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A mass scale law connecting cosmophysics to microphysics

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We introduce different mass scales corresponding to the Universe, fermion stars, mini boson stars, Planck black holes, the electron, the neutrino, and the cosmon. These mass scales are obtained by combining the maximum mass of fermion stars, boson stars and soliton stars set by general relativity with the Eddington relation connecting the mass m_e of the electron to the cosmological constant Λ . In this manner, we can express the mass of these objects in terms of the fundamental constants of physics G , c , \hbar and Λ . By normalizing the mass M_a of these objects by the Planck mass M_P , we find that $M_a/M_P \sim \chi^{a/6}$, where $\chi \sim \rho_P/\rho_\Lambda \sim 10^{120}$ is the “largest large number” in Nature (the ratio of the Planck density on the cosmological density) and $a = 3, 2, 1, 0, -1, -2, -3$ for the Universe, fermion stars, mini boson stars, Planck black holes, the electron, the neutrino, and the cosmon respectively. This formula suggests an interesting symmetrical mass scale law connecting cosmophysics (Universe, fermion stars, mini boson stars) to microphysics (electron, neutrino, cosmon). A generalization of this law including the earth mass and another neutrino mass scale is proposed. We also highlight an accurate form of Eddington relation $m_e \simeq \alpha(\Lambda\hbar^4/G^2)^{1/6}$ or $\Lambda \simeq G^2 m_e^6 / \alpha^6 \hbar^4 = \alpha^{-6} (m_e/M_P)^6 l_P^{-2}$, where $\alpha = e^2/\hbar c \simeq 1/137$ is the fine structure constant, given in a previous paper [P.H. Chavanis, Phys. Dark Univ. **24**, 100271 (2019)] and provide different heuristic derivations of this relation. We suggest that the dark energy particle (cosmon) of mass $m_\Lambda = \hbar\sqrt{\Lambda}/c$ could be a quantum of mass and entropy and that $\Lambda = (m_\Lambda/M_P)^2 l_P^{-2}$. Finally, we give hints how to possibly solve the cosmological constant problem, the cosmic coincidence problem, and the large number coincidence problem.

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I. INTRODUCTION

Physics has revealed that Nature follows a kind of organization which can be expressed in terms of simple mathematical laws. Several physicists have attempted to determine the mass of celestial bodies and elementary particles in terms of the fundamental constants of physics (the gravitational constant $G = 6.67 \times 10^{-14} \text{ m}^3 \text{ g}^{-1} \text{ s}^{-2}$, the speed of light $c = 3.00 \times 10^8 \text{ m s}^{-1}$, the Planck constant $\hbar = 1.05 \times 10^{-31} \text{ m}^2 \text{ g s}^{-1}$ and the cosmological constant $\Lambda = 1.11 \times 10^{-52} \text{ m}^{-2}$). This ambitious project was pioneered by Eddington [1] who tried to deduce the value of all the constants of physics (e.g. the fine structure constant, the electron mass, the proton mass...) from theoretical considerations. This project included as a crucial assumption that atomic and cosmic constants are related. In that respect, introducing together with Weyl [2] and Dirac [3, 4] the notion of large dimensionless numbers, Eddington [5–7] tried to link the properties of the smallest particles to those of the whole universe. He obtained an intriguing formula,

$$\Lambda = \left(\frac{m_e}{\alpha\hbar}\right)^4 \left(\frac{2Gm_p}{\pi}\right)^2, \quad (1)$$

relating the cosmological constant Λ to the mass of the electron m_e , the mass of the proton m_p and the fine-structure constant $\alpha = e^2/\hbar c \simeq 1/137$. This relation expresses a link between cosmological and atomic constants, i.e., between the parameters of macro and microphysics that at first sight seem unconnected. The qualitative approach of Eddington relying sometimes on very obscure arguments was very much criticized by contem-

porary physicists such as Pauli and Born but it is fair to say that Eddington unraveled remarkable connections between particle physics and cosmology that remain mysterious at this day and possibly point towards a future theory of quantum gravity.¹ The present paper pursues his initial goal with, however, a less ambitious objective.

We introduce different mass scales corresponding to the Universe, fermion stars, mini boson stars, Planck black holes, the electron, the neutrino, and the cosmon. These mass scales are obtained from simple arguments (close to dimensional analysis) by combining the maximum mass of fermion stars, boson stars and soliton stars set by general relativity with the Eddington relation. In this manner, we can express the mass of these objects in terms of the fundamental constants of physics G , c , \hbar and Λ . By normalizing the mass M_a of these objects by the Planck mass $M_P = (\hbar c/G)^{1/2}$, we find that

$$\frac{M_a}{M_P} \sim \chi^{a/6}, \quad (2)$$

where

$$\chi = \frac{\rho_P}{\rho_\Lambda} = \frac{c^3}{G\hbar\Lambda} \sim 10^{120} \quad (3)$$

is the “largest large number” in Nature (the fundamental ratio between the Planck density $\rho_P = c^5/G^2\hbar = 5.16 \times$

¹ As noted by Jordan [8] “It is an outstanding merit of Eddington that he emphasized that we should never believe in meaningless coincidences”.

10^{99} g m^{-3} and the cosmological density $\rho_\Lambda = \Lambda c^2/G = 1.50 \times 10^{-22} \text{ g m}^{-3}$) and $a = 3, 2, 1, 0, -1, -2, -3$ for the Universe, fermion stars, mini boson stars, Planck black holes, the electron, the neutrino, and the cosmon respectively. This formula suggests an interesting mass scale law connecting cosmophysics (Universe, fermion stars, mini boson stars) to microphysics (electron, neutrino, cosmon). The Planck mass ($a = 0$) makes a clear separation between these two groups of objects (macroscopic and microscopic). Furthermore, this separation is symmetrical: Celestial bodies correspond to $a > 0$ (namely $a = 1, 2, 3$) and elementary particles correspond to $a < 0$ (namely $a = -1, -2, -3$). We note, however, that our mass scale law presents an anomaly. General arguments predict a neutrino mass scale corresponding to $a = -3/2$ instead of -2 . This suggests that there exist several neutrino mass scales. By using a reciprocity property, we introduce a cosmic body corresponding to $a = 3/2$. Its mass scale turns out to be of the order of the earth mass. These considerations provide a generalization of our original mass scale law.

The object of this contribution is to highlight this mass scale law. We also explain in Appendix A how we came to these considerations by studying the connection between the universal surface density of dark matter halos and the surface density of the electron (this connection may be related to the holographic principle) [9–12]. More details and references (retracing in particular the rich history of the Eddington relation) will be given in a forthcoming paper [13].

II. THE EDDINGTON RELATION

At the beginning of the 20th century, several eminent scientists noticed the occurrence of large dimensionless numbers in Nature and were intrigued by certain coincidences and some mysterious relations between atomic and cosmic constants.

Weyl [2] observed that the ratio of the electric force e^2/r^2 to the gravitational force $Gm_p m_e/r^2$ between the proton and the electron is equal to

$$F = \frac{e^2}{Gm_e m_p} \sim 10^{40}. \quad (4)$$

We will call it the Weyl number. This large number shows that gravitational forces are remarkably weak by the standards of atomic and nuclear physics.²

² In our qualitative discussion, we will consider the fine-structure constant $\alpha = e^2/\hbar c \simeq 1/137$ and the ratio of the mass of the proton to that of the electron $\mu = m_p/m_e \simeq 1836$ to be of order unity. As a result, the Weyl number (force ratio) can be written as $F \sim e^2/Gm_e^2 \sim \hbar c/Gm_e^2 \sim 1/\alpha_g$ where $\alpha_g = Gm_e^2/\hbar c = (m_e/M_P)^2 \sim 10^{-40}$ is the gravitational coupling constant. The force ratio is of order unity ($F \sim 1$) when $Gm^2 \sim e^2 \sim \hbar c$, i.e.,

Eddington [14–16] noticed that the number of protons in the Einstein universe of mass $M_E = \pi c^2/(2G\sqrt{\Lambda})$ [17] is of order

$$N_p = \frac{M_E}{m_p} \sim 10^{80}. \quad (5)$$

This is called the Eddington number. Eddington [15] argued that the two large numbers (4) and (5) are fundamentally related by

$$F \sim \sqrt{N_p}. \quad (6)$$

In Eddington's approach, this relation explains why F is large as it is related to the number of protons in the universe which is obviously large. By combining Eqs. (4)-(6) and taking $m_p \sim m_e$ and $e^2 \sim \hbar c$ (see footnote 2), one obtains a formula,

$$\Lambda \sim \frac{G^2 m_e^6}{\hbar^4}, \quad (7)$$

relating the cosmological constant Λ to the mass of the electron m_e . This is the so-called Eddington relation. This relation expresses a link between cosmological and atomic constants, i.e., between the parameters of macrophysics and microphysics. It links the properties of the smallest particles to those of the whole universe. In Refs. [5–7], Eddington tried to motivate this relation from more physical (albeit obscure) considerations based on the Dirac and the Einstein equations.³

Dirac [3, 4] developed a cosmological theory based on a large number hypothesis. He had the idea to measure the age of the universe $t_0 \sim t_H^{(0)} = 1/H_0$ (Hubble time) in terms of a unit of time $t_a = e^2/(m_e c^3)$ provided by the atomic constants,⁴ and he obtained the large dimensionless number

$$\frac{t_0}{t_a} = \frac{t_H^{(0)}}{e^2/(m_e c^3)} \sim 10^{40}. \quad (9)$$

This number characterizes the present epoch in a natural way, independent of man-made standards. Dirac

when $m \sim M_P$. At the Planck scale we have the connection of gravity, electromagnetism and quantum mechanics.

³ The Eddington relation (7) was re-derived later by Zel'dovich [18] in the context of quantum field theory. He wrote the vacuum energy density $\rho_\Lambda c^2 = \Lambda c^4/G$ under the form

$$\rho_\Lambda c^2 \sim \frac{Gm_e^2}{\lambda_e} \times \frac{1}{\lambda_e^3}, \quad (8)$$

where $\lambda_e = \hbar/(m_e c)$ is the Compton wavelength of an elementary particle (\sim electron). This expression assumes that the vacuum contains virtual pairs of particles with effective density $n_e \sim 1/\lambda_e^3$ and that these pairs have a gravitational energy of interaction Gm_e^2/λ_e . One can check that Eq. (8) is equivalent to the Eddington relation (7).

⁴ Taking $e^2 \sim \hbar c$, this timescale is also of the order of $\hbar/m_e c^2$ as provided by the Heisenberg uncertainty principle $t_a \times m_e c^2 \sim \hbar$.

proposed that all the very large dimensionless numbers which can be constructed from the important natural constants of cosmology and atomic theory are connected by simple mathematical relations involving coefficients of the order of magnitude unity. Dirac's large number hypothesis is motivated by certain scaling laws that relate the parameters of macro and microphysics. Comparing Eqs. (5) and (9) we see that

$$N_p \sim \left(\frac{t_0}{t_a}\right)^2. \quad (10)$$

Comparing Eqs. (4) and (9) we see that

$$\frac{e^2}{Gm_em_p} \sim \frac{t_0}{t_a}. \quad (11)$$

These relations depend on the age t_0 of the universe. Therefore, in Dirac's theory, these dimensionless parameters are large simply because the universe is old. Furthermore, these relations seem to be valid only at the present epoch t_0 ("now"). In order to solve this large number coincidence problem, Dirac argued that the physical constants (like the gravitational constant G or the number of protons N_p) vary with time so that relations (10) and (11) are always valid. Therefore, the number of protons in the universe must be increasing proportionally to t^2 (implying a continuous creation of matter) and the gravitational 'constant' must decrease with time proportionally to t^{-1} . However, the variation $G \sim t^{-1}$ predicted by Dirac was found to be in strong conflict with observations [19] and precise measurements [20], so his theory was abandoned.

In his famous monograph *Gravitation and Cosmology*, Weinberg [21] reported an empirical relation of the form

$$m_e \sim \left(\frac{H_0 \hbar^2}{Gc}\right)^{1/3}, \quad (12)$$

where $H_0 = 2.195 \times 10^{-18} \text{ s}^{-1}$ is the present value of the Hubble parameter, whose inverse gives the typical age of the universe. This relation expresses the mass of the electron as a function of the Hubble constant. Weinberg considered this relation as "so far unexplained" and having "a real though mysterious significance". This relation is clearly related to the Eddington relation (7) which can be rewritten as

$$m_e \sim \left(\frac{\Lambda \hbar^4}{G^2}\right)^{1/6}. \quad (13)$$

However, the two relations (12) and (13) are not exactly the same because the Eddington relation (13) involves Λ while the Weinberg relation (12) involves H_0 . The Weinberg relation is disturbing because it involves the *present* value of the Hubble parameter H_0 which determines the age of the universe. Since the Hubble parameter $H(t)$ changes with time (it decreases as t^{-1} in the matter era) this suggests that the Weinberg relation is

valid only "now". This seems to give to our present epoch a special place in the history of the universe.

One way to solve this coincidence problem is to invoke Dirac's theory according to which Newton's gravitational constant G changes with time as $G \sim 1/t$. Since $H \sim 1/t$ (in the matter era), the Weinberg relation would be valid at any time, not just now. However, no observation has evidenced the slightest variation of G and Dirac's theory of variation of fundamental constants is well known to lead to many problems. Alternatively, Dicke [22] argued that our epoch is particular and that the relation (12) is valid only at the present time. Written under the form

$$t_0 \sim \frac{1}{H_0} \sim \frac{\hbar^2}{Gm_e^3 c}, \quad (14)$$

it determines the present Hubble constant, hence the age of the universe $t_0 \sim 1/H_0$, as a function of the electron mass m_e and the fundamental constants of physics \hbar , G and c .⁵ Dicke then developed an anthropic argument based on stellar physics explaining why the age of the universe should have this order of magnitude. Therefore, the anthropic principle may qualitatively explain the Weinberg relation (12). No hypothesis of varying constants is necessary to explain it. In this sense, the relation (12) is not as mysterious as Weinberg thought. However, the Eddington and the Weinberg relations may have a different status. In Appendix D and in our companion paper [13] we discuss their interrelation in connection to the cosmic coincidence problem of modern cosmology (Λ CDM model).

III. PLANCK DENSITY AND COSMOLOGICAL DENSITY

In cosmology, there are two important density scales that can be constructed in terms of the fundamental constants of physics G , c , \hbar and Λ . These are the Planck density

$$\rho_P = \frac{c^5}{G^2 \hbar} = 5.16 \times 10^{99} \text{ g m}^{-3}, \quad (15)$$

and the cosmological density⁶

$$\rho_\Lambda = \frac{\Lambda c^2}{G} = 1.50 \times 10^{-22} \text{ g m}^{-3}. \quad (16)$$

The Planck density is very large. It is expected to be important in the early universe during the phase of inflation.

⁵ Note that the Weinberg relation (12) is implicitly contained in the Dirac relations (9)-(11). This is why Dicke [22] came to discuss these problems in relation to Dirac's theory [3, 4] before Weinberg [21] explicitly wrote the relation (12).

⁶ The cosmological density is usually defined as $\rho_\Lambda = \Lambda c^2 / (8\pi G) = 5.96 \times 10^{-24} \text{ g m}^{-3}$. Here, we define it without the 8π in order to avoid the presence of numerical factors in the expressions of the fundamental scales.

The cosmological density is very small. It is expected to be important in the late universe. It may represent a form of dark energy explaining the accelerated expansion of the universe that we observe today. The ratio between the Planck density and the cosmological density is huge

$$\chi = \frac{\rho_P}{\rho_\Lambda} = \frac{c^3}{G\hbar\Lambda} \sim 10^{120}. \quad (17)$$

They differ by about 120 orders of magnitude.⁷ The fact that the observed value of the cosmological density ρ_Λ is 120 orders of magnitude smaller than the Planck density ρ_P leads to the so-called ‘‘cosmological constant problem’’ [23, 24]. This problem occurs when the cosmological constant is interpreted as the vacuum energy. However, if we interpret Λ as a fundamental constant of physics, there may not be any problem. Indeed, it is quite natural that ρ_P and ρ_Λ differ by 120 orders of magnitude if the Planck density represents the density of the universe in the far past and if the cosmological density represents the density of the universe in the far future. This is, for example, the outcome of a simple cosmological model based on a quadratic equation of state [25–28] describing the evolution of the Universe from an early de Sitter era characterized by the Planck density ρ_P to a late de Sitter era characterized by the cosmological density ρ_Λ (see in particular Fig 16 of [27]). This universe has no beginning and no end, i.e., it exists eternally in the past and in the future and does not display any singularity (aioniotic universe). This model leads furthermore to an extremely symmetric evolution, the cosmological constant Λ in the late universe playing the role of the Planck constant \hbar in the early universe. In Appendix B and in our companion paper [13], we motivate from entropic considerations (based on the holographic principle) the fact that 10^{120} is probably the largest dimensionless number in Nature from which we can deduce all the other ‘‘large numbers’’ found by previous investigators.

A long time ago, motivated by Dirac’s paper [3], Chandrasekhar [29] discussed some coincidences that he had observed based on dimensional arguments. With the Planck mass M_P and the proton mass m_p , he formed the following combination

$$M_a = \frac{M_P^{a+1}}{m_p^a}, \quad (18)$$

which has the dimension of a mass, where a is an arbitrary numerical constant (we have used a slightly different notation). He then considered particular values of a in relation to the size of the Sun ($M_a = 1.82 M_\odot$ for $a = 2$), the mass of the galaxies ($M_a = 6.68 \times 10^9 M_\odot$ for $a = 5/2$), and the mass of the Universe ($M_a =$

$2.41 \times 10^{19} M_\odot$ for $a = 3$). We shall take up his approach in a slightly different angle and generalize it. Since we are interested by orders of magnitude, we shall disregard the difference of mass between the electron and the proton (a factor ~ 1000) and replace m_p by m_e in the foregoing formula.⁸ Then, using the Eddington relation (13), we can express M_a in terms of the fundamental constants of physics as

$$M_a = \frac{\hbar^{(3-a)/6} c^{(a+1)/2}}{\Lambda^{a/6} G^{(3+a)/6}}. \quad (19)$$

Normalizing the mass M_a by the Planck mass M_P we obtain the ratio

$$\frac{M_a}{M_P} = \chi^{a/6} \sim (10^{20})^a, \quad (20)$$

where χ is the ‘‘largest large number’’ in Nature defined by Eq. (17).

Remark: The gravitational (Schwarzschild) radius $R_S = 2GM/c^2$ and the Compton radius $\lambda_C = \hbar/Mc$ of an object of mass M normalized by the Planck length l_P can be expressed in terms of M/M_P as

$$\frac{R_S}{l_P} = \frac{2M}{M_P}, \quad \frac{\lambda_C}{l_P} = \frac{M_P}{M}. \quad (21)$$

We have the general relations

$$\frac{\lambda_C}{R_S} = \frac{M_P^2}{2M^2}, \quad R_S \lambda_C = 2l_P^2. \quad (22)$$

Knowing M/M_P in terms of χ , we can easily get R_S/l_P and λ_C/l_P in terms of χ . In particular, using Eqs. (19) and (20), we get

$$\frac{R_{S,a}}{2} = \frac{\hbar^{(3-a)/6} G^{(3-a)/6}}{\Lambda^{a/6} c^{(3-a)/2}}, \quad \frac{R_{S,a}}{2l_P} = \chi^{a/6}, \quad (23)$$

$$\lambda_{C,a} = \frac{\Lambda^{a/6} \hbar^{(3+a)/6} G^{(3+a)/6}}{c^{(3+a)/2}}, \quad \frac{\lambda_{C,a}}{l_P} = \chi^{-a/6}. \quad (24)$$

IV. FUNDAMENTAL MASS SCALES OF MICRO AND MACRO OBJECTS

In this section, we introduce fundamental mass scales of micro and macro objects encountered in our previous investigations [11, 30] and show that they correspond to a simple series of mass M_a where a is an integer ranging from -3 to $+3$.

⁷ The relation (17) can also be written $\Lambda = 1/(\chi l_P^2) \sim 10^{-120} l_P^{-2}$, where $l_P = (G\hbar/c^3)^{1/2}$ is the Planck length.

⁸ The dimensionless large numbers that we manipulate are so enormously large that these distinctions are of little consequence. Only orders of magnitude are significant.

A. The Planck mass ($a = 0$)

In 1899, Planck [31] introduced a fundamental constant of physics \hbar which governs the laws of quantum mechanics. He also constructed units of mass, length and time based on G (gravity), c (electromagnetism) and \hbar (quantum mechanics). These are the Planck scales. The Planck density is $\rho_P = c^5/(G^2\hbar) = 5.16 \times 10^{99} \text{ g m}^{-3}$, the Planck time is $t_P = 1/(G\rho_P)^{1/2} = (\hbar G/c^5)^{1/2} = 5.39 \times 10^{-44} \text{ s}$, the Planck length is $l_P = ct_P = (G\hbar/c^3)^{1/2} = 1.62 \times 10^{-35} \text{ m}$, and the Planck mass is $M_P = \rho_P l_P^3 = (\hbar c/G)^{1/2} = 2.18 \times 10^{-5} \text{ g}$. It is expected that quantum mechanics plays a fundamental role in the early universe where the scale factor is small and the density is high. Therefore, the Planck scales are expected to be important in the primordial universe. In particular, the Planck time gives the typical duration of the inflation and the Planck length is the size at which quantum effects become important for understanding the physics of the early universe. At that scale, general relativity breaks down and we need to develop a theory of quantum gravity. We note that the Planck density is extremely large while the Planck length and the Planck time are very small. By contrast, the Planck mass

$$M_P = \left(\frac{\hbar c}{G}\right)^{1/2} = 2.18 \times 10^{-5} \text{ g}, \quad (25)$$

i.e., $M_P = 1.22 \times 10^{19} \text{ GeV}/c^2$ is neither very large nor very small. It is at an intermediate scale which is commensurate with human scales. It separates macrophysics from microphysics (see below).

The Planck mass is equal to the mass of an object whose Compton wavelength $\lambda_C = \hbar/mc$ is equal to its semi gravitational (or semi Schwarzschild) radius $R_S/2 = Gm/c^2$. These in turn are equal to the Planck length:

$$\lambda_C = \frac{\hbar}{M_P c} = \frac{R_S}{2} = \frac{GM_P}{c^2} = l_P = 1.62 \times 10^{-35} \text{ m}. \quad (26)$$

Therefore, the Planck scales satisfy the relation $l_P = GM_P/c^2$, similar to the Schwarzschild radius. The Planck mass M_P may correspond to the mass of the primordial universe, when it had a size $\sim l_P$. It also characterizes a Planck black hole of radius $2l_P$. In principle, this black hole should evaporate by Hawking radiation [32] on a timescale of the order of the Planck time t_P . However, it is conceivable that a Planck black hole of mass M_P and radius $2l_P$ represents the ground state of a pure quantum black hole (see Appendix B). As a result, it is fully stable and does not evaporate. It is natural (and conventional) to take the Planck mass as a mass of reference. This is why we have normalized the mass M_a by M_P in Eq. (20).

Remark: Equilibrium states of nonrelativistic bosons stars with an attractive $|\varphi|^4$ self-interaction (axions) exist only below a maximum mass [33–35]

$$M_{\text{max}}^{\text{NR}} = 5.07 \frac{M_P}{\sqrt{|\lambda|}}. \quad (27)$$

This is the maximum mass of dilute axion stars [36]. Above that mass, the star collapses towards a dense axion star [37] or a black hole [38], or explodes in a bosonova by emitting relativistic axions [39]. For $|\lambda| \sim 1$, the maximum mass from Eq. (27) is of the order of the Planck mass (25). This may represent an elementary particle of mass M_P (“planckion”) or the mass of inflaton stars [40]. However, in the case of axions, $|\lambda|$ can be as small as 10^{-100} (!) leading to much larger objects [41]. For the QCD axion with $\lambda \sim -7.39 \times 10^{-49}$, we obtain a mass $M_{\text{max}}^{\text{NR}} = 6.46 \times 10^{-14} M_\odot$ of the order of the mass of asteroids. QCD axion stars are thus called “axteroids”. For ultralight axions with $\lambda = -1.18 \times 10^{-96}$, we obtain a mass $M_{\text{max}}^{\text{NR}} = 5.10 \times 10^{10} M_\odot$ of the order of the mass of the galaxies. The corresponding axion “stars” may represent ultracompact dwarf spheroidals or the quantum core (soliton) of DM halos.

B. The mass of the universe ($a = 3$)

In 1917, Einstein [17] introduced a cosmological constant Λ in his equations of general relativity in order to have a static universe. After the discovery of the expansion of the universe, he considered the cosmological constant as his “biggest blunder” [42] and banished it.⁹ However, the importance of the cosmological constant was revived with the discovery of the present accelerating expansion of the universe [43–46]. The cosmological constant could be the source of the dark energy fueling this acceleration. Therefore, it may play a fundamental role in the late universe where the scale factor is large and the density is low. We shall regard the cosmological constant Λ as a fundamental constant of physics that describes the late universe (cosmophysics) in the same manner that the Planck constant \hbar describes the early universe (microphysics). From the cosmological constant Λ , the speed of light c , and the constant of gravity G , we can construct a mass scale

$$M_\Lambda = \frac{c^2}{G\sqrt{\Lambda}} = 1.28 \times 10^{56} \text{ g}, \quad (28)$$

i.e., $M_\Lambda = 6.43 \times 10^{22} M_\odot$. It corresponds to the Einstein mass $M_E = \pi c^2/(2G\sqrt{\Lambda})$ in his static model of universe [17]. In the standard Λ CDM model, M_Λ represents the typical mass of the present universe. Indeed, by using general arguments based on physical considerations and dimensional analysis, we can introduce cosmological scales. The cosmological density $\rho_\Lambda = \Lambda c^2/G = 1.50 \times 10^{-22} \text{ g m}^{-3}$ is of the order of the density of the universe, the cosmological time $t_\Lambda = 1/(G\rho_\Lambda)^{1/2} = 1/(c\sqrt{\Lambda}) = 3.16 \times 10^{17} \text{ s}$ is of the order of the age of the universe, the

⁹ In contrast, other authors such as Lemaitre and Eddington were more cautious and considered the cosmological constant as a fundamental ingredient of general relativity and cosmology [25].

cosmological length $R_\Lambda = ct_\Lambda = 1/\sqrt{\Lambda} = 9.49 \times 10^{25}$ m is of the order of the size of the visible universe (the distance travelled by a photon on a timescale t_Λ), and the cosmological mass $M_\Lambda = \rho_\Lambda R_\Lambda^3 = c^2/(G\sqrt{\Lambda}) = 1.28 \times 10^{56}$ g is of the order of the mass of the universe.¹⁰ In astronomical units, $t_\Lambda = 10.0$ Gyrs, $R_\Lambda = 3.07$ Gpc and $M_\Lambda = 6.42 \times 10^{22} M_\odot$. As indicated above, we consider that Λ is a fundamental constant of physics like \hbar . Therefore, the cosmological scales ρ_Λ , t_Λ , l_Λ , M_Λ are fundamental scales determined by Λ in the same sense that the Planck scales ρ_P , t_P , l_P , M_P are fundamental scales determined by \hbar (and the other constants of physics G and c). We note that the cosmological scales satisfy the relation $R_\Lambda = GM_\Lambda/c^2$, similar to the Schwarzschild radius. This suggests that the Universe may be a huge black hole inside which we live (this idea is further developed in [13]). The evaporation time of a black hole due to Hawking radiation is $t_{\text{evap}} \sim G^2 M^3/\hbar c^4$ [32]. The evaporation time of the Universe viewed as a huge black hole is $t_{\text{evap}}^\Lambda \sim c^2/(G\hbar\Lambda^3) \sim 10^{217}$ s. The relation $R_\Lambda \sim GM_\Lambda/c^2$ can also be written as $M_\Lambda c^2 \sim GM_\Lambda^2/R_\Lambda$ implying that the total energy of the universe $E_\Lambda = M_\Lambda c^2 - GM_\Lambda^2/R_\Lambda$ (rest mass + gravitational) is equal to zero.¹¹

Lee [47] has introduced the notion of soliton stars and showed that general relativity determines a maximum mass above which a soliton star collapses towards a black hole. This maximum mass is of order

$$M_{\text{max}}^S \sim \left(\frac{\hbar c}{G}\right)^2 \frac{1}{m^3} \sim \frac{M_P^4}{m^3}. \quad (29)$$

We will call it the Lee mass. If we replace m by m_e and use the Eddington relation (13), we obtain the mass of the universe M_Λ [see Eq. (28)]:

$$M_\Lambda = \frac{M_P^4}{m_e^3} = \frac{c^2}{G\sqrt{\Lambda}} = 1.28 \times 10^{56} \text{ g}. \quad (30)$$

In this sense, the Universe can be viewed as a huge soliton star made of particles of mass $\sim m_e$. Of course, this interpretation is too naive and oversimplified but it allows us to construct fundamental mass scales of physical interest. We will use similar qualitative arguments below to construct other fundamental mass scales.

¹⁰ These results can be derived from the Friedmann equations [13] by using the fact that the present density of the universe is of the order of the cosmological density on account of the cosmic coincidence (see Appendix D).

¹¹ It is possible that the relation $M = Rc^2/2G$ is valid at each time t , so the universe has always been a black hole [13]. Since the radius of the visible universe is $R = ct$, the mass of the universe increases with time as $M = c^3 t/2G$ in agreement with Milne's model [48]. It was of the order M_P at the Planck time t_P and is of the order M_Λ now at $t \sim t_\Lambda$. The mass of the universe is not conserved but the total energy $Mc^2 - GM^2/R = 0$ is conserved (it vanishes). This is a form of virial theorem $v^2 \sim GM/R$ with $v \sim c$. This is also connected to the Mach principle [13].

The semi gravitational radius of the universe is

$$R_\Lambda = \frac{GM_\Lambda}{c^2} = \frac{1}{\sqrt{\Lambda}} = 9.49 \times 10^{25} \text{ m}. \quad (31)$$

It is of the order of its size. The Compton wavelength of the universe is

$$\lambda_\Lambda = \frac{\hbar}{M_\Lambda c} = \frac{G\hbar\sqrt{\Lambda}}{c^3} = 2.75 \times 10^{-96} \text{ m}. \quad (32)$$

If we normalize the mass of the Universe by the Planck mass we obtain the large dimensionless number

$$\frac{M_\Lambda}{M_P} = \chi^{1/2} \sim 10^{60}, \quad (33)$$

which corresponds to Eq. (20) with $a = 3$. On the other hand, the ratio of the Compton wavelength of the universe on its semi gravitational radius scales as

$$\frac{\lambda_\Lambda}{R_\Lambda} = \frac{M_P^2}{M_\Lambda^2} \sim \chi^{-1} \sim 10^{-120}. \quad (34)$$

Remark: If we define the gravitational Bohr radius by

$$a'_B = \frac{\hbar^2}{Gm^3}, \quad (35)$$

take $m = m_e$, and use the Eddington relation (13), we find that $a'_B \sim 1/\sqrt{\Lambda} \sim R_\Lambda$. Therefore, the gravitational Bohr radius associated with a particle of mass m_e is of the same order of magnitude as the radius of the universe. This is because the typical radius $R_* \sim GM_{\text{max}}^S/c^2$ of a soliton star is equal to the gravitational Bohr radius: $R_* \sim a'_B$.¹² Conversely, if we suppose $R_\Lambda \sim a'_B$, we obtain the Eddington relation (13).

C. The mass of the electron ($a = -1$)

According to the Eddington relation (13), we can express the mass of the electron in terms of the cosmological constant as¹³

$$m_e \sim \left(\frac{\Lambda\hbar^4}{G^2}\right)^{1/6} \sim 1.21 \times 10^{-25} \text{ g}, \quad (36)$$

i.e., $m_e \sim 6.77 \times 10^7$ eV/ c^2 . As we showed previously, this mass scale can also be obtained by identifying the Lee mass (29) with the mass of the Universe (28) yielding

$$m_e \sim \left(\frac{M_P^4}{M_\Lambda}\right)^{1/3} \sim \left(\frac{\Lambda\hbar^4}{G^2}\right)^{1/6} \sim 1.21 \times 10^{-25} \text{ g}. \quad (37)$$

¹² To our knowledge, this remark was not made by Lee [47].

¹³ The Eddington relation (13) is not very accurate. It gives the mass of the electron up to a factor ~ 100 . Therefore, all the numerical values given in this section are approximate. Eq. (36) should be regarded as “defining” a mass scale m_e which is of the order of the electron mass. We refer to Appendix A for more accurate results.

If we normalize the electron mass by the Planck mass we obtain the small dimensionless number

$$\frac{m_e}{M_P} \sim \chi^{-1/6} \sim 10^{-20}, \quad (38)$$

which corresponds to Eq. (20) with $a = -1$.

The Compton wavelength of the electron is

$$\lambda_e \sim \frac{\hbar}{m_e c} \sim \left(\frac{G^2 \hbar^2}{\Lambda c^6} \right)^{1/6} = 2.92 \times 10^{-15} \text{ m} \quad (39)$$

and its semi gravitational (or semi Schwarzschild) radius is

$$r_g \sim \frac{G m_e}{c^2} \sim \left(\frac{\Lambda \hbar^4 G^4}{c^{12}} \right)^{1/6} \sim 8.96 \times 10^{-56} \text{ m}. \quad (40)$$

Their ratio scales as

$$\frac{\lambda_e}{r_g} = \frac{M_P^2}{m_e^2} \sim \chi^{1/3} \sim 10^{40}. \quad (41)$$

The numerical density of electrons in the universe (of typical density ρ_Λ) is

$$n_e \sim \frac{\rho_\Lambda}{m_e} \sim \left(\frac{\Lambda^5 c^{12}}{G^4 \hbar^4} \right)^{1/6}. \quad (42)$$

The total number of electrons is therefore

$$N_e \sim n_e R_\Lambda^3 \sim \frac{M_\Lambda}{m_e} \sim \chi^{2/3} \sim 10^{80}. \quad (43)$$

The average distance between the electrons is

$$d_e \sim n_e^{-1/3} \sim \left(\frac{G^4 \hbar^4}{\Lambda^5 c^{12}} \right)^{1/18} \sim 0.0931 \text{ m}. \quad (44)$$

It is of the order of 1 m, the unit of length in the International System of Units.

Remark: We note that, according to the Eddington relation (13), the mass of the electron is given by [30]

$$m_e \sim (m_\Lambda M_P^2)^{1/3}, \quad (45)$$

where m_Λ is the cosmon mass (see Sec. IV F) and M_P is the Planck mass (see Sec. IV A). Accordingly, the gravitational energy of the electron is of the order of the rest mass energy of the cosmon:

$$E_e \sim \frac{G m_e^2}{\lambda_e} \sim \frac{G m_e^3 c}{\hbar} \sim m_\Lambda c^2. \quad (46)$$

This equation can be written under the form of an Heisenberg relation: $E_e \times t_\Lambda \sim \hbar$. Using the Eddington relation (13) we also find that

$$\frac{e^2}{G m_e^2} \sim \frac{R_\Lambda}{a_B}, \quad (47)$$

where $F \sim e^2/Gm_e^2$ is the Weyl number, R_Λ is the cosmological radius and $a_B = \hbar^2/(e^2 m_e)$ is the Bohr radius. Conversely, the relations (46) and (47) imply the Eddington relation (13).

D. The mass of fermion stars ($a = 2$)

General relativity determines a maximum mass above which a fermion star at $T = 0$ becomes unstable. This maximum mass is of order

$$M_{\text{max}}^F \sim \left(\frac{\hbar c}{G} \right)^{3/2} \frac{1}{m^2} \sim \frac{M_P^3}{m^2}. \quad (48)$$

The exact expression involves the prefactor 0.384. This maximum mass was obtained by Oppenheimer and Volkoff (OV) [49] in the context of neutron stars. The corresponding radius is $R_* \sim GM_{\text{max}}^F/c^2 \sim (\hbar^3/Gm^4c)^{1/2}$. The exact expression involves the prefactors 8.73 and 3.35. The scaling of the Oppenheimer-Volkoff mass is the same as the scaling of the Chandrasekhar mass for special relativistic white dwarf stars [50]. In a relativistic fermion star the interparticle distance $d \sim n^{-1/3} \sim (R^3/N)^{1/3}$ is of the order of the Compton wavelength $\lambda_C = \hbar/mc$ of the fermions which is a measure of their ‘‘size’’. In this sense, the fermions are closely packed in a fermion star at $T = 0$. Below the mass M_{max}^F , the fermion star is stabilized by the Pauli exclusion principle. Above that mass it collapses towards a black hole.

If we replace m by m_e and use the Eddington relation (13), we obtain¹⁴

$$M_2 = \frac{M_P^3}{m_e^2} = \left(\frac{\hbar c^9}{G^5 \Lambda^2} \right)^{1/6} = 7.08 \times 10^{35} \text{ g}, \quad (49)$$

i.e., $M_2 = 356 M_\odot$. This gives a mass scale of the order of the solar mass (recall that our orders of magnitudes are valid up to a factor ~ 100). This is the typical mass of white dwarfs and neutron stars. The evaporation time of a stellar mass black hole (resulting from the collapse of a fermion star) is $t_{\text{evap}} \sim G^2 M_2^3/\hbar c^4 \sim (c/\hbar G \Lambda^2)^{1/2} \sim 1.86 \times 10^{78} \text{ s}$, which is much larger than the age of the universe t_Λ .

The semi gravitational radius associated with the mass M_2 is

$$R_2 = \frac{G M_2}{c^2} = \left(\frac{G \hbar}{\Lambda^2 c^3} \right)^{1/6} = 5.26 \times 10^5 \text{ m}. \quad (50)$$

It is of the order of the km. The Compton wavelength of a fermion star is

$$\lambda_2 = \frac{\hbar}{M_2 c} = \left(\frac{\Lambda^2 \hbar^5 G^5}{c^{15}} \right)^{1/6} = 4.96 \times 10^{-76} \text{ m}. \quad (51)$$

¹⁴ The Oppenheimer-Volkoff mass involves the neutron mass m_n and the Chandrasekhar mass involves the proton mass m_p (multiplied by the molecular weight μ). As mentioned previously, in our qualitative approach, we do not make a distinction between m_e and m_p or m_n .

If we normalize the mass of fermion stars by the Planck mass we obtain the large dimensionless number

$$\frac{M_2}{M_P} = \chi^{1/3} \sim 10^{40}, \quad (52)$$

which corresponds to Eq. (20) with $a = 2$. On the other hand, the ratio of the Compton wavelength of a fermion star on its semi gravitational radius scales as

$$\frac{\lambda_2}{R_2} = \frac{M_P^2}{M_2^2} \sim \chi^{-2/3} \sim 10^{-80}. \quad (53)$$

We can consider that a fermion star is at $T = 0$ (i.e. the fermions are completely degenerate) if its actual temperature T is much smaller than the Fermi temperature $T_F \sim \rho^{2/3} \hbar^2 / m^{5/3}$. Using $\rho \sim M/R^3$ and $R \sim GM/c^2$, we get $\rho \sim c^6 / (G^3 M^2)$ hence $k_B T_F \sim \hbar^2 c^4 / (G^2 m^{5/3} M^{4/3})$. Combined with Eq. (49) we obtain $\rho \sim m_e^4 c^3 / \hbar^3 \sim m_e / \lambda_e^3 \sim 4.92 \times 10^{18} \text{ g m}^{-3}$ and $k_B T_F \sim m_e c^2 \sim 7.88 \times 10^{11} \text{ K}$. This degeneracy temperature can be written $k_B T_F \sim (\Lambda \hbar^4 / G^2)^{1/6} c^2$ or $T_F / T_P \sim \chi^{-1/6} \sim 10^{-20}$. In general $T \ll T_F$ so the $T = 0$ limit is a good approximation.

E. The mass of mini-boson stars ($a = 1$)

General relativity determines a maximum mass above which a boson star at $T = 0$ becomes unstable. This maximum mass is of order

$$M_{\text{max}}^B \sim \frac{\hbar c}{Gm} \sim \frac{M_P^2}{m}. \quad (54)$$

It is called the Kaup mass [51, 52]. The exact expression involves the prefactor 0.633. The corresponding radius is $R_*' \sim GM_{\text{max}}^B / c^2 \sim \hbar / mc$. The exact expression involves the prefactors 9.53 and 6.03. The size R_*' of a noninteracting boson star is of the order of the Compton wavelength $\lambda_C = \hbar / mc$ of the bosons. Below the mass M_{max}^B , the boson star is stabilized by the Heisenberg uncertainty principle. Above that mass it collapses towards a black hole.

If we replace m by m_e and use the Eddington relation (13), we obtain the mass scale¹⁵

$$M_1 = \frac{M_P^2}{m_e} = \left(\frac{\hbar^2 c^6}{G^4 \Lambda} \right)^{1/6} = 3.93 \times 10^{15} \text{ g}, \quad (55)$$

¹⁵ Of course, we are not claiming that boson stars are made of electrons (which are fermions). We are considering the case where they are made of bosons whose mass is of the order of m_e . As mentioned previously, in our qualitative approach, we do not make a distinction between m_e and m_p or m_n . For example, we could consider the case where fermions of mass $m_e \sim m_p \sim m_n$ form Cooper pairs and behave as bosons. In that case, we could have boson stars (or BEC stars) made of pseudo bosons of mass $m_e \sim m_p \sim m_n$ [53].

i.e., $M_1 = 1.97 \times 10^{-18} M_\odot$. This is the typical mass of a mini-boson star.¹⁶ We can easily see that M_1 is the mass of a black hole (resulting from the collapse of a boson star) whose evaporation time is of the order of the age of the universe t_Λ (i.e. $t_{\text{evap}} \sim G^2 M_1^3 / \hbar c^4 \sim 1 / (c\sqrt{\Lambda}) \sim t_\Lambda$). Conversely, if we require that $t_{\text{evap}} \sim t_\Lambda$ we obtain the Eddington relation (13).

The semi gravitational radius associated with the mass M_1 is

$$R_1 = \frac{GM_1}{c^2} = \left(\frac{G^2 \hbar^2}{\Lambda c^6} \right)^{1/6} = 2.92 \times 10^{-15} \text{ m}. \quad (56)$$

It coincides with the Compton wavelength (39) of the electron. The Compton wavelength of a mini boson star is

$$\lambda_1 = \frac{\hbar}{M_1 c} = \left(\frac{\Lambda \hbar^4 G^4}{c^{12}} \right)^{1/6} = 8.96 \times 10^{-56} \text{ m}. \quad (57)$$

It coincides with the semi gravitational radius (40) of the electron. If we normalize the mass of mini-boson stars by the Planck mass we obtain the large dimensionless number

$$\frac{M_1}{M_P} = \chi^{1/6} \sim 10^{20}, \quad (58)$$

which corresponds to Eq. (20) with $a = 1$. On the other hand, the ratio of the Compton wavelength of a mini boson star on its semi gravitational radius scales as

$$\frac{\lambda_1}{R_1} = \frac{M_P^2}{M_1^2} \sim \chi^{-1/3} \sim 10^{-40}. \quad (59)$$

We can consider that a boson star is at $T = 0$ (i.e. the bosons are completely condensed) if its actual temperature T is much smaller than the condensation temperature $T_c \sim \rho^{2/3} \hbar^2 / m^{5/3}$. Using $\rho \sim M/R^3$ and $R \sim GM/c^2$, we get $\rho \sim c^6 / (G^3 M^2)$ hence $k_B T_c \sim \hbar^2 c^4 / (G^2 m^{5/3} M^{4/3})$. Combined with Eq. (55) we obtain $\rho \sim m_e^2 c^4 / (G \hbar^2) \sim 1.59 \times 10^{59} \text{ g m}^{-3}$ and $k_B T_c \sim \hbar^{2/3} c^{8/3} / (G^{2/3} m_e^{1/3}) \sim 8.00 \times 10^{38} \text{ K}$. This condensation temperature can be written $k_B T_c \sim (\hbar^8 c^{12} / \Lambda G^{10})^{1/18} c^2$ or $T_c / T_P \sim \chi^{1/18} \sim 10^{20/3}$. It is much larger than the Planck temperature. Therefore, we clearly have $T \ll T_c$ in all situations of physical interest so the $T = 0$ limit is an excellent approximation.

Remark: Equilibrium states of general relativistic bosons stars with a repulsive $|\varphi|^4$ self-interaction in the

¹⁶ The maximum mass of fermion stars scales as M_P^3 / m^2 [see Eq. (48)] while the maximum mass of mini boson stars scales as M_P^2 / m [see Eq. (54)]. For the same particle mass m , the mass of fermion stars is much bigger than the mass of mini boson stars by a factor $M_P / m \gg 1$. This is because fermion stars are stabilized by the Pauli exclusion principle while mini boson stars are stabilized by the Heisenberg uncertainty principle.

Thomas-Fermi limit exist only below the maximum mass [53, 54]

$$M_{\max}^{\text{TF}} = 0.0612\sqrt{\lambda}\frac{M_P^3}{m^2}. \quad (60)$$

This is the maximum mass of massive boson stars. Above that mass, the star collapses towards a black hole. For $\lambda \sim 1$, the maximum mass from Eq. (60) is of the order of the Oppenheimer-Volkoff or Chandrasekhar mass (48) like in the case of relativistic fermion stars. Therefore, a repulsive self-interaction between bosons allows us to build up bigger objects. This is because it plays a role similar to the Pauli exclusion principle for fermions. Massive boson stars could describe dark matter stars or even the superfluid core of neutron stars where neutrons form Cooper pairs and behave as bosons of mass $2m_n$ [53]. This may explain the large mass $\sim 2 - 2.4 M_\odot$ of recently observed neutron stars. The TF approximation is valid when $M_{\max}^{\text{TF}} \gg M_{\max}^B$, i.e., when $\lambda \gg (m/M_P)^2$ [40]. In general, the gravitational coupling constant $\alpha_g = Gm^2/\hbar c = (m/M_P)^2$ is extremely small, e.g., $\alpha_g \sim 10^{-100}$ for a boson of mass $m \sim 10^{-22} \text{ eV}/c^2$, so the TF approximation is valid even when λ is of order 10^{-100} (!) The same remarks apply to bosons stars with an attractive self-interaction (see the end of Sec. IV A). In that case, the nonrelativistic expression of the maximum mass (27) is valid when $M_{\max}^{\text{NR}} \ll M_{\max}^B$, i.e., when $|\lambda| \gg (m/M_P)^2$ [40].

F. The mass of the cosmon ($a = -3$)

The maximum mass of a mini boson star is given by Eq. (54). If we identify the Kaup mass (54) with the mass of the universe (28), we obtain a particle mass

$$m_\Lambda = \frac{M_P^2}{M_\Lambda} = \frac{\hbar\sqrt{\Lambda}}{c} = 3.71 \times 10^{-66} \text{ g}, \quad (61)$$

i.e., $m_\Lambda = 2.08 \times 10^{-33} \text{ eV}/c^2$. This mass is often interpreted as the smallest mass of the bosons predicted by string theory [55, 56]. It fixes their fundamental mass scale. The mass m_Λ also represents the quantum of mass in theories of extended supergravity [55, 57]. It could be the mass of the elementary particle of dark energy called the cosmon [11, 58].¹⁷ The cosmon mass can be obtained by writing that the Compton wavelength $\lambda_C = \hbar/mc$ of the particle is equal to the cosmological length $R_\Lambda \sim 1/\sqrt{\Lambda}$ (the typical size of the visible universe).¹⁸ This means that the cosmon is completely delo-

calized. In Appendix F we obtain the cosmon mass from a scalar field theory which is equivalent to the Λ CDM model interpreted as a unified dark matter and dark energy (UDME) model. If we normalize the cosmon mass by the Planck mass we obtain the small dimensionless number

$$\frac{m_\Lambda}{M_P} = \chi^{-1/2} \sim 10^{-60}, \quad (62)$$

which corresponds to Eq. (20) with $a = -3$.

The Compton wavelength of the cosmon is

$$\lambda'_\Lambda = \frac{\hbar}{m_\Lambda c} = R_\Lambda = \frac{1}{\sqrt{\Lambda}} = 9.49 \times 10^{25} \text{ m}. \quad (63)$$

It coincides with the semi gravitational radius (31) of the universe. The semi gravitational (or semi Schwarzschild) radius of the cosmon is

$$r_\Lambda = \frac{Gm_\Lambda}{c^2} = \frac{G\hbar\sqrt{\Lambda}}{c^3} = 2.75 \times 10^{-96} \text{ m}. \quad (64)$$

It coincides with the Compton wavelength (32) of the universe. The ratio of the Compton wavelength of the cosmon on its semi gravitational radius scales as

$$\frac{\lambda'_\Lambda}{r_\Lambda} = \frac{M_P^2}{m_\Lambda^2} = \frac{M_\Lambda}{m_\Lambda} \sim \chi \sim 10^{120}. \quad (65)$$

The numerical density of cosmons (bosons) in the universe (of typical density ρ_Λ) is

$$n_\Lambda = \frac{\rho_\Lambda}{m_\Lambda} = \frac{c^3\sqrt{\Lambda}}{G\hbar}. \quad (66)$$

The total number of cosmons is therefore

$$N_B \sim n_\Lambda R_\Lambda^3 \sim \frac{M_\Lambda}{m_\Lambda} \sim \chi \sim 10^{120}. \quad (67)$$

It is equal to the fundamental number $\chi \sim 10^{120}$. The average distance between the cosmons is¹⁹

$$d_\Lambda \sim n_\Lambda^{-1/3} \sim \left(\frac{G^2\hbar^2}{c^6\Lambda}\right)^{1/6} \sim 2.92 \times 10^{-15} \text{ m}. \quad (68)$$

It is of the order of the classical radius of the electron $r_e = e^2/m_e c^2 = 2.82 \times 10^{-15} \text{ m}$ (see Appendix A). This is a consequence of the accurate Eddington relation (A9). Conversely, if we identify these two lengths ($d_\Lambda \sim r_e$) we obtain the accurate Eddington relation (A9).

Remark: We encountered the mass scale (61) in Ref. [30] by trying to determine from a Jeans stability analysis the mass of the bosonic particle that may compose dark matter halos. We found that the typical mass $m \sim 10^{-22} \text{ eV}/c^2$ of the bosonic dark matter particle is equal to m_Λ multiplied by a huge prefactor of order 10^{11} .

¹⁷ Alternatively, some authors [59, 60] interpret the mass from Eq. (61) as the upper bound on the mass of the photon or on the mass of the graviton.

¹⁸ It can also be obtained from the Heisenberg uncertainty principle $m_\Lambda c^2 \times t_\Lambda \sim \hbar$, equivalent to $m_\Lambda c^2 \sim \hbar c/R_\Lambda$, or by equating the smallest Schwarzschild radius and the smallest Compton wavelength: $Gm_\Lambda/c^2 \sim \hbar/M_\Lambda c$.

¹⁹ This is also the radius of a particle of mass m_Λ and density ρ_Λ .

G. The mass of the neutrino ($a = -3/2$)

The maximum mass of a fermion star is given by Eq. (48). If we identify the Oppenheimer-Volkoff mass (48) with the mass of the universe (28) we obtain a particle mass

$$m_{\Lambda}^* = \left(\frac{M_P^3}{M_{\Lambda}}\right)^{1/2} = \left(\frac{\Lambda \hbar^3}{Gc}\right)^{1/4} = 8.98 \times 10^{-36} \text{ g}, \quad (69)$$

i.e., $m_{\Lambda}^* = 5.04 \times 10^{-3} \text{ eV}/c^2$. This mass may be interpreted as the smallest mass of the fermions. It fixes their fundamental mass scale. It could be connected to the mass of the neutrino. If we normalize this ‘‘neutrino’’ mass by the Planck mass we obtain the small dimensionless number

$$\frac{m_{\Lambda}^*}{M_P} = \chi^{-1/4} \sim 10^{-30}, \quad (70)$$

which corresponds to Eq. (20) with $a = -3/2$.

The Compton wavelength of the neutrino is

$$\lambda_{\Lambda}^* = \frac{\hbar}{m_{\Lambda}^* c} = \left(\frac{G\hbar}{\Lambda c^3}\right)^{1/4} = 3.91 \times 10^{-5} \text{ m} \quad (71)$$

and its semi gravitational (or semi Schwarzschild) radius is

$$r_{\Lambda}^* = \frac{Gm_{\Lambda}^*}{c^2} = \left(\frac{\Lambda \hbar^3 G^3}{c^9}\right)^{1/4} = 6.67 \times 10^{-66} \text{ m}. \quad (72)$$

Their ratio scales as

$$\frac{\lambda_{\Lambda}^*}{r_{\Lambda}^*} = \frac{M_P^2}{(m_{\Lambda}^*)^2} = \chi^{1/2} \sim 10^{60}. \quad (73)$$

The numerical density of neutrinos (fermions) in the universe (of typical density ρ_{Λ}) is

$$n_{\Lambda}^* = \frac{\rho_{\Lambda}}{m_{\Lambda}^*} = \left(\frac{\Lambda c^3}{G\hbar}\right)^{3/4}. \quad (74)$$

The total number of neutrinos is therefore

$$N_F \sim n_{\Lambda}^* R_{\Lambda}^3 \sim \frac{M_{\Lambda}}{m_{\Lambda}^*} \sim \chi^{3/4} \sim 10^{90}. \quad (75)$$

The average distance between the neutrinos is

$$d_{\Lambda}^* \sim (n_{\Lambda}^*)^{-1/3} \sim \left(\frac{G\hbar}{\Lambda c^3}\right)^{1/4} \sim 3.91 \times 10^{-5} \text{ m}. \quad (76)$$

It is of the order of their Compton wavelength (71) which is a measure of their ‘‘size’’. In this sense, the neutrinos are densely packed in the universe.²⁰ Conversely, if we

²⁰ Of course, this statement should not be taken literally. This is just an order of magnitude in the case where the universe would be made only of neutrinos of mass m_{Λ}^* uniformly distributed over space. This picture is of course oversimplified.

require that the universe is made of particles whose interparticle distance is equal to their Compton wavelength ($d_{\Lambda}^* \sim \lambda_{\Lambda}^*$) we find that $m \sim m_{\Lambda}^*$.

Remark: We encountered the mass scale (69) in Ref. [30] by trying to determine from a Jeans stability analysis the mass of the fermionic particle that may compose dark matter halos.²¹ We found that the typical mass $m \sim 1 \text{ keV}/c^2$ of the fermionic dark matter particle is equal to m_{Λ}^* multiplied by a large prefactor of order 10^6 . We also obtained the relation

$$m_{\Lambda}^* = \sqrt{m_{\Lambda} M_P}, \quad (77)$$

showing that the fundamental fermion mass (neutrino) is the geometric mean of the cosmon (boson) mass and the Planck mass [30]. This kind of general arguments explains the huge difference of scales between the mass of the fermion particle and the mass of the boson particle that may compose dark matter. Indeed from Eqs. (62) and (70) we get

$$\frac{m_{\Lambda}^*}{m_{\Lambda}} \sim \chi^{1/4} \sim 10^{30}, \quad (78)$$

in qualitative agreement with the ratio between the values $m \sim 10^{-22} \text{ eV}/c^2$ (for bosons) and $m \sim 1 \text{ keV}/c^2$ (for fermions) usually reported in the literature [30].

H. Mass scale law and an anomaly: Another neutrino mass scale ($a = -2$)

Interestingly, when we regroup the foregoing results and compare them with Eq. (19), we find that the mass M_a of the universe corresponds to $a = 3$, the mass of fermion stars corresponds to $a = 2$, the mass of mini boson stars corresponds to $a = 1$, the Planck mass corresponds to $a = 0$, the mass of the electron corresponds to $a = -1$, the mass of the neutrino corresponds to $a = -3/2$ (close to -2), and the mass of the cosmon corresponds to $a = -3$. Therefore, the parameter a takes integer values between $+3$ and -3 . These two extreme values have a simple interpretation. For $a = 3$ (universe) the mass scale defined by Eq. (19) is independent of \hbar and for $a = -3$ (cosmon) it is independent of G . We also note that M_a is independent of Λ for $a = 0$ (Planck mass) and independent of c for $a = -1$ (electron).

When the mass M_a is normalized by the Planck mass, using Eq. (20), we find that $M_{\Lambda}/M_P \sim 10^{60}$ for the universe ($a = 3$), $M_2/M_P \sim 10^{40}$ for fermion stars ($a = 2$), $M_1/M_P \sim 10^{20}$ for mini boson stars ($a = 1$), $m_e/M_P \sim 10^{-20}$ for the electron ($a = -1$), $m_{\Lambda}^*/M_P \sim 10^{-30}$ for the neutrino ($a = -3/2$), and $m_{\Lambda}/M_P \sim 10^{-60}$ for the cosmon ($a = -3$). It is relevant to normalize the mass

²¹ Recently, Resca [60] obtained a similar result based on different arguments.

M_a by the Planck mass M_P because: (i) it is the conventional reference mass scale in physics; (ii) it depends on the fundamental constants G , c and \hbar but not on Λ whose status as a fundamental constant of physics is still controversial; (iii) it makes a clear separation between micro and macro scales.²² Furthermore, this separation is symmetrical: micro-objects (cosmon, neutrino, electron) correspond to $a < 0$ and macro-objects (mini boson stars, fermion stars, Universe) correspond to $a > 0$. These mass scales were introduced in our previous papers [11, 30] before realizing that they follow a simple law. The mass scale law from Eq. (20) was realized *a posteriori*.

Actually, this mass scale law presents an anomaly. General considerations lead to a fermionic mass scale m_Λ^* corresponding to the half-integer value $a = -3/2$ instead of the integer value $a = -2$. We have suggested that the fundamental mass scale m_Λ^* may be connected to the mass of the neutrino but this suggestion remains of course highly speculative. If we take $a = -2$ in order to respect the integer nature of the mass scale law, we obtain the mass

$$m_2 = \left(\frac{\Lambda^2 \hbar^5}{G c^3} \right)^{1/6} = 6.69 \times 10^{-46} \text{ g}, \quad (79)$$

i.e., $m_2 = 3.75 \times 10^{-13} \text{ eV}/c^2$ which may be associated to another type of neutrino. If we normalize this mass by the Planck mass we obtain

$$\frac{m_2}{M_P} \sim \chi^{-1/3} \sim 10^{-40}. \quad (80)$$

The Compton wavelength of the neutrino is

$$\lambda'_2 = \frac{\hbar}{m_2 c} = \left(\frac{G \hbar}{\Lambda^2 c^3} \right)^{1/6} = 5.26 \times 10^5 \text{ m}. \quad (81)$$

It coincides with the semi gravitational radius (50) of fermion stars. On the other hand, the semi gravitational (or semi Schwarzschild) radius of the neutrino is

$$r_2 = \frac{G m_2}{c^2} = \left(\frac{\Lambda^2 \hbar^5 G^5}{c^{15}} \right)^{1/6} = 4.96 \times 10^{-76} \text{ m}. \quad (82)$$

It coincides with the Compton wavelength (51) of fermion stars. The ratio of the Compton wavelength of the neutrino on its semi gravitational radius scales as

$$\frac{\lambda'_2}{r_2} = \frac{M_P^2}{m_2^2} = \chi^{2/3} \sim 10^{80}. \quad (83)$$

The numerical density of neutrinos (fermions) in the universe (of typical density ρ_Λ) is

$$n_2 = \frac{\rho_\Lambda}{m_2} = \left(\frac{\Lambda^4 c^{15}}{G^5 \hbar^5} \right)^{1/6}. \quad (84)$$

The total number of neutrinos is therefore

$$N_2 \sim n_2 R_\Lambda^3 \sim \frac{M_\Lambda}{m_2} \sim \chi^{5/6} \sim 10^{100}. \quad (85)$$

The average distance between the neutrinos is

$$d_2 \sim n_2^{-1/3} \sim \left(\frac{G^5 \hbar^5}{\Lambda^4 c^{15}} \right)^{1/18} \sim 1.65 \times 10^{-8} \text{ m}. \quad (86)$$

Remark: For convenience, the fermion of mass m_Λ^* ($a = -3/2$) will be called “neutrino A” and the fermion of mass m_2 ($a = -2$) will be called “neutrino B”. The two mass scales m_Λ^* and m_2 differ by a factor $m_\Lambda^*/m_2 \sim \chi^{1/12} \sim 10^{10}$ (we have the same ratio $m_e/m_\Lambda^* \sim \chi^{1/12} \sim 10^{10}$ between the mass of the electron and the mass of the neutrino A). The anomaly in our mass scale law and the existence of (at least) two mass scales for fermions (+ one if we include the electron) may be related to the fact that there exist different (three) types of neutrinos: electron neutrinos, muon neutrinos and tau neutrinos.

I. The earth mass ($a = 3/2$)

By the reciprocity property of Appendix B 3 it is natural to introduce a macro object of index $a = 3/2$ associated with the neutrino of index $a = -3/2$. According to Eq. (19) this object has a mass

$$M_{3/2} = \left(\frac{\hbar c^5}{\Lambda G^3} \right)^{1/4} = 5.27 \times 10^{25} \text{ g}, \quad (87)$$

i.e., $M_{3/2} = 2.65 \times 10^{-8} M_\odot$. Interestingly, this argument gives a mass scale of the order of the earth mass: $M_{3/2} = 8.83 \times 10^{-3} M_\oplus$ (recall that our orders of magnitudes are valid up to a factor ~ 100). The evaporation time of a terrestrial mass black hole is $t_{\text{evap}} \sim G^2 M_{3/2}^3 / \hbar c^4 \sim (1/\hbar G c \Lambda^3)^{1/4} \sim 7.67 \times 10^{47} \text{ s}$, which is much larger than the age of the universe t_Λ .

The semi gravitational radius associated with the mass $M_{3/2}$ is

$$R_{3/2} = \frac{G M_{3/2}}{c^2} = \left(\frac{G \hbar}{\Lambda c^3} \right)^{1/4} = 3.91 \times 10^{-5} \text{ m}. \quad (88)$$

It is equal to the Compton wavelength (71) of the neutrino A. The Compton wavelength associated with the mass $M_{3/2}$ is

$$\lambda_{3/2} = \frac{\hbar}{M_{3/2} c} = \left(\frac{\Lambda \hbar^3 G^3}{c^9} \right)^{1/4} = 6.67 \times 10^{-66} \text{ m}. \quad (89)$$

It coincides with the semi gravitational radius (72) of the neutrino A. If we normalize the mass (87) by the Planck mass we obtain the large dimensionless number

$$\frac{M_{3/2}}{M_P} = \chi^{1/4} \sim 10^{30}, \quad (90)$$

²² Roughly speaking, microphysics corresponds to particle physics and macrophysics corresponds to astrophysics and cosmology.

which corresponds to Eq. (20) with $a = 3/2$. On the other hand, the ratio of the Compton wavelength of an object of mass $M_{3/2}$ on its semi gravitational radius scales as

$$\frac{\lambda_{3/2}}{R_{3/2}} = \frac{M_P^2}{M_{3/2}^2} \sim \chi^{-1/2} \sim 10^{-60}. \quad (91)$$

Remark: By analogy with the results of Secs. IV B, IV D and IV E, we may conjecture that the mass $M_{3/2}$ corresponds to the general relativistic maximum mass of a new type of “stars” made of particles of mass $\sim m_e$. Specifically, we conjecture the existence of a new type of stars, that we call “solid stars”, with a general relativistic maximum mass

$$M_{\max} \sim \left(\frac{\hbar c}{G}\right)^{5/4} \frac{1}{m^{3/2}} \sim \frac{M_P^{5/2}}{m^{3/2}}. \quad (92)$$

The corresponding radius is $R_* \sim GM_{\max}/c^2 \sim (\hbar^5/Gm^6c^3)^{1/4}$. If we replace m by m_e and use the Edington relation (13), we obtain the mass scale of terrestrial objects [see Eq. (87)]:

$$M_{3/2} = \frac{M_P^{5/2}}{m_e^{3/2}} = \left(\frac{\hbar c^5}{\Lambda G^3}\right)^{1/4} = 5.27 \times 10^{25} \text{ g}. \quad (93)$$

Alternatively, if we identify the mass of a solid star (92) with the mass of the universe (28), we obtain a particle mass

$$m_2 = \frac{M_P^{5/3}}{M_\Lambda^{2/3}} = \left(\frac{\Lambda^2 \hbar^5}{Gc^3}\right)^{1/6} = 6.69 \times 10^{-46} \text{ g}, \quad (94)$$

corresponding to the neutrino B.

V. FINAL REMARKS

In this section, we reformulate the previous results in a slightly different manner by calculating the fundamental general relativistic mass scales (Chandrasekhar, Kaup and Lee) with the characteristic masses of elementary particles (electron, neutrino, cosmon) and provide general rules.

A. Combinations of fundamental scales

The maximum mass of boson stars (Kaup mass) is given by²³

$$\frac{M_{\max}^B}{M_P} \sim \frac{M_P}{m}. \quad (95)$$

²³ In this section all the masses are normalized by the Planck mass.

For the cosmon of mass $m_\Lambda/M_P \sim \chi^{-1/2}$ ($a = -3$) we get the mass of the universe $M_\Lambda/M_P \sim \chi^{1/2}$ ($a = 3$), for the neutrino B of mass $m_2/M_P \sim \chi^{-1/3}$ ($a = -2$) we get the mass of a fermion star $M_2/M_P \sim \chi^{1/3}$ ($a = 2$), for the neutrino A of mass $m_\Lambda^*/M_P \sim \chi^{-1/4}$ ($a = -3/2$) we get the earth mass $M_{3/2}/M_P \sim \chi^{1/4}$ ($a = 3/2$), and for the electron of mass $m_e/M_P \sim \chi^{-1/6}$ ($a = -1$) we get the mass of mini boson stars $M_1/M_P \sim \chi^{1/6}$ ($a = 1$). These correspondances are a direct consequence of the reciprocity property (see Appendix B 3).

The maximum mass of solid stars is given by

$$\frac{M_{\max}}{M_P} \sim \frac{M_P^{3/2}}{m^{3/2}}. \quad (96)$$

For the neutrino B of mass $m_2/M_P \sim \chi^{-1/3}$ ($a = -2$) we get the mass of the universe $M_\Lambda/M_P \sim \chi^{1/2}$ ($a = 3$), for the neutrino A of mass $m_\Lambda^*/M_P \sim \chi^{-1/4}$ ($a = -3/2$) we get a new mass scale $M/M_P \sim \chi^{3/8}$ ($a = 9/4$) which is of the order of the galactic mass,²⁴ and for the electron of mass $m_e/M_P \sim \chi^{-1/6}$ ($a = -1$) we get the mass of solid stars (earth’s mass) $M_{3/2}/M_P \sim \chi^{1/4}$ ($a = 3/2$). In the case of the cosmon, the mass (96) is larger than the mass of the universe.

The maximum mass of fermion stars (Chandrasekhar’s mass) is given by

$$\frac{M_{\max}^F}{M_P} \sim \frac{M_P^2}{m^2}. \quad (97)$$

For the neutrino A of mass $m_\Lambda^*/M_P \sim \chi^{-1/4}$ ($a = -3/2$) we get the mass of the universe $M_\Lambda/M_P \sim \chi^{1/2}$ ($a = 3$), for the electron of mass $m_e/M_P \sim \chi^{-1/6}$ ($a = -1$) we get the mass of fermion stars $M_2/M_P \sim \chi^{1/3}$ ($a = 2$). In the other cases (cosmon and neutrino B) the Chandrasekhar mass is larger than the mass of the universe.

The maximum mass of soliton stars (Lee’s mass) is given by

$$\frac{M_{\max}^S}{M_P} \sim \frac{M_P^3}{m^3}. \quad (98)$$

For the electron of mass $m_e/M_P \sim \chi^{-1/6}$ ($a = -1$) we get the mass of the universe $M_\Lambda/M_P \sim \chi^{1/2}$ ($a = 3$). In the other cases (cosmon and neutrinos) the Lee mass is larger than the mass of the universe.

Remark: In the above examples, the obvious condition $M_{\max} > m$ implies $m < M_P$. This is why we have only considered the mass m of micro objects.

B. Rule based on the mass of universe

The mass of the universe M_Λ corresponds to:

²⁴ We find $M_{9/4} = (\hbar c^{13}/\Lambda^3 G^7)^{1/8} = 4.13 \times 10^7 M_\odot$. Interestingly, the index $a = 9/4$ is close to the index $5/2$ considered by Chandrasekhar [29] for the galaxies (see Sec. III).

- (i) the maximum mass of boson stars (Kaup mass) calculated with the cosmon mass $m_\Lambda (= m_3)$;
- (ii) the maximum mass of solid stars calculated with the mass m_2 of the neutrino B;
- (iii) the maximum mass of fermion stars (Oppenheimer-Volkoff mass) calculated with the mass $m_\Lambda^* (= m_{3/2})$ of the neutrino A;
- (iv) the maximum mass of soliton stars (Lee mass) calculated with the electron mass $m_e (= m_1)$.

This is how we have introduced our fundamental micro objects: the cosmon ($a = -3$), the neutrinos B ($a = -2$) and A ($a = -3/2$) and the electron ($a = -1$).

C. Rule based on the electron mass

Taking as a reference the mass m_e of the electron:

- (i) the maximum mass of soliton stars (Lee mass) corresponds to the mass of the universe M_Λ ;
- (ii) the maximum mass of fermion stars (Oppenheimer-Volkoff mass) corresponds to the solar mass M_2 ;
- (iii) the maximum mass of solid stars corresponds to the mass of the earth $M_{3/2}$;
- (iv) the maximum mass of boson stars (Kaup mass) corresponds to the mass of mini boson stars M_1 .

This is how we have introduced our fundamental macro objects: the universe ($a = 3$), the fermion stars (sun) ($a = 2$), the solid stars (earth) ($a = 3/2$) and the mini boson stars ($a = 1$).

VI. CONCLUSION

In this paper, by combining the maximum mass of fermion stars (Chandrasekhar), boson stars (Kaup) and soliton stars (Lee) with the Eddington relation, and making a sort of “cosmic numerology” [13], we have shown that the typical mass of the macro and micro objects in the Universe can be expressed in terms of the fundamental constants of physics including the cosmological constant. When normalized by the Planck mass, these mass scales can be expressed as some powers of the fundamental dimensionless number

$$\chi = \frac{\rho_P}{\rho_\Lambda} = \frac{c^3}{G\hbar\Lambda} \sim 10^{120}. \quad (99)$$

Specifically, the mass of the Universe, fermion stars, mini boson stars, Planck black holes, the electron, the neutrino and the cosmon follow a simple law

$$\frac{M_a}{M_P} \sim \chi^{a/6} \sim (10^{20})^a \quad (100)$$

with $a = 3, 2, 1, 0, -1, -2, -3$ (see Table I).²⁵ This mass scale law makes a bridge between “infinitely” large scales

and “infinitely” small scales.²⁶ Actually, those scales do not extend *ad infinitum* because of the large but finite value of χ . There exists a maximum mass M_Λ corresponding to the mass of the Universe and a minimum mass m_Λ corresponding to the particle of dark energy (cosmon). The Planck mass M_P can be seen as an intermediate (or “middle”) mass separating the macro-world (stars and galaxies) from the micro-world (elementary particles).

The dimensionless number χ is extremely important because it involves *all* the fundamental constants of physics G (gravity), c (electromagnetism and relativity), \hbar (quantum mechanics) and Λ (cosmology). Furthermore, this number $\chi \sim 10^{120}$ is gigantic. We conjectured that it is the “largest large number” occurring in Nature. This claim is supported by the holographic principle when applied to the Universe as a whole (see Appendix B). The number $\chi \sim 10^{120}$ corresponds to the entropy S_Λ/k_B of the universe which is equal to the number $N_B = M_\Lambda/m_\Lambda$ of cosmons (quanta of dark energy). The greatness of χ can be understood by the fact that the number of degrees of freedom in the universe is huge, i.e., the universe contains a huge number of cosmons. It can also be understood by the fact that χ is equal to the ratio between the Planck density ρ_P which is the largest density of the universe (it represents the density of the early universe) and the cosmological density ρ_Λ which is the smallest density of the universe (it represents the density of the late universe).

By using the Eddington relation, all the large numbers identified in the past literature can be expressed in terms of powers of χ . For example, the typical age of the universe measured in units of the atomic time (Dirac number) is given by $t_\Lambda/t_a \sim \chi^{1/3} \sim 10^{40}$, the force ratio (Weyl’s number) is given by $F \sim e^2/Gm_p m_e \sim \chi^{1/3} \sim 10^{40}$, the number of protons (or electrons) in the universe (Eddington’s number) is given by $N_p \sim M_\Lambda/m_p \sim \chi^{2/3} \sim 10^{80}$, the number of bosons (cosmons) in the universe is given by $N_B \sim M_\Lambda/m_\Lambda \sim \chi \sim 10^{120}$, and the number of fermions (neutrinos) in the universe is given by $N_F \sim M_\Lambda/m_\Lambda^* \sim \chi^{3/4} \sim 10^{90}$ ($N_F \sim N_B^{3/4}$). Similarly, the typical number of bosons in a mini boson star

finity of mass scales by adopting arbitrary values of a . In our approach, instead of trying to fit the values of a to some characteristic masses observed in Nature, we introduced some relevant mass scales from general considerations based on the fundamental mass limits of fermion, boson and soliton stars and we deduced the corresponding values of a . These considerations showed that Nature follows a certain organization by selecting particular values of $a = 3, 2, 1, 0, -1, -2, -3$ that are more relevant than others.

²⁶ The wonders of the micro and macro cosmos, and their connections, have often amazed scientists, philosophers and poets. In this connection, we would like to mention the beautiful text of Blaise Pascal on *The two infinities* [61] which is one of the monuments of the French literature. The quadricentenary of his birth is commemorated this year.

²⁵ Of course, from the general expression (19) of the mass in terms of the fundamental constants of physics, we can construct an in-

is $\mathcal{N}_B \sim M_{\max}^B/m_e \sim M_P^2/m_e^2 \sim \chi^{1/3} \sim 10^{40}$ and the typical number of fermions in a fermion star is $\mathcal{N}_F \sim M_{\max}^F/m_e \sim M_P^3/m_e^3 \sim \chi^{1/2} \sim 10^{60}$ ($\mathcal{N}_F \sim \mathcal{N}_B^{3/2}$). These numbers are large because they are equal to some powers of $\chi = \rho_P/\rho_\Lambda$ which is large.²⁷ We suggest that the fundamental origin of the large (and small) numbers in Nature is related to the large value of χ , i.e., the fact that the Planck density and the cosmological density differ by 120 orders of magnitude. On the other hand, the fine structure constant $\alpha = e^2/\hbar c \simeq 1/137$ and the mass ratio of proton and electron $\mu = m_p/m_e \simeq 1836$ are of order unity (i.e. χ^0) or, possibly scale as (powers of) $\ln \chi$. The following scalings have been suggested: $1/\alpha \sim (1/2) \ln \chi$ or $1/\alpha \propto (\ln \chi)^{1/2}$ and $\mu \propto 1/\alpha$ or $\mu \propto 1/\alpha^2$ or even $\mu \propto 1/\alpha^{2/3}$ (see [11, 13] and Appendix A).

By eliminating χ between the different large (and small) numbers we can explain the numerous mysterious coincidences observed by previous authors and find new ones. Indeed, the large numbers are related to each other by simple power laws. For example, we have the relations

$$\begin{aligned} \chi &\sim \left(\frac{t_\Lambda}{t_a}\right)^3 \sim F^3 \sim N_p^{3/2} \sim N_B \sim N_F^{4/3} \sim \mathcal{N}_F^2 \sim \mathcal{N}_B^3 \\ &\sim \left(\frac{M_\Lambda}{M_P}\right)^2 \sim \left(\frac{M_2}{M_P}\right)^3 \sim \left(\frac{M_{3/2}}{M_P}\right)^4 \sim \left(\frac{M_1}{M_P}\right)^6 \\ &\sim \left(\frac{M_P}{m_e}\right)^6 \sim \left(\frac{M_P}{m_\Lambda^*}\right)^4 \sim \left(\frac{M_P}{m_2}\right)^3 \sim \left(\frac{M_P}{m_\Lambda}\right)^2 \\ &\sim \left(\frac{t_\Lambda}{t_P}\right)^2 \sim \frac{\rho_P}{\rho_\Lambda} \sim \left(\frac{R_\Lambda}{r_e}\right)^3 \sim \left(\frac{R_\Lambda}{l_P}\right)^2 \sim \left(\frac{m_\Lambda^*}{m_\Lambda}\right)^4 \end{aligned} \quad (101)$$

and many others. These numerical coincidences are different manifestations of a unique fundamental relation between atomic and cosmic structural constants, the Eddington relation (13), which remains highly enigmatic. The Eddington relation is a relation between the elementary mass of the electron m_e and the cosmological constant Λ . In a previous paper [11] we have introduced the refined formula

$$m_e \simeq \alpha \left(\frac{\Lambda \hbar^4}{G^2}\right)^{1/6} \quad \text{or} \quad \Lambda \simeq \frac{G^2 m_e^6}{\alpha^6 \hbar^4}, \quad (102)$$

which is relatively accurate [see Eq. (37) in [11] and Eq. (A9) of Appendix A] and we have given several heuristic justifications of this relation. We view the accurate Eddington relation (102) as a fundamental relation that connects atomic physics (microphysics) and cosmology

(macrophysics). This is one of the most intriguing equations of physics because it is numerically obvious but completely mysterious.²⁸ Since it involves G , \hbar and Λ it is intimately connected to quantum mechanics and general relativity. It may be explained by a future theory of quantum gravity which remains to be constructed. Surprisingly, the Eddington relation is not very well-known among physicists and one motivation of the present contribution was to publicize it (see also our companion paper [13]). The Eddington relation and the mass scale law that we deduced from it forms a bridge between the “macroscopic” and the “microscopic” properties of the universe or between cosmology and particle physics. These connections still remain to be fully understood.

Finally, our study gives some hints how to solve long standing problems in cosmology such as the cosmological constant problem [23, 24], the cosmic coincidence problem [62, 63], and the large number coincidence problem [3, 4]. It is usually argued in particle physics that the cosmological constant should be of the order of the Planck scale (i.e. $\rho_\Lambda \sim \rho_P$ or $\Lambda \sim l_P^{-2}$). The fact that the measured cosmological density ρ_Λ differs from the Planck density ρ_P by about 120 orders of magnitude leads to the so-called cosmological constant problem. However, the Eddington relation (102) suggests that Λ is connected to the mass of the electron instead of the Planck mass, yielding $\rho_\Lambda/\rho_P \sim 10^{-120} \ll 1$ in agreement with the observations. Indeed, the Eddington relation (102) may be rewritten as

$$\Lambda \simeq \frac{1}{\alpha^6 l_P^2} \left(\frac{m_e}{M_P}\right)^6, \quad (103)$$

showing that the small value of the cosmological constant comes from the attenuation factor $(m_e/M_P)^6 \sim \chi^{-1} \sim 10^{-120}$. On the other hand, if we combine the Eddington relation (13) with the Weinberg relation (12), which can be explained by Dicke’s anthropic principle leading to Eq. (14), we find that $H_0 \sim c\sqrt{\Lambda}$ meaning that the age of the universe $t_0 \sim 1/H_0$ is of the order of the cosmological time $t_\Lambda \sim 1/(c\sqrt{\Lambda})$. This implies that the proportions of DM and DE are comparable in the present universe (i.e. we live at the transition between the Einstein-de Sitter era and the de Sitter era). It could even be that $\Omega_{\text{de},0}/\Omega_{\text{dm},0} \simeq e = 2.71828\dots$ [12]. These considerations may solve the cosmic coincidence problem (see Appendix D). Finally, the large number coincidences essentially result from the Eddington relation (102) and the Friedmann equations of cosmology implying the relation $R_\Lambda = GM_\Lambda/c^2$ between the size and the mass of the universe. Depending on the interpretation, the large dimensionless numbers are (enormously) large because (i) the universe contains a large number of protons $N_p \sim 10^{80}$ [15], (ii) the universe is old $t_0/t_a \sim 10^{40}$

²⁷ Of course, by the same argument, very small numbers like the mass of elementary particles normalized by the Planck mass are small because they are equal to some inverse powers of $\chi = \rho_P/\rho_\Lambda$.

²⁸ Weinberg [21] emphasized the fundamental importance of this type of relations and the fact that they are largely unexplained.

Object	a	M_a	M_a/M_P
Universe (soliton star)	3	$M_\Lambda = \frac{c^2}{G\sqrt{\Lambda}}$	$\chi^{1/2} \sim 10^{60}$
fermion star (sun)	2	$M_2 = \left(\frac{\hbar c^9}{G^5 \Lambda^2}\right)^{1/6}$	$\chi^{1/3} \sim 10^{40}$
solid star (earth)	3/2	$M_{3/2} = \left(\frac{\hbar c^5}{G^3 \Lambda}\right)^{1/4}$	$\chi^{1/4} \sim 10^{30}$
mini-boson star	1	$M_1 = \left(\frac{\hbar^2 c^6}{G^4 \Lambda}\right)^{1/6}$	$\chi^{1/6} \sim 10^{20}$
Planck black hole	0	$M_P = \left(\frac{\hbar c}{G}\right)^{1/2}$	$\chi^0 \sim 1$
electron (fermion)	-1	$m_e \sim \left(\frac{\Lambda \hbar^4}{G^2}\right)^{1/6}$	$\chi^{-1/6} \sim 10^{-20}$
neutrino A (fermion)	-3/2	$m_\Lambda^* = \left(\frac{\Lambda \hbar^3}{Gc}\right)^{1/4}$	$\chi^{-1/4} \sim 10^{-30}$
neutrino B (fermion)	-2	$m_2 = \left(\frac{\Lambda^2 \hbar^5}{Gc^3}\right)^{1/6}$	$\chi^{-1/3} \sim 10^{-40}$
cosmon (boson)	-3	$m_\Lambda = \frac{\hbar \sqrt{\Lambda}}{c}$	$\chi^{-1/2} \sim 10^{-60}$

TABLE I: Mass scale law $M_a/M_P \sim \chi^{a/6} \sim (10^{20})^a$ connecting macro objects (universe, fermion stars, mini-boson stars) and micro objects (electron, neutrino, cosmon). The mass of these objects is normalized by the Planck mass $M_P = (\hbar c/G)^{1/2} = 2.18 \times 10^{-5}$ g.

[3, 4], (iii) its entropy is large, i.e., the universe contains a large number of cosmons (quanta of dark energy): $S_\Lambda/k_B \sim M_\Lambda/m_\Lambda \sim N_B \sim \chi \sim 10^{120}$ (see Appendix B). This may be related to a form of H -theorem.

Appendix A: Universal surface density of dark matter halos and the surface density of the electron

It is an observational evidence that the surface density of dark matter halos has a universal value [64]

$$\Sigma_{\text{obs}} \equiv \rho_0 r_h \simeq 295_{-107}^{+173} \text{ g m}^{-2} \simeq 141_{-52}^{+83} M_\odot \text{ pc}^{-2}, \quad (\text{A1})$$

even if their sizes and masses vary by several orders of magnitude (the mass of dark matter halos typically ranges from $10^8 M_\odot$ and $10^{14} M_\odot$). Here, ρ_0 is the central density of a DM halo and r_h is the halo radius at which the central density has been divided by 4. Therefore, the mass of DM halos scales with their radius as $M_h \sim \Sigma r_h^2$. We shall call this mass-radius relation the M - R^2 relation.

In Refs. [9–12], we have developed a cosmological model based on a logotropic equation of state leading to a prediction of the universal surface density of DM halos without free parameter.²⁹ We derived the formula

$$\Sigma_{\text{th}} = 0.01955 \frac{c^2 \sqrt{\Lambda}}{G} = 278 \text{ g m}^{-2} = 133 M_\odot \text{ pc}^{-2}, \quad (\text{A2})$$

²⁹ The density of the logotropic sphere decreases as $\rho \sim r^{-1}$ yielding a constant surface density $\Sigma \sim \rho R \sim 1$ and a mass-radius relation $M \sim R^2$. In comparison, the density of the isothermal sphere decreases as $\rho \sim r^{-2}$ yielding a constant M/R ratio and a mass-radius relation $M \sim R$ like for black holes (see [65] for the analogy between isothermal spheres and black holes). In a sense, the logotropic equation of state can be interpreted as a sort of isothermal equation of state in the context of generalized thermodynamics [9–12].

which turns out to be in good agreement with the observational (empirical) value from Eq. (A1). We stress that the prefactor in Eq. (A2) is determined by the theory³⁰ so there is no adjustable parameter in our model. This relation expresses the universal surface density of DM halos in terms of the fundamental constants of physics G , c and Λ .

We also predicted that the present ratio of dark energy (DE) and dark matter (DM) is equal to the Euler number:

$$\frac{\Omega_{\text{de},0}^{\text{th}}}{\Omega_{\text{dm},0}^{\text{th}}} = e = 2.71828\dots \quad (\text{A3})$$

This prediction is in good agreement with the observational value $\Omega_{\text{de},0}^{\text{obs}}/\Omega_{\text{dm},0}^{\text{obs}} = 2.669 \pm 0.08$ within the error bars. This result is disturbing because the ratio of DE and DM changes with time so it is only at the present epoch (“now”) that it is equal (or close) to e . This suggests that our epoch plays a particular role in the history of the universe (see Fig. 3 in [12]). This is a form of “strong cosmic coincidence” (see Appendix D). In the absence of an explanation, we called this intriguing result “dark magic” [12].³¹

It is not clear if our logotropic theory is correct or if the two predictions (A2) and (A3) are pure coincidences. Anyway, the relation from Eq. (A2), which can be checked by a direct numerical application, is interesting in itself. It shows that the order of magnitude of the surface density of DM halos can be expressed in terms of

³⁰ It is given by $(B/32)^{1/2} \xi_h / \pi$ where $B = 1/\ln(\rho_P/\rho_\Lambda) = 3.53 \times 10^{-3}$ and $\xi_h = 5.85$ is the normalized radius of a polytrope of index $n = -1$ (logotrope) derived from the Lane-Emden equation [9–12]. Note that $B = 1/\ln \chi$.

³¹ More generally, the expression “dark magic” could refer to all the mysteries of dark matter and dark energy.

the fundamental constants of physics as³²

$$\Sigma_\Lambda = \frac{c^2\sqrt{\Lambda}}{G} = 1.42 \times 10^4 \text{ g m}^{-2} = 6800 M_\odot/\text{pc}^2. \quad (\text{A4})$$

This quantity is of the order of the surface density of the universe $\Sigma_\Lambda = M_\Lambda/R_\Lambda^2 = c^2\sqrt{\Lambda}/G$ (see Sec. IV B).

We then remarked that the surface density of DM halos (or the surface density of the Universe) is of the same order of magnitude as the surface density of the electron. The classical radius of the electron r_e can be obtained qualitatively by writing that the electrostatic energy of the electron, e^2/r_e , is equal to its rest-mass energy $m_e c^2$ i.e.³³

$$m_e c^2 = \frac{e^2}{r_e}. \quad (\text{A5})$$

This relation was originally obtained in connection to the Abraham-Lorentz [66, 67] model of the extended electron, in which the mass has an electromagnetic origin, and in the Born-Infeld [68, 69] theory of nonlinear electrodynamics (see a short history of these old models in Appendix F of [40]). Recalling the value of the charge of the electron $e = 4.80 \times 10^{-13} \text{ g}^{1/2} \text{ m}^{3/2} \text{ s}^{-1}$ and its mass $m_e = 9.11 \times 10^{-28} \text{ g}$, we obtain $r_e = e^2/m_e c^2 = 2.82 \times 10^{-15} \text{ m}$.³⁴ As a result, the typical surface density of the electron is

$$\Sigma_e = \frac{m_e}{r_e^2} = \frac{m_e^3 c^4}{e^4} = 115 \text{ g m}^{-2} = 54.9 M_\odot \text{ pc}^{-2}. \quad (\text{A6})$$

As we can see, the surface density of the electron is of the same order of magnitude as the surface density of dark matter halos from Eq. (A2) or as the surface density of the universe from Eq. (A4). They differ by a factor 2 – 100 which can be regarded as being of order unity in our qualitative approach. Therefore

$$\Sigma_e \sim \Sigma \sim \Sigma_\Lambda. \quad (\text{A7})$$

³² We note that pure dimensional analysis or heuristic considerations leading to Eq. (A4) are not sufficient to explain the observational result from Eq. (A1). We need the value of the prefactor 0.01955 in Eq. (A2) which is predicted by our model. However, if we use the Einstein gravitational constant $8\pi G$ (or $\kappa = 8\pi G/c^4$) instead of the Newton constant G , we obtain $\Sigma_{\text{th}} = 0.491 c^2\sqrt{\Lambda}/(8\pi G)$ where the prefactor is of order unity (actually close to 1/2). Therefore, the formula $\Sigma'_\Lambda = c^2\sqrt{\Lambda}/(8\pi G) = \sqrt{\Lambda}/(\kappa c^2)$ provides a relatively accurate order of magnitude of the surface density of DM halos.

³³ Similarly, the condition $m_e c^2 \sim Gm^2/r$ gives the (semi) Schwarzschild radius $r = Gm/c^2$.

³⁴ We note that the classical radius of the electron can be written as $r_e = a\hbar/m_e c = \alpha\lambda_e$ where $\lambda_e = \hbar/m_e c = 3.86 \times 10^{-13} \text{ m}$ is the Compton wavelength of the electron and $\alpha = e^2/\hbar c \simeq 1/137$ is the fine-structure constant. Therefore, the classical electron radius is smaller than the scale at which quantum effects come into play. We also note that the semi gravitational radius of the electron is $r_g = Gm_e/c^2 = 6.76 \times 10^{-58} \text{ m}$.

This coincidence is amazing in view of the different scales (atomic versus cosmological) involved.³⁵ This provides a curious connection between microphysics and cosmophysics.

Equating Eqs. (A4) and (A6), and using $e^2 \sim \hbar c$, we obtain

$$m_e \sim \left(\frac{\Lambda\hbar^4}{G^2}\right)^{1/6}. \quad (\text{A8})$$

This equation relates the mass of the electron to the cosmological constant (or the converse) and the other fundamental constants of physics. It corresponds to the relation (13) that Eddington obtained in a different manner.³⁶ The fact that the Eddington relation (A8) expresses the commensurability of the surface density of the Universe and the surface density of the electron [see Eq. (A7)] may help elucidating its physical meaning. Other derivations of this relation are given in Appendix C based on entropic arguments and the holographic principle. The Eddington relation (A8) is not very accurate: the left and side and the right hand side differ by a factor ~ 100 . However, the refined relation [11, 13]

$$m_e \simeq \alpha \left(\frac{\Lambda\hbar^4}{G^2}\right)^{1/6}, \quad (\text{A9})$$

where $\alpha = e^2/\hbar c \simeq 1/137$ is the Sommerfeld fine-structure constant, gives a relatively accurate determination of the mass of the electron, or conversely, of the cosmological constant (the value of the right hand side of Eq. (A9) is $8.80 \times 10^{-28} \text{ g}$ which is close to $m_e = 9.11 \times 10^{-28} \text{ g}$). *This close agreement suggests that relation (A9) may have a fundamental significance [11, 13].* Furthermore, the form of this equation suggests an expansion in powers of $\alpha \ll 1$. Eq. (A9) could be the leading term of this expansion and subsequent (perturbative) terms may further improve the agreement with the mass of the electron. Using the accurate Eddington relation (A9), the cosmological density can be written as

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G} \simeq \frac{Gm_e^6 c^2}{8\pi\alpha^6 \hbar^4} \simeq \frac{Gm_e^6 c^8 \hbar^2}{8\pi e^{12}} \quad (\text{A10})$$

³⁵ By comparison, the volume density of the electron $\rho_e \sim m_e/r_e^3 = 4.07 \times 10^{16} \text{ g m}^{-3}$ differs from the volume density of the universe $\rho_\Lambda = 5.96 \times 10^{-24} \text{ g m}^{-3}$ by 40 orders of magnitude (by using the results of the main text, we can easily establish that $\rho_e/\rho_\Lambda \sim \chi^{1/3} \sim 10^{40}$).

³⁶ We arrived at this relation by ourselves in Ref. [11] by developing the logotropic model along the lines described previously. We then remembered having seen a similar relation in the book of Weinberg [21]. He mentioned that this kind of relation was established by several scientists before him but did not give references. We made an exhaustive bibliographic work to retrace the history of this relation that finds its origin in the work of Eddington [5]. A detailed history of the Eddington relation will be presented in a forthcoming paper [13].

and the electron radius as

$$r_e = \alpha \lambda_e \simeq \left(\frac{G^2 \hbar^2}{\Lambda c^6} \right)^{1/6}. \quad (\text{A11})$$

We have the relation

$$r_e^3 \simeq l_p^2 R_\Lambda. \quad (\text{A12})$$

Finally, if we account for the $B = 1/\ln \chi$ factor in Eq. (A2) (see footnote 30) and repeat the above arguments, we get [11]

$$\alpha \propto B^{1/2} \propto (\ln \chi)^{-1/2}, \quad (\text{A13})$$

suggesting that the fine-structure constant is related to the logarithm of the cosmological constant.

The accurate Eddington relation (A9) is equivalent to the expression

$$\frac{r_e^2}{R_\Lambda r_g} \simeq \frac{1}{\alpha}, \quad (\text{A14})$$

where $r_e = e^2/m_e c^2$ is the classical radius of the electron, $r_g = Gm_e/c^2$ is the semi gravitational radius of the electron and $R_\Lambda = 1/\sqrt{\Lambda}$ is the radius of the universe. It is also equivalent to

$$R_\Lambda^2 \simeq \alpha N_e r_e^2, \quad (\text{A15})$$

where $N_e = M_\Lambda/m_e$ is the number of electrons in the universe. Equation (A15) can be rewritten as

$$\frac{m_e}{r_e^2} \simeq \alpha \frac{M_\Lambda}{R_\Lambda^2}. \quad (\text{A16})$$

This formula shows that the surface density of the electron is equal to the surface density of the universe up to a factor α (i.e. $\Sigma_e \simeq \alpha \Sigma_\Lambda$).

Instead of working with the surface density of the electron, we could consider working with the surface density of the proton of mass $m_p = 1.67 \times 10^{-24}$ g. The proton radius is of the order of its Compton wavelength $\lambda_p = \hbar/m_p c = 2.10 \times 10^{-16}$ m. Therefore, its surface density is

$$\Sigma_p = \frac{m_p}{\lambda_p^2} = \frac{m_p^3 c^2}{\hbar^2}. \quad (\text{A17})$$

The surface density of the electron can be written as

$$\Sigma_e = \frac{m_e}{r_e^2} = \frac{m_e^3 c^4}{e^4} = \frac{m_e^3 c^2}{\alpha^2 \hbar^2}. \quad (\text{A18})$$

Using $m_p = \mu m_e$, we get

$$\frac{m_p}{\lambda_p^2} = \mu^3 \alpha^2 \frac{m_e}{r_e^2}. \quad (\text{A19})$$

The prefactor is $\mu^3 \alpha^2 = 3.30 \times 10^5$. Combining this relation with Eq. (A16) we get

$$\frac{m_p}{\lambda_p^2} \simeq \mu^3 \alpha^3 \frac{M_\Lambda}{R_\Lambda^2}. \quad (\text{A20})$$

The prefactor is $\mu^3 \alpha^3 = 2.40 \times 10^3$. If we require that the prefactor is independent of α we find that

$$\mu = \frac{m_p}{m_e} = \frac{k}{\alpha} \quad (\text{A21})$$

with $k \simeq 13.4$ (empirical). This suggests that $\mu \propto 1/\alpha$.³⁷ We find the same scaling if we compare the accurate Eddington relation (A9) with the original Eddington relation (1). The accurate Eddington relation (A9) can therefore be written as

$$\frac{m_p}{\lambda_p^2} = k^3 \frac{M_\Lambda}{R_\Lambda^2} \quad (\text{A22})$$

or, equivalently, as

$$R_\Lambda^2 = k^3 N_p \lambda_p^2, \quad (\text{A23})$$

where $N_p = M_\Lambda/m_p$ is the number of protons in the universe. Finally, combining the relation $r_e = e^2/m_e c^2 = \alpha \hbar \mu / m_p c = \alpha \mu \lambda_p$ with Eq. (A21), we find that

$$r_e = k \lambda_p. \quad (\text{A24})$$

Therefore, the electron radius is of the order of the Compton wavelength of the proton $r_e \sim \lambda_p$ (up to a factor $k \simeq 13.4$).

Remark: The universality of the surface density of DM halos is equivalent to the universality of their surface gravity (or gravitational acceleration)

$$a = \frac{GM_h}{r_h^2} \sim G\Sigma. \quad (\text{A25})$$

Using Eq. (A4), we get

$$a_{\text{th}} \sim a_\Lambda \sim G\Sigma_\Lambda \sim c^2 \sqrt{\Lambda} \sim c^2 / R_\Lambda \sim H_0 c. \quad (\text{A26})$$

The surface gravity of DM halos a is of the order of the surface gravity of the universe $a_\Lambda = GM_\Lambda/R_\Lambda^2 = R_\Lambda/t_\Lambda^2 = c/t_\Lambda$. To obtain the last equality in Eq. (A26) we have used the relation $H_0 \sim c\sqrt{\Lambda}$ coming from the cosmic coincidence (see Appendix D). A more careful analysis [11, 12] based on Eq. (A2) gives $a_{\text{th}} \simeq H_0 c/4 = 1.65 \times 10^{-10}$ m/s² for the universal surface gravity of DM halos, which is close to the value of the fundamental acceleration appearing in the MOND theory [70]. We note that a universal surface density $\Sigma \sim M_h/r_h^2$ (or a universal surface gravity $a \sim GM_h/r_h^2$) combined with the virial relation $v_c^2 \sim GM_h/r_h$, where v_c is the circular velocity, leads to the Tully-Fisher relation $v_c^4 \sim \Sigma G^2 M_h \sim a GM_h$ (see [11, 12] for a more precise discussion). We also note

³⁷ The value of k is close to $1/(10\alpha)$ so we also have $\mu \simeq 1/(10\alpha^2)$ which suggests $\mu \propto 1/\alpha^2$. It is not easy to distinguish between the two scalings $\mu \propto 1/\alpha$ and $\mu \propto 1/\alpha^2$ [13]. We could also require that the prefactor in Eq. (A19) is independent of α . In that case, we find that $\mu = k'/\alpha^{2/3}$ with $k' \simeq 69.1$ (empirical).

that the surface gravity $a = GM/R^2$ of a Schwarzschild black hole with radius $R = 2GM/c^2$ is

$$a = \frac{c^4}{4GM}. \quad (\text{A27})$$

If we apply this relation to the Universe as a whole (interpreted as a huge black hole) of mass $M \sim M_\Lambda \sim c^2/G\sqrt{\Lambda}$, we recover Eq. (A26). This is because the Universe satisfies both the M - R^2 relation and the Schwarzschild relation (see Appendix B). Finally, Eq. (A7) implies that

$$a_e \sim a \sim a_\Lambda. \quad (\text{A28})$$

The surface gravity of the electron $a_e = Gm_e/r_e^2$ is of the order of the surface gravity of DM halos $a = GM_h/r_h^2$ and of the surface gravity of the Universe $a_\Lambda = GM_\Lambda/R_\Lambda^2$. This is equivalent to the Eddington relation (A8).

Appendix B: Holographic principle and reciprocity property

1. Holographic principle

The holographic principle [71–73] states that the entropy S of a physical system subject to gravity is bounded from above by a quarter of its boundary area $A = 4\pi R^2$ in Planck units:

$$S \leq k_B \frac{A}{4l_P^2}. \quad (\text{B1})$$

As a result, the physical degrees of freedom are not proportional to the volume in the presence of the gravitational field, but reside on the bounding surface like a hologram. This form of entropy was initially introduced in the context of black hole physics by Bekenstein [74] and Hawking [32] where A is the area of the horizon. In the following, we shall only work with orders of magnitude and assume that the entropy of an object scales as

$$S \sim k_B \left(\frac{R}{l_P} \right)^2. \quad (\text{B2})$$

The associated temperature $T \sim E/S \sim Mc^2/S$ reads

$$k_B T \sim \frac{GM\hbar}{R^2 c} \sim \frac{G\Sigma\hbar}{c} \sim \frac{\hbar a}{c}, \quad (\text{B3})$$

where we have introduced the surface density $\Sigma \sim M/R^2$ and the surface gravity (acceleration) $a \sim GM/R^2 \sim G\Sigma$ of the object. This is the so-called Unruh temperature [75].

For a Schwarzschild black hole, using the relation

$$R = \frac{2GM}{c^2} \quad (\text{B4})$$

between its mass and its radius, we obtain the Bekenstein-Hawking entropy³⁸

$$S_{\text{BH}} \sim k_B \left(\frac{M}{M_P} \right)^2. \quad (\text{B5})$$

It is proportional to the square of the mass of the black hole. The Bekenstein-Hawking temperature $T_{\text{BH}} \sim E/S_{\text{BH}} \sim Mc^2/S_{\text{BH}}$ reads

$$k_B T_{\text{BH}} \sim \frac{M_P^2 c^2}{M} \sim \frac{\hbar c^3}{GM} \sim \frac{\hbar c}{R}, \quad (\text{B6})$$

or, in Planck units,

$$\frac{T_{\text{BH}}}{T_P} \sim \frac{M_P}{M} \sim \frac{l_P}{R}. \quad (\text{B7})$$

These expressions can also be obtained from the Unruh temperature (B3) by using Eq. (A27) valid for a black hole. Since $E = Mc^2 \sim \hbar c^5/(Gk_B T)$ the black holes have a negative specific heat

$$C = \frac{dE}{dT} \sim -\frac{\hbar c^5}{Gk_B T^2} < 0. \quad (\text{B8})$$

By losing heat, they grow hotter. This strange property is not limited to black holes. Self-gravitating systems [76–78] and, more generally, systems with long-range interactions [79] may also have negative specific heats in the microcanonical ensemble (the statistical ensembles are inequivalent for non additive systems).

On the other hand, we have seen that certain objects such as the electron, the DM halos and the universe satisfy the mass-radius relation

$$M \sim \Sigma_\Lambda R^2 \quad (\text{B9})$$

with

$$\Sigma_\Lambda = \frac{c^2 \sqrt{\Lambda}}{G}. \quad (\text{B10})$$

We called this law the M - R^2 relation. Combining Eqs. (B2) and (B9), and noting that $m_\Lambda = \Sigma_\Lambda l_P^2$, we find that the entropy of the objects satisfying the M - R^2 relation scales as

$$\tilde{S} \sim k_B \frac{M}{m_\Lambda} \sim k_B N. \quad (\text{B11})$$

Interestingly, their entropy turns out to be proportional to the number $N = M/m_\Lambda$ of cosmons that they contain

³⁸ Coincidentally, the initials stand either for “black hole” or “Bekenstein-Hawking”.

(in a real or in an effective sense).³⁹ The corresponding temperature $\tilde{T} \sim E/\tilde{S} \sim Mc^2/\tilde{S}$ reads

$$k_B \tilde{T} \sim \hbar c \sqrt{\Lambda} \sim m_\Lambda c^2, \quad (\text{B14})$$

or, in Planck units,

$$\frac{\tilde{T}}{T_P} \sim \frac{m_\Lambda}{M_P} \sim \chi^{-1/2} \sim 10^{-60}. \quad (\text{B15})$$

These expressions can also be obtained from the Unruh temperature (B3) by using Eq. (A26) valid for objects that satisfy the M - R^2 relation. These results apply to the Universe, the DM halos, and the electron (and possibly others systems). These objects have a universal temperature \tilde{T} equal to the rest mass energy of the cosmon. This is the temperature of the vacuum. It can be regarded as the smallest temperature in the universe. Numerically, $\tilde{T} \sim 2.41 \times 10^{-29}$ K.

2. Examples

Let us specifically discuss different examples:

(i) The Universe (which can be interpreted as a huge black hole) satisfies both the Schwarzschild relation (B4) and the M - R^2 relation (B9). Using either the entropy (B5) or the entropy (B11) with $M = M_\Lambda$, we obtain

$$S_\Lambda/k_B \sim \left(\frac{M_\Lambda}{M_P}\right)^2 \sim \frac{M_\Lambda}{m_\Lambda} \sim \chi \sim 10^{120}. \quad (\text{B16})$$

This is the largest value of the entropy that we can obtain. This result suggests that 10^{120} may be interpreted

³⁹ Writing $M = Nm_\Lambda$, we can interpret the cosmon as the quantum of mass. Indeed, for $N = 1$ (ground state), we have $M_{\text{ground}} = m_\Lambda$. This is consistent with the quantization of mass proposed by Wesson [55]. Comparing Eqs. (B2) and (B11) we also have $R \sim \sqrt{N} l_P$. This suggests that the minimum length scale in nature ($N = 1$) is the Planck length l_P . We therefore propose the quantization of mass, length and entropy as

$$M \sim N m_\Lambda, \quad R \sim \sqrt{N} l_P, \quad \tilde{S} \sim N k_B, \quad (\text{B12})$$

where N is an integer. This is valid for objects that respect the $M - R^2$ relation (this is not valid for black holes except for the universe as a whole). The ground state ($N = 1$) is the cosmon with mass m_Λ , size l_P and entropy $\tilde{S}_c \sim k_B$ (one bit). These results can be compared with the quantization of black holes proposed by He and Ma [80] leading to

$$R \sim \sqrt{n} l_P, \quad M \sim \sqrt{n} M_P, \quad S_{\text{BH}} \sim n k_B, \quad (\text{B13})$$

where n is an integer. The ground state ($n = 1$) is the Planck black hole with mass M_P , size l_P and entropy $S_P \sim k_B$ (one bit). This is the smallest stable black hole. There should not be black holes with a mass smaller than the Planck mass. This conjecture supports the existence of primordial black holes. They range from mini black holes of the Planck scale produced at $t_P \sim 5.39 \times 10^{-44}$ s to very big ones with large n .

as the ‘‘largest large number’’ in Nature.⁴⁰ Using Eq. (B6) with $M = M_\Lambda$ or Eq. (B14), the temperature of the universe is

$$k_B T_\Lambda \sim \hbar c \sqrt{\Lambda} \sim m_\Lambda c^2, \quad (\text{B17})$$

i.e., $T_\Lambda \sim 2.41 \times 10^{-29}$ K. It is equal to the rest mass energy of the cosmon.⁴¹ In Planck units, we have

$$\frac{T_\Lambda}{T_P} \sim \frac{M_P}{M_\Lambda} \sim \frac{l_P}{R_\Lambda} \sim \chi^{-1/2} \sim 10^{-60}. \quad (\text{B18})$$

(ii) For a black hole resulting from the collapse of a fermion star, using Eq. (B5) with $M = M_2$, we obtain

$$S_2/k_B \sim \left(\frac{M_2}{M_P}\right)^2 \sim \chi^{2/3} \sim 10^{80}. \quad (\text{B19})$$

Using Eq. (B6), its temperature is

$$k_B T_2 \sim \left(\frac{\Lambda^2 \hbar^5}{G c^3}\right)^{1/6} c^2 \sim m_2 c^2, \quad (\text{B20})$$

i.e., $T_2 \sim 4.35 \times 10^{-9}$ K. It is equal to the rest mass energy of the neutrino B. In Planck units, we have

$$\frac{T_2}{T_P} \sim \frac{M_P}{M_2} \sim \frac{l_P}{R_2} \sim \chi^{-1/3} \sim 10^{-40}. \quad (\text{B21})$$

For a dark energy star of mass M_2 and radius $\tilde{R}_2 \sim (\hbar G/c^3 \Lambda^5)^{1/12} \sim \chi^{5/12} l_P \sim 7.07 \times 10^{15}$ m satisfying the $M - R^2$ relation we get

$$\tilde{S}_2/k_B \sim \frac{M_2}{m_\Lambda} \sim \chi^{5/6} \sim 10^{100}. \quad (\text{B22})$$

(iii) For a black hole resulting from the collapse of a solid star, using Eq. (B5) with $M = M_{3/2}$, we obtain

$$S_{3/2}/k_B \sim \left(\frac{M_{3/2}}{M_P}\right)^2 \sim \chi^{1/2} \sim 10^{60}. \quad (\text{B23})$$

⁴⁰ Since the radius of the visible universe increases as $R = ct$ the mass of the universe increases as $M \sim c^2 R/G \sim c^3 t/G$ for $t \leq t_\Lambda$ [13]. It goes from the Planck mass M_P at t_P (primordial universe) to the cosmological mass M_Λ at t_Λ (now). As a result, the entropy of the universe $S_{\text{BH}}/k_B \sim (R/l_P)^2 \sim (M/M_P)^2$ increases as $S_{\text{BH}} \sim t^2$ (the corresponding temperature $T_{\text{BH}}/T_P \sim l_P/R \sim M_P/M$ decreases as $T_{\text{BH}} \sim 1/t$). This forms an H -theorem and explains the large value of the entropy $S_\Lambda/k_B \sim (R_\Lambda/l_P)^2 \sim (M_\Lambda/M_P)^2 \sim 10^{120}$ of the universe at the present epoch (it was of order $S_P/k_B \sim (l_P/l_P)^2 \sim (M_P/M_P)^2 \sim 1$ at the Planck time). By contrast, the usual entropy is conserved by the Friedmann equations and has the value $S/k_B \sim (t_\Lambda/t_P)^{3/2} \sim \chi^{3/4} \sim 10^{90}$ [25].

⁴¹ Using the relation $H_0 \sim c\sqrt{\Lambda}$ resulting from the cosmic coincidence (see Appendix D) we can rewrite Eq. (B17) as $k_B T_H \sim \hbar H_0$. This is the so-called Gibbons-Hawking [81] temperature. It can be obtained from the Heisenberg uncertainty principle $E \times t_0 \sim \hbar$ by taking $E \sim k_B T$ and $t_0 \sim 1/H_0$.

Using Eq. (B6), its temperature is

$$k_B T_{3/2} \sim \left(\frac{\Lambda \hbar^3}{Gc} \right)^{1/4} c^2 \sim m_\Lambda^* c^2, \quad (\text{B24})$$

i.e., $T_{3/2} \sim 58.4 \text{ K}$.⁴² It is equal to the rest mass energy of the neutrino A. In Planck units, we have

$$\frac{T_{3/2}}{T_P} \sim \frac{M_P}{M_{3/2}} \sim \frac{l_P}{R_{3/2}} \sim \chi^{-1/4} \sim 10^{-30}. \quad (\text{B25})$$

For a dark energy star of mass $M_{3/2}$ and radius $\tilde{R}_{3/2} \sim (\hbar G/c^3 \Lambda^3)^{1/8} \sim \chi^{3/8} l_P \sim 6.10 \times 10^{10} \text{ m}$ satisfying the $M - R^2$ relation we get

$$\tilde{S}_{3/2}/k_B \sim \frac{M_{3/2}}{m_\Lambda} \sim \chi^{3/4} \sim 10^{90}. \quad (\text{B26})$$

(iv) For a black hole resulting from the collapse of a mini boson star, using Eq. (B5) with $M = M_1$, we obtain

$$S_1/k_B \sim \left(\frac{M_1}{M_P} \right)^2 \sim \chi^{1/3} \sim 10^{40}. \quad (\text{B27})$$

Using Eq. (B6), its temperature is

$$k_B T_1 \sim \left(\frac{\Lambda \hbar^4}{G^2} \right)^{1/6} c^2 \sim m_e c^2, \quad (\text{B28})$$

i.e., $T_1 \sim 7.88 \times 10^{11} \text{ K}$. It is equal to the rest mass energy of the electron. In Planck units, we have

$$\frac{T_1}{T_P} \sim \frac{M_P}{M_1} \sim \frac{l_P}{R_1} \sim \chi^{-1/6} \sim 10^{-20}. \quad (\text{B29})$$

For a dark energy star of mass M_1 and radius $\tilde{R}_1 \sim (\hbar G/c^3 \Lambda^2)^{1/6} \sim \chi^{1/3} l_P \sim 5.26 \times 10^5 \text{ m}$ satisfying the $M - R^2$ relation we get

$$\tilde{S}_1/k_B \sim \frac{M_1}{m_\Lambda} \sim \chi^{2/3} \sim 10^{80}. \quad (\text{B30})$$

(v) For a Planck black hole, using Eq. (B5) with $M = M_P$, we obtain

$$S_P/k_B \sim 1, \quad (\text{B31})$$

corresponding to one bit (see footnote 39). Using Eq. (B6) the temperature of a Planck black hole is the Planck temperature

$$k_B T_P \sim M_P c^2, \quad (\text{B32})$$

i.e., $T_P \sim 1.42 \times 10^{32} \text{ K}$. It is equal to its rest mass energy. For a dark energy star of mass M_P and radius

$\tilde{R}_P \sim (\hbar G/c^3 \Lambda)^{1/4} \sim \chi^{1/4} l_P \sim 3.92 \times 10^{-5} \text{ m}$ satisfying the $M - R^2$ relation we get

$$\tilde{S}_P/k_B \sim \frac{M_P}{m_\Lambda} \sim \chi^{1/2} \sim 10^{60}. \quad (\text{B33})$$

(vi) For a black hole with the mass m_e of the electron and a radius r_g , using Eq. (B5) with $M = m_e$, we obtain

$$S_e/k_B \sim \left(\frac{m_e}{M_P} \right)^2 \sim \chi^{-1/3} \sim 10^{-40}. \quad (\text{B34})$$

Using Eq. (B6), its temperature is

$$k_B T_e \sim \left(\frac{\hbar^4 c^6}{G^4 \Lambda} \right)^{1/6} c^2 \sim M_1 c^2, \quad (\text{B35})$$

i.e., $T_e \sim 2.56 \times 10^{52} \text{ K}$. It is equal to the rest mass energy of a mini boson star. In Planck units, we have

$$\frac{T_e}{T_P} \sim \frac{M_P}{m_e} \sim \frac{l_P}{r_g} \sim \chi^{1/6} \sim 10^{20}. \quad (\text{B36})$$

We also know that the electron of mass m_e and radius r_e satisfies the $M - R^2$ relation (see Appendix A). This suggests that the electron is made of quanta of DE (cosmons). Using Eq. (B11) with $M = m_e$, we get

$$\tilde{S}_e/k_B \sim \frac{m_e}{m_\Lambda} \sim \chi^{1/3} \sim 10^{40}. \quad (\text{B37})$$

(vii) For a black hole with the mass m_Λ^* of the neutrino A and a radius r_Λ^* , using Eq. (B5) with $M = m_\Lambda^*$, we obtain

$$S'_{3/2}/k_B \sim \left(\frac{m_\Lambda^*}{M_P} \right)^2 \sim \chi^{-1/2} \sim 10^{-60}. \quad (\text{B38})$$

Using Eq. (B6), its temperature is

$$k_B T'_{3/2} \sim \left(\frac{\hbar c^5}{G^3 \Lambda} \right)^{1/4} c^2 \sim M_{3/2} c^2, \quad (\text{B39})$$

i.e., $T'_{3/2} \sim 3.43 \times 10^{62} \text{ K}$. It is equal to the rest mass energy of a solid star. In Planck units, we have

$$\frac{T'_{3/2}}{T_P} \sim \frac{M_P}{m_\Lambda^*} \sim \frac{l_P}{r_\Lambda^*} \sim \chi^{1/4} \sim 10^{30}. \quad (\text{B40})$$

If we consider that the neutrino A of mass m_Λ^* is made of quanta of DE (cosmons) and that it satisfies the $M - R^2$ relation it would have an effective radius $\tilde{R}'_{3/2} \sim (\hbar^3 G^3/c^9 \Lambda)^{1/8} \sim \chi^{1/8} l_P \sim 2.52 \times 10^{-20} \text{ m}$ and an entropy

$$\tilde{S}'_{3/2}/k_B \sim \frac{m_\Lambda^*}{m_\Lambda} \sim \chi^{1/4} \sim 10^{30}. \quad (\text{B41})$$

(viii) For a black hole with the mass m_2 of the neutrino B and a radius r_2 , using Eq. (B5) with $M = m_2$, we obtain

$$S'_2/k_B \sim \left(\frac{m_2}{M_P} \right)^2 \sim \chi^{-2/3} \sim 10^{-80}. \quad (\text{B42})$$

⁴² Interestingly, it is of the order of the earth temperature within a factor 100.

Using Eq. (B6), its temperature is

$$k_B T'_2 \sim \left(\frac{\hbar c^9}{G^5 \Lambda^2} \right)^{1/6} c^2 \sim M_2 c^2, \quad (\text{B43})$$

i.e., $T'_2 \sim 4.61 \times 10^{72}$ K. It is equal to the rest mass energy of a fermion star. In Planck units, we have

$$\frac{T'_2}{T_P} \sim \frac{M_P}{m_2} \sim \frac{l_P}{r_2} \sim \chi^{1/3} \sim 10^{40}. \quad (\text{B44})$$

If we consider that the neutrino B of mass m_2 is made of quanta of DE (cosmons) and that it satisfies the $M - R^2$ relation it would have an effective radius $\tilde{R}'_2 \sim (\hbar^5 G^5 / c^{15} \Lambda)^{1/12} \sim \chi^{1/12} l_P \sim 2.17 \times 10^{-25}$ m and an entropy

$$\tilde{S}'_2 / k_B \sim \frac{m_2}{m_\Lambda} \sim \chi^{1/6} \sim 10^{20}. \quad (\text{B45})$$

(ix) For a black hole with the mass m_Λ of the cosmon and a radius r_Λ , using Eq. (B5) with $M = m_\Lambda$, we obtain

$$S_c / k_B \sim \left(\frac{m_\Lambda}{M_P} \right)^2 \sim \chi^{-1} \sim 10^{-120}. \quad (\text{B46})$$

This is the “smallest small number” of Nature. It is equal to the inverse of the entropy of the universe: $S_c = 1/S_\Lambda$. Using Eq. (B6), the temperature of a black hole with the mass of the cosmon is

$$k_B T_c \sim \frac{c^4}{G\sqrt{\Lambda}} \sim M_\Lambda c^2, \quad (\text{B47})$$

i.e., $T_c \sim 8.33 \times 10^{92}$ K. It is equal to the rest mass energy of the universe. In Planck units, we have

$$\frac{T_c}{T_P} \sim \frac{M_P}{m_\Lambda} \sim \frac{l_P}{r_\Lambda} \sim \chi^{1/2} \sim 10^{60}. \quad (\text{B48})$$

We may also hypothesize that the cosmon of mass m_Λ satisfies the $M - R^2$ relation (B9). In that case, since $m_\Lambda = \Sigma_\Lambda l_P^2$, its effective radius is of the order of the Planck length

$$r_c \sim l_P \quad (\text{B49})$$

and its entropy is

$$\tilde{S}_c / k_B \sim 1, \quad (\text{B50})$$

corresponding to one bit. This is consistent with the fact that the cosmon is the quantum of mass (see footnote 39).

Remark: For the sake of completeness (and curiosity), we have considered the possibility that elementary particles of mass $m < M_P$ could be black holes. However, it is usually believed that the Planck black hole is the smallest stable black hole (see footnote 39). Indeed, the condition that the gravitational radius $R_S = 2GM/c^2$

be larger than the Compton wavelength $\lambda_C = \hbar/Mc$ requires that $M \geq M_P$. On the other hand, an entropy smaller than 1 bit is unlikely or even impossible. Finally, the Hawking evaporation time of black holes with a mass $m < M_P$ would be smaller than the Planck time so they would have too short lifetimes (for the cosmon interpreted as a black hole one finds $t_{\text{evap}} \sim 10^{-226}$ s). Therefore, the particles of mass $m < M_P$ are probably not black holes. For example, it is more likely that the cosmon satisfies the $M - R^2$ relation (B9) rather than the black hole relation (B4). In that case, it would be described by the entropy (B11) giving 1 bit while the Bekenstein-Hawking entropy (B5) gives the unphysical number 10^{-120} bit. This is consistent with the fact that the cosmon is the quantum of mass. Similarly, we suggest that other elementary particles and other macroscopic objects (DE stars) could be made of quanta of DE and satisfy the $M - R^2$ relation (B9).⁴³ This relation is valid at least for the electron, the DM halos and the Universe (see Appendix A). The condition that the radius $R \sim (M/\Sigma_\Lambda)^{1/2}$ be larger than the Compton wavelength $\lambda_C = \hbar/(Mc)$ requires that $M \geq m_e$. However, for lighter particles like the cosmon, the radius $R \sim (M/\Sigma_\Lambda)^{1/2}$ may still be relevant as an “effective” radius. The fact that the Compton wavelength of the cosmon is of the order of the size of the universe means that it cannot be localized, but the cosmon could still be characterized by an effective radius $r_c \sim (m_\Lambda/\Sigma_\Lambda)^{1/2}$. Interestingly, this radius turns out to be of the order of the Planck length l_P . Our approach is clearly very speculative (and perfectible) but it points out the possible fundamental importance of the $M - R^2$ relation (B9) and (B10). In a sense, the relation $R \sim (G^2 M^2 / \Lambda c^4)^{1/4}$ for objects made of DE quanta is the counterpart of the Schwarzschild relation $R = 2GM/c^2$ for black holes. It involves the cosmological constant Λ in addition to the gravitational constant G and the speed of light c .

3. Reciprocity property

The above results and the mass scale law (20) suggest a reciprocity property between micro and macro objects. Indeed, according to Eq. (20) we have

$$\frac{M_{-a}}{M_P} \sim \frac{M_P}{M_a}. \quad (\text{B52})$$

⁴³ In that case, using Eqs. (19) and (20), their radius $\tilde{R}_a \sim (M_a/\Sigma_\Lambda)^{1/2} \sim (GM_a/c^2\sqrt{\Lambda})^{1/2}$ would be given by

$$\tilde{R}_a \sim \frac{\hbar^{(3-a)/12} G^{(3-a)/12}}{c^{(3-a)/4} \Lambda^{(3+a)/12}}, \quad \tilde{R}_a / l_P \sim \chi^{(3+a)/12}. \quad (\text{B51})$$

This relation highlights the particular indices $a = 3$ (Universe) where $\tilde{R} = R_\Lambda$ and $a = -3$ (cosmon) where $\tilde{R} = l_P$. We also have $M_a/M_P \sim (\tilde{R}_a/\lambda_\Lambda^*)^2$.

Using Eqs. (B5) and (B7) this implies

$$S_{-a}/k_B \sim \frac{1}{S_a/k_B} \quad (\text{B53})$$

and $T_{-a}/T_P \sim M_a/M_P$, i.e.,

$$k_B T_{-a} \sim M_a c^2. \quad (\text{B54})$$

Therefore, the mass of a micro (macro) object normalized by the Planck mass is equal to the inverse of the mass of the corresponding macro (micro) object normalized by the Planck mass.⁴⁴ As a result, the entropy of a black hole with the mass of a micro (macro) object is equal to the inverse of the entropy of the black hole with the mass of the corresponding macro (micro) object and its temperature is equal to the rest mass energy of the corresponding macro (micro) object. Specifically:

(i) The entropy of a black hole with the cosmon mass ($a = -3$) is equal to the inverse of the entropy of the universe ($a = 3$) and its temperature is equal to the rest-mass energy of the universe. Inversely, the temperature of the universe is equal to the rest-mass energy of the cosmon. We have $S_c \sim 1/S_\Lambda$ (taking $k_B = 1$), $k_B T_c \sim M_\Lambda c^2$ and $k_B T_\Lambda \sim m_\Lambda c^2$.

(ii) The entropy of a black hole with the neutrino B mass ($a = -2$) is equal to the inverse of the entropy of a black hole resulting from the collapse of a fermion star ($a = 2$) and its temperature is equal to the rest-mass energy of a fermion star. Inversely, the temperature of a black hole resulting from the collapse of a fermion star is equal to the rest-mass energy of the neutrino B. We have $S'_2 \sim 1/S_2$ (taking $k_B = 1$), $k_B T'_2 \sim M_2 c^2$ and $k_B T_2 \sim m_2 c^2$.

(iii) The entropy of a black hole with the neutrino A mass ($a = -3/2$) is equal to the inverse of the entropy of a black hole resulting from the collapse of a solid star ($a = 3/2$) and its temperature is equal to the rest-mass energy of a solid star. Inversely, the temperature of a black hole resulting from the collapse of a solid star is equal to the rest-mass energy of the neutrino A. We have $S'_{3/2} \sim 1/S_{3/2}$ (taking $k_B = 1$), $k_B T'_{3/2} \sim M_{3/2} c^2$ and $k_B T_{3/2} \sim m_\Lambda^* c^2$.

(iv) The entropy of a black hole with the electron mass ($a = -1$) is equal to the inverse of the entropy of a black hole resulting from the collapse of a mini boson star ($a = 1$) and its temperature is equal to the rest-mass energy of a mini boson star. Inversely, the temperature of a black hole resulting from the collapse of a mini boson

star is equal to the rest-mass energy of the electron. We have $S_e \sim 1/S_1$ (taking $k_B = 1$), $k_B T_e \sim M_1 c^2$ and $k_B T_1 \sim m_e c^2$.

Remark: There is another consequence of the reciprocity property. From Eq. (B52) we easily establish that

$$(\lambda_C)_{-a} = \frac{1}{2}(R_S)_a, \quad (\text{B55})$$

namely the Compton wavelength $\lambda_C = \hbar/mc$ of a micro (macro) object is equal to the semi gravitational radius $R_S/2 = GM/c^2$ of the corresponding macro (micro) object. This is a sort of duality. Indeed, we have indicated in the main text that the Compton radius of the cosmon is equal to the semi gravitational radius of the Universe, the Compton radius of the neutrino B is equal to the semi gravitational radius of a fermion star, the Compton radius of the neutrino A is equal to the semi gravitational radius of the earth, and the Compton radius of the electron is equal to the semi gravitational radius of a mini boson star (and conversely). In this respect, the Planck black hole is a sort of “fixed point” ($-a = a = 0$) between micro and macro objects. For the Planck black hole, $\lambda_C = R_S/2 = l_P$, $S_P = k_B$ and $k_B T_P = M_P c^2$.

4. Entropic origin of the large numbers

By using the holographic principle (B2) and the $M-R^2$ relation (B9), we have introduced a new form of entropy [see [11] and Eq. (B11)]

$$\tilde{S} \sim k_B \frac{M}{m_\Lambda} \sim k_B N. \quad (\text{B56})$$

The number $N = M/m_\Lambda$ may be interpreted as the number of quanta of DE (cosmons) in the object of mass M . It represents the number of physical degrees of freedom. The ground state (cosmon) corresponds to $N = 1$, hence $\tilde{S} = k_B$ (1 bit). The entropy of the universe, which is proportional to the total number $N_{\max} = M_\Lambda/m_\Lambda$ of cosmons (dark energy particle), is equal to

$$S_\Lambda/k_B \sim \frac{M_\Lambda}{m_\Lambda} \sim N_B \sim \chi \sim 10^{120}. \quad (\text{B57})$$

This is the “largest large number” in Nature from which other large numbers can be deduced [see Sec. VI and the mass scale law from Eq. (100)]. From this point of view, the origin of the large numbers in Nature is due to the fact that the number of physical degrees of freedom $N_{\max} = M_\Lambda/m_\Lambda$ is large, i.e., there is a large number of cosmons (quanta of dark energy) in the Universe. Equivalently, it is due to the fact that the entropy of the Universe is large. This is consistent with a form of H -theorem (see footnote 40).

⁴⁴ By definition, a macro object of index a is associated with a micro object of index $-a$ (and conversely). The Universe ($a = 3$) is associated with the cosmon ($a = -3$): $M_\Lambda/M_P \sim M_P/m_\Lambda$. A fermion star ($a = 2$) is associated with the neutrino B ($a = -2$): $M_2/M_P \sim M_P/m_2$. A solid star ($a = 3/2$) is associated with the neutrino A ($a = -3/2$): $M_{3/2}/M_P \sim M_P/m_\Lambda^*$. A mini boson star ($a = 1$) is associated with the electron ($a = -1$): $M_1/M_P \sim M_P/m_e$.

Appendix C: Heuristic derivation of the Eddington relation based on simple entropic considerations

In this Appendix we present three simple arguments based on the holographic principle leading to the Eddington relation. Other justifications of the Eddington relation from the holographic principle are given in Refs. [11, 13].

1. First argument

Let us assume that the universe is made of N_e particles of mass m_e .⁴⁵ The total mass and the total entropy of the Universe are equal to the sum of the masses and entropies of its constituents. Therefore, $M_\Lambda \sim N_e m_e$ and $S_\Lambda \sim N_e S_e$. Using a form of holographic principle (see Appendix B), we assume that the entropy of the Universe and the entropy of the electron are proportional to their area: $S_\Lambda \sim R_\Lambda^2$ and $S_e \sim r_e^2$. We then obtain $R_\Lambda^2 \sim N_e r_e^2$. Using $N_e \sim M_\Lambda/m_e$ this relation can be rewritten as $M_\Lambda/R_\Lambda^2 \sim m_e/r_e^2$ which expresses the equality of the surface density of the Universe and the surface density of the electron: $\Sigma_\Lambda \sim \Sigma_e$ (or, equivalently, the equality between their surface gravity $GM_\Lambda/R_\Lambda^2 \sim Gm_e/r_e^2$). As shown in Appendix A, this equality is equivalent to the Eddington relation (A8). As a result, the holographic principle [73], which conjectures by analogy with black hole thermodynamics [32, 74] that the maximum entropy of an object is proportional to its area (instead of its volume), may provide a justification of the Eddington relation.

2. Second argument

Following [11, 13] we postulate that the entropy of an object of mass M is⁴⁶

$$\tilde{S} \sim N k_B \sim \frac{M}{m_\Lambda} k_B. \quad (\text{C1})$$

In other words \tilde{S}/k_B is equal to the number of ‘‘cosmons’’ (quanta of entropy and mass) in the body. On the other hand, according to the holographic principle, we have

$$S \sim k_B \left(\frac{R}{l_P} \right)^2, \quad (\text{C2})$$

⁴⁵ To be specific, we assume that these particles are electrons but we should of course account for other kinds of particles. Since we are only interested in orders of magnitude, our assumption is sufficient to give the main idea.

⁴⁶ As discussed below, this relation not valid for all types of objects. It may be limited to objects related, in one way or another, to dark energy. The conditions of applicability of this relation should be given further consideration in future works.

where R is the radius of the object. Equating these two entropies in order of magnitude and noting that $m_\Lambda/l_P^2 = \Sigma_\Lambda$, we find that

$$\frac{M}{R^2} \sim \Sigma_\Lambda \sim \frac{c^2 \sqrt{\Lambda}}{G}. \quad (\text{C3})$$

This argument suggests that the surface density of the objects is constant and has a universal value Σ_Λ equal (in order of magnitude) to the surface density of the universe. We have seen that this is indeed the case for DM halos and for the electron. However, this is not true for all the objects in the universe. For example, the mass-radius relation of black holes, namely $M = Rc^2/2G$, is linear⁴⁷ while Eq. (C3) implies a quadratic relation $M \sim \Sigma_\Lambda R^2$. Therefore, black holes do not generally satisfy Eq. (C1). The only exception is the universe as a whole which satisfies both the black hole mass-radius relation $M \sim Rc^2/2G$ and the quadratic relation (C3). Indeed, these relations are simultaneously satisfied when $R \sim 1/\sqrt{\Lambda} \sim R_\Lambda$. If we now apply Eq. (C3) to the electron we get $\Sigma_e \sim \Sigma_\Lambda$ which leads to the Eddington relation (A8) as we have seen in Appendix A. Therefore, the Eddington relation results from Eq. (C1) and from the holographic principle (C2) applied to the electron.

3. Third argument

Bekenstein [82] obtained an upper bound for the ratio of the entropy S to the energy $E = Mc^2$ of any bounded system with effective size R :

$$S/E \leq 2\pi k_B R/\hbar c. \quad (\text{C4})$$

This inequality is valid for all objects, not only for black holes. In order of magnitude, the Bekenstein entropy reads

$$S_B \sim k_B \frac{ER}{\hbar c} \sim k_B \frac{MRc}{\hbar}. \quad (\text{C5})$$

If we apply it to black holes with the mass-radius relation from Eq. (B4), we recover the Bekenstein-Hawking entropy (B2) and (B5). In that case, it saturates the bound from Eq. (C4). Alternatively, if we apply it to objects which satisfy the $M - R^2$ relation from Eqs. (B9) and (B10) we obtain

$$S_B \sim k_B \frac{c^3 \sqrt{\Lambda}}{G\hbar} R^3 \sim k_B \left(\frac{R}{r_e} \right)^3, \quad (\text{C6})$$

where r_e is the classical electron radius given by Eq. (A11). We note that this entropy scales with the volume

⁴⁷ In this respect, we recall that isothermal spheres described by a linear equation of state in Newtonian gravity and general relativity also have a linear mass-radius relation (see [65] for the analogy between isothermal spheres and black holes).

(as for usual systems) instead of the surface (as for black holes). There is no contradiction with the holographic principle since Eq. (C4) represents only an upper bound for the entropy. The number

$$\mathcal{N} \sim \left(\frac{R}{r_e}\right)^3 \quad (\text{C7})$$

may be interpreted as the number of physical degrees of freedom in a spatial region of volume V . Therefore, we could introduce the entropy

$$S \sim k_B \mathcal{N} \sim k_B \left(\frac{R}{r_e}\right)^3 \quad (\text{C8})$$

without reference to the previous arguments. We thus have three forms of entropy given by Eqs. (B2), (B11) and (C8). If we apply them to the universe as a whole (representing an upper bound for the entropy) and equate them we obtain

$$\left(\frac{R_\Lambda}{l_P}\right)^2 \sim \frac{M_\Lambda}{m_\Lambda} \sim \left(\frac{R_\Lambda}{r_e}\right)^3. \quad (\text{C9})$$

This leads to the relation $r_e^3 \simeq l_P^2 R_\Lambda$ which is equivalent to the accurate Eddington relation (A9) [see Eq. (A12) in Sec. A]. Therefore, this argument provides a justification of the accurate Eddington relation (A9).

Appendix D: About the connection between the Eddington and Weinberg relations: A clue to the cosmic coincidence problem?

Let us recall for clarity the Weinberg relation [see Eq. (12)]

$$m_e \sim \left(\frac{H_0 \hbar^2}{Gc}\right)^{1/3} \quad (\text{D1})$$

and the accurate form of the Eddington relation [see Eq. (A9)]

$$m_e \simeq \alpha \left(\frac{\Lambda \hbar^4}{G^2}\right)^{1/6}. \quad (\text{D2})$$

The Eddington relation involves a true *constant* – the cosmological constant Λ – while the Weinberg relation involves the *present* value H_0 of the Hubble parameter (recall that $H(t)$ changes with time except in the de Sitter era). We can have two different points of view concerning the connection between the Eddington relation and the Weinberg relation.

According to cosmological observations, and their interpretation in terms of the Λ CDM model, we live at a period in the history of the universe where DM and DE (assumed to correspond to the cosmological constant Λ) have a comparable proportion (differing by a factor ~ 2.7). The fact that DE is dominant accounts for the

present acceleration of the universe. The reason why DM and DE have a comparable proportion at present is not obvious because the density of DM decreases with the scale factor as a^{-3} while the density of DE is constant. Therefore, it is only at a very particular epoch that the two densities are comparable. Why do we live precisely at this epoch? This intriguing observation is known as the cosmic coincidence problem [62, 63].⁴⁸ Using the Friedmann equation $H^2 = 8\pi G\epsilon/3c^2$ and the expression $\rho_\Lambda = \Lambda c^2/8\pi G$ of the cosmological density, the empirical relation $\epsilon_0 \sim \rho_\Lambda c^2$ (which is a consequence of the cosmic coincidence $\epsilon_{\text{dm},0} \sim \epsilon_{\text{de},0}$) implies that the present value of the Hubble parameter is related to the cosmological constant by

$$H_0 \sim \sqrt{G\rho_\Lambda} \sim c\sqrt{\Lambda}. \quad (\text{D3})$$

This relation, which can be written as

$$t_0 \sim t_\Lambda, \quad (\text{D4})$$

expresses the fact that we live precisely at the transition between the DM era and the DE era.⁴⁹ In other words, the age of the universe $t_0 = 1/H_0$ (present Hubble time) is of the order of the cosmological time $t_\Lambda = 1/(c\sqrt{\Lambda})$. If we use Eq. (D3), we see that Eqs. (D1) and (D2) have the same order of magnitude. In this point of view, the Eddington and the Weinberg relations are essentially equivalent.⁵⁰ However, this equivalence relies on the cosmic coincidence (D3) which remains mysterious.

We can have a different point of view in which the Eddington relation and the Weinberg relation have a different status. We can view the Eddington relation (D2) as an accurate fundamental (yet unexplained) relation between the mass of the electron and the cosmological constant and, using Dicke’s anthropic principle (see Sec. II), interpret the Weinberg relation (D1) as an approximate formula just giving an order of magnitude of the age of the universe $t_0 \sim 1/H_0$ [see Eq. (14)]. Then, if we compare these two relations, we conclude that $H_0 \sim c\sqrt{\Lambda}$, i.e., $\epsilon_0 \sim \rho_\Lambda c^2$. Therefore, we find that the present density of the Universe is of the order of the cosmological density.⁵¹ This may explain why the Universe is accelerating *at present*. This may also alleviate the cosmic coincidence problem. Indeed, it appears to result from the

⁴⁸ In this sense, the relation (A3) which suggests that the present ratio of DE and DM is equal (or close) to the Euler number e leads to a form of “strong cosmic coincidence” problem [12].

⁴⁹ The relation $H \sim c\sqrt{\Lambda}$ was not true in the past (in the matter era where $\epsilon \gg \rho_\Lambda c^2$ and $H \sim 2/3t$) but it is marginally valid now and it will be more and more true in the future (in the de Sitter era where $\epsilon = \rho_\Lambda c^2$ and $H^2 = \Lambda c^2/3$ exactly).

⁵⁰ For that reason, one sometimes talk about the “Eddington-Weinberg” relation.

⁵¹ This basically comes from the fact that the age of the Universe, hence the present value of the Hubble constant, can be expressed in terms of m_e according to Dicke’s anthropic principle [see Eq. (14)] and that m_e can be expressed in terms of Λ according to the Eddington relation [see Eq. (13)].

anthropic principle and the Eddington relation. However, in this interpretation, we have to justify the accurate Eddington relation (D2). This justification may come from the holographic principle or from a theory of quantum gravity which remains to be constructed.

Appendix E: The chronon

The characteristic (dynamical) time associated with the electron is

$$t_e = \frac{r_e}{c} = \frac{e^2}{m_e c^3} \simeq \left(\frac{G^2 \hbar^2}{\Lambda c^{12}} \right)^{1/6} \sim \chi^{1/6} t_P \sim 9.40 \times 10^{-24} \text{ s.} \quad (\text{E1})$$

It is sometimes interpreted as a quantum of time [83] called the ‘‘chronon’’ (see Appendix F of [40]). A black hole whose Hawking (evaporation) time is of the order of t_e has a mass

$$M \sim \left(\frac{\hbar^8 c^{12}}{\Lambda G^{10}} \right)^{1/18} \sim \chi^{1/18} M_P \sim 123 \text{ g} \quad (\text{E2})$$

and a radius

$$R \sim \left(\frac{\hbar^8 G^8}{\Lambda c^{24}} \right)^{1/18} \sim \chi^{1/18} l_P \sim 9.13 \times 10^{-29} \text{ m.} \quad (\text{E3})$$

The mass scale (E2) appears in the condensation temperature evaluated in Sec. IV E. This type of black holes may be produced in the early universe at $t_e \sim 10^{-24}$ s after the formation of Planck black holes.

Appendix F: The cosmon mass from a scalar field theory

The Friedmann equations determining the evolution of the homogeneous Universe read

$$\frac{d\epsilon}{dt} + 3H(\epsilon + P) = 0 \quad (\text{F1})$$

and

$$H^2 = \frac{8\pi G}{3c^2} \epsilon - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}, \quad (\text{F2})$$

where $H = \dot{a}/a$ is the Hubble parameter, Λ is the cosmological constant and k determines the curvature of space. In the following we shall take $k = 0$ in agreement with the inflation paradigm [84] and the observations of the cosmic microwave background [85, 86].

The Λ CDM model assumes that the Universe is made of pressureless DM ($P_m = 0$) and that DE is associated with a nonzero cosmological constant Λ or with a fluid characterized by an equation of state $P_{\text{de}} = -\epsilon_{\text{de}}$. According to the energy conservation equation (F1), the energy density of DM decreases as $\epsilon_m = \epsilon_{m,0}/a^3$ (where $\epsilon_{m,0}$ is the present energy density of DM) while the DE

density is constant: $\epsilon_{\text{de}} = \rho_\Lambda c^2$ with $\rho_\Lambda = \Lambda c^2/8\pi G$. The total energy density $\epsilon = \epsilon_m + \epsilon_{\text{de}}$ is given by⁵²

$$\epsilon = \frac{\epsilon_{m,0}}{a^3} + \epsilon_\Lambda. \quad (\text{F3})$$

Introducing the present value of the energy density ϵ_0 and the present proportions of DM and DE denoted $\Omega_{m,0}$ and $\Omega_{\Lambda,0} = 1 - \Omega_{m,0}$, we can rewrite Eq. (F3) as

$$\frac{\epsilon}{\epsilon_0} = \frac{\Omega_{m,0}}{a^3} + 1 - \Omega_{m,0}, \quad (\text{F4})$$

where $\Omega_{m,0} = 0.309$. The equality between DM and DE in the Λ CDM model is achieved for a value of the scale factor

$$a_t = \left(\frac{\Omega_{m,0}}{1 - \Omega_{m,0}} \right)^{1/3} = 0.765. \quad (\text{F5})$$

We now take $\Lambda = 0$ in Eq. (F2) and assume that the Universe is filled with a single dark fluid (DF) characterized by a constant equation of state

$$P = -\epsilon_\Lambda, \quad (\text{F6})$$

where $\epsilon_\Lambda = \rho_\Lambda c^2$ is identified with the cosmological density. Since the pressure is constant, the squared speed of sound $c_s^2 = P'(\epsilon)c^2$ vanishes. Integrating the energy conservation equation (F1) with the equation of state (F6), we obtain an equation equivalent to Eq. (F3) where $\epsilon_{m,0}$ appears as a constant of integration. Remarkably, we obtain the same equation as in the Λ CDM model. The single DF model, based on the equation of state (F6), provides the simplest unification of DM and DE that one can imagine and it coincides with the usual Λ CDM model.⁵³ In this connection, the first term in Eq. (F3) plays the role of DM and the second term plays the role of DE. As shown in [9] at a general level, the effective DM term corresponds to the rest-mass energy $\rho_m c^2 \propto a^{-3}$ of the DF and the effective DE term corresponds to its internal energy $u = \rho_m \int^{\rho_m} [P(\rho')/\rho'^2] d\rho'$ determined by the equation of state $P(\rho_m)$. In other words: $\epsilon = \rho_m c^2 + u$. This decomposition provides a simple and nice interpretation of DM and DE in terms of the rest-mass energy and internal energy of a single DF and elucidates their mysterious nature [9, 10]. In the present case, the internal energy is constant: $u = \rho_\Lambda c^2$.

The relation (F3) between the energy density and the scale factor can be rewritten as

$$\epsilon = \rho_\Lambda c^2 \left[\left(\frac{a_t}{a} \right)^3 + 1 \right], \quad (\text{F7})$$

⁵² We can treat baryonic matter as an additional species or include it in DM if it is pressureless. We adopt this second approach for convenience and simplicity.

⁵³ It can be shown that the two descriptions are equivalent not only for the evolution of the background but to all orders in perturbation theory, even in the nonlinear clustering regime.

where a_t is the transition scale factor defined by Eq. (F5). Solving the Friedmann equation (F2) with $k = \Lambda = 0$ and the energy density from Eq. (F7), we find that the temporal evolution of the scale factor is given by

$$\frac{a}{a_t} = \sinh^{2/3} \left(\sqrt{6\pi G \rho_\Lambda t} \right). \quad (\text{F8})$$

Let us now assume that the DF corresponds to a spatially homogeneous real SF evolving according to the KG equation

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0 \quad (\text{F9})$$

coupled to the Friedmann equation (F2). Here, the time variable stands for ct . The SF tends to run down the potential $V(\varphi)$ towards lower energies. It is also submitted to an Hubble friction. The density and the pressure of the SF are given by

$$\epsilon = \frac{1}{2}\dot{\varphi}^2 + V(\varphi), \quad (\text{F10})$$

$$P = \frac{1}{2}\dot{\varphi}^2 - V(\varphi). \quad (\text{F11})$$

We can easily check that these equations imply the energy conservation equation (F1) [87]. For a general equation of state $P(\epsilon)$, using standard techniques (see [26] and references therein), we can obtain the SF potential as follows. From Eqs. (F10) and (F11), we get

$$\dot{\varphi}^2 = (w + 1)\epsilon, \quad (\text{F12})$$

where we have introduced the equation of state parameter $w = P/\epsilon$. Using $\dot{\varphi} = (d\varphi/da)Ha$ and the Friedmann equation (F2) with $k = \Lambda = 0$, we find that the relation between the SF and the scale factor is given by⁵⁴

$$\frac{d\varphi}{da} = \left(\frac{3c^4}{8\pi G} \right)^{1/2} \frac{\sqrt{1+w}}{a}. \quad (\text{F13})$$

On the other hand, according to Eqs. (F10) and (F11), the potential of the SF is given by

$$V = \frac{1}{2}(1-w)\epsilon. \quad (\text{F14})$$

Therefore, the potential of the SF is determined in parametric form by the equations

$$\varphi(a) = \left(\frac{3c^4}{8\pi G} \right)^{1/2} \int \sqrt{1+w(a)} \frac{da}{a}, \quad (\text{F15})$$

$$V(a) = \frac{1}{2} [1 - w(a)] \epsilon(a). \quad (\text{F16})$$

⁵⁴ We assume $w > -1$, i.e., a non-phantom scalar field.

For the constant equation of state (F6) corresponding to the Λ CDM model in its UDME interpretation, the equation of state parameter reads

$$w(a) = \frac{P}{\epsilon} = \frac{-\epsilon_\Lambda}{\frac{\epsilon_{m,0}}{a^3} + \epsilon_\Lambda}. \quad (\text{F17})$$

Eqs. (F15) and (F16) with Eq. (F17) are readily integrated leading to the hyperbolic potential

$$V(\psi) = \frac{1}{2}\rho_\Lambda c^2 (\cosh^2 \psi + 1), \quad (\text{F18})$$

where

$$\psi = - \left(\frac{8\pi G}{3c^4} \right)^{1/2} \frac{3}{2}\varphi. \quad (\text{F19})$$

The SF is related to the scale factor by

$$(a/a_t)^{-3/2} = \sinh \psi, \quad (\text{F20})$$

where a_t is the transition scale factor defined by Eq. (F5) and $\psi \geq 0$.⁵⁵

Expanding Eq. (F18) for $\varphi \rightarrow 0$, we find that

$$V(\varphi) = \rho_\Lambda c^2 + \frac{9\Lambda}{8}\varphi^2 + \dots \quad (\text{F22})$$

If we compare this expression with the normal form of a SF potential

$$V(\varphi) = V_0 + \frac{m^2 c^2}{2\hbar^2}\varphi^2 + \dots \quad (\text{F23})$$

we see that the minimum of the potential is equal to the cosmological density

$$V_0 = \rho_\Lambda c^2 \quad (\text{F24})$$

and that the mass of the SF is given by

$$m = \frac{\sqrt{3}}{2} m_\Lambda, \quad (\text{F25})$$

where m_Λ is the cosmon mass from Eq. (61). Our approach provides therefore a physical interpretation of the cosmon mass as being the mass (up to a factor $\sqrt{3}/2$ of order unity) of the SF responsible for the DE in the late universe. This SF model, which is associated with the constant equation of state (F6), is equivalent to the Λ CDM model in its UDME interpretation. If we admit the existence of a particle (SF) with the fundamental mass m_Λ we can write the cosmological constant as

$$\Lambda = \left(\frac{m_\Lambda}{M_P} \right)^2 l_P^{-2}. \quad (\text{F26})$$

This formula can be compared with Eq. (103) [see also footnote 7 and Eq. (101)].

⁵⁵ We can also associate to the Λ CDM model a tachyonic SF with a potential (see [26] for details)

$$V(\psi) = \frac{\rho_\Lambda c^2}{\cos \psi}, \quad (\text{F21})$$

where $\psi = -\sqrt{6\pi G \rho_\Lambda / c^2} \varphi$. The SF is related to the scale factor by $(a/a_t)^{-3/2} = \tan \psi$ with $0 \leq \psi \leq \pi/2$.

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