transferable utility games in which players have several activity levels. Unlike in transferable utility games, there is no straightforward way to interpret the support of a coalition in a multi-choice game. This makes it more tedious to distribute the Harsanyi dividend of a multi-choice coalition. We consider three possible interpretations of the support of a multi-choice coalition. Based on these interpretations, we derive three families of Sharing values for multi-choice games. To conduct this study, we discuss novel and classical axioms for multi-choice games. This allows us to provide an axiomatic foundation for each of these families of values.

Keywords: Multi-choice games, Sharing values, Harsanyi set

JEL code: C71

1. Introduction

In cooperative games with transferable utilities, or TU-games for short, players form coalitions to generate some worth. Each coalition's worth is measured by a characteristic function. In addition, each coalition's net surplus can be measured by its Harsanyi dividend (see Harsanyi (1959)). Such dividends can be defined inductively: the dividend of the empty coalition (the coalition in which no one cooperates) is null and the dividend of any other coalition is equal to its worth minus the sum of all dividends of proper subsets of that coalition. A payoff vector for TU-games assigns a payoff to each player. A single-valued solution on a class of TU-games assigns a unique payoff vector to each game in this class, while a set-valued solution assigns a set (possibly empty) of payoff vectors. The Shapley value (see Shapley (1953)) is probably the most prominent single-valued solution for TU-games. It distributes the Harsanyi dividend of each coalition equally among the members of that coalition, i.e., the support of that coalition.

A Sharing value for TU-games is a single-valued solution that distributes the Harsanyi dividend of each coalition among the players in that coalition's support. This distribution is based on a sharing system that determines the Sharing value. A sharing system assigns, to each coalition,

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unit-normalized weights defined over the player set. If a player is in the coalition's support, he is assigned a positive weight, otherwise, his weight is null. The Shapley value can be viewed as a specific Sharing value whose sharing system associates, to each coalition, uniform weights to the players in the coalition's support. Derks et al. (2000) show that, given a TU-game, the set of all the Sharing values of that game coincides with its Harsanyi set (also known as Selectope), which is a set-valued solution originally introduced by Hammer et al. (1977) and, independently, by Vasil'ev (1978) and Vasil'ev (1981). Axiomatic approaches are proposed by Derks et al. (2000), Vasil'ev and van der Laan (2001), Derks et al. (2006), Besner (2020a), and Besner (2020b) to study Sharing values. In particular, Derks et al. (2000) show that the family of Sharing values can be characterized by using four axioms: Efficiency, Additivity, Null player, and Positivity. Efficiency ensures that the worth of the grand coalition is fully distributed among the players. Additivity allows us to decompose the original game without affecting the outcome. Null player stands for the idea that players with null marginal contributions to the game should receive no payoff. Positivity indicates that the payoffs of the players should be positive as long as all the Harsanyi dividends of the game are positive.

In this paper, we generalize the family of Sharing values from TU-games to multi-choice games. Multi-choice games, or MC-games for short, introduced by Hsiao and Raghavan (1992) and van den Nouweland (1993), are a natural extension of TU-games. In TU-games, each player in N has two choices. It can either cooperate by joining a coalition or not cooperate. Such binary choice can be represented by $\{0, 1\}$. In MC-games, each player $i \in N$ has access to a finite set of activity levels $\{0, 1, \dots, m_i\}$ to cooperate within a coalition. For convenience, a player i and any of its (strictly positive) activity level $j \in \{1, \dots, m_i\}$ may be denoted by a player-activity level pair (i, j) . Recently, MC-games have been successfully applied to economic theory and operations research. For instance, Techer (2021) addresses the social cost problem, originally introduced by Coase (1960), by using MC-games. The author studies situations in which one polluter, which has several levels of pollution, interacts with several potential victims, and aims at negotiating a stable agreement regarding the level of pollution; and Branzei et al. (2021) introduce the model of multi-choice two-sided market games related to two-sided market situations. The authors study situations in which sellers and buyers trade goods. Each seller may have multiple units of the good and each buyer may need multiple units of the good. The trading of goods generates a certain utility surplus. The authors investigate how to allocate this surplus among the sellers and the buyers.

Consider a finite and fixed set of players N . In an MC-game, a (multi-choice) coalition s is a vector describing each player i activity level $s_i \in \{0, 1, \dots, m_i\}$ within this coalition. A characteristic function v for MC-games measures the worth $v(s)$ of each coalition s . An MC-game is denoted by a couple (m, v) , where $m = (m_1, \dots, m_{|N|})$ plays the role of the grand coalition. The net surplus generated by a coalition s can be measured by its (multi-choice) Harsanyi dividend $\Delta_v(s)$ (see Hsiao and Raghavan (1992)). Such dividends can be defined inductively: the dividend of the empty coalition (the coalition in which all players are inactive) is null and the dividend of any other coalition is equal to its worth minus the sum of all dividends of smaller coalitions (in the Euclidean sense). A (multi-choice) payoff vector x associates a payoff x_{ij} to each (strictly positive) activity level $j \in \{1, \dots, m_i\}$ of each player i , i.e., to each pair (i, j) . In other words, a payoff vector describes how much each player's total payoff varies according to their activity level. By convention, a player receives no payoff for his activity level 0. A single-valued solution, or a value for short, on a class of MC-games, assigns a unique payoff vector to each game in this

class, while a set-valued solution assigns a set (possibly empty) of payoff vectors. We introduce a running example to illustrate concepts related to MC-games. Let $N = \{a, b, c\}$ be a team of consultants collaborating on a common project. The project generates an income that depends on the involvement of the consultants in the collaboration. Each consultant $i \in N$ has a finite set of activity levels, which can be interpreted as the maximal number of workdays they are willing to invest in the project. Let us pick $m_a = 4, m_b = 3$ and $m_c = 4$. In this context, a coalition s represents a situation in which each consultant $i \in N$ invests exactly s_i workdays into the project. A characteristic function v describes the hypothetical income $v(s)$ that can be generated by each s . The Harsanyi dividend $\Delta_v(s)$ can be interpreted as the net surplus in the income generated by s . A payoff vector x associates a daily payoff x_{ij} to each workday j of each consultant i .

In TU-games, the support of a coalition is simple to interpret: it is the set of players cooperating in that coalition. In MC-games, however, there is no straightforward interpretation of the support of a (multi-choice) coalition. Three possible interpretations have been proposed by Klijn et al. (1999), Peters and Zank (2005) and Lowing and Techer (2022a) to define values for MC-games.¹ Each of these interpretations assigns to each coalition a certain set of player-activity level pairs.

Klijn et al. (1999) interpret the support of a coalition s as the set of activity levels necessary to form s , which we denote by $B(s)$. Formally, the set $B(s)$ contains all the pairs (i, j) such that $i \in N$ and $0 < j \leq s_i$. The authors proposed a value for MC-games that distributes the dividend $\Delta_v(s)$ of each coalition s equally on $B(s)$. For instance, in the context of the running example, the coalition $(3, 2, 4)$ represents a situation in which consultant a works for three days, b works for two days, and c works for four days. The set $B(3, 2, 4)$ contains all the workdays that a has invested in $(3, 2, 4)$, i.e., his first, second, and third workdays. It also contains all the workdays that b and c have invested in $(3, 2, 4)$. An illustration of $B(3, 2, 4)$ is given by Figure 1. The workdays contained in $B(3, 2, 4)$ are highlighted in blue. According to the value of Klijn et al. (1999), each workday of each consultant contained in $B(3, 2, 4)$ should receive an equal share of $\Delta_v(3, 2, 4)$.

Peters and Zank (2005) interpret the support of a coalition s as the set of activity levels at which the players are active in s , which we denote by $C(s)$. Formally, the set $C(s)$ contains all the pairs (i, j) such that $i \in N$ and $j = s_i, s_i > 0$. The authors proposed a value for MC-games that distributes the dividend $\Delta_v(s)$ of each coalition s equally on $C(s)$. For instance, in the context of the running example, consider again the coalition $(3, 2, 4)$. The set $C(3, 2, 4)$ contains the workday at which a is currently working in $(3, 2, 4)$, i.e., his third workday. It also contains the second and fourth workdays of b and c , respectively. An illustration of $C(3, 2, 4)$ is given by Figure 1. According to the value of Peters and Zank (2005), each workday of each consultant contained in $C(3, 2, 4)$ should receive an equal share of $\Delta_v(3, 2, 4)$.

Lowling and Techer (2022a) assume that the players are always competing in coalitions. Only the highest activity level within a coalition s can be part of its support, which we denote by $T(s)$. Formally, the set $T(s)$ contains all the pairs (i, j) such that $i \in N$ and $j = s_i, s_i = \max_{k \in N} s_k$. The authors proposed a value for MC-games that distributes the dividend $\Delta_v(s)$ of each coalition s equally on $T(s)$. For instance, in the context of the running example, consider again the coalition $(3, 2, 4)$. In this coalition, c has invested more workdays than the other consultants. Therefore, the set $T(3, 2, 4)$ only contains the fourth workday of c . An illustration of $T(3, 2, 4)$ is given by Figure 1. The value of Lowling and Techer (2022a) requires that the fourth workday of c receives $\Delta_v(3, 2, 4)$.

¹The value defined by Klijn et al. (1999) was originally introduced in a more general context by Derks and Peters (1993).

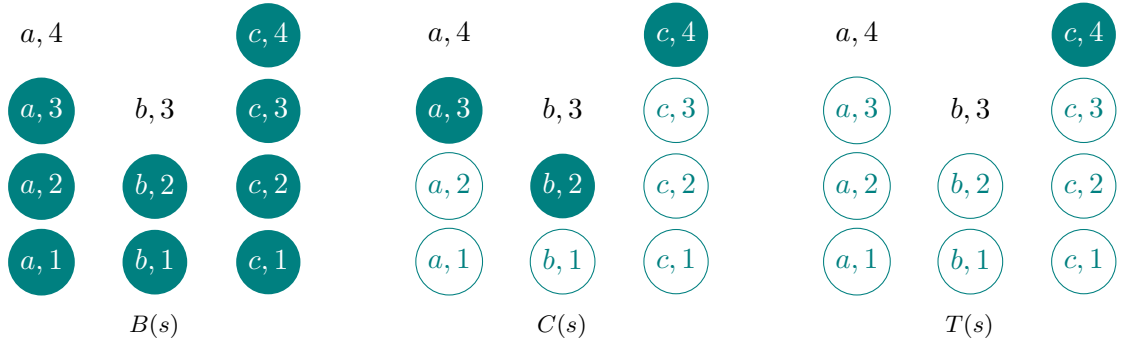


Figure 1: Interpretations of coalition $(3, 2, 4)$ support

Each of the above-mentioned values has its own interpretation of coalition support. Such interpretation has a direct impact on how the coalitions' dividends are distributed. Based on these interpretations, we propose three generalizations of Sharing values from TU-games to MC-games: B -Sharing values, C -Sharing values, and T -Sharing values.

A B -Sharing value for MC-games distributes the Harsanyi dividend of each coalition s among the player-activity level pairs in the support $B(s)$. Such distribution is determined by a sharing system. This sharing system assigns, to each coalition s , unit-normalized weights defined on the whole set of player-activity level pairs. If a pair is in the support $B(s)$, it is assigned a positive weight, otherwise, its weight is null. The value proposed by Klijn et al. (1999) is a specific B -Sharing value whose sharing system associates, to each coalition, uniform weights to the pairs in $B(s)$. The set of all B -Sharing values forms the B -Harsanyi set, which generalizes the Harsanyi set from TU-games to MC-games. Similarly, we define C -Sharing values and T -Sharing values based on the two other interpretations of coalition support. The value proposed by Peters and Zank (2005) is a specific C -Sharing value, whereas the value proposed by Lowing and Techer (2022a) is a specific T -Sharing value. The set of all C -Sharing values forms the C -Harsanyi set, and the set of all T -Sharing values forms the T -Harsanyi set. Both sets generalize the Harsanyi set from TU-games to MC-games. Moreover, the T -Harsanyi set is included in the C -Harsanyi set, which itself is included in the B -Harsanyi set (see Figure 3).

We axiomatically characterize B -Sharing values, C -Sharing values, and T -Sharing values on the full class of MC-games. In this class, the player set is fixed, but the grand coalition and the characteristic function may vary. Each of these axiomatic characterizations relies on four axioms for solutions of MC-games: Efficiency, Additivity, Null pair out, and Positivity. Efficiency, Additivity, and Positivity are straightforward adaptations of their counterparts for TU-games. Null pair out ensures that if a player i has null contributions for his maximal activity level m_i , then it can just stop cooperating at this level without affecting the payoffs of the other players and his own payoffs for his remaining activity levels. These four axioms characterize B -Sharing values on the full class of MC-games (see Theorem 2). We discuss two other axioms to characterize C -Sharing values and T -Sharing values. The first axiom is Independence of the maximal activity level and was originally introduced by Hwang and Liao (2009) and Béal et al. (2012). This axiom ensures that the reduction of a player's maximal activity level should have no impact on the payoff of his remaining activity levels. The second axiom is Independence of higher activity levels and was originally introduced by Lowing and Techer (2022a). This axiom requires that if the maximal activity levels of all players reduce to a certain level, then the payoff of each player for this activity level remains unchanged. Adding Independence of the maximal activity level to Efficiency, Additivity, Null pair

out and Positivity allows us to characterize C -Sharing values on the full class of MC-games (see Theorem 3). Adding Independence of higher activity levels to Efficiency, Additivity, Null pair out and Positivity allows us to characterize T -Sharing values on the full class of MC-games (see Theorem 4). In addition to the above-mentioned axioms for MC-games, we introduce other axioms and discuss their relationships with each other. This allows us to provide additional axiomatic characterizations of B -Sharing values, C -Sharing values, and T -Sharing values on any sub-class of MC-games with a fixed grand coalition (see the Corollaries 1, 2 and 3).

The rest of the paper is organized as follows. After dealing with preliminaries on TU-games and MC-games in Section 2, we introduce B -Sharing values, C -Sharing values, and T -Sharing values in Section 3. Section 4 contains the axiomatic study. In particular, Subsection 4.1 discusses standard axioms for solutions of MC-games. Subsection 4.2 provides two axiomatic characterizations of B -Sharing values: one on the full class of MC-games and one on any sub-class of MC-games with a fixed grand coalition. Similar results are provided for C -Sharing values and T -Sharing values in the Subsections 4.3 and 4.4, respectively. Section 5 contains additional remarks. Finally, Section 6 concludes the paper. Section 7 is an appendix containing the proofs of the results and the logical independence of the axioms invoked in the characterization results.

2. Preliminaries

This section contains some preliminaries on TU-games and MC-games. First, we present the axiomatic characterization of Sharing values for TU-games proposed by Derks et al. (2000). Then, we present three solutions for MC-games and discuss different interpretations of coalition support in MC-games.

We denote by $|A|$ the number of elements in a finite set $A \subset \mathbb{N}$. For each non-empty $B \subseteq A$, we denote by $e_B \in \mathbb{R}^{|A|}$ the vector such that $(e_B)_i = 1$ if $i \in B$ and $(e_B)_i = 0$ otherwise. The power set of A is denoted by 2^A . Pick any $s \in \mathbb{R}^{|A|}$. The vector $(s_{-i}, j) \in \mathbb{R}^{|A|}$ is defined as $(s_{-i}, j)_i = j$ and $(s_{-i}, j)_{i'} = s_{i'}$ for each $i' \neq i$.

2.1. TU-games

Let $N = \{1, \dots, n\}$ be a **fixed set of players**. Each $E \in 2^N$ is called a **coalition** of cooperating players. The grand coalition N represents a situation in which all players cooperate. The empty coalition \emptyset represents a situation in which no player cooperates. A TU-game is a couple (N, v) where $v : 2^N \rightarrow \mathbb{R}$, $v(\emptyset) = 0$, is a characteristic function. The real number $v(E)$ can be interpreted as the worth the players in E generate when they cooperate. The class of TU-games on N is denoted by \mathbb{G} . A player $i \in N$ is a null player in $(N, v) \in \mathbb{G}$ if $v(E \cup \{i\}) = v(E)$ for each $E \in 2^N \setminus \{i\}$.

Pick any $(N, v) \in \mathbb{G}$. It is known that v can be expressed as a linear combination of $2^{|N|} - 1$ linearly independent games, since $v(\emptyset) = 0$. For any $E \in 2^N \setminus \{\emptyset\}$, the unanimity game (N, u_E) with ruling coalition E is defined as

$$\forall T \in 2^N, \quad u_E(T) = \begin{cases} 1 & \text{if } E \subseteq T, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

There exists a unique decomposition of v in terms of unanimity games given by

$$v = \sum_{E \in 2^N \setminus \{\emptyset\}} \Delta_v(E) u_E. \quad (2)$$

The coordinate $\Delta_v(E)$ is called the **Harsanyi dividend** (see Harsanyi (1959)) of coalition $E \in 2^N$. This dividend reflects the net surplus generated by E . It can be defined inductively: the dividend of the empty coalition is null and the dividend of any other coalition is equal to its worth minus the sum of all dividends of proper subsets of that coalition.² Formally,

$$\forall E \in 2^N \setminus \{\emptyset\}, \quad \Delta_v(E) = v(E) - \sum_{T \subset E} \Delta_v(T) \quad (3)$$

$$\Delta_v(\emptyset) = 0.$$

In TU-games, a **payoff vector** $x \in \mathbb{R}^{|N|}$ assigns a payoff $x_i \in \mathbb{R}$ to each player $i \in N$. A single-valued solution, or a value for short, on \mathbb{G} is a map f that assigns a unique payoff vector $f(N, v)$ to each $(N, v) \in \mathbb{G}$, whereas a set-valued solution is a map F that assigns a set (possibly empty) of payoff vectors $F(N, v)$ to each $(N, v) \in \mathbb{G}$.

The **Shapley value** (see Shapley (1953)) is probably the most prominent single-valued solution for TU-games. It distributes the dividend of each coalition equally among the members of that coalition, i.e., on the **support** of that coalition.

Definition 1 (Shapley value). *Pick any $(N, v) \in \mathbb{G}$. The Shapley value is defined as*

$$\forall i \in N, \quad Sh_i(N, v) = \sum_{\substack{E \in 2^N \\ E \ni i}} \frac{\Delta_v(E)}{|E|}.$$

Next, we define the family of **Sharing values** (see Derks et al. (2000)). Each Sharing value is determined by a **sharing system**. A sharing system is a collection $q = (q(E))_{E \in 2^N}$ such that for each $E \in 2^N \setminus \emptyset$, $q(E) \in \mathbb{R}^N$ verifies the following conditions

1. $q_i(E) = 0$ if $i \notin E$;
2. $q_i(E) \geq 0$ if $i \in E$;
3. and $\sum_{i \in N} q_i(E) = 1$.

We use the convention $q(\emptyset) = (0, \dots, 0)$.³ Let Q^N denote the set of all possible sharing systems on N . A Sharing value distributes the dividend of each coalition $E \in 2^N$ on its support according to a certain sharing system in Q^N .

Definition 2 (Sharing values). *To each sharing system $q \in Q^N$, one can associate a Sharing value φ^q on \mathbb{G} defined, for each $(N, v) \in \mathbb{G}$, as*

$$\forall i \in N, \quad \varphi_i^q(N, v) = \sum_{E \in 2^N} q_i(E) \Delta_v(E).$$

The Shapley value is the only Sharing value that verifies $q_i(E) = 1/|E|$ for each $E \in 2^N \setminus \emptyset$ and each $i \in E$. The Harsanyi set is a set-valued solution introduced by Hammer et al. (1977), Vasil'ev (1978) and Vasil'ev (1981). Derks et al. (2000) show that the Sharing values of a given TU-game fill the Harsanyi set of that game.

²The Harsanyi dividends $(\Delta_v(E))_{E \in 2^N}$ associated with a TU-game (N, v) correspond to the Möbius transform of $v : 2^N \rightarrow \mathbb{R}$.

³Each $q(E)$ can be viewed as a probability distribution over $E \in 2^N$.

Definition 3 (Harsanyi set). Pick any $(N, v) \in \mathbb{G}$. The Harsanyi set is defined as

$$H(N, v) = \{\varphi^q(N, v) : q \in Q^N\}.$$

Next, we present an axiomatic characterization of Sharing values on \mathbb{G} by Derks et al. (2000). To that end, we introduce four axioms that define properties for a solution f on \mathbb{G} :

1. **Efficiency (E^{TU}).** For each $(N, v) \in \mathbb{G}$, $\sum_{i \in N} f_i(N, v) = v(N)$;
2. **Additivity (A^{TU}).** For each $(N, v), (N, w) \in \mathbb{G}$, $f(N, v + w) = f(N, v) + f(N, w)$;
3. **Null player (N^{TU}).** For each $(N, v) \in \mathbb{G}$, if $i \in N$ is a null player, then $f_i(N, v) = 0$;
4. **Positivity (P^{TU}).** For each $(N, v) \in \mathbb{G}$, where $\Delta_v(E) \geq 0$ for each $E \in 2^N$, $f(N, v) \in \mathbb{R}_+^N$.

Theorem 1 (Derks et al. (2000)). A solution f on \mathbb{G} is a Sharing value if and only if it satisfies (E^{TU}) , (A^{TU}) , (N^{TU}) and (P^{TU}) .

2.2. MC-games

Let $N = \{1, \dots, n\}$ be a **fixed set of players**. Each player $i \in N$ has a finite set of pairwise distinct and linearly ordered activity levels $\{0, \dots, m_i\}$, where m_i is the **maximal activity level** of i . For the rest of the paper, we fix $K \in \mathbb{N}$ an upper bound for maximal activity levels, i.e., $m_i \leq K$ for each $i \in N$. Let \mathcal{M} be the Cartesian product $\prod_{i \in N} \{0, \dots, m_i\}$. Each vector $s \in \mathcal{M}$ is called a (multi-choice) **coalition**. The vector $m = (m_1, \dots, m_n)$ plays the role of the **grand coalition**, whereas $\mathbf{0} = (0, \dots, 0)$ plays the role of the **empty coalition**. The set \mathcal{M} endowed with the usual binary relation \leq on \mathbb{R}^n induces a (complete) lattice with greatest element m and least element $\mathbf{0}$. For any two coalitions $a, b \in \mathcal{M}$, $a \vee b$ and $a \wedge b$ denote their least upper bound and their greatest lower bound over \mathcal{M} , respectively. Assume that all players agree on forming a coalition in which everyone plays the same activity level, let us say j . Players unable to cooperate at such a level play at their maximal activity level. We call such coalition a **j-synchronized coalition** and we denote it by $(j \wedge m_i)_{i \in N}$.

Let \mathcal{K} be the fixed Cartesian product $\prod_{i \in N} \{0, \dots, K\}$. The set \mathcal{K} can be viewed as the set of all thinkable coalitions in our framework, whereas the set $\mathcal{M} \subseteq \mathcal{K}$ only contains the coalition induced by a grand coalition m . An **MC-game** on N and \mathcal{K} is a couple (m, v) where $m \in \mathcal{K}$ and $v : \mathcal{M} \rightarrow \mathbb{R}$ is a characteristic function, $v(\mathbf{0}) = 0$, that specifies a worth $v(s)$ to each coalition $s \in \mathcal{M}$. Pick any $m \in \mathcal{K}$. The class of MC-games with grand coalition m is denoted by \mathcal{G}^m . Denote by $\mathcal{G} = \cup_{m \in \mathcal{K}} \mathcal{G}^m$ the **class of all MC-games** on N and \mathcal{K} . Observe that \mathbb{G} , the class of TU-games on N , can be viewed as the class of MC-games $\mathcal{G}^{(1, \dots, 1)}$ with grand coalition $(1, \dots, 1)$. The sub-game (t, v^t) of $(m, v) \in \mathcal{G}$ induced by $t \in \mathcal{M}$ is the MC-game such that v^t is the restriction of v to the subset of coalitions $\{y \in \mathcal{M} \mid y \leq t\}$. When no confusion arises, we simply denote the sub-game (t, v^t) by (t, v) . For any two MC-games $(m, v), (m, w) \in \mathcal{G}$ and for any $c \in \mathbb{R}$, the MC-game $(m, v + cw)$ is such that, for each $s \in \mathcal{M}$, $(v + cw)(s) = v(s) + cw(s)$. An MC-game $(m, v) \in \mathcal{G}$ is a null game if $v(s) = 0$ for each $s \in \mathcal{M}$.

Pick any $(m, v) \in \mathcal{G}$. A **player-activity level pair** (i, j) represents a player $i \in N$ and one of his (strictly positive) activity levels $0 < j \leq m_i$. We use $M^+ = \bigcup_{i \in N} (\{i\} \times \{1, \dots, m_i\})$ to denote the set of all player-activity level pairs induced by m . Similarly, we use $K^+ = \bigcup_{i \in N} (\{i\} \times \{1, \dots, K\})$ to denote the set of all thinkable player-activity level pairs.

A pair $(i, j) \in M^+$ is **unproductive** in (m, v) if $v(s + e_i) - v(s) = 0$ for each $s \in \mathcal{M}$ such that $s_i = j - 1$. In the context of the running example from the Introduction, an unproductive pair (i, j) may represent a specific workday j on which the consultant i produces nothing. A pair $(i, j) \in M^+$ is **null** in (m, v) if each pair (i, j') , such that $j \leq j' \leq m_i$, is an unproductive pair. In the context of the running example, a null pair (i, j) may represent a specific workday j from which the consultant i produces nothing. Obviously, if a pair is null in a game $(m, v) \in \mathcal{G}$, then it is unproductive in (m, v) . The converse is not true.

An analogue of (1) in the multi-choice setting is the concept of (multi-choice) **unanimity games**. Pick any $m \in \mathcal{K}$. For any $t \leq m$, the unanimity game (m, u_t) with ruling coalition t is defined as

$$\forall s \in \mathcal{M}, \quad u_t(s) = \begin{cases} 1 & \text{if } s \geq t, \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

In a unanimity game (m, u_t) , each player $i \in N$ is required to be active at least at his level t_i to allow a coalition to generate a non-null worth. For each MC-game $(m, v) \in \mathcal{G}$, the characteristic function v admits a unique **linear decomposition** in terms of unanimity games (see Hsiao and Raghavan (1992)) given by

$$v = \sum_{t \in \mathcal{M}} \Delta_v(t) u_t, \quad (5)$$

where $\Delta_v(t)$ is called the (multi-choice) **Harsanyi dividend** of t .⁴ Harsanyi dividends are defined recursively as

$$\begin{aligned} \forall t \in \mathcal{M}, \quad \Delta_v(t) &= v(t) - \sum_{s < t} \Delta_v(s), \\ \text{and } \Delta_v(0, \dots, 0) &= 0. \end{aligned} \quad (6)$$

In an MC-game $(m, v) \in \mathcal{G}$, a **payoff vector** $x \in \mathbb{R}^{|M^+|}$ assigns a payoff $x_{ij} \in \mathbb{R}$ to each (positive) activity level j of each player i , or equivalently, to each pair $(i, j) \in M^+$. The payoff x_{ij} can be interpreted as the increment in player i 's total payoff when he increases his activity level from $j - 1$ to j . A single-valued solution, or a **value** for short, on any sub-class $G \subseteq \mathcal{G}$ is a map f that assigns a unique payoff vector $f(m, v)$ to each $(m, v) \in G$. A set-valued solution F on $G \subseteq \mathcal{G}$ assigns a set (possibly empty) of payoff vectors $F(m, v)$ to each $(m, v) \in G$.

Next, we present three values for MC-games. Each of these values can be viewed as an extension of the Shapley value from TU-games to MC-games. To define these values, we introduce three interpretations of the **support** of a (multi-choice) coalition. For each value, we provide an expression in terms of Harsanyi dividends.

First, let us define the **KSZ value** proposed by Klijn et al. (1999) (introduced in a more general context in Derks and Peters (1993)). To that end, let us define the map $B : \mathcal{K} \rightarrow 2^{K^+}$ as

$$\forall s \in \mathcal{K}, \quad B(s) = \{(i, j) \in K^+ : j \leq s_i\}.$$

The subset $B(s) \subseteq K^+$ can be interpreted as the **support** of coalition s . It gathers all the pairs whose activity level can be considered necessary to form s . The *KSZ* value distributes the dividend $\Delta_v(s)$ of each coalition s equally among the pairs contained in the support $B(s)$.

⁴The Harsanyi dividends $(\Delta_v(t))_{t \leq m}$ associated with an MC-game (m, v) correspond to the Möbius transform of $v : \mathcal{M} \rightarrow \mathbb{R}$.

Definition 4 (KSZ value). For each $(m, v) \in \mathcal{G}$, the KSZ value is defined as

$$\forall (i, j) \in M^+, \quad KSZ_{ij}(m, v) = \sum_{\substack{s \in \mathcal{M} \\ (i, j) \in B(s)}} \frac{\Delta_v(s)}{|B(s)|}.$$

Second, let us define the **PZ value** introduced by Peters and Zank (2005). To that end, let us define the map $C : \mathcal{K} \rightarrow 2^{K^+}$ as

$$\forall s \in \mathcal{K}, \quad C(s) = \{(i, j) \in B(s) : j = s_i\}.$$

The subset $C(s) \subseteq K^+$ is another possible interpretation of the **support** of coalition s . It gathers all the pairs featuring the activity levels at which the players are active in s . Observe that $C(s) \subseteq B(s)$. The *PZ* value distributes the dividend $\Delta_v(s)$ of each coalition s equally among the pairs contained in the support $C(s)$.

Definition 5 (PZ value). For each $(m, v) \in \mathcal{G}$, the PZ value is defined as

$$\forall (i, j) \in M^+, \quad PZ_{ij}(m, v) = \sum_{\substack{s \in \mathcal{M} \\ (i, j) \in C(s)}} \frac{\Delta_v(s)}{|C(s)|}.$$

Finally, let us define the **LT value** introduced by Lowing and Techer (2022a). To that end, let us define the map $T : \mathcal{K} \rightarrow 2^{K^+}$ as

$$\forall s \in \mathcal{K}, \quad T(s) = \{(i, j) \in C(s) : j \geq s_k, \forall k \in N\}. \quad (7)$$

The subset $T(s) \subseteq K^+$ is another possible interpretation of the **support** of coalition s . It contains all the pairs featuring the players with the highest activity level in s . Observe that $T(s) \subseteq C(s)$. The *LT* value distributes the dividend $\Delta_v(s)$ of each coalition s equally among the pairs contained in the support $T(s)$.

Definition 6 (LT value). For each $(m, v) \in \mathcal{G}$, the LT value is defined as

$$\forall (i, j) \in M^+, \quad LT_{ij}(m, v) = \sum_{\substack{s \in \mathcal{M} \\ (i, j) \in T(s)}} \frac{\Delta_v(s)}{|T(s)|}.$$

The three values *KSZ*, *PZ* and *LT* all boil down to the Shapley value on the class of TU-games $\mathcal{G}^{(1, \dots, 1)}$ (see Klijn et al. (1999), Peters and Zank (2005) and Lowing and Techer (2022a)). Observe that the three values only differ in their interpretation of the support of a coalition, which affects the way the dividends are distributed among the player-activity level pairs. These interpretations of the support of the coalitions are central to this study. We provide an example to better illustrate their differences.

Example 1. Consider the running example from the Introduction. Recall that $N = \{a, b, c\}$ and $m = (4, 3, 4)$. Assume that the three consultants struggle to collaborate and generate income. However, if a is willing to work for at least 3 days, b for at least 2 days and c for at least 4 days, then an income of 9 can be generated. Otherwise, their collaboration does not produce anything. This situation can be represented by the characteristic function v defined as

$$\forall t \in \mathcal{M}, \quad v(t) = 9 \times u_s(t) = \begin{cases} 9 & \text{if } t \geq s, \\ 0 & \text{otherwise,} \end{cases}$$

where $s = (3, 2, 4)$. In that case, $\Delta_v(s) = 9$ and $\Delta_v(t) = 0$ for each $t \neq s$. The different interpretations of the support of $(3, 2, 4)$ are

$$\begin{aligned} B(s) &= \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (c, 1), (c, 2), (c, 3), (c, 4)\}, \\ C(s) &= \{(a, 3), (b, 2), (c, 4)\}, \\ \text{and } T(s) &= \{(c, 4)\}. \end{aligned}$$

See also Figure 1 for an illustration of the different interpretations of the support of $(3, 2, 4)$. Since there is only one non-null Harsanyi dividend in the MC-game (m, v) , the computation of the values is direct. The *KSZ* value distributes $\Delta_v(s)$ among all the pairs in $B(s)$. Hence, each pair in $B(s)$ shall receive $9/9 = 1$. The *PZ* value distributes $\Delta_v(s)$ among all the pairs in $C(s)$. Hence, each pair in $C(s)$ shall receive $9/3 = 3$. The *LT* value distributes 9 among all the pairs in $T(s)$. Hence, the pair $(c, 4)$ shall receive 9. The daily payoffs obtained by applying the *KSZ*, *PZ*, and *LT* values are given in Figure 2.

	(a,1)	(a,2)	(a,3)	(a,4)	(b,1)	(b,2)	(b,3)	(c,1)	(c,2)	(c,3)	(c,4)
<i>KSZ</i> (m, v)	1	1	1	0	1	1	0	1	1	1	1
<i>PZ</i> (m, v)	0	0	3	0	0	3	0	0	0	0	3
<i>LT</i> (m, v)	0	0	0	0	0	0	0	0	0	0	9

Figure 2: Example of daily payoffs

3. Sharing values and Harsanyi sets for MC-games

In this section, we define the families of *B*-Sharing values, *C*-Sharing values, and *T*-Sharing values. These are defined using the three interpretations of the support of a coalition introduced in the previous section. Each of these families extends Sharing values from TU-games to MC-games. To properly define our Sharing values for MC-games, we need to define sharing systems on K^+ . First, let us define *B*-sharing systems and *B*-Sharing values.

Definition 7. A *B*-sharing system is a collection $q = (q(s))_{s \in \mathcal{K}}$ such that for each $s \in \mathcal{K} \setminus \{\mathbf{0}\}$, $q(s) \in \mathbb{R}^{|K^+|}$ verifies the following conditions

- (a) $q_{ij}(s) = 0$ if $(i, j) \notin B(s)$;
- (b) $q_{ij}(s) \geq 0$ if $(i, j) \in B(s)$;
- (c) and $\sum_{(i,j) \in K^+} q_{ij}(s) = 1$.

We use the convention $q_{ij}(\mathbf{0}) = 0$ for each $(i, j) \in K^+$.

Let Q^B denote the set of all B -sharing systems.

Definition 8 (B-Sharing values). Pick any $q \in Q^B$. For each $(m, v) \in \mathcal{G}$, the B -Sharing value φ^q is defined as

$$\forall (i, j) \in M^+, \quad \varphi_{ij}^q(m, v) = \sum_{s \in \mathcal{M}} q_{ij}(s) \Delta_v(s).$$

A B -Sharing value distributes the dividend $\Delta_v(s)$ of each coalition s among the pairs contained in the support $B(s)$. This distribution is done according to the positive weights given by $q(s)$. Observe that each $q(s)$, $s \in \mathcal{M}$, is independent from the grand coalition m and from the characteristic function v . The KSZ value is a specific B -Sharing value that verifies $q_{ij}(s) = 1/|B(s)|$ for each $s \in \mathcal{K} \setminus \{\mathbf{0}\}$ and each $(i, j) \in B(s)$. The B -Harsanyi set H^B is defined as the set of all B -Sharing values, i.e.,

$$\forall (m, v) \in \mathcal{G}, \quad H^B(m, v) = \{\varphi^q(m, v) : q \in Q^B\}.$$

Observe that H^B boils down to the Harsanyi set of TU-games on the sub-class of TU-games. Next, let us define C -sharing systems and C -Sharing values.

Definition 9. A C -sharing system $q = (q(s))_{s \in \mathcal{K}}$ is a B -sharing system such that for each $s \in \mathcal{K} \setminus \{\mathbf{0}\}$, $q(s) \in \mathbb{R}^{K^+}$ verifies the following condition

$$(d) \quad q_{ij}(s) = 0 \text{ if } (i, j) \notin C(s).$$

Let Q^C denote the set of all C -sharing systems. Observe that (d) implies (a). Consequently, $Q^C \subseteq Q^B$.

Definition 10 (C-Sharing values). A C -Sharing value φ^q is a B -Sharing value associated with a C -sharing system.

A C -Sharing value distributes the dividend $\Delta_v(s)$ of each coalition s among the pairs contained in the support $C(s)$. The PZ value is a specific C -Sharing value that verifies $q_{ij}(s) = 1/|C(s)|$ for each $s \in \mathcal{K} \setminus \{\mathbf{0}\}$, and each $(i, j) \in C(s)$. The C -Harsanyi set H^C is defined as the set of all C -Sharing values, i.e.,

$$\forall (m, v) \in \mathcal{G}, \quad H^C(m, v) = \{\varphi^q(m, v) : q \in Q^C\}.$$

Observe that H^C boils down to the Harsanyi set of TU-games on the sub-class of TU-games. Finally, let us define T -sharing systems and T -Sharing values.

Definition 11. A T -sharing system $q = (q(s))_{s \in \mathcal{K}}$ is a C -sharing system such that for each $s \in \mathcal{K} \setminus \{\mathbf{0}\}$, $q(s) \in \mathbb{R}^{K^+}$ verifies the following condition

$$(e) \quad q_{ij}(s) = 0 \text{ if } (i, j) \notin T(s).$$

Let Q^T denote the set of all T -sharing systems. Observe that (e) implies (d). Consequently, $Q^T \subseteq Q^C$.

Definition 12 (T-Sharing values). A *T-Sharing value* φ^q is a *C-Sharing value* associated with a *T-sharing system*.

A *T-Sharing value* distributes the dividend $\Delta_v(s)$ of each coalition s among the pairs contained in the support $T(s)$. The *LT value* is a specific *T-Sharing value*, which verifies $q_{ij}(s) = 1/|T(s)|$ for each $s \in \mathcal{K} \setminus \{\mathbf{0}\}$, and each $(i, j) \in T(s)$. The *T-Harsanyi set* H^T is defined as the set of all *T-Sharing values*, i.e.,

$$\forall (m, v) \in \mathcal{G}, \quad H^T(m, v) = \{\varphi^q(m, v) : q \in Q^T\}.$$

Observe that H^T boils down to the Harsanyi set of TU-games on the sub-class of TU-games. Since $Q^T \subseteq Q^C \subseteq Q^B$, the *T-Harsanyi set* is included in the *C-Harsanyi set*, which itself is included in the *B-Harsanyi set* (see Figure 3). Observe that the *B-sharing system* of the *KSZ value* fails to satisfy condition (d). Therefore, the *KSZ value* is not a *C-Sharing value*. Similarly, the *PZ value* is not a *T-Sharing value*.

Proposition 1. For each $(m, v) \in \mathcal{G}$,

$$H^T(m, v) \subseteq H^C(m, v) \subseteq H^B(m, v).$$

Proof. The proof is direct and will be omitted. □

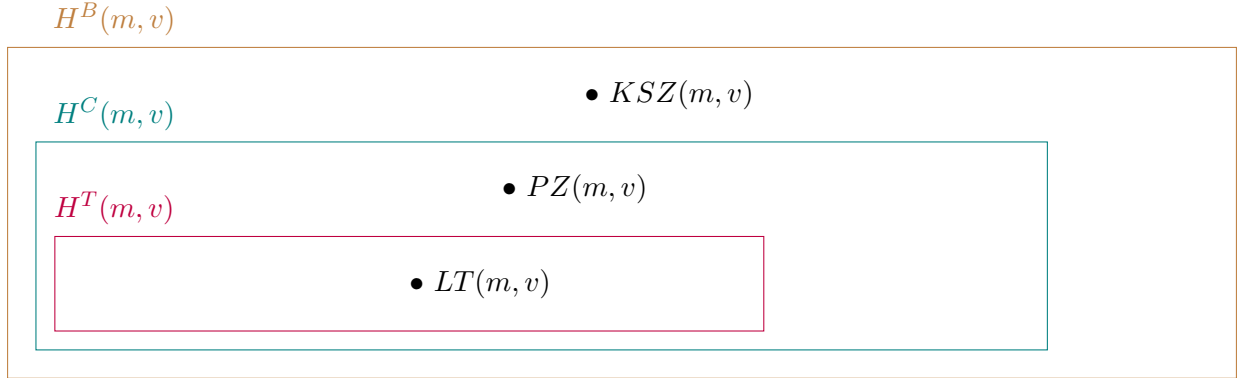


Figure 3: Inclusions of the Harsanyi sets

4. Axiomatic study and characterizations

In this section, we discuss new and classical axioms for MC-games. Several propositions highlight the relationships between the axioms. We also provide axiomatic characterizations of each family of values introduced in Section 3.

4.1. Standard axioms

Let f be a value on \mathcal{G} . First, we introduce two classical axioms for MC-games.

Efficiency (E). For each $(m, v) \in \mathcal{G}$, $\sum_{(i,j) \in M^+} f_{ij}(m, v) = v(m)$.

Additivity (A). For each $(m, v), (m, w) \in \mathcal{G}$, $f(m, v + w) = f(m, v) + f(m, w)$.

Obviously, (E) and (A) reduce to (E^{TU}) and (A^{TU}) on TU-games, respectively. The next axiom, originally introduced by Klijn et al. (1999), requires that any null pair receives a null payoff. In the context of Example 1, this axiom guarantees that if a consultant is unproductive as of a certain workday, he shall receive a null payoff for that workday. For instance, consultant b shall receive nothing for his third and fourth workdays.

Null pair (N). For each $(m, v) \in \mathcal{G}$, if $(i, j) \in M^+$ is a null pair, then $f_{ij}(m, v) = 0$.

Observe that (N) reduces to (N^{TU}) on TU-games. The next axiom indicates that the payoff of each pair should be null in a null game. In the context of Example 1, this axiom indicates that if no income is generated by the consultants' cooperation, then each of the consultant's daily payoffs should be null.

Null game (NG). If $(m, v) \in \mathcal{G}$ is a null game, then $f_{ij}(m, v) = 0$ for each $(i, j) \in M^+$.

In a null game, all pairs are null pairs. It follows that if a solution satisfies (N), then it also satisfies (NG).

Proposition 2. *If a solution f on \mathcal{G} satisfies (N), then it satisfies (NG).*

Proof. The proof is direct and will be omitted. □

Remark 1. The converse of Proposition 2 is not true. To see this, consider the value f^1 defined, for each $(m, v) \in \mathcal{G}$, as

$$\forall (i, j) \in M^+ \quad f_{ij}^1(m, v) = \frac{v(m)}{|M^+|}. \quad (8)$$

Clearly, f^1 satisfies (NG) since any null game (m, v) satisfies $v(m) = 0$. However, f^1 fails to satisfy (N) since a null pair in game $(m, v) \in \mathcal{G}$ may receive a non null payoff as long as $v(m) \neq 0$. □

The next axiom indicates that if the Harsanyi dividend of each coalition is positive in a game, then the payoff of each pair should also be positive. In the context of Example 1, this axiom guarantees that if the consultants' project generates positive surpluses, then each of the consultant's daily payoffs should be positive, which is the case in this example.

Positivity (P). For each $(m, v) \in \mathcal{G}$, where $\Delta_v(s) \geq 0$ for each $s \in \mathcal{M}$, $f(m, v) \in \mathbb{R}_+^{|M^+|}$.

Observe that (P) reduces to (P^{TU}) on TU-games. In a null game, the worth of the grand coalition is null. Moreover, in a null game, the Harsanyi dividend of each coalition is null and therefore is positive. It follows that (E) and (P) imply (NG).

Proposition 3. *If a solution f on \mathcal{G} satisfies (E) and (P), then it satisfies (NG).*

Proof. The proof is direct and will be omitted. □

Remark 2. The converse of Proposition 3 is not true. To see this, consider the value f^2 defined, for each $(m, v) \in \mathcal{G}$, as

$$\forall (i, j) \in M^+, \quad f_{ij}^2(m, v) = \begin{cases} 0 & \text{if } v(m) = 0 \\ -1 & \text{otherwise.} \end{cases}$$

This value satisfies (NG), but fails to satisfy both (E) and (P). □

The next axiom deals with null pairs differently from (N). Pick any $(m, v) \in \mathcal{G}$. The axiom requires that if (i, m_i) is a null pair in (m, v) , then player i can reduce his maximal activity level by one unit without affecting the payoffs of the remaining pairs. In the context of Example 1, this axiom ensures that if a consultant is unproductive on his last workday, then he can take this day off without affecting the payoffs of the other consultants as well as his own payoffs for his remaining workdays. For instance, consultant a can take his fourth workday off.

Null pair out (NO). For each $(m, v) \in \mathcal{G}$, if $(i, m_i) \in M^+$ is a null pair, then

$$\forall (k, l) \in M^+ \setminus \{(i, m_i)\}, \quad f_{kl}(m - e_i, v) = f_{kl}(m, v).$$

Lowing and Techer (2022b) show that combining this last axiom with (E) implies (N).

Proposition 4. *If a solution f on \mathcal{G} satisfies (E) and (NO), then it satisfies (N).*

From Proposition 2 and Proposition 4, it directly follows that if a solution satisfies (E) and (NO), then it satisfies (NG).

Remark 3. The converse of Proposition 4 is not true. In particular, (N) does not imply (NO). To see this, consider the solution f^3 defined, for any $(m, v) \in \mathcal{G}$, as

$$\forall (i, j) \in M^+, \quad f_{ij}^3(m, v) = \sum_{\substack{s \in \mathcal{M} \\ (i, j) \in T(s)}} \frac{m_i}{\sum_{(k, l) \in T(s)} m_k} \Delta_v(s). \quad (9)$$

This solution distributes the dividend of each coalition s among the pairs in $T(s)$ according to some weights given by the grand coalition m . These weights will vary if m varies. In that respect, the solution f^3 is not a B -Sharing value since a B -Sharing value must be defined according to a sharing system independent from the grand coalition. Clearly, f^3 satisfies (E) and (N). However, f^3 fails to satisfy (NO). Indeed, reducing the maximal activity level of a player will inevitably affect the distribution of the dividends since such distribution depends on the maximal activity levels of the players.

Additionally, to see that (N) does not imply (E), consider the value f^4 defined, for any $(m, v) \in \mathcal{G}$, as

$$\forall (i, j) \in M^+, \quad f_{ij}^4(m, v) = 0. \quad (10)$$

The value f^4 is called the null value. It satisfies (N) and (NO) but fails to satisfy (E). \square

4.2. Characterizations of B -Sharing values

We have the material to characterize B -Sharing values on the full class of MC-games. This characterization relies on (E), (A), (NO) and (P). In this respect, it is conceptually close to Theorem 1.

Theorem 2. *A solution f on \mathcal{G} is a B -Sharing value if and only if it satisfies (E), (A), (NO) and (P).*

Proof. See Appendix 7.2. \square

It is possible to replace (NO) by (N) in the statement of Theorem 2 to obtain a characterization of B -Sharing values on any sub-class of MC-games with a fixed grand coalition.

Corollary 1. *Pick any $m \in \mathcal{K}$. A solution f on \mathcal{G}^m is a B -Sharing value if and only if it satisfies (E), (A), (N) and (P).*

Proof. See Appendix 7.3 □

Observe that Corollary 1 coincides with Theorem 1 on the sub-class of TU-games, i.e., on $\mathcal{G}^{(1, \dots, 1)}$.

Remark 4. Observe that the axioms invoked in Corollary 1 do not characterize B -Sharing values on \mathcal{G} . To see this, it suffices to consider f^3 given by (9). This solution satisfies (E), (A), (N) and (P) on \mathcal{G} , but it cannot be viewed as a B -Sharing value on \mathcal{G} . However, f^3 can be viewed as a B -Sharing value on \mathcal{G}^m , $m \in \mathcal{K}$, since the grand coalition is fixed. □

4.3. Characterizations of C -Sharing values

Next, we propose an axiomatic characterization of C -Sharing values on the full class of MC-games. To that end, we need an additional axiom: Independence of the maximal activity level. This axiom indicates that the variation of a player's maximal activity level should have no impact on the payoff of his remaining activity levels. The axiom was first introduced by Hwang and Liao (2009) and Béal et al. (2012). In the context of Example 1, the axiom ensures that a consultant can reduce his total number of workdays (by taking a day off for instance) and keep his original payoffs for his remaining workdays.

Independence of the maximal activity level (IM). For each $(m, v) \in \mathcal{G}$, each $i \in N$ and each $j < m_i$, $f_{ij}(m, v) = f_{ij}(m - e_i, v)$.

Any B -Sharing value that is not a C -Sharing value fails to satisfy (IM). Adding (IM) to the set of axioms invoked in Theorem 2 leads to a characterization of C -Sharing values on the full class of MC-games.

Theorem 3. *A solution f on \mathcal{G} is a C -Sharing value, if and only if it satisfies (E), (A), (NO), (P) and (IM).*

Proof. See Appendix 7.4. □

Next, we propose an alternative characterization of C -Sharing values on any sub-class of MC-games with a fixed grand coalition. To that end, we consider an additional axiom: Unproductive pair. This axiom was originally introduced by Peters and Zank (2005) to characterize the PZ value. It states that any unproductive pair should receive a null payoff. In the context of Example 1, this axiom guarantees that if a consultant is unproductive on a certain workday, then it shall receive no payoff for that workday.

Unproductive pair (U). Pick any $(m, v) \in \mathcal{G}$. If $(i, j) \in M^+$ is an unproductive pair in (m, v) , then $f_{ij}(m, v) = 0$.

Observe that (U) reduces to (N^{TU}) on TU-games. It is clear that (U) implies (N). But the converse is not true. Indeed, the KSZ value satisfies (N) but fails to satisfy (U). The next proposition shows that (N) coupled with (IM) implies (U).

Proposition 5. *If a solution f on \mathcal{G} satisfies (N) and (IM), then it satisfies (U).*

Proof. See Appendix 7.5. □

Remark 5. The converse of Proposition 5 is not true. To see this, it suffices to consider the alternative solution f^3 defined by (9). This solution satisfies (U), but fails to satisfy (IM). □

From Proposition 4 and Proposition 5, it is clear that any solution satisfying (E), (NO) and (IM) satisfies (U). One can replace (NO) and (IM) by (U) in the statement of Theorem 3 to obtain a characterization of C -Sharing values on any sub-class of MC-games with a fixed grand coalition.

Corollary 2. *Pick any $m \in \mathcal{K}$. A solution f on \mathcal{G}^m is a C -Sharing value if and only if it satisfies (E), (A), (U) and (P).*

Proof. See Appendix 7.6 □

Observe that Corollary 2 coincides with Theorem 1 on the sub-class of TU-games.

Remark 6. Similarly to Remark 4, the axioms invoked in Corollary 2 do not characterize C -Sharing values on the full class of MC-games \mathcal{G} . Remark 5 corroborates this statement. □

4.4. Characterizations of T -Sharing values

Next, we propose an axiomatic characterization of T -Sharing values on the full class of MC-games. To that end, we need an additional axiom: Independence of higher activity levels. This axiom requires that if the maximal activity level of all players reduces to a certain level, then the payoff of each player for this activity level remains unchanged. This axiom was originally introduced in Lowing and Techer (2022a) to characterize the LT value. In the context of Example 1, the axiom ensures that the income received by a consultant on a given workday does not depend on future workdays.

Independence of higher activity levels (IH). For each $(m, v) \in \mathcal{G}$,

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v) = f_{ij}((j \wedge m_k)_{k \in N}, v).$$

Proposition 6. *If a solution f on \mathcal{G} satisfies (IH), then it satisfies (IM).*

Proof. See Appendix 7.7. □

The converse of Proposition 6 is not true. For instance, the PZ value satisfies (IM), but fails to satisfy (IH). Adding (IH) to the set of axioms invoked in Theorem 2 leads to a characterization of T -Sharing values on the full class of MC-games.

Theorem 4. *A solution f on \mathcal{G} is a T -Sharing value, if and only if it satisfies (E), (A), (NO), (P) and (IH).*

Proof. See Appendix 7.8. □

Finally, we propose an alternative characterization of T -Sharing values on any sub-class of MC-games with a fixed grand coalition. To that end, we consider a new axiom: Null and synchronized coalition. Unlike (N), (NO), and (U), which focus on the productivity of player-activity level pairs, Null and synchronized coalition is an axiom that focuses on the productivity of coalitions. Consider an MC-game in which there exists a synchronized coalition s whose worth is null. The Null and synchronized coalition axiom requires that the sum of the payoffs of all the pairs in $B(s)$ should be null. For instance, in the context of Example 1, if all consultants agree on collaborating together for one day but still fail to generate any income, then the sum of their payoffs should be null.

Null and synchronized coalition (NS). Pick any $(m, v) \in \mathcal{G}$. If there exists a $j \leq \max_{i \in N} m_i$ such that $v((j \wedge m_i)_{i \in N}) = 0$, then

$$\sum_{(i', j') \in B((j \wedge m_i)_{i \in N})} f_{ij}(m, v) = 0.$$

Proposition 7. *If a solution f on \mathcal{G} satisfies (E) and (IH), then it satisfies (NS).*

Proof. See Appendix 7.9. □

Remark 7. The converse of Proposition 7 is not true. In particular, (NS) does not imply (E). To see this, observe that the null value f^4 defined by (10) satisfies (NS) but fails to satisfy (E). Moreover, (NS) does not imply (IH). To see this, consider the value f^8 defined for each $(m, v) \in \mathcal{G}$ as

$$\forall (i, j) \in M^+, \quad f_{ij}^8(m, v) = \begin{cases} \frac{v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N})}{|T(m)|} & \text{if } (i, m_i) \in T(m), \\ 0 & \text{otherwise.} \end{cases}$$

The value f^8 satisfies (NS), but fails to satisfy (IH). □

Adding (NS) to the axioms invoked in Corollary 1, we obtain a characterization of T -Sharing values on any sub-class of MC-games with a fixed grand coalition.

Corollary 3. *Pick any $m \in \mathcal{K}$. A solution f on \mathcal{G}^m is a T -Sharing value if and only if it satisfies (E), (A), (N), (NS) and (P).*

Proof. See Appendix 7.10. □

Observe that (E) implies (NS) on the sub-class of TU-games. Consequently, Corollary 3 coincides with Theorem 1 on the sub-class of TU-games.

Remark 8. Similarly to Remark 4, the axioms invoked in Corollary 3 do not characterize T -Sharing values on the full class of MC-games \mathcal{G} . Remark 7 corroborates this statement. □

To conclude this section, we summarize all the relationships between the axioms in Figure 4.

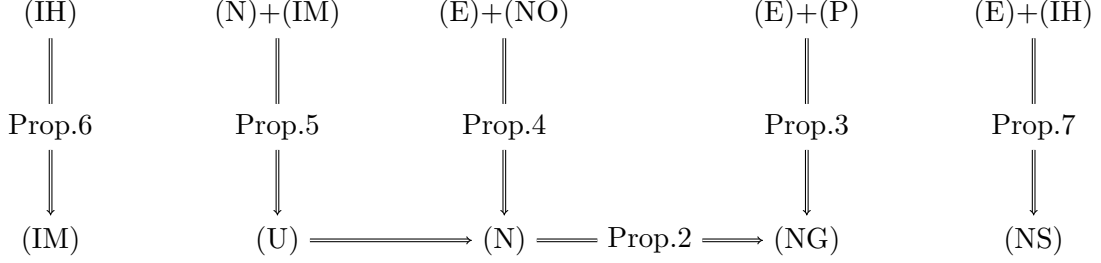


Figure 4: Relationships between the axioms

5. Additional remarks

In this section, we compare our results with existing axiomatic characterizations of the KSZ , PZ and LT values. To be specific, in Corollary 1, (P) can be replaced by an alternative axiom to obtain a characterization of the KSZ value (see Theorem 4.2 in Klijn et al. (1999)). Similarly, in Corollary 2, (P) can be replaced by two alternative axioms to obtain a characterization of the PZ value (see Theorem 3.1 in Peters and Zank (2005)). Finally, in Theorem 4, by replacing (P) by an alternative axiom, by weakening (NO), and by strengthening (A) we obtain a characterization of the LT value (see Theorem 1 in Lowing and Techer (2022a)).

Klijn et al. (1999) provide a characterization of the KSZ value on \mathcal{G} that relies on four axioms, three of which are already introduced in this paper: (E), (A) and (N). The fourth axiom focuses on the so-called veto pairs. A pair (i, j) is **veto** in $(m, v) \in \mathcal{G}$ if $v(s) = 0$ for each $s \in \mathcal{M}$ such that $s_i < j$. In other words, a veto pair represents an activity level at which a player must participate for a coalition to have a non-null worth. The axiom, originally introduced by van den Nouweland (1993), states that veto pairs are all equally necessary and should receive the same payoff.

Veto pair (V). For each $(m, v) \in \mathcal{G}$, if $(i, j), (i', j') \in M^+$ are two distinct veto pairs, then $f_{ij}(m, v) = f_{i'j'}(m, v)$.

In the context of Example 1, (V) advocates that all the necessary workdays (which correspond to the pairs in $B(3, 2, 4)$) should receive the same payoff. This seems reasonable considering that no income can be generated without these workdays. By substituting (P) by (V) in the statement of Corollary 1, one obtains the characterization of the KSZ value on \mathcal{G} proposed by Klijn et al. (1999). Observe that this characterization holds on the full class of MC-games and not only on sub-classes of MC-games with a fixed grand coalition as Corollary 1 does.

Peters and Zank (2005) provide a characterization of the PZ value that relies on (E), (A), (U), and two other axioms that we detail below. The first axiom generalizes the well-known Anonymity axiom from TU-games to MC-games. It is defined using permutations on the player set N . However, in the MC-game framework, a permutation on N would only make sense if the players all have the same maximal activity level (which is assumed by Peters and Zank (2005)). Denote by $\bar{\mathcal{G}} \subseteq \mathcal{G}$ the subset of MC-games in which all players have the same maximal activity level.

Anonymity (AN) Pick any $(m, v) \in \bar{\mathcal{G}}$. For each $t \in \mathcal{M}$ and each permutation σ on N , define σt as $\sigma t_{\sigma(i)} = t_i$ for each $i \in N$, and σv as $\sigma v(\sigma t) = v(t)$. Then, it holds that $f_{ij}(m, v) = f_{\sigma(i)j}(m, \sigma v)$.

In the context of Example 1, (AN) ensures that even if consultants a and c exchange their names, this should not affect their payoffs. This means that there can be no discrimination in payoff between two consultants who work equally hard. The second axiom focuses on the so-called intra-equal pairs. Two distinct pairs $(i, j), (i, j') \in M^+$ are **intra-equal** in a game $(m, v) \in \mathcal{G}$ if they contain the same player and if $v(s_{-i}, j) - v(s_{-i}, j - 1) = v(s_{-i}, j') - v(s_{-i}, j' - 1)$ for each $s \in \mathcal{M}$. In other words, two intra-equal pairs represent two activity levels at which a given player is contributing equally. The axiom indicates that such two pairs should obtain the same payoff.

Symmetry for intra-equal pairs (SI). For each $(m, v) \in \mathcal{G}$ and two distinct intra-equal pairs $(i, j), (i, j') \in M^+$, $f_{ij}(m, v) = f_{ij'}(m, v)$.

In the context of Example 1, (SI) guarantees that a consultant who is equally productive on two separate days (e.g. a on his first two workdays) will receive the same payoff for those workdays. By substituting (P) by (AN) and (SI) in the statement of Corollary 2, one obtains the characterization of the PZ value on $\bar{\mathcal{G}}$ proposed by Peters and Zank (2005).

Finally, Lowing and Techer (2022a) provide a characterization of the LT value on \mathcal{G} that relies on (E), (IH), (N) and two other axioms that we detail below. The first axiom is the classical Linearity axiom for solutions of MC-games.

Linearity (L). For each $(m, v), (m, w) \in \mathcal{G}$ and each $\lambda \in \mathbb{R}$, $f(m, v + \lambda w) = f(m, v) + \lambda f(m, w)$.

Obviously, this axiom strengthens (A). The second axiom is based on the concept of equal pairs. Two distinct pairs $(i, j), (i', j) \in M^+$ are **equal** in $(m, v) \in \mathcal{G}$ if they contain the same activity level and if $v(s + e_i) = v(s + e_{i'})$ for each $s \in \mathcal{M}$ such that $s_i = s_{i'} = j - 1$. The axiom states that two equal pairs should receive a payoff of the same sign. To properly present the axiom we define the $sign$ function as $sign : \mathbb{R} \rightarrow \{-1, 0, 1\}$, where $sign(x) = 1$ for each $x > 0$, $sign(0) = 0$, and $sign(x) = -1$ for each $x < 0$. In the context of Example 1, two equal pairs may represent two consultants that are equally productive on a given workday (e.g. a and c on their second workday). These consultants should receive a payoff of the same sign on that particular workday.

Sign symmetry for equal pairs (SSE). For each $(m, v) \in \mathcal{G}$ and two distinct equal pairs $(i, j), (i', j) \in M^+$, $sign(f_{ij}(m, v)) = sign(f_{i'j}(m, v))$.

By substituting (P) by (SSE), by weakening (NO) into (N) and by strengthening (A) into (L) in the statement of Theorem 4, one obtains the characterization of the LT value on \mathcal{G} proposed by Lowing and Techer (2022a). Observe that the characterization still holds even if we keep (N) and (A) instead of (NO) and (L).

6. Conclusion

In this paper, we proposed three extensions of Sharing values from TU-games to MC-games: *B*-Sharing values, *C*-Sharing values, and *T*-Sharing values. The main difference between these extensions is the interpretation of coalition support. We have introduced several axioms and made some propositions to highlight the relationships between them. Invoking these axioms, we characterized *B*-Sharing values, *C*-Sharing values, and *T*-Sharing values on the full class of MC-games and on any sub-class of MC-games with a fixed grand coalition. Our characterization results are summarized in the following table. Symbol “.” means that the value satisfies the axiom, but such an axiom is not used to characterize it. Symbol “-” means that the value does not satisfy the axiom. Symbol “×” indicates that the axiom is used for the two characterizations of the value. Symbol “⊗” indicates that the axiom is only used for the characterization on the full class of MC-games. Symbol “⊠” indicates that the axiom is only used for the characterization on any sub-class of MC-games with a fixed grand coalition.

Values\Axioms	(E)	(A)	(N)	(NG)	(P)	(NO)	(IM)	(IH)	(U)	(NS)
<i>B</i> -sharing	×	×	⊠	·	×	⊗	-	-	-	-
<i>C</i> -sharing	×	×	·	·	×	⊗	⊗	-	⊠	-
<i>T</i> -sharing	×	×	⊠	·	×	⊗	·	⊗	·	⊠

Some questions remain of interest for future research. First, in the context of TU-games, a consistent sharing system is such that the ratio of the weights of two players is the same for all coalitions containing them (see Derks et al. (2000)). A Sharing value associated with a consistent sharing system can be viewed as a Weighted Shapley value (see Kalai and Samet (1987) and Monderer et al. (1992)). It can be interesting to extend the notion of consistency from TU-games to MC-games, and investigate if this leads to new extensions of Weighted Shapley values from TU-games to MC-games. If this investigation is successful, it may be interesting to enrich or modify the axiomatic characterizations proposed in this paper to characterize these new extensions of Weighted Shapley values.

Second, this paper adopts a value-theoretic point of view to study Sharing values for MC-games. Instead, it can be interesting to adopt a set-theoretic point of view to study Harsanyi sets for MC-games. For instance, Grabisch and Xie (2007) extend the Weber set and the Core from TU-games to MC-games. Recently, Lowing and Techer (2022a) show that the *LT* value is the centroid of the Weber set of MC-games and belongs to the Core of super-modular MC-games (a super-modular MC-game can be viewed as a game in which the incentives for being active in a coalition increase as the coalition grows). Since the *LT* value belongs to the *T*-Harsanyi set, it can be interesting to investigate how the *T*-Harsanyi set relates to the Weber set and the Core. In TU-games, it is known that the Weber set, which always contains the Core (see Weber (1988) and Derks (1992)), is included in the Harsanyi set (see Derks et al. (2000)). One could investigate if an analogous result holds in the context of MC-games, i.e., if the Weber set of MC-games is included in the *T*-Harsanyi set.

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7. Appendix

This section contains all the proofs of our results. In order to perform these proofs, we introduce some preliminary results.

7.1. Preliminary results

Remark 9. Pick any $(m, v) \in \mathcal{G}$. From the definition of Harsanyi dividends (see (6)), the following hold.

1. If $(i, j) \in M^+$ is a null pair, then $\Delta_v(s) = 0$ whenever $s_i \geq j$.
2. For each sub-game (t, v^t) , $t \leq m$, each Harsanyi dividend $\Delta_{v^t}(s)$, $s \leq t$, in (t, v^t) coincides with the Harsanyi dividend $\Delta_v(s)$ in (m, v) .
3. $\sum_{t \in \mathcal{M}} \Delta_v(t) = v(m)$.
4. For each $(m, w) \in \mathcal{G}$ and each $t \leq m$, $\Delta_v(t) + \Delta_w(t) = \Delta_{v+w}(t)$.

Pick any $(m, u_s) \in \mathcal{G}$. From the definition of unanimity games (see (4)), the following properties hold:

1. Each pair $(i, j) \notin B(s)$ is a null pair in (m, u_s) ;
2. Each pair $(i, j) \in B(s) \setminus C(s)$ is an unproductive pair in (m, u_s) . □

Next, we show that any value satisfying (A), (N) and (P) satisfies a linearity requirement.

Proposition 8. *Pick any $(m, v) \in \mathcal{G}$. Any value f satisfying (A), (NG) and (P) verifies*

$$f(m, v) = \sum_{s \in \mathcal{M}} \Delta_v(s) f(m, u_s).$$

Proof. Pick any $(m, v) \in \mathcal{G}$. Consider a value f satisfying (A), (NG) and (P). Recall that the characteristic function of an MC-game can be decomposed the following way:

$$v = \sum_{s \in \mathcal{M}} \Delta_v(s) u_s.$$

By (A),

$$f(m, v) = \sum_{s \in \mathcal{M}} f(m, \Delta_v(s) u_s). \tag{11}$$

Pick any $s \in \mathcal{M}$. Let us show that for any $\alpha \in \mathbb{R}$, $f(m, \alpha u_s) = \alpha f(m, u_s)$ holds. Pick any $\alpha \in \mathbb{R}$. If $\alpha \in \mathbb{N}$, then by (A), $f(m, \alpha u_s) = \alpha f(m, u_s)$. If $\alpha = 0$, then by (NG), $f(m, \alpha u_s) = 0 = \alpha f(m, u_s)$. For any $\alpha \in \mathbb{R}$, observe that

$$\begin{aligned} f((\alpha - \alpha) u_s) &= f(\alpha u_s) + f(-\alpha u_s) \\ &= 0. \end{aligned}$$

Therefore,

$$f(-\alpha u_s) = -f(\alpha u_s).$$

If $\alpha \in \mathbb{Q}$, then there exists $\beta, \gamma \in \mathbb{N}$ such that $\alpha = \beta/\gamma$. By (A), we obtain $\gamma f(m, \beta/\gamma u_s) = \beta f(m, u_s)$, so that $f(m, \alpha u_s) = \alpha f(m, u_s)$ follows after dividing by γ . Now, if $\alpha \in \mathbb{R}$, let us consider $\zeta, v \in \mathbb{Q}$ with $\zeta < \alpha < v$. By (P), it holds that $f(m, (\alpha - \zeta)u_s) \geq 0$ and $f(m, (v - \alpha)u_s) \geq 0$, hence

$$\zeta f(m, u_s) = f(m, \zeta u_s) \leq f(m, \alpha u_s) \leq f(m, v u_s) = v f(m, u_s).$$

By letting ζ and v tend to α , we obtain the desired result: $f(m, \alpha u_s) = \alpha f(m, u_s)$. From this, (11) becomes

$$f(m, v) = \sum_{s \in \mathcal{M}} \Delta_v(s) f(m, u_s),$$

which shows the desired result. The proof of Proposition 8 is completed. \square

7.2. Proof of Theorem 2

To prove Theorem 2, we proceed in two steps. First, let us show that any B -Sharing value satisfies (E), (A), (NO) and (P). Pick any $q \in Q^B$ and consider the B -Sharing value φ^q defined for each $(m, v) \in \mathcal{G}$ as

$$\forall (i, j) \in M^+, \quad \varphi_{ij}^q(m, v) = \sum_{s \in \mathcal{M}} q_{ij}(s) \Delta_v(s).$$

Let us show that φ^q satisfies each axiom.

(E): Pick any $(m, v) \in \mathcal{G}$. By definition of a B -sharing system, $\sum_{(i,j) \in M^+} q_{ij}(s) = 1$ for each $s \in \mathcal{M}$. By Remark 9, $\sum_{s \in \mathcal{M}} \Delta_v(s) = 1$. It follows that

$$\begin{aligned} \sum_{(i,j) \in M^+} \varphi_{ij}^q(m, v) &= \sum_{(i,j) \in M^+} \sum_{s \in \mathcal{M}} q_{ij}(s) \Delta_v(s) \\ &= \sum_{s \in \mathcal{M}} \Delta_v(s) \sum_{(i,j) \in M^+} q_{ij}(s) \\ &= v(m), \end{aligned}$$

which shows that φ^q satisfies (E).

(A): Pick any two $(m, v), (m, w) \in \mathcal{G}$. By Remark 9, $\Delta_{v+w}(s) = \Delta_v(s) + \Delta_w(s)$ for each $s \in \mathcal{M}$. It follows that

$$\begin{aligned} \forall (i, j) \in M^+, \quad \varphi_{ij}^q(m, v) + \varphi_{ij}^q(m, w) &= \sum_{s \in \mathcal{M}} q_{ij}(s) \Delta_v(s) + \sum_{s \in \mathcal{M}} q_{ij}(s) \Delta_w(s) \\ &= \sum_{s \in \mathcal{M}} q_{ij}(s) (\Delta_v(s) + \Delta_w(s)) \\ &= \sum_{s \in \mathcal{M}} q_{ij}(s) \Delta_{v+w}(s) \\ &= \varphi_{ij}^q(m, v + w), \end{aligned}$$

which shows that φ^q satisfies (A).

(NO): Pick any $(m, v) \in \mathcal{G}$. Assume that $(i, m_i) \in M^+$ is a null pair. By Remark 9, $\Delta_v(s) = 0$ for each $s \in \mathcal{M}$ such that $s_i = m_i$. It follows that

$$\begin{aligned}
\forall (k, l) \in M^+ \setminus \{(i, m_i)\}, \quad \varphi_{kl}^q(m - e_i, v) &= \sum_{\substack{s \in \mathcal{M} \\ s_i < m_i}} q_{kl}(s) \Delta_v(s) \\
&= \sum_{s \in \mathcal{M}} q_{kl}(s) \Delta_v(s) - \sum_{\substack{s \in \mathcal{M} \\ s_i = m_i}} q_{kl}(s) \Delta_v(s) \\
&= \sum_{s \in \mathcal{M}} q_{kl}(s) \Delta_v(s) - \sum_{\substack{s \in \mathcal{M} \\ s_i = m_i}} 0 \\
&= \varphi_{kl}^q(m, v),
\end{aligned}$$

which shows that φ^q satisfies (NO).

(P): It is direct to show that φ^q satisfies (P) since $q_{ij} \geq 0$ for each $(i, j) \in M^+$.

We have shown that any B -Sharing value satisfies (E), (A), (NO) and (P) on \mathcal{G} . Next, pick any value f on \mathcal{G} satisfying (E), (A), (NO) and (P). Let us show that f is a B -Sharing value. By Proposition 4, f satisfies (N). By Proposition 8, f verifies

$$\forall (m, v) \in \mathcal{G}, \quad f(m, v) = \sum_{s \in \mathcal{M}} \Delta_v(s) f(m, u_s). \quad (12)$$

Pick any game $(m, v) \in \mathcal{G}$ such that $m = (K, \dots, K)$. For such a game, f verifies

$$f(m, v) = \sum_{s \in \mathcal{K}} \Delta_v(s) f(m, u_s).$$

We show that there exists a B -sharing system $q^B \in Q^B$ such that $f(m, u_s) = q^B(s)$ for each $s \in \mathcal{K}$. To that end, we show that each element of the collection $(f(m, u_s))_{s \in \mathcal{K} \setminus \{\mathbf{0}\}}$ satisfies the conditions (a), (b) and (c) that define a B -sharing system. Pick any $s \in \mathcal{K}$ and any $(i, j) \in K^+$.

(a) By Remark 9, any pair $(i, j) \notin B(s)$ is a null pair in (m, u_s) . By (N), $f_{ij}(m, u_s) = 0$.

(b) If $(i, j) \in B(s)$, then by (P), $f_{ij}(m, u_s) \geq 0$.

(c) Finally, by (E), $\sum_{(i, j) \in K^+} f_{ij}(m, u_s) = 1$.

This shows that each element of the collection $(f(m, u_s))_{s \in \mathcal{K} \setminus \{\mathbf{0}\}}$ satisfies the conditions (a), (b) and (c). Since $\Delta_v(\mathbf{0}) = 0$, we can set $f_{ij}(m, u_{\mathbf{0}}) = 0$ for each $(i, j) \in K^+$.

By successive applications of (NO), we obtain $f(m, u_s) = f(s, u_s)$ for each $s \in \mathcal{K}$. This shows that $f(m, u_s) = f(s, u_s)$ is independent from the grand coalition m , and depends on s only. It follows that there exists a B -sharing system $q^B \in Q^B$ such that $q^B(s) = f(s, u_s)$, for each $s \in \mathcal{K}$. Hence, by applying this result to (12), we obtain

$$\forall (m, v) \in \mathcal{G}, \quad f(m, v) = \sum_{s \in \mathcal{M}} \Delta_v(s) q^B(s).$$

Therefore, f is a B -Sharing value. This concludes the proof of Theorem 2. \square

The axioms invoked in Theorem 2 are logically independent, as shown by the following alternative solutions:

- The null value f^4 defined in (10) satisfies all the axioms except (E).
- The value f^5 defined, for each $(m, v) \in \mathcal{G}$, as

$$\forall (i, j) \in M^+, \quad f_{ij}^5(m, v) = \sum_{\substack{s \in \mathcal{M} \\ (i, j) \in T(s)}} \frac{(v((j-1 \wedge m_h)_{h \in N}) + e_i)^2 + 1}{\sum_{(k, s_k) \in T(s)} (v((j-1 \wedge m_h)_{h \in N}) + e_k)^2 + 1} \Delta_v(s),$$

satisfies all the axioms except (A).

- The value f^6 defined, for each $(m, v) \in \mathcal{G}$, as

$$\forall (i, j) \in M^+, \quad f_{ij}^6(m, v) = \frac{1}{|Q(j)|} \left[v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N}) \right], \quad (13)$$

where $Q(j) = \{i \in N : m_i \geq j\}$, satisfies all the axioms except (NO).⁵

- Fix the n -dimensional vector $w = (-1, 2, \dots, 2)$. The value f^7 defined, for each $(m, v) \in \mathcal{G}$, as

$$\forall (i, j) \in M^+, \quad f_{ij}^7(m, v) = \sum_{\substack{s \in \mathcal{M} \\ (i, j) \in T(s)}} \frac{w_i}{\sum_{(k, j) \in T(s)} w_k} \Delta_v(s),$$

satisfies all the axioms except (P).

7.3. Proof of Corollary 1

Pick any $m \in \mathcal{K}$ and consider the sub-class of games \mathcal{G}^m . From Theorem 2, it is direct to infer that any B -Sharing value on \mathcal{G}^m satisfies (E), (A), (NO) and (P). By Proposition 4, any B -sharing value on \mathcal{G}^m value satisfies (N).

We have shown that any B -Sharing value on \mathcal{G}^m satisfies (E), (A), (N) and (P). Next, pick any value f on \mathcal{G}^m satisfying (E), (A), (N) and (P). Let us show that f is a B -Sharing value. By Proposition 3, f satisfies (NG). By Proposition 8, f verifies

$$\forall (m, v) \in \mathcal{G}^m, \quad f(m, v) = \sum_{s \in \mathcal{M}} \Delta_v(s) f(m, u_s).$$

Similarly to the proof of Theorem 1, each element of the collection $(f(m, u_s))_{s \in \mathcal{M} \setminus \{0\}}$ satisfies the conditions (a), (b) and (c). Since the grand coalition is fixed, there is no need to show that $(f(m, u_s))_{s \in \mathcal{M} \setminus \{0\}}$ is independent from m . It follows that there exists a B -sharing system $q^B \in Q^B$ such that $q^B(s) = f(m, u_s)$, for each $s \in \mathcal{M}$. Therefore, f is a B -Sharing value. This concludes the proof of Corollary 1. \square

The solutions employed in the logical independence of Theorem 2 can be used to show the logical independence of the axioms invoked in Corollary 1.

⁵This value was originally introduced by Lowing and Techer (2022a) under the name of multi-choice Equal division value.

7.4. Proof of Theorem 3

To prove Theorem 3, we proceed in two steps. First, let us show that any C -Sharing value satisfies (E), (A), (NO), (P) and (IM). Since $H^C(m, v) \subseteq H^B(m, v)$ for each $(m, v) \in \mathcal{G}$, it is clear that any C -Sharing value satisfies (E), (A), (NO) and (P). It remains to show that any C -Sharing value satisfies (IM).

(IM): Pick any $q \in Q^C$ and consider its associated C -Sharing value φ^q . For each $(m, v) \in \mathcal{G}$, each $i \in N$ and each $j < m_i$,

$$\begin{aligned} \varphi_{ij}^q(m - e_i, v) &= \sum_{s \leq m - e_i} q_{ij}(s) \Delta_v(s) \\ &= \sum_{s \leq m} q_{ij}(s) \Delta_v(s) - \sum_{\substack{s \leq m \\ s_i = m_i}} q_{ij}(s) \Delta_v(s) \\ &= \sum_{s \in \mathcal{M}} q_{ij}(s) \Delta_v(s) - \sum_{\substack{s \leq m \\ s_i = m_i}} 0 \Delta_v(s) \\ &= \varphi_{ij}^q(m, v). \end{aligned}$$

The third equality follows from condition (d) satisfied by any C -sharing systems. This shows that φ^q satisfies (IM).

We have shown that any C -Sharing value satisfies (E), (A), (NO), (P) and (IM). Next, pick any value f satisfying (E), (A), (NO), (P) and (IM). Let us show that f is a C -Sharing value. By Proposition 2, f satisfies (NG). By Theorem 2, any value f satisfying (E), (A), (NO) and (P) is a B -Sharing value. Therefore, f verifies, for any $(m, v) \in \mathcal{G}$,

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v) = \sum_{s \in \mathcal{M}} q_{ij}(s) \Delta_v(s), \quad (14)$$

where q is a B -sharing system, i.e., for any $s \in \mathcal{K}$, $q_{ij}(s) = 0$ if $(i, j) \notin B(s)$. Let us show that q is actually a C -sharing system, i.e., for any $s \in \mathcal{K}$, $q_{ij}(s) = 0$ if $(i, j) \in B(s) \setminus C(s)$. Pick any $s \in \mathcal{K}$. Consider the unanimity game $(m, u_s) \in \mathcal{G}$. Observe that $\Delta_{u_s}(s) = 1$ and $\Delta_{u_s}(t) = 0$ for each $t \neq s$. By (14),

$$f_{ij}(m, u_s) = q_{ij}(s).$$

Pick any pair $(i, j) \in B(s) \setminus C(s)$. Let us show that $q_{ij}(s) = 0$. To that end, let us show that $f_{ij}(m, u_s) = 0$. Since $(i, j) \in B(s) \setminus C(s)$, it holds that $j < s_i$. By (IM), $f_{ij}(m, u_s) = f_{ij}(s - e_i, u_s)$. Observe that $(s - e_i, u_s)$ is a null game. By (NG), $f_{ij}(s - e_i, u_s) = 0$, and thus $f_{ij}(m, u_s) = 0$. This directly leads to the desired result: $q_{ij}(s) = 0$. This concludes the proof of Theorem 3. \square

The axioms invoked in Theorem 3 are logically independent. To see this, observe that the solutions employed to show the logical independence of the axioms invoked in Theorem 2 all satisfy (IM). Thus, they can be used to show the logical independence of the axioms invoked in Theorem 3 aside from (IM). Observe that the KSZ value satisfies all the axioms invoked in Theorem 3 aside from (IM). Thus, the axioms invoked in Theorem 3 are logically independent.

7.5. Proof of Proposition 5

Pick any $(m, v) \in \mathcal{G}$ and any unproductive pair $(i, j) \in M^+$ in (m, v) . If $j = m_i$, then by (N), $f_{ij}(m, v) = 0$. If $j < m_i$, then by successive applications of (IM), $f_{ij}(m, v) = f_{ij}((m_{-i}, j), v)$. Observe that (i, j) is a null pair in $((m_{-i}, j), v)$. By (N), $f_{ij}((m_{-i}, j), v) = 0$, and so $f_{ij}(m, v) = 0$. This concludes the proof of Proposition 5. \square

7.6. Proof of Corollary 2

Pick any $m \in \mathcal{K}$ and consider the sub-class of games \mathcal{G}^m . To prove Corollary 2, we proceed in two steps. Pick any C -Sharing value f . First, let us show that f satisfies (E), (A), (U) and (P) on \mathcal{G}^m . By Theorem 3, we know that any C -Sharing value satisfies (E), (A), (NO), (P) and (IM) on $\mathcal{G}^m \subseteq \mathcal{G}$. By Proposition 4, f satisfies (N). By Proposition 5, f satisfies (U).

We have shown that any C -Sharing value satisfies (E), (A), (U) and (P) on \mathcal{G}^m . Next, pick any value f on \mathcal{G}^m satisfying (E), (A), (U) and (P). Let us show that f is a C -Sharing value. Recall that if f satisfies (U), then it satisfies (N). By Corollary 1, any value f on \mathcal{G}^m satisfying (E), (A), (N) and (P) is a B -Sharing value. Therefore, f verifies, for any $(m, v) \in \mathcal{G}^m$,

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v) = \sum_{s \in \mathcal{M}} q_{ij}(s) \Delta_v(s), \quad (15)$$

where q is a B -sharing system, i.e., for any $s \in \mathcal{K}$, $q_{ij}(s) = 0$ if $(i, j) \notin B(s)$. Let us show that q is actually a C -sharing system, i.e., for any $s \in \mathcal{K}$, $q_{ij}(s) = 0$ if $(i, j) \in B(s) \setminus C(s)$. Pick any $s \in \mathcal{K}$. Consider the unanimity game $(m, u_s) \in \mathcal{G}^m$. Observe that $\Delta_{u_s}(s) = 1$ and $\Delta_{u_s}(t) = 0$ for each $t \neq s$. By (15),

$$f_{ij}(m, u_s) = q_{ij}(s).$$

Pick any pair $(i, j) \in B(s) \setminus C(s)$. Let us show that $q_{ij}(s) = 0$. To that end, let us show that $f_{ij}(m, u_s) = 0$. By Remark 9, any $(i, j) \in B(s) \setminus C(s)$ is an unproductive pair in (m, u_s) . By (U), $f_{ij}(m, u_s) = 0$. This directly leads to the desired result: $q_{ij}(s) = 0$. This concludes the proof of Corollary 2. \square

The same solutions employed to show the logical independence of the axioms invoked in Theorem 3 can be used to show the logical independence of the axioms invoked in Corollary 2.

7.7. Proof of Proposition 6

Let f be a value satisfying (IH). Pick any game $(m, v) \in \mathcal{G}$ and any pair $(i, j) \in M^+$. Consider the two sub-games $((m_{-i}, j), v)$ and $((j \wedge m_k)_{k \in N}, v)$ of (m, v) . By (IH),

$$f_{ij}(m, v) = f_{ij}((j \wedge m_k)_{k \in N}, v). \quad (16)$$

Similarly, by (IH),

$$f_{ij}((m_{-i}, j), v) = f_{ij}((j \wedge m_k)_{k \in N}, v). \quad (17)$$

Combining (16) with (17), one obtains

$$f_{ij}(m, v) = f_{ij}((m_{-i}, j), v),$$

which shows that f satisfies (IM). \square

7.8. Proof of Theorem 4

To prove Theorem 4, we proceed in two steps. First, let us show that any T -Sharing value satisfies (E), (A), (NO), (P) and (IH). Since $H^T(m, v) \subseteq H^C(m, v)$ for each $(m, v) \in \mathcal{G}$, it is clear that any T -Sharing value satisfies (E), (A), (NO) and (P). It remains to show that any T -Sharing value satisfies (IH).

(IH): Pick any $q \in Q^T$ and consider its associated T -Sharing value. For each $(m, v) \in \mathcal{G}$ and each $(i, j) \in M^+$,

$$\begin{aligned} \varphi_{ij}^q((j \wedge m_k)_{k \in N}, v) &= \sum_{s \leq (j \wedge m_k)_{k \in N}} q_{ij}(s) \Delta_v(s) \\ &= \sum_{s \leq m} q_{ij}(s) \Delta_v(s) - \sum_{\substack{s \leq m \\ s \not\leq (j \wedge m_k)_{k \in N}}} q_{ij}(s) \Delta_v(s) \\ &= \sum_{s \in \mathcal{M}} q_{ij}(s) \Delta_v(s) - \sum_{\substack{s \leq m \\ s \not\leq (j \wedge m_k)_{k \in N}}} 0 \Delta_v(s) \\ &= \varphi_{ij}^q(m, v). \end{aligned}$$

The third equality follows from condition (e) satisfied by any T -sharing systems. This shows that φ^q satisfies (IH).

We have shown that any T -Sharing value satisfies (E), (A), (NO), (P) and (IH). Next, pick any value f satisfying (E), (A), (NO), (P) and (IH). Let us show that f is a T -Sharing value. By Proposition 2, f satisfies (NG). By Theorem 2, any value f satisfying (E), (A), (NO) and (P) is a B -Sharing value. Therefore, f verifies, for any $(m, v) \in \mathcal{G}$,

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v) = \sum_{s \in \mathcal{M}} q_{ij}(s) \Delta_v(s), \quad (18)$$

where q is a B -sharing system, i.e., for any $s \in \mathcal{K}$, $q_{ij}(s) = 0$ if $(i, j) \notin B(s)$. Let us show that q is actually a T -sharing system, i.e., for any $s \in \mathcal{K}$, $q_{ij}(s) = 0$ if $(i, j) \in B(s) \setminus T(s)$. Pick any $s \in \mathcal{K}$. Consider the unanimity game $(m, u_s) \in \mathcal{G}$. Observe that $\Delta_{u_s}(s) = 1$ and $\Delta_{u_s}(t) = 0$ for each $t \neq s$. By (18),

$$f_{ij}(m, u_s) = q_{ij}(s).$$

Pick any pair $(i, j) \in B(s) \setminus T(s)$. Let us show that $q_{ij}(s) = 0$. To that end, let us show that $f_{ij}(m, u_s) = 0$. Since $(i, j) \in B(s) \setminus T(s)$, it holds that $j < \max_{i \in N} s_i$. By (IH), $f_{ij}(m, u_s) = f_{ij}((j \wedge m_k)_{k \in N}, u_s)$. Observe that $((j \wedge m_k)_{k \in N}, u_s)$ is a null game. By (NG), $f_{ij}((j \wedge m_k)_{k \in N}, u_s) = 0$, and thus $f_{ij}(m, u_s) = 0$. We directly obtain the desired result: $q_{ij}(s) = 0$. This concludes the proof of Theorem 4. \square

The axioms invoked in Theorem 4 are logically independent. To see this, observe that all the alternative solutions used to show the logical independence of the axioms invoked in Theorem 2 satisfy (IH), and thus can be used to show the independence of the axioms invoked in Theorem 4. Moreover, observe that the PZ value satisfies all the axioms invoked in Theorem 4 aside from (IH). Thus, the axioms invoked in Theorem 4 are logically independent.

7.9. Proof of Proposition 7

Pick any $(m, v) \in \mathcal{G}$. Assume there exists an activity level $j \leq \max_{i \in N} m_i$ such that

$$v((j \wedge m_i)_{i \in N}) = 0. \quad (19)$$

Pick any solution f on \mathcal{G} that satisfies (E) and (IH). According to Proposition 2 in Lowing and Techer (2022a),

$$\sum_{(i', j') \in B((j \wedge m_i)_{i \in N})} f_{i' j'}(m, v) = v((j \wedge m_i)_{i \in N}). \quad (20)$$

The desired result follows from (19) and (20). This concludes the proof of Proposition 7. \square

7.10. Proof of Corollary 3

Pick any $m \in \mathcal{K}$ and consider the sub-class of games \mathcal{G}^m . To prove Corollary 3, we proceed in two steps. Pick any T -Sharing value f . First, let us show that f satisfies (E), (A), (N), (NS) and (P) on \mathcal{G}^m . By Theorem 4, we know that f satisfies (E), (A), (NO), (IH) and (P) on $\mathcal{G}^m \subseteq \mathcal{G}$. By Proposition 4, f satisfies (N) on $\mathcal{G}^m \subseteq \mathcal{G}$. By Proposition 7, f satisfies (NS) on $\mathcal{G}^m \subseteq \mathcal{G}$.

We have shown that any T -Sharing value satisfies (E), (A), (N), (NS) and (P) on \mathcal{G}^m . Next, pick any value f on \mathcal{G}^m satisfying (E), (A), (N), (NS) and (P). Let us show that f is a T -Sharing value.

By Corollary 1, any value f on \mathcal{G}^m satisfying (E), (A), (N) and (P) is a B -Sharing value. Therefore, f verifies, for any $(m, v) \in \mathcal{G}^m$,

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v) = \sum_{s \in \mathcal{M}} q_{ij}(s) \Delta_v(s), \quad (21)$$

where q is a B -sharing system, i.e., for any $s \in \mathcal{K}$, $q_{ij}(s) = 0$ if $(i, j) \notin B(s)$. Let us show that q is actually a T -sharing system, i.e., for any $s \in \mathcal{K}$, $q_{ij}(s) = 0$ if $(i, j) \in B(s) \setminus T(s)$. Pick any $s \in \mathcal{K}$. Consider the unanimity game $(m, u_s) \in \mathcal{G}^m$. Observe that $\Delta_{u_s}(s) = 1$ and $\Delta_{u_s}(t) = 0$ for each $t \neq s$. By (21),

$$f_{ij}(m, u_s) = q_{ij}(s).$$

Pick any pair $(i, j) \in B(s) \setminus T(s)$. Let us show that $q_{ij}(s) = 0$. To that end, let us show that $f_{ij}(m, u_s) = 0$. Observe that $u_s((j \wedge m_i)_{i \in N}) = 0$. By (NS),

$$\sum_{(i', j') \in B((j \wedge m_i)_{i \in N})} f_{i' j'}(m, u_s) = 0.$$

The Harsanyi dividends of the MC-game (m, u_s) are all positive. Hence, by (P), $f_{i' j'}(m, u_s) \geq 0$ for each $(i', j') \in B((j \wedge m_i)_{i \in N})$. Therefore, $f_{i' j'}(m, u_s) = 0$ for each $(i', j') \in B((j \wedge m_i)_{i \in N})$. In particular, $f_{ij}(m, u_s) = 0$. We have shown that $f_{ij}(m, u_s) = 0$ for each $(i, j) \in B(s) \setminus T(s)$. This directly leads to the desired result: $q_{ij}(s) = 0$. This concludes the proof of Corollary 3. \square

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