



HAL
open science

A linear bound for the Colin de Verdière parameter μ for graphs embedded on surfaces

Camille Lanuel, Francis Lazarus, Rudi Pendavingh

► **To cite this version:**

Camille Lanuel, Francis Lazarus, Rudi Pendavingh. A linear bound for the Colin de Verdière parameter μ for graphs embedded on surfaces. EuroCG'23 (39th European Workshop on Computational Geometry), Mar 2023, Barcelona, Spain. hal-04018060

HAL Id: hal-04018060

<https://hal.science/hal-04018060v1>

Submitted on 7 Mar 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A linear bound for the Colin de Verdière parameter μ for graphs embedded on surfaces

Camille Lanuel^{*1}, Francis Lazarus^{†2}, and Rudi Pendavingh³

¹ Université de Lorraine, CNRS, Inria, LORIA, F-54000 Nancy, France

² G-SCOP, CNRS, UGA, Grenoble, France

³ Department of Mathematics and Computer Science, Eindhoven University of Technology, 5600 MB Eindhoven, Netherlands

Abstract

The Colin de Verdière graph parameter $\mu(G)$ was introduced in 1990 by Y. Colin de Verdière. It is defined via spectral properties of a certain type of matrices, called Schrödinger operators, associated to a graph G . We provide a combinatorial and self-contained proof that for all graphs G embedded on a surface S , the Colin de Verdière parameter $\mu(G)$ is upper bounded by $7 - 2\chi(S)$, where $\chi(S)$ is the Euler characteristic of S .

Related Version <https://arxiv.org/abs/2303.00556>

1 Introduction

In this paper, we establish an upper bound for the Colin de Verdière’s graph parameter μ for graphs that can be embedded on a fixed surface. This parameter was introduced by Colin de Verdière [4] in analogy with the multiplicity of the second eigenvalue of Schrödinger operators on a Riemannian surface. The exact definition of $\mu(G)$ resorts to a transversality condition between the space of so-called discrete Schrödinger operators on a graph $G = (V, E)$ and a certain stratification of the space of symmetric matrices of dimension $V \times V$. This *Strong Arnold Hypothesis* (SAH), as coined by Colin de Verdière [3], expresses a stability property and ensures that μ is minor-monotone. It can thus be applied to the graph minor theory of Robertson and Seymour. We refer to the survey by van der Holst, Lovász and Schrijver [13] for more properties on μ .

Here, we are interested in an upper bound for the parameter μ of the minor-closed family of graphs that can be embedded on a surface S . It is relatively easy to show that $\mu(K_n) = n - 1$ for K_n , the complete graph with n vertices [13]. On the other hand, the largest n such that K_n embeds on S is known as the Heawood number

$$\gamma(S) = \left\lfloor \frac{7 + \sqrt{49 - 24\chi(S)}}{2} \right\rfloor,$$

where $\chi(S)$ is the Euler characteristic of S . Colin de Verdière [2] conjectured that the maximum of μ for all graphs that can be embedded in S is attained at $K_{\gamma(S)}$. In other words, μ is upper bounded by $\gamma(S) - 1$. In practice, the known upper bounds have been proved in the realm of Riemannian surfaces where μ is defined for each Riemannian metric

^{*} This work was conducted when this author was at the G-SCOP laboratory in Grenoble. This author is (partially) supported by the grant ANR-17-CE40-0033 of the French National Research Agency ANR (project SoS)

[†] This author is partially supported by LabEx PERSYVAL-Lab (ANR-11-LABX-0025-01) funded by the French program Investissement d’avenir.

as the maximum multiplicity of the second eigenvalue of Schrödinger differential operators (based on the Laplace-Beltrami operator associated to the given metric). In this framework, Besson [1] obtained the bound $7 - 2\chi(S)$. When the Euler characteristic is negative, this bound was further decreased by 2 by Nadirashvili [8], who thus showed the upper bound $5 - 2\chi(S)$. Sévenec [10] eventually divided by two the dependency in the characteristic to obtain the upper bound $5 - \chi(S)$. Note that all those bounds are linear in the genus of S and remain far from the square root bound in the conjecture of Colin de Verdière. It can be proved that any upper bound for μ in the Riemannian world holds for graphs, cf. [6, Th. 6.3] and [2, Th. 7.1]. However, the proof relies on the construction of Schrödinger differential operators from combinatorial ones and is not particularly illuminating from the combinatorial viewpoint. The goal of this paper is to propose a purely combinatorial and self-contained proof of the bound of Besson in the combinatorial framework of graphs. In the following, S is a compact surface, orientable or not.

► **Theorem 1.1.** *Let G be a graph that can be embedded on a surface S , then*

$$\mu(G) \leq 7 - 2\chi(S).$$

Our proof completes and slightly simplifies a proof of Pendavingh that appeared in his PhD thesis [9]. Before describing the general strategy of the proof, we provide some relevant definitions and basic facts.

2 Background

2.1 Schrödinger operators and μ

Let $G = (V, E)$ be a connected simplicial¹ graph with at least two vertices. A *Schrödinger operator* on G , sometimes called a generalized Laplacian, is a symmetric $V \times V$ matrix such that for $i \neq j \in V$, its ij coefficient is negative if $ij \in E$ and zero otherwise. (There is no condition on the diagonal coefficients.) We say that a Schrödinger operator L satisfies the *Strong Arnold Hypothesis (SAH)* if there is no nonzero symmetric matrix $X = (X_{ij})$ such that $LX = 0$ and $X_{ij} = 0$ whenever $i = j$ or $ij \notin E$, in other words, if the only symmetric matrix satisfying these conditions is zero. This is an algebraic translation of a transversality condition between the space of the discrete Schrödinger operators on G and a certain stratification of the space of symmetric matrices of dimension $V \times V$. It is not necessary to understand the SAH for the purpose of this paper.

It follows from Perron–Frobenius theorem that the first (smallest) eigenvalue of a Schrödinger operator L has multiplicity one [5]. Now, if λ_2 is the second eigenvalue of L , then the first eigenvalue of $L - \lambda_2 \text{Id}$, where Id is the identity matrix, is negative and its second eigenvalue is zero. This translation by $-\lambda_2 \text{Id}$ does not change the sequence of multiplicities of the eigenvalues of L nor the stability of L with respect to the SAH. Consequently, we can safely restrict to Schrödinger operators whose second eigenvalue is zero. We can now define the *Colin de Verdière graph parameter* $\mu(G)$ as the maximal corank (dimension of the kernel $\ker(L)$) of any Schrödinger operator L satisfying the Strong Arnold Hypothesis. In other words, $\mu(G)$ is the largest integer p such that there exists a Schrödinger operator L with $\dim \ker L = p$ and L satisfies the SAH.

Recall that a *minor* of G is a graph obtained from G by deleting edges, vertices or contracting edges. As a fundamental property, μ is minor-monotone.

¹ A graph is *simplicial*, or *simple*, if it has no loops or multiple edges.

► **Theorem 2.1** ([4]). *If H is a minor of G , then $\mu(H) \leq \mu(G)$.*

It also characterizes planar graphs.

► **Theorem 2.2** ([4]). *A graph G is planar if and only if $\mu(G) \leq 3$.*

Colin de Verdière made the following conjecture in [4].

► **Conjecture 2.3.** *The chromatic number of a graph G satisfies $\text{chr}(G) \leq \mu(G) + 1$.*

This conjecture contains the four colour theorem thanks to Theorem 2.2. It is implied by Hadwiger's conjecture (a graph which is not colourable with k colours has the complete graph K_{k+1} as a minor), using the fact that $\mu(K_n) = n - 1$ [13] and the minor-monotone property of μ .

We view a vector of \mathbb{R}^V as a discrete map $V \rightarrow \mathbb{R}$, so that a Schrödinger operator acts linearly on the set of discrete maps. For $f : V \rightarrow \mathbb{R}$, we denote by V_f^+, V_f^0, V_f^- the subsets of vertices where f takes respectively positive, null and negative values. The *support* of f is the subset $V_f^+ \cup V_f^-$ of vertices with nonzero values. As a simple property of Schrödinger operators we have

► **Lemma 2.4** ([13]). *Let L be a Schrödinger operator of G and let $f \in \ker L$. Then, a vertex $v \in V_f^0$ is adjacent to a vertex of V_f^+ if and only if v is adjacent to a vertex of V_f^- .*

A discrete version of the nodal theorem of Courant reads as follows.

► **Theorem 2.5** ([5, 12]). *Let L be a Schrödinger operator of G and let $f \in \ker L$ be a nonzero map with minimal support². Then, the subgraphs of G induced respectively by V_f^+ and V_f^- are nonempty and connected.*

2.2 Surfaces and Euler characteristic

By a *surface* of finite type we mean a topological space homeomorphic to a compact two dimensional manifold minus a finite number of points. A surface may have nonempty boundary and each of the finitely many boundary components is homeomorphic to a circle. We shall only consider surfaces of finite type and omit to specify this condition. A *closed* surface means a compact surface without boundary. By a *triangulation* of a surface, we mean a simplicial complex together with a homeomorphism between its underlying space and the surface.

The *Euler characteristic* $\chi(X)$ of a finite simplicial complex (and more generally a finite CW complex) is the alternating sum of the numbers of cells of each dimension. In particular, the Euler characteristic of a graph $\Gamma = (V, E)$ is $\chi(\Gamma) = |V| - |E|$. For a surface S we define $\chi(S)$ as the Euler characteristic of a polygonisation of S , that is, a description of S as a CW complex. The Euler characteristic is homotopy invariant: two spaces with the same homotopy type have the same Euler characteristic.

The following property will be needed in our proof. It is a corollary of the Inclusion-exclusion formula [11, p.205]. The proof is not completely trivial and can be found in [7, Section 2].

² A discrete map $f \in \ker L$ has minimal support if it is nonzero and for every nonzero $g \in \ker L$, if $V_g^+ \cup V_g^- \subseteq V_f^+ \cup V_f^-$, then $V_g^+ \cup V_g^- = V_f^+ \cup V_f^-$.

18:4 A linear bound for μ for graphs embedded on a surface

► **Proposition 2.6.** *Let X be a triangulated compact surface and let Y be a subcomplex of X . Then*

$$\chi(X) = \chi(Y) + \chi(X \setminus Y).$$

Here, X and Y should be considered as topological spaces, forgetting about their triangulations. In general, $X \setminus Y$ is not a subcomplex of X .

3 Overview of the proof

In this section, we give the main ideas of the proof of Theorem 1.1. The detailed proof can be found in [7, Section 5].

Let G be a graph embedded on a surface S , which we can assume closed without loss of generality. We may also assume that S is not homeomorphic to a sphere as otherwise Theorem 1.1 follows directly from Theorem 2.2. In a first step, we remove an open disk $D \subset S \setminus G$ whose boundary avoids G , and build a graph³ H embedded in $S \setminus D$ such that

- C1.** H triangulates $S \setminus D$ and subdivides ∂D into a cycle of $\mu(G) - 1$ edges,
- C2.** G is a minor of H , and
- C3.** the length of the shortest closed walk in H that is non-contractible in $S \setminus D$, i.e. the *edgewidth* of H in $S \setminus D$, is $\mu(G) - 1$.

We denote by W the set of vertices of H .

We next choose a Schrödinger operator L for H whose corank achieves $\mu(H)$. Condition C2 and the monotonicity of μ imply $\mu(H) \geq \mu(G)$, so that $\ker L$ has dimension at least $\mu(G)$. By C1, ∂D has $\mu(G) - 1$ vertices, so the vectors of the basis of $\ker L$ restricted to ∂D are linearly dependent. Thus there exists a nonzero vector $f \in \ker L$ such that f cancels on the vertices of ∂D . We pick such an f with minimal support so that by Theorem 2.5 the subsets of vertices W_f^+ and W_f^- induce connected subgraphs of H . We connect the vertices of ∂D by inserting $\mu(G) - 4$ edges in D to obtain a graph H' with the same vertices as H and that triangulates S . We can now extend f linearly on each face of H' to get a piecewise linear map $\bar{f} : S \rightarrow \mathbb{R}$. Let S_f^+, S_f^0, S_f^- denote the subspaces of S where \bar{f} is respectively positive, null, and negative. By Theorem 2.5, S_f^+ and S_f^- are connected open subsurfaces of S , while S_f^0 is a closed subcomplex of some subdivision of the triangulation induced by H' . We can thus apply Proposition 2.6 to write

$$\chi(S) = \chi(S_f^0) + \chi(S_f^+ \cup S_f^-) = \chi(S_f^0) + \chi(S_f^+) + \chi(S_f^-).$$

The subsets S_f^+ and S_f^- cannot be homeomorphic to spheres because they are both proper subsets of S . We deduce from the classification of surfaces, that $\chi(S_f^+) \leq 1$ and $\chi(S_f^-) \leq 1$. It ensues that $\chi(S_f^0) \geq \chi(S) - 2$. The goal is now to provide an upper bound for $\chi(S_f^0)$ in terms of $\mu(G)$ in order to obtain the desired upper bound for $\mu(G)$.

Start by observing S_f^0 : it is formed of plain triangles adjacent by an edge or a vertex, or connected together, via their vertices, by a piece of dimension 1. To provide the upper bound for $\chi(S_f^0)$, we build a graph Γ whose Euler characteristic is larger than S_f^0 by contracting its two dimensional parts, that are the parts formed by the plain triangles. To ensure that this operation results in the desired property on the Euler characteristic of Γ , precautions are taken before the contraction to remove the singularities of the two dimensional parts while keeping the same homotopy type. By definition of \bar{f} , $D \subset S_f^0$, so we can define K as the two

³ See [7, Section 4] for a detailed construction of such a graph.

dimensional part containing D . Because of condition C3 on H , we can argue that K has a non-contractible boundary in $S \setminus D$. It follows that this boundary has length at least the edgewidth of H , hence at least $\mu(G) - 1$ by condition C3. Thanks to Lemma 2.4 we may infer that K contracts to a vertex of degree at least $\mu(G) - 1$ in Γ . We also argue thanks to Lemma 2.4 that Γ has no vertex of degree one. By the handshaking lemma applied to Γ , we deduce that $\chi(S_f^0) \leq \chi(\Gamma) \leq (3 - \mu(G))/2$. We finally conclude that $\chi(S) - 2 \leq (3 - \mu(G))/2$, hence $\mu(G) \leq 7 - 2\chi(S)$.

References

- 1 Gérard Besson. Sur la multiplicité de la première valeur propre des surfaces riemanniennes. *Annales de l'institut Fourier*, 30(1):109–128, 1980. URL: <http://www.numdam.org/item/10.5802/aif.777.pdf>.
- 2 Yves Colin de Verdière. Construction de laplaciens dont une partie finie du spectre est donnée. *Annales scientifiques de l'école normale supérieure*, 20(4):599–615, 1987.
- 3 Yves Colin de Verdière. Sur une hypothèse de transversalité d'Arnol'd. *Comment. Math. Helv.*, 63(2):184–193, 1988.
- 4 Yves Colin de Verdière. Sur un nouvel invariant des graphes et un critère de planarité. *Journal of Combinatorial Theory, Series B*, 50(1):11–21, 1990.
- 5 Yves Colin de Verdière. Théorèmes de Courant et de Cheng combinatoires. *Séminaire de théorie spectrale et géométrie*, 13:9–13, 1994.
- 6 Yves Colin de Verdière. *Spectres de graphes*. Number 4 in Cours Spécialisés. Société Mathématique de France, 1998.
- 7 Camille Lanuel, Francis Lazarus, and Rudi Pendavingh. A linear bound for the Colin de Verdière parameter μ for graphs embedded on surfaces, 2023. Preprint. URL: <https://arxiv.org/abs/2303.00556>, doi:10.48550/ARXIV.2303.00556.
- 8 Nikolai Semenovitch Nadirashvili. Multiple eigenvalues of the Laplace operator. *Mathematics of the USSR-Sbornik*, 61(1):225, 1988.
- 9 Rudolf Anton Pendavingh. *Spectral and Geometrical Graph Characterizations*. PhD thesis, Amsterdam University, 1998.
- 10 Bruno Sévenec. Multiplicity of the second Schrödinger eigenvalue on closed surfaces. *Mathematische Annalen*, 324(1):195–211, 2002.
- 11 Edwin H. Spanier. *Algebraic topology*. Springer Verlag, corrected reprint of the 1966 original edition, 1989.
- 12 Hein Van der Holst. A short proof of the planarity characterization of Colin de Verdière. *Journal of Combinatorial Theory, Series B*, 65(2):269–272, 1995. URL: <https://www.sciencedirect.com/science/article/pii/S0095895685710544/pdf?md5=cb7b57b58c203666ae179c8de5e87e19&pid=1-s2.0-S0095895685710544-main.pdf>.
- 13 Hein Van der Holst, László Lovász, and Alexander Schrijver. The Colin de Verdière graph parameter. *Graph Theory and Computational Biology (Balatonlelle, 1996)*, pages 29–85, 1999.