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# A linear bound for the Colin de Verdière parameter $\mu$ for graphs embedded on surfaces

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## Abstract

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The Colin de Verdière graph parameter  $\mu(G)$  was introduced in 1990 by Y. Colin de Verdière. It is defined via spectral properties of a certain type of matrices, called Schrödinger operators, associated to a graph  $G$ . We provide a combinatorial and self-contained proof that for all graphs  $G$  embedded on a surface  $S$ , the Colin de Verdière parameter  $\mu(G)$  is upper bounded by  $7 - 2\chi(S)$ , where  $\chi(S)$  is the Euler characteristic of  $S$ .

**Related Version** <https://arxiv.org/abs/2303.00556>

## 1 Introduction

In this paper, we establish an upper bound for the Colin de Verdière’s graph parameter  $\mu$  for graphs that can be embedded on a fixed surface. This parameter was introduced by Colin de Verdière [4] in analogy with the multiplicity of the second eigenvalue of Schrödinger operators on a Riemannian surface. The exact definition of  $\mu(G)$  resorts to a transversality condition between the space of so-called discrete Schrödinger operators on a graph  $G = (V, E)$  and a certain stratification of the space of symmetric matrices of dimension  $V \times V$ . This *Strong Arnold Hypothesis* (SAH), as coined by Colin de Verdière [3], expresses a stability property and ensures that  $\mu$  is minor-monotone. It can thus be applied to the graph minor theory of Robertson and Seymour. We refer to the survey by van der Holst, Lovász and Schrijver [13] for more properties on  $\mu$ .

Here, we are interested in an upper bound for the parameter  $\mu$  of the minor-closed family of graphs that can be embedded on a surface  $S$ . It is relatively easy to show that  $\mu(K_n) = n - 1$  for  $K_n$ , the complete graph with  $n$  vertices [13]. On the other hand, the largest  $n$  such that  $K_n$  embeds on  $S$  is known as the Heawood number

$$\gamma(S) = \left\lfloor \frac{7 + \sqrt{49 - 24\chi(S)}}{2} \right\rfloor,$$

where  $\chi(S)$  is the Euler characteristic of  $S$ . Colin de Verdière [2] conjectured that the maximum of  $\mu$  for all graphs that can be embedded in  $S$  is attained at  $K_{\gamma(S)}$ . In other words,  $\mu$  is upper bounded by  $\gamma(S) - 1$ . In practice, the known upper bounds have been proved in the realm of Riemannian surfaces where  $\mu$  is defined for each Riemannian metric

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as the maximum multiplicity of the second eigenvalue of Schrödinger differential operators (based on the Laplace-Beltrami operator associated to the given metric). In this framework, Besson [1] obtained the bound  $7 - 2\chi(S)$ . When the Euler characteristic is negative, this bound was further decreased by 2 by Nadirashvili [8], who thus showed the upper bound  $5 - 2\chi(S)$ . Sévenec [10] eventually divided by two the dependency in the characteristic to obtain the upper bound  $5 - \chi(S)$ . Note that all those bounds are linear in the genus of  $S$  and remain far from the square root bound in the conjecture of Colin de Verdière. It can be proved that any upper bound for  $\mu$  in the Riemannian world holds for graphs, cf. [6, Th. 6.3] and [2, Th. 7.1]. However, the proof relies on the construction of Schrödinger differential operators from combinatorial ones and is not particularly illuminating from the combinatorial viewpoint. The goal of this paper is to propose a purely combinatorial and self-contained proof of the bound of Besson in the combinatorial framework of graphs. In the following,  $S$  is a compact surface, orientable or not.

► **Theorem 1.1.** *Let  $G$  be a graph that can be embedded on a surface  $S$ , then*

$$\mu(G) \leq 7 - 2\chi(S).$$

Our proof completes and slightly simplifies a proof of Pendavingh that appeared in his PhD thesis [9]. Before describing the general strategy of the proof, we provide some relevant definitions and basic facts.

## 2 Background

### 2.1 Schrödinger operators and $\mu$

Let  $G = (V, E)$  be a connected simplicial<sup>1</sup> graph with at least two vertices. A *Schrödinger operator* on  $G$ , sometimes called a generalized Laplacian, is a symmetric  $V \times V$  matrix such that for  $i \neq j \in V$ , its  $ij$  coefficient is negative if  $ij \in E$  and zero otherwise. (There is no condition on the diagonal coefficients.) We say that a Schrödinger operator  $L$  satisfies the *Strong Arnold Hypothesis (SAH)* if there is no nonzero symmetric matrix  $X = (X_{ij})$  such that  $LX = 0$  and  $X_{ij} = 0$  whenever  $i = j$  or  $ij \notin E$ , in other words, if the only symmetric matrix satisfying these conditions is zero. This is an algebraic translation of a transversality condition between the space of the discrete Schrödinger operators on  $G$  and a certain stratification of the space of symmetric matrices of dimension  $V \times V$ . It is not necessary to understand the SAH for the purpose of this paper.

It follows from Perron–Frobenius theorem that the first (smallest) eigenvalue of a Schrödinger operator  $L$  has multiplicity one [5]. Now, if  $\lambda_2$  is the second eigenvalue of  $L$ , then the first eigenvalue of  $L - \lambda_2 \text{Id}$ , where  $\text{Id}$  is the identity matrix, is negative and its second eigenvalue is zero. This translation by  $-\lambda_2 \text{Id}$  does not change the sequence of multiplicities of the eigenvalues of  $L$  nor the stability of  $L$  with respect to the SAH. Consequently, we can safely restrict to Schrödinger operators whose second eigenvalue is zero. We can now define the *Colin de Verdière graph parameter*  $\mu(G)$  as the maximal corank (dimension of the kernel  $\ker(L)$ ) of any Schrödinger operator  $L$  satisfying the Strong Arnold Hypothesis. In other words,  $\mu(G)$  is the largest integer  $p$  such that there exists a Schrödinger operator  $L$  with  $\dim \ker L = p$  and  $L$  satisfies the SAH.

Recall that a *minor* of  $G$  is a graph obtained from  $G$  by deleting edges, vertices or contracting edges. As a fundamental property,  $\mu$  is minor-monotone.

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<sup>1</sup> A graph is *simplicial*, or *simple*, if it has no loops or multiple edges.

► **Theorem 2.1** ([4]). *If  $H$  is a minor of  $G$ , then  $\mu(H) \leq \mu(G)$ .*

It also characterizes planar graphs.

► **Theorem 2.2** ([4]). *A graph  $G$  is planar if and only if  $\mu(G) \leq 3$ .*

Colin de Verdière made the following conjecture in [4].

► **Conjecture 2.3.** *The chromatic number of a graph  $G$  satisfies  $\text{chr}(G) \leq \mu(G) + 1$ .*

This conjecture contains the four colour theorem thanks to Theorem 2.2. It is implied by Hadwiger's conjecture (a graph which is not colourable with  $k$  colours has the complete graph  $K_{k+1}$  as a minor), using the fact that  $\mu(K_n) = n - 1$  [13] and the minor-monotone property of  $\mu$ .

We view a vector of  $\mathbb{R}^V$  as a discrete map  $V \rightarrow \mathbb{R}$ , so that a Schrödinger operator acts linearly on the set of discrete maps. For  $f : V \rightarrow \mathbb{R}$ , we denote by  $V_f^+, V_f^0, V_f^-$  the subsets of vertices where  $f$  takes respectively positive, null and negative values. The *support* of  $f$  is the subset  $V_f^+ \cup V_f^-$  of vertices with nonzero values. As a simple property of Schrödinger operators we have

► **Lemma 2.4** ([13]). *Let  $L$  be a Schrödinger operator of  $G$  and let  $f \in \ker L$ . Then, a vertex  $v \in V_f^0$  is adjacent to a vertex of  $V_f^+$  if and only if  $v$  is adjacent to a vertex of  $V_f^-$ .*

A discrete version of the nodal theorem of Courant reads as follows.

► **Theorem 2.5** ([5, 12]). *Let  $L$  be a Schrödinger operator of  $G$  and let  $f \in \ker L$  be a nonzero map with minimal support<sup>2</sup>. Then, the subgraphs of  $G$  induced respectively by  $V_f^+$  and  $V_f^-$  are nonempty and connected.*

## 2.2 Surfaces and Euler characteristic

By a *surface* of finite type we mean a topological space homeomorphic to a compact two dimensional manifold minus a finite number of points. A surface may have nonempty boundary and each of the finitely many boundary components is homeomorphic to a circle. We shall only consider surfaces of finite type and omit to specify this condition. A *closed* surface means a compact surface without boundary. By a *triangulation* of a surface, we mean a simplicial complex together with a homeomorphism between its underlying space and the surface.

The *Euler characteristic*  $\chi(X)$  of a finite simplicial complex (and more generally a finite CW complex) is the alternating sum of the numbers of cells of each dimension. In particular, the Euler characteristic of a graph  $\Gamma = (V, E)$  is  $\chi(\Gamma) = |V| - |E|$ . For a surface  $S$  we define  $\chi(S)$  as the Euler characteristic of a polygonisation of  $S$ , that is, a description of  $S$  as a CW complex. The Euler characteristic is homotopy invariant: two spaces with the same homotopy type have the same Euler characteristic.

The following property will be needed in our proof. It is a corollary of the Inclusion-exclusion formula [11, p.205]. The proof is not completely trivial and can be found in [7, Section 2].

<sup>2</sup> A discrete map  $f \in \ker L$  has minimal support if it is nonzero and for every nonzero  $g \in \ker L$ , if  $V_g^+ \cup V_g^- \subseteq V_f^+ \cup V_f^-$ , then  $V_g^+ \cup V_g^- = V_f^+ \cup V_f^-$ .

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► **Proposition 2.6.** *Let  $X$  be a triangulated compact surface and let  $Y$  be a subcomplex of  $X$ . Then*

$$\chi(X) = \chi(Y) + \chi(X \setminus Y).$$

Here,  $X$  and  $Y$  should be considered as topological spaces, forgetting about their triangulations. In general,  $X \setminus Y$  is not a subcomplex of  $X$ .

### 3 Overview of the proof

In this section, we give the main ideas of the proof of Theorem 1.1. The detailed proof can be found in [7, Section 5].

Let  $G$  be a graph embedded on a surface  $S$ , which we can assume closed without loss of generality. We may also assume that  $S$  is not homeomorphic to a sphere as otherwise Theorem 1.1 follows directly from Theorem 2.2. In a first step, we remove an open disk  $D \subset S \setminus G$  whose boundary avoids  $G$ , and build a graph<sup>3</sup>  $H$  embedded in  $S \setminus D$  such that

- C1.**  $H$  triangulates  $S \setminus D$  and subdivides  $\partial D$  into a cycle of  $\mu(G) - 1$  edges,
- C2.**  $G$  is a minor of  $H$ , and
- C3.** the length of the shortest closed walk in  $H$  that is non-contractible in  $S \setminus D$ , i.e. the *edgewidth* of  $H$  in  $S \setminus D$ , is  $\mu(G) - 1$ .

We denote by  $W$  the set of vertices of  $H$ .

We next choose a Schrödinger operator  $L$  for  $H$  whose corank achieves  $\mu(H)$ . Condition C2 and the monotonicity of  $\mu$  imply  $\mu(H) \geq \mu(G)$ , so that  $\ker L$  has dimension at least  $\mu(G)$ . By C1,  $\partial D$  has  $\mu(G) - 1$  vertices, so the vectors of the basis of  $\ker L$  restricted to  $\partial D$  are linearly dependent. Thus there exists a nonzero vector  $f \in \ker L$  such that  $f$  cancels on the vertices of  $\partial D$ . We pick such an  $f$  with minimal support so that by Theorem 2.5 the subsets of vertices  $W_f^+$  and  $W_f^-$  induce connected subgraphs of  $H$ . We connect the vertices of  $\partial D$  by inserting  $\mu(G) - 4$  edges in  $D$  to obtain a graph  $H'$  with the same vertices as  $H$  and that triangulates  $S$ . We can now extend  $f$  linearly on each face of  $H'$  to get a piecewise linear map  $\bar{f} : S \rightarrow \mathbb{R}$ . Let  $S_f^+, S_f^0, S_f^-$  denote the subspaces of  $S$  where  $\bar{f}$  is respectively positive, null, and negative. By Theorem 2.5,  $S_f^+$  and  $S_f^-$  are connected open subsurfaces of  $S$ , while  $S_f^0$  is a closed subcomplex of some subdivision of the triangulation induced by  $H'$ . We can thus apply Proposition 2.6 to write

$$\chi(S) = \chi(S_f^0) + \chi(S_f^+ \cup S_f^-) = \chi(S_f^0) + \chi(S_f^+) + \chi(S_f^-).$$

The subsets  $S_f^+$  and  $S_f^-$  cannot be homeomorphic to spheres because they are both proper subsets of  $S$ . We deduce from the classification of surfaces, that  $\chi(S_f^+) \leq 1$  and  $\chi(S_f^-) \leq 1$ . It ensues that  $\chi(S_f^0) \geq \chi(S) - 2$ . The goal is now to provide an upper bound for  $\chi(S_f^0)$  in terms of  $\mu(G)$  in order to obtain the desired upper bound for  $\mu(G)$ .

Start by observing  $S_f^0$ : it is formed of plain triangles adjacent by an edge or a vertex, or connected together, via their vertices, by a piece of dimension 1. To provide the upper bound for  $\chi(S_f^0)$ , we build a graph  $\Gamma$  whose Euler characteristic is larger than  $S_f^0$  by contracting its two dimensional parts, that are the parts formed by the plain triangles. To ensure that this operation results in the desired property on the Euler characteristic of  $\Gamma$ , precautions are taken before the contraction to remove the singularities of the two dimensional parts while keeping the same homotopy type. By definition of  $\bar{f}$ ,  $D \subset S_f^0$ , so we can define  $K$  as the two

<sup>3</sup> See [7, Section 4] for a detailed construction of such a graph.

dimensional part containing  $D$ . Because of condition C3 on  $H$ , we can argue that  $K$  has a non-contractible boundary in  $S \setminus D$ . It follows that this boundary has length at least the edgewidth of  $H$ , hence at least  $\mu(G) - 1$  by condition C3. Thanks to Lemma 2.4 we may infer that  $K$  contracts to a vertex of degree at least  $\mu(G) - 1$  in  $\Gamma$ . We also argue thanks to Lemma 2.4 that  $\Gamma$  has no vertex of degree one. By the handshaking lemma applied to  $\Gamma$ , we deduce that  $\chi(S_f^0) \leq \chi(\Gamma) \leq (3 - \mu(G))/2$ . We finally conclude that  $\chi(S) - 2 \leq (3 - \mu(G))/2$ , hence  $\mu(G) \leq 7 - 2\chi(S)$ .

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