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# NILPOTENT MATRICES, INTEGRAL CURVES AND JACOBIAN CONJECTURE

JEAN-YVES CHARBONNEL

ABSTRACT. Following the results of L.M. Drużkowski, we are interested in the Jacobian Conjecture.

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## 1. INTRODUCTION

In this note, we are interested in the jacobian conjecture, that is: for  $n$  positive integer, an étale polynomial endomorphism of  $\mathbb{C}^n$  is a polynomial automorphism. Some results are known for a long time. For example, an étale polynomial endomorphism is an automorphism if it is proper or birational [Bass 82, Theorem 2.1]. According to the results of L.M. Drużkowski, it is sufficient to prove that the following conjecture is true.

**Conjecture 1.1.** *Let  $n$  be a positive integer and  $F_1, \dots, F_n$  in the dual of  $\mathbb{C}^n$ . Denote by  $\Phi$  the polynomial endomorphism of  $\mathbb{C}^n$ :*

$$\mathbb{C}^n \xrightarrow{\Phi} \mathbb{C}^n, \quad x = (x_1, \dots, x_n) \mapsto (x_1 + F_1(x)^3, \dots, x_n + F_n(x)^3).$$

*If  $\Phi$  is an étale morphism, then  $\Phi$  is a polynomial automorphism.*

As a result, we are interested in the proof of conjecture 1.1.

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**1.1. Sketch of proof.** Let  $n$  be a positive integer and  $F_1, \dots, F_n, \Phi$  as in Conjecture 1.1. Suppose that  $\Phi$  is étale. For  $x$  in  $\mathbb{C}^n$ , let  $X_x$  be the vector field on  $\mathbb{C}^n$ ,

$$y \longmapsto \Phi'(y)^{-1}(x)$$

with  $\Phi'(y)$  the differential of  $\Phi$  at  $y$ . If  $\gamma$  is an integral curve of  $X_x$  with initial datum  $\gamma(0) = y$ , then for all  $t$  in the interval of definition of  $\gamma$ ,

$$\Phi \circ \gamma(t) = \Phi(y) + tx.$$

The key step of the proof is the following proposition:

**Proposition 1.2.** *Let  $x$  be in  $\mathbb{C}^n$  and  $\gamma_x$  the maximal integral curve of  $X_x$  with initial datum  $\gamma_x(0) = 0$ . Then  $\mathbb{R}$  is the interval of definition of  $\gamma_x$ .*

For  $x$  in  $\mathbb{C}^n$ , let  $\varphi_x$  be the map from  $\mathbb{C}$  to  $\mathbb{C}^n$  such that

$$\varphi_x(0) := 0 \quad \text{and} \quad \varphi_x(t) := \gamma_{ux}(|t|) \quad \text{with} \quad u := \frac{t}{|t|},$$

for all  $t$  in  $\mathbb{C}^*$ . The second step is the following result:

**Proposition 1.3.** *For all  $x$  in  $\mathbb{C}^n$ ,  $\varphi_x$  is a polynomial map such that  $\Phi \circ \varphi_x(t) = tx$  for all  $t$  in  $\mathbb{C}$ .*

We deduce from this proposition that the map  $x \mapsto \varphi_x(1)$  is a polynomial endomorphism. Then  $\Phi$  is a polynomial automorphism and its inverse is the map  $x \mapsto \varphi_x(1)$ .

**1.2. Notations and Definition.** • As usual, for  $k$  positive integer and for  $i = (i_1, \dots, i_k)$  in  $\mathbb{N}^k$ ,

$$|i| := i_1 + \dots + i_k.$$

For  $j$  nonnegative integer, denote by  $\mathbb{N}_j^k$ , the subset of elements  $i$  of  $\mathbb{N}^k$  such that  $|i| = j$ .

- For  $E$  subset,  $|E|$  is its cardinality when it is finite and  $+\infty$  when it is not finite.
- The topology on  $\mathbb{C}^n$  is the usual topology and  $\|\cdot\|$  is an euclidean norm on  $\mathbb{C}^n$ . Let  $d$  be the distance on  $\mathbb{C}^n$  such that  $d(x, y) := \|x - y\|$ .
- For  $\Phi$  a holomorphic endomorphism of  $\mathbb{C}^n$  and  $x$  in  $\mathbb{C}^n$ , its differential at  $x$  is denoted by  $\Phi'(x)$ .
- For  $n$  positive integer and  $x, y$  in  $\mathbb{C}^n$ , set:

$$xy := (x_1y_1, \dots, x_ny_n) \quad \text{with} \quad x = (x_1, \dots, x_n) \quad \text{and} \quad y = (y_1, \dots, y_n).$$

As a result, by induction on  $k$ , for  $v_1, \dots, v_k$  in  $\mathbb{C}^n$ ,  $v_1 \cdots v_k$  is a well defined element of  $\mathbb{C}^n$ . For  $F$  a map from  $\mathbb{C}^n$  to  $\mathbb{C}^n$  and  $k$  positive integer, denote by  $F^{(k)}$  the map

$$\mathbb{C}^n \xrightarrow{F^{(k)}} \mathbb{C}^n, \quad x \longmapsto F(x)^k.$$

• As usual, for  $n$  positive integer,  $M_n(\mathbb{C})$  is the algebra of square matrices of order  $n$  with complex coefficients. Denote by  $\mathbf{1}_n$  the identity element of  $M_n(\mathbb{C})$ . For  $A$  in  $M_n(\mathbb{C})$  and  $1 \leq i, j \leq n$ ,  $A_{i,j}$  is the coefficient of  $A$  at its  $i$ -row and  $j$ -column. The matrix  $A$  is identified with the endomorphism of  $\mathbb{C}^n$ ,

$$(x_1, \dots, x_n) \longmapsto (A_{i,1}x_1 + \dots + A_{i,n}x_n, i = 1, \dots, n).$$

• Let  $A$  be in  $\mathbb{C}^n$ . Denote by  $F$  the linear endomorphism of  $\mathbb{C}^n$  identified with  $A$  and  $\Phi$  the polynomial endomorphism of  $\mathbb{C}^n$ ,

$$\mathbb{C}^n \xrightarrow{\Phi} \mathbb{C}^n, \quad x \mapsto x + F^{(3)}(x).$$

For  $i = 1, \dots, n$ , let  $F_i$  be the  $i$ -th coordinate of  $F$  and for  $x$  in  $\mathbb{C}^n$ , denote by  $D_x$  the diagonal matrix whose entries are  $F_1(x)^2, \dots, F_n(x)^2$ .

**Definition 1.4.** The matrix  $A$  is said AJC (adapted to Jacobian conjecture) if  $D_x A$  is a nilpotent matrix for all  $x$  in  $\mathbb{C}^n$ .

By [Dru83, Proposition 2],  $A$  is AJC if and only if  $\Phi$  is an étale morphism.

**Lemma 1.5.** *Suppose that  $\Phi$  is étale. Then  $\Phi^{-1}(v) = \{v\}$  for all  $v$  in the kernel of  $F$ .*

*Proof.* As  $v$  is in the kernel of  $F$ ,  $\Phi(v) = v$ . Let  $x$  be in  $\mathbb{C}^n$  such that  $\Phi(x) = v$ . Then  $v$  is the image of  $x$  by  $\mathbf{1}_n + D_x A$ . As  $\Phi$  is étale, for some positive integer  $m$ ,  $(D_x A)^m = 0$ , whence

$$(\mathbf{1}_n + D_x A)^{-1} = \sum_{i=0}^{m-1} (-1)^i (D_x A)^i.$$

Since  $v$  is in the kernel of  $F$ , for  $i$  positive integer,  $(D_x A)^i(v) = 0$ , whence

$$x = (\mathbf{1}_n + D_x A)^{-1}(v) = v,$$

and the lemma. □

In this note,  $A$  is an AJC matrix different from 0. As  $\Phi$  is an étale morphism, for all  $x$  in  $\mathbb{C}^n$ , there exists an open neighborhood  $\omega$  of  $x$  in  $\mathbb{C}^n$  such that the restriction of  $\Phi$  to  $\omega$  is a holomorphic isomorphism onto the open neighborhood  $\Phi(\omega)$  of  $\Phi(x)$  in  $\mathbb{C}^n$ . Denote by  $\psi$  the inverse of the restriction of  $\Phi$  to  $\omega$  and call the pair  $(\omega, \psi)$  an adapted pair to  $x$ .

## 2. SEMIALGEBRAIC SETS AND MAPS

We recall some results of semialgebraic geometry (see [Cos02]). Let  $E$  and  $F$  be finite dimensional real vector spaces. A subset of  $E$  is semialgebraic if it is a finite union of subsets defined by a system of a finite number of equalities and inequalities of polynomials. Denoting by  $\mathfrak{S}(E)$  the set of semialgebraic subsets of  $E$ ,  $\mathfrak{S}(E)$  is stable under finite union and finite intersection. For  $S$  in  $\mathfrak{S}(E)$ , its closure in  $E$ , its interior and  $E \setminus S$  are in  $\mathfrak{S}(E)$ .

For  $X$  subset of  $E$  and  $\varphi$  a map from  $X$  to  $F$ ,  $\varphi$  is a semialgebraic map if its graph is a semialgebraic subset of  $E \times F$ . By Tarski-Seidenberg's theorem,  $X$  is a semialgebraic subset of  $E$  and the image of  $\varphi$  is a semialgebraic subset of  $F$ .

**Lemma 2.1.** *Let  $X$  be a subset of  $E$ ,  $Y$  a subset of  $F$  and  $G$  a real vector space of finite dimension. If  $\varphi$  is a semialgebraic map from  $X$  to  $Y$  and  $\psi$  a semialgebraic map from  $Y$  to  $G$  then  $\psi \circ \varphi$  is a semialgebraic map from  $X$  to  $G$ .*

*Proof.* Let  $\Gamma_\varphi$  be the graph of  $\varphi$  and  $\Gamma_\psi$  the graph of  $\psi$ . Denote by  $\Gamma$  the intersection of  $\Gamma_\varphi \times G$  and  $E \times \Gamma_\psi$ . Then the graph of  $\psi \circ \varphi$  is the image of  $\Gamma$  by the projection

$$E \times F \times G \longrightarrow E \times G, \quad (x, y, z) \mapsto (x, z).$$

By hypothesis,  $\Gamma$  is a semialgebraic subset of  $E \times F \times G$  and by Tarski-Seidenberg's theorem,  $\psi \circ \varphi$  is a semialgebraic map.  $\square$

**2.1. Some results on semialgebraic sets.** According to [Cos02, Section 2], a semialgebraic subset  $S$  of  $\mathbb{R}^n$  has a decomposition into a finite disjoint union of semialgebraic subsets semialgebraically homeomorphic to open hypercubes  $]0, 1[^d$  (by convention  $]0, 1[^0$  is a point). By [Cos02, Proposition 3.15], the dimension of  $S$  is the maximum of these  $d$  and is independent of the decomposition. Moreover, by [Cos02, Theorem 2.23],  $S$  has a finite number of connected components and its connected components are semialgebraic.

**Lemma 2.2.** *Let  $S$  be in  $\mathfrak{S}(\mathbb{R}^n)$  and  $\overline{S}$  its closure in  $\mathbb{R}^n$ .*

(i) *The sets  $S$  and  $\overline{S}$  have the same dimension.*

(ii) *The dimension of  $\overline{S} \setminus S$  is smaller than  $\dim S$ .*

(iii) *The set  $S$  has an empty interior if and only if  $\dim S \leq n - 1$ .*

(iv) *If  $S$  is a Zariski closed subset of  $\mathbb{R}^n$ , then  $\dim S$  is the dimension of  $S$  as an algebraic subset.*

(v) *For  $m$  positive integer and  $\varphi$  semialgebraic map from  $S$  to  $\mathbb{R}^m$ ,  $\dim \varphi(S) \leq \dim S$ .*

*Proof.* Assertions (i) and (ii) are given by [Cos02, Proposition 3.16], Assertion (iv) is given by [Cos02, Theorem 3.20] and Assertion (v) is given by [Cos02, Theorem 3.18].

(iii) By definition, a nonempty semialgebraic open subset of  $\mathbb{R}^n$  has dimension  $n$ . For  $T$  semialgebraic subset  $T$  of  $S$ ,  $\dim T \leq \dim S$  by [Cos02, Corollary 3.8]. Then  $S$  has an empty interior if  $\dim S \leq n - 1$ . Conversely if  $\dim S = n$ , it contains a semialgebraic subset homeomorphic to  $]0, 1[^n$ . Hence  $S$  has a nonempty interior.  $\square$

**Proposition 2.3.** *Let  $Y$  be a semialgebraic subset of  $\mathbb{R}^n$ , containing 0 and having an empty interior. Then there exists a continuous semialgebraic arc  $\theta$ ,*

$$[0, 1] \xrightarrow{\theta} \mathbb{R}^n, \quad t \mapsto \theta(t)$$

*such that  $\theta(0) = 0$  and  $\theta(t) \notin Y$  for all  $t$  in  $]0, 1]$ .*

*Proof.* As  $Y$  has an empty interior,  $\mathbb{R}^n \setminus Y$  is a dense semialgebraic open subset of  $\mathbb{R}^n$ , not containing 0, whence the proposition by the Curve selection lemma [Cos02, Theorem 3.13].  $\square$

**2.2. A result on a semialgebraic function.** Let  $\Omega$  be a connected semialgebraic open subset of  $\mathbb{C}^n$  and  $a$  a continuous semialgebraic function on  $\Omega$  with real values.

**Lemma 2.4.** *Let  $\Omega'$  be the subset of elements  $x$  of  $\Omega$  such that  $a$  is constant in a neighborhood of  $x$ . Then  $\Omega'$  is a semialgebraic open subset of  $\mathbb{C}^n$ .*

*Proof.* By definition  $\Omega'$  is a union of open subsets of  $\mathbb{C}^n$ . Hence it is open. Let  $Z$  be the subset of elements  $(x, x_1, x_2, \varepsilon)$  of  $\Omega \times \Omega \times \Omega \times \mathbb{R}_+^*$  such that

$$d(x, x_1) \leq \varepsilon, \quad d(x, x_2) \leq \varepsilon, \quad a(x_1) \neq a(x_2).$$

Since  $\Omega, a, d$  are semialgebraic,  $Z$  is a semialgebraic subset of  $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{R}$ . Denote by  $\bar{Z}$  the closure of  $Z$  in  $\Omega \times \Omega \times \Omega \times \mathbb{R}_+$  and  $Z_0$  the intersection of  $\bar{Z}$  and  $\Omega \times \Omega \times \Omega \times \{0\}$ . Then  $\Omega \setminus \Omega'$  is the image of  $Z_0$  by the first projection. As a result,  $\Omega \setminus \Omega'$  and  $\Omega'$  are semialgebraic subsets of  $\mathbb{C}^n$ , whence the lemma.  $\square$

**Corollary 2.5.** *Suppose that the interior of  $\Omega \setminus \Omega'$  is empty. Then the function  $a$  is constant.*

*Proof.* By hypothesis and Lemma 2.4,  $\Omega'$  is a dense semialgebraic open subset of  $\Omega$ . Then, by [Cos02, Theorem 2.23],  $\Omega'$  has a finite number of connected components. Denote them by  $O_1, \dots, O_m$ . For  $i = 1, \dots, m$ ,  $a$  is constant on  $O_i$  since  $a$  is continuous. Denote by  $a_i$  its value. As  $\Omega \setminus \Omega'$  has an empty interior, for  $x$  in  $\Omega \setminus \Omega'$ , for some  $i$ ,  $x$  is in the closure of  $O_i$  so that  $a(x) = a_i$ . As a result, the image of  $a$  is equal to  $\{a_1, \dots, a_m\}$ . Hence  $a$  is constant since  $a$  is continuous and  $\Omega$  is connected.  $\square$

**2.3. On semialgebraic curves.** In this subsection we give some results on continuous semialgebraic curves.

**Lemma 2.6.** *Let  $\gamma$  be a continuous semialgebraic function from the interval  $]a, b[$  to  $\mathbb{R}$ .*

(i) *Suppose that  $\gamma$  has a boundary value when  $t$  goes to  $a$ . Then  $\gamma$  has a continuous extension to  $[a, b]$ .*

(ii) *If  $\gamma$  is not bounded, then for some positive rational number  $r$  and a continuous semialgebraic function  $\gamma_0$  on  $[a, b]$  such that  $\gamma_0(a) \neq 0$ ,  $\gamma(t) = (t - a)^{-r} \gamma_0(t)$  for all  $t$  in  $]a, b[$ .*

*Proof.* (i) Let  $\Gamma$  be the graph of the function  $\gamma$  and  $\bar{\Gamma}$  its closure in  $[a, b] \times \mathbb{R}$ . By hypothesis,  $\bar{\Gamma} \setminus \Gamma$  is not empty and contained in  $\{a\} \times \mathbb{R}$  since  $\gamma$  is continuous. Let  $x$  be in  $\mathbb{R}$  such that  $(a, x)$  is in  $\bar{\Gamma}$  and  $\varepsilon$  a positive real number. The intersection of  $\bar{\Gamma}$  and  $[a, b] \times ]x - \varepsilon, x + \varepsilon[$  is a semialgebraic subset of  $\mathbb{R}^2$ . Then it has a finite number of connected components by [Cos02, Theorem 2.23]. Denote them by  $C_1, \dots, C_m$ . Since  $(a, x)$  is in  $\bar{\Gamma}$ , for some positive number  $\eta$  and for some  $i$  in  $\{1, \dots, m\}$ ,  $]a, a + \eta[$  is contained in the image of  $C_i$  by the first projection since a semialgebraic subset of  $\mathbb{R}$  is a finite union of intervals and isolated points. As a result, for all  $t$  in  $]a, a + \eta[$ ,  $|\gamma(t) - x| < \varepsilon$ . Hence the extension  $\bar{\gamma}$  of  $\gamma$  to  $[a, b]$  such that  $\bar{\gamma}(a) = x$  is continuous.

(ii) Suppose that  $\gamma$  is not bounded. By (i),

$$\lim_{t \rightarrow a} |\gamma(t)| = +\infty.$$

Moreover, the function  $t \mapsto |\gamma(t)|$  is a semialgebraic function. Let  $P_1, \dots, P_m$  be the polynomials corresponding to the equalities of a system defining the graph of the function  $|\gamma|$ . Denote by  $Q$  the polynomial

$$Q(t, s) := \prod_{i=1}^m P_i(t + a, s).$$

Then, for some prime divisor  $Q_*$  of  $Q$ ,  $Q_*(t, \gamma(t+a)) = 0$  for all  $t$ , different from 0, in a neighborhood of 0 in  $]0, b-a]$ . So, by a Puiseux's expansion, for some positive rational number  $r$  and positive real number  $c$ ,

$$\lim_{t \rightarrow a} (t-a)^r |\gamma(t)| = c.$$

Setting:

$$\gamma_0(t) = (t-a)^r \gamma(t)$$

for  $t$  in  $]a, b]$ , the function  $\gamma_0$  on  $]a, b]$  is bounded, continuous and semialgebraic. Then, by (i),  $\gamma_0$  has a continuous expansion to  $[a, b]$  such that  $\gamma_0(a) \neq 0$ .  $\square$

**Corollary 2.7.** *Let  $E$  be a finite dimensional real vector space and  $\gamma$  a continuous semialgebraic map from the interval  $[a, b]$  to  $E$ . For  $k$  nonnegative integer, there exist an increasing sequence  $r_1 < \dots < r_{k+1}$  of positive rational numbers and  $x_1, \dots, x_k$  in  $E$  such that the map*

$$[a, b] \longrightarrow E, \quad t \longmapsto (t-a)^{-r_{k+1}} (\gamma(t) - (\gamma(a) + (t-a)^{r_1} x_1 + \dots + (t-a)^{r_k} x_k))$$

is bounded, continuous and semialgebraic.

*Proof.* Considering the coordinates of  $\gamma$  in a basis of  $E$ , it is sufficient to prove the corollary for  $E = \mathbb{R}$ . The map  $t \mapsto \gamma(t) - \gamma(a)$  has a zero at  $a$ . Moreover, the set of zeros of this map in  $[a, b]$  is a closed semialgebraic subset of  $[a, b]$ . As a result, it is a union of a finite number of closed intervals and isolated points. Hence  $\gamma(t) - \gamma(a) = 0$  for all  $t$  in a neighborhood of  $a$  in  $[a, b]$  or  $\gamma(t) - \gamma(a) \neq 0$  for all  $t$  different from  $a$  in a neighborhood of  $a$  in  $[a, b]$ . In the first case, there is nothing to prove. Suppose that the second case is satisfied. For some positive number  $\eta$  such that  $a + \eta \leq b$ , the function

$$t \longmapsto \frac{1}{\gamma(t) - \gamma(a)}$$

on  $]a, a+\eta]$  is a continuous semialgebraic function and it is not bounded. Hence, by Lemma 2.6(ii), for some positive rational number  $r_1$  and for some continuous semialgebraic function  $\gamma_1$  on  $[a, b]$  such that  $\gamma_1(a) \neq 0$ ,

$$\gamma(t) = \gamma(a) + (t-a)^{r_1} \gamma_1(t)$$

for all  $t$  in  $[a, b]$ , whence the corollary by induction on  $k$ .  $\square$

*Remark 2.8.* By Corollary 2.7 and an induction argument, we can find an increasing sequence  $r_1 < r_2 < \dots$  of positive rational numbers and a sequence  $x_1, x_2, \dots$  in  $E \setminus \{0\}$  satisfying one of the two following conditions:

- (1) if the sequences are finite of cardinality  $k$ , then the function

$$t \longmapsto \gamma(t) - (\gamma(a) + (t-a)^{r_1} x_1 + \dots + (t-a)^{r_k} x_k)$$

is equal to 0 in a neighborhood of  $a$  in  $[a, b]$ ,

- (2) if the first sequence is not finite so is the second one and for all positive integer  $k$ , the map

$$t \longmapsto (t-a)^{-r_{k+1}} (\gamma(t) - (\gamma(a) + (t-a)^{r_1} x_1 + \dots + (t-a)^{r_k} x_k))$$

is bounded and semialgebraic.

In the second case, by [Duv89], as the rational numbers  $r_1, r_2, \dots$  are the exponents of a Puiseux's serie, for some positive integer  $m$ ,  $mr_1, mr_2, \dots$  are positive integers so that the sequence  $r_1, r_2, \dots$  is not bounded.

### 3. VECTOR FIELDS AND INTEGRAL CURVES

For  $x$  in  $\mathbb{C}^n$ , denote by  $X_x$  the polynomial vector field on  $\mathbb{C}^n$ ,

$$y \mapsto X_x(y) := \Phi'(y)^{-1}(x).$$

**Definition 3.1.** An integral curve of the vector field  $X_x$  is a derivable map  $\gamma$  from an open interval  $I$  of  $\mathbb{R}$  to  $\mathbb{C}^n$  such that its derative  $\gamma'$  satisfies the equality

$$\gamma'(t) = X_x(\gamma(t))$$

for all  $t$  in  $I$ .

**3.1. Elementary properties.** Let  $x$  be in  $\mathbb{C}^n$ .

**Lemma 3.2.** Denote by  $\gamma$  a continuous map from an open interval  $I$  of  $\mathbb{R}$  to  $\mathbb{C}^n$ . Then  $\gamma$  is an integral curve of  $X_x$  if and only if for some  $t_0$  in  $I$

$$\Phi \circ \gamma(t) = \Phi \circ \gamma(t_0) + (t - t_0)x$$

for all  $t$  in  $I$ .

*Proof.* Suppose that  $\gamma$  is an integral curve of  $X_x$ . Then  $\Phi \circ \gamma$  is an integral curve of the vector field  $z \mapsto \Phi'(z)(X_x(z))$ . By definition,  $\Phi'(z)(X_x(z)) = x$  for all  $z$  in  $\mathbb{C}^n$ . As a result, by the unicity of the integral curves with a given initial datum,

$$\Phi \circ \gamma(t) = \Phi \circ \gamma(t_0) + (t - t_0)x$$

for all  $(t, t_0)$  in  $I \times I$ . Conversely, suppose that  $\gamma$  is a continuous map from  $I$  to  $\mathbb{C}^n$  such that for some  $t_0$  in  $I$ ,

$$\Phi \circ \gamma(t) = \Phi \circ \gamma(t_0) + (t - t_0)x \quad \forall t \in I.$$

Let  $s$  be in  $I$  and  $(\omega, \psi)$  an adapted pair to  $\gamma(s)$ . Then, for all  $t$  in  $\gamma^{-1}(\omega)$ ,

$$\gamma(t) = \psi(\Phi \circ \gamma(t_0) + (t - t_0)x).$$

As a result,  $\gamma$  is a derivable map and  $\Phi'(\gamma(t))(\gamma'(t)) = x$  for all  $t$  in  $I$ , whence the assertion by definition of  $X_x$ . □

Denote by  $\mathcal{C}_x$  the set of integral curves of  $X_x$  whose interval of definition contains 0. For  $\gamma$  and  $\delta$  in  $\mathcal{C}_x$ , we say that  $\gamma$  is smaller than  $\delta$  if the interval of definition of  $\gamma$  is contained in the interval of definition  $\delta$  and if  $\delta$  is an expansion of  $\gamma$ . This order is inductive so that any element of  $\mathcal{C}_x$  is smaller than a maximal one.



**3.2. On the integral curves.** Denote by  $\Psi$  the map

$$\mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^n \xrightarrow{\Psi} \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^n, \quad (x, t, z) \mapsto (x, t, \Phi(z)).$$

As  $\Phi$  is étale so is  $\Psi$ . Let  $\Delta$  be the graph of the map

$$\mathbb{C}^n \times \mathbb{C} \longrightarrow \mathbb{C}^n, \quad (x, t) \mapsto tx.$$

**Lemma 3.3.** *Denote by  $\Gamma$  the inverse image of  $\Delta$  by  $\Psi$  and for  $x$  in  $\mathbb{C}^n$  denote by  $\Gamma_x$  the intersection of  $\Gamma$  with  $\{x\} \times \mathbb{C} \times \mathbb{C}^n$ .*

(i) *The subset  $\Gamma$  of  $\mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^n$  is a smooth algebraic variety of dimension  $n + 1$ , containing  $\mathbb{C}^n \times \{0, 0\}$ .*

(ii) *For  $x$  in  $\mathbb{C}^n$ ,  $\Gamma_x$  is a smooth algebraic variety of dimension 1, containing  $(x, 0, 0)$ .*

*Proof.* (i) As  $\Psi(x, 0, 0) = (x, 0, 0)$ ,  $(x, 0, 0)$  is in  $\Gamma$  for all  $x$  in  $\mathbb{C}^n$ . Let  $(x, t, y)$  be in  $\Gamma$ . Since  $\Psi$  is étale and  $\Delta$  is smooth of dimension  $n + 1$ , the tangent space of  $\Gamma$  at  $(x, t, y)$  has dimension at most  $n + 1$ . Moreover, the image of some open neighborhood of  $(x, t, y)$  in  $\mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^n$  by  $\Psi$  is an open neighborhood of  $(x, t, \Phi(y))$  in  $\mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^n$ . As a result,  $\Gamma$  has dimension  $n + 1$ . Hence  $\Gamma$  is a smooth algebraic variety, equidimensional of dimension  $n + 1$ .

(ii) Let  $x$  be in  $\mathbb{C}^n$ . By definition,  $\Gamma_x$  is the inverse image by  $\Psi$  of the graph  $\Delta_x$  of the map

$$\{x\} \times \mathbb{C} \times \mathbb{C}^n \longrightarrow \mathbb{C}^n, \quad (x, t) \mapsto tx.$$

As the restriction of  $\Psi$  to  $\{x\} \times \mathbb{C} \times \mathbb{C}^n$  is an étale endomorphism of  $\{x\} \times \mathbb{C} \times \mathbb{C}^n$ ,  $\Gamma_x$  is a smooth algebraic variety of dimension 1 since so is  $\Delta_x$ . Moreover,  $(x, 0, 0)$  is in  $\Gamma_x$ .  $\square$

**Proposition 3.4.** *Let  $y$  be in  $\mathbb{C}^n$  and  $\gamma$  in  $\mathcal{C}_x$  such that  $\gamma(0) = y$ . Then  $\gamma$  is a semialgebraic map from an interval of  $\mathbb{R}$  to  $\mathbb{C}^n$ .*

*Proof.* Let  $I$  be the interval of definition of  $\gamma$ . According to Lemma 3.2,

$$\Phi \circ \gamma(t) = \Phi(y) + tx$$

for all  $t$  in  $I$ . Denote by  $\Delta_\gamma$  the graph of  $\gamma$  and set:

$$\Delta_I := \{(t, z) \in I \times \mathbb{C}^n \mid \Phi(z) = \Phi(y) + tx\}.$$

Since  $\Phi$  is a polynomial map,  $\Delta_I$  is a semialgebraic subset of  $\mathbb{R} \times \mathbb{C}^n$ . Moreover,  $\Delta_\gamma$  is a connected closed subset of  $\Delta_I$ . For all  $t$  in  $I$ , denote by  $(\omega_t, \psi_t)$  an adapted pair to  $\gamma(t)$ . Let  $\delta_t$  be the inverse image of  $\omega_t$  by  $\gamma$ . Then, for  $(s, z)$  in the intersection of  $\delta_t \times \omega_t$  and  $\Delta_I$ ,

$$\Phi(z) = \Phi(y) + sx \quad \text{and} \quad \Phi \circ \gamma(s) = \Phi(y) + sx,$$

whence  $z = \gamma(s)$ . As a result,  $\Delta_\gamma$  is an open subset of  $\Delta_I$ . Then  $\Delta_\gamma$  is a connected component of  $\Delta_I$ . So, by [Cos02, Theorem 2.23],  $\Delta_\gamma$  is a semialgebraic subset of  $\mathbb{R} \times \mathbb{C}^n$  since so is  $\Delta_I$ .  $\square$

**Corollary 3.5.** *Let  $y$  be in  $\mathbb{C}^n$  and  $\gamma$  in  $\mathcal{C}_x$  such that  $\gamma(0) = y$ . Suppose that  $\gamma$  is defined over the relatively compact interval  $]a, b[$  of  $\mathbb{R}$ . If  $\|\gamma\|$  is bounded then there exists  $\delta$  in  $\mathcal{C}_x$ , bigger than  $\gamma$  and such that its domain of definition contains  $[a, b]$ .*

*Proof.* By Proposition 3.4 and Lemma 2.6(i),  $\gamma$  has a continuous extension to  $[a, b]$ . Denote again by  $\gamma$  this extension. For some positive real number  $\eta$ , there exists an integral curve  $\gamma_a$  of  $X_x$  defined over  $]a - \eta, a + \eta[$  and such that  $\gamma_a(a) = \gamma(a)$ . Let  $(\omega, \psi)$  be an adapted pair to  $\gamma(a)$ . For some positive real number  $\eta'$ , smaller than  $\eta$ , for all  $t$  in  $]a, a + \eta'[$ ,  $\gamma(t)$  is in  $\omega$  and  $\Phi \circ \gamma(t) = \Phi(y) + tx$ , whence

$$\Phi \circ \gamma(a) = \Phi(y) + ax \quad \text{and} \quad \Phi \circ \gamma(t) = \Phi \circ \gamma(a) + (t - a)x.$$

By Lemma 3.2, for all  $t$  in  $[a, a + \eta[$ ,

$$\Phi \circ \gamma_a(t) = \Phi \circ \gamma(a) + (t - a)x.$$

As a result, for some  $t$  in  $]a, a + \eta'[$ ,  $\gamma(t) = \gamma_a(t)$ . So, by the unicity of the integral curve with initial datum,  $\gamma_a$  and  $\gamma$  have the same restriction to  $[a, a + \eta[$ . As a result, there exists an integral curve of  $X_x$ , extending  $\gamma$  and such that its interval of definition is  $]a - \eta, b[$ . In the same way, we prove that for some positive real number  $\eta''$ , there exists an integral curve of  $X_x$ , extending  $\gamma$  and such that its interval of definition is  $]a, b + \eta''[$ , whence the corollary.  $\square$

#### 4. SPECIAL INTEGRAL CURVES

For  $x$  in  $\mathbb{C}^n$ , denote by  $\gamma_x$  the maximal element of  $\mathcal{C}_x$  such that  $\gamma_x(0) = 0$ . The goal of the section is the following proposition:

**Proposition 4.1.** *For all  $x$  in  $\mathbb{C}^n$ , the interval of definition of  $\gamma_x$  is equal to  $\mathbb{R}$ .*

**4.1. Intervals of definition.** For  $x$  in  $\mathbb{C}^n$ , we denote by  $I_x$  the interval of definition of  $\gamma_x$ .

**Lemma 4.2.** *Let  $x$  be in  $\mathbb{C}^n$ .*

(i) *The interval  $I_x$  is invariant under the homeomorphism  $t \mapsto -t$ . In particular, for some  $a(x)$  in  $\mathbb{R}_+^* \cup \{+\infty\}$ ,  $I_x = ]-a(x), a(x)[$ .*

(ii) *For  $s$  in  $\mathbb{R}^*$ ,  $\gamma_{sx}(t) = \gamma_x(st)$ . Moreover,  $I_{sx} = s^{-1}I_x$  and  $a(sx) = |s|^{-1}a(x)$ .*

*Proof.* (i) By the maximality of  $\gamma_x$  and the unicity of an integral curve with initial datum it is sufficient to prove that the map  $\delta_x$ ,

$$-I \xrightarrow{\delta_x} \mathbb{C}^n, \quad t \mapsto -\gamma_x(-t)$$

is an integral curve of  $X_x$  since 0 is in  $I$  and  $\gamma_x(0) = 0$ . By definition, for all  $t$  in  $-I$ ,

$$\delta'_x(t) = \gamma'_x(-t) = X_x(\gamma_x(-t)) = \Phi'(\gamma_x(-t))^{-1}(x).$$

As  $\Phi$  is an odd map,  $\Phi'(y) = \Phi'(-y)$  for all  $y$  in  $\mathbb{C}^n$ . As a result,  $\delta_x$  is an integral curve of  $X_x$ , whence the assertion.

(ii) Denote by  $\delta$  the map

$$s^{-1}I_x \xrightarrow{\delta} \mathbb{C}^n, \quad t \mapsto \gamma_x(st).$$

Then

$$\delta'(t) = s\gamma'_x(st) = s\Phi'(\gamma_x(st))^{-1}(x) = X_{sx}(\delta(t)).$$

As a result,  $\delta$  is an integral curve of the vector field  $X_{sx}$  such that  $\delta(0) = 0$ . So by the unicity of the integral curve with initial datum,  $s^{-1}I_x$  is contained in  $I_{sx}$  and  $\delta$  is the restriction of  $\gamma_{sx}$  to  $s^{-1}I_x$ . Considering the map

$$sI_{sx} \xrightarrow{\tau} \mathbb{C}^n, \quad t \mapsto \gamma_{sx}(s^{-1}t),$$

$sI_{sx}$  is contained in  $I_x$  by the same argument, whence  $I_{sx} = s^{-1}I_x$  and the assertion.  $\square$

According to Lemma 4.2(i), a map  $a$  from  $\mathbb{C}^n$  to  $\mathbb{R}_+^* \cup \{+\infty\}$  is well defined. Let  $\Omega$  be the subset of elements  $(x, t)$  of  $\mathbb{C}^n \times \mathbb{R}$  such that  $-a(x) < t < a(x)$  and  $G$  the graph of the map

$$\Omega \xrightarrow{\gamma} \mathbb{C}^n, \quad (x, t) \mapsto \gamma(x, t) := \gamma_x(t).$$

**Lemma 4.3.** *Let  $(x, t)$  be in  $\Omega$  and  $t'$  a positive real number such that  $|t| < t' < a(x)$ . Then for some open neighborhood  $\omega$  of  $x$  in  $\mathbb{C}^n$ ,  $\omega \times [-t', t']$  is contained in  $\Omega$  and the restriction of  $\gamma$  to  $\omega \times [-t', t']$  is continuous.*

*Proof.* Since  $[-t', t']$  is a compact interval contained in  $I_x$  and the map  $(y, z) \mapsto X_z(y)$  is polynomial, for some open neighborhood  $\omega$  of  $x$  in  $\mathbb{C}^n$ ,  $\omega \times [-t', t']$  is contained in  $\Omega$  and the restriction of  $\gamma$  to  $\omega \times [-t', t']$  is continuous by [Car67, Théorème 3.6.1].  $\square$

**Corollary 4.4.** (i) *The subset  $\Omega$  of  $\mathbb{C}^n \times \mathbb{R}$  is open and the map  $\gamma$  is continuous.*

(ii) *The function  $a$  is continuous. In particular,  $a^{-1}(\{+\infty\})$  is a closed subset of  $\mathbb{C}^n$ . Moreover  $a^{-1}(\{+\infty\})$  contains the kernel of  $F$ .*

*Proof.* (i) By Lemma 4.3,  $\Omega$  is the union of open subsets of  $\mathbb{C}^n \times \mathbb{R}$  at which the restriction of  $\gamma$  is continuous, whence the assertion.

(ii) By (i), the function  $a$  is lower continuous. Hence  $a^{-1}(\{+\infty\})$  is a closed subset of  $\mathbb{C}^n$ . By Lemma 4.3,  $a$  is upper continuous. Hence  $a$  is continuous. By Lemma 1.5 and Lemma 3.2, for  $x$  in the kernel of  $F$ ,  $\gamma_x(t) = tx$  for all  $t$  in  $\mathbb{R}$ , whence the assertion.  $\square$

**4.2. Properties of the maps  $\gamma$  and  $a$ .** Let  $\Gamma$  be as in Subsection 3.2 and  $\Gamma_{\mathbb{R}}$  its intersection with  $\mathbb{C}^n \times \mathbb{R} \times \mathbb{C}^n$ .

**Lemma 4.5.** *Let  $\Gamma_0$  be the connected component of  $\Gamma_{\mathbb{R}}$  containing  $\mathbb{C}^n \times \{0, 0\}$ .*

(i) *The subset  $\Gamma_0$  of  $\mathbb{C}^n \times \mathbb{R} \times \mathbb{C}^n$  is closed and semialgebraic. Moreover, it contains  $G$ .*

(ii) *The subset  $G$  of  $\Gamma_0$  is open.*

*Proof.* (i) As  $\mathbb{C}^n \times \{(0, 0)\}$  is contained in  $\Gamma$ , it is contained in a connected component of  $\Gamma_{\mathbb{R}}$ . Since  $\Gamma_{\mathbb{R}}$  is a semialgebraic closed subset of  $\mathbb{C}^n \times \mathbb{R} \times \mathbb{C}^n$ ,  $\Gamma_0$  is semialgebraic by [Cos02, Theorem 2.23]. The set  $G$  is contained in  $\Gamma_{\mathbb{R}}$  since for all  $(x, t)$  in  $\Omega$ ,  $\Phi \circ \gamma_x(t) = tx$  by Lemma 3.2. As  $\Omega$  is a connected subset of  $\mathbb{C}^n \times \mathbb{R}$ ,  $G$  is a connected subset of  $\mathbb{C}^n \times \mathbb{R} \times \mathbb{C}^n$  by Corollary 4.4(i). Hence  $G$  is contained in  $\Gamma_0$  since  $\mathbb{C}^n \times \{(0, 0)\}$  is contained in  $G$ .

(ii) Let  $(x, t, y)$  be in  $G$  and  $(\omega, \psi)$  an adapted pair to  $y$ . As  $\gamma$  is continuous,  $\gamma^{-1}(\omega) \times \omega$  is an open subset of  $\mathbb{C}^n \times \mathbb{R} \times \mathbb{C}^n$ . Its intersection with  $\Gamma_{\mathbb{R}}$  is the graph of the map

$$\gamma^{-1}(\omega) \longrightarrow \mathbb{C}^n, \quad (z, s) \mapsto \psi(sz).$$

Since for all  $(z, s)$  in  $\gamma^{-1}(\omega)$ ,  $\gamma(z, s)$  is in  $\omega$  and  $\Phi \circ \gamma(z, s) = sz$ ,  $\gamma^{-1}(\omega) \times \omega \cap \Gamma_{\mathbb{R}}$  is contained in  $G$ , whence the assertion.  $\square$

**Proposition 4.6.** (i) *The graph  $G$  is equal to  $\Gamma_0$ .*

(ii) *The map  $\gamma$  is continuous and semialgebraic.*

(iii) *The function  $a$  is continuous and semialgebraic. In particular,  $a^{-1}(\{+\infty\})$  is a closed semialgebraic subset of  $\mathbb{C}^n$ .*

*Proof.* (i) Let  $\overline{G}$  be the closure of  $G$  in  $\mathbb{C}^n \times \mathbb{R} \times \mathbb{C}^n$ . Then  $\overline{G}$  is contained in  $\Gamma_0$  since  $\Gamma_0$  is a closed subset of  $\mathbb{C}^n \times \mathbb{R} \times \mathbb{C}^n$ . By Lemma 4.5(ii),  $G$  is an open subset of  $\Gamma_0$ . Hence it remains to prove  $G = \overline{G}$  since  $\Gamma_0$  is connected by definition.

Suppose that  $G$  is strictly contained in  $\overline{G}$ . A contradiction is expected. Let  $(x, t, y)$  be in  $\overline{G} \setminus G$ . As  $\gamma$  is continuous by Corollary 4.4(i),  $G$  is a closed subset of  $\Omega \times \mathbb{C}^n$ . Hence  $t = \pm a(x)$  since  $a$  is continuous by Corollary 4.4(ii). By Lemma 4.2(i),  $G$  and  $\overline{G}$  are invariant under the homeomorphism  $(x, t, y) \mapsto (x, -t, y)$ . So we can suppose  $t = a(x)$ . For some sequence  $(x_m, t_m)$ ,  $m = 1, 2, \dots$  in  $\Omega$ ,

$$\lim_{m \rightarrow +\infty} \gamma(x_m, t_m) = y.$$

Let  $(\omega, \psi)$  be an adapted pair to  $y$ . For  $z$  in  $\omega$ , denote by  $\tau_z$  the integral curve of the vector field  $X_{a(x)^{-1}\Phi(z)}$  such that  $\tau_z(a(x)) = z$ . In particular,  $\tau_y$  is the integral curve of the vector field  $X_x$  such that  $\tau_y(a(x)) = y$  and by Lemma 3.2, for all  $z$  in  $\omega$  and for all  $t$  in the interval of definition of  $\tau_z$ ,

$$\Phi \circ \tau_z(t) = \Phi(z) + (t - a(x))a(x)^{-1}\Phi(z) = ta(x)^{-1}\Phi(z).$$

For some positive real number  $\eta$ ,  $[a(x) - \eta, a(x) + \eta]$  is contained in the interval of definition of  $\tau_y$ . Then, by [Car67, Théorème 3.6.1], for some open neighborhood  $\omega'$  of  $y$  in  $\omega$ ,  $[a(x) - \eta, a(x) + \eta]$  is contained in the interval of definition of  $\tau_z$  for all  $z$  in  $\omega'$  since the map  $(y, z) \mapsto X_{a(x)^{-1}\Phi(z)}(y)$  is polynomial. Moreover, the map

$$\omega' \times [a(x) - \eta, a(x) + \eta] \xrightarrow{\tau} \mathbb{C}^n, \quad (z, t) \mapsto \tau_z(t)$$

is continuous. Hence we can suppose that the image of  $\tau$  is contained in  $\omega$ .

For some positive integer  $M$  and for  $m$  bigger than  $M$ ,  $\gamma(x_m, t_m)$  is in  $\omega'$  and for  $s$  positive real number smaller than  $\eta/2$ ,  $t_m - s$  is in  $[a(x) - \eta, a(x) + \eta]$ . Since  $\Phi(\omega')$  is a neighborhood of  $\Phi(y)$  and  $\Phi \circ \gamma(x_m, t_m) = t_m x_m$ , we can suppose that for  $m$  bigger than  $M$ ,  $a(x)x_m$  is in  $\Phi(\omega')$ . Setting  $z_m := \psi(a(x)x_m)$ ,  $\tau_{z_m}$  is an integral curve of  $X_{x_m}$  and  $\tau_{z_m}(t_m) = \gamma(z_m, t_m)$  since

$$\Phi \circ \tau_{z_m}(t_m) = \Phi \circ \gamma(z_m, t_m) = t_m z_m.$$

So, by the unicity of the integral curve with initial datum,  $\tau_{z_m}(t) = \gamma_{z_m}(t)$  for all  $t$  in  $[a(x) - \eta, a(x) + \eta]$ . As a result, for  $s$  positive real number smaller than  $\eta/2$ ,

$$\gamma(x, a(x) - s) = \lim_{m \rightarrow +\infty} \gamma(x_m, t_m - s) = \lim_{m \rightarrow +\infty} \tau_{z_m}(t_m - s) = \tau_y(a(x) - s)$$

by continuity of  $\gamma$  and  $\tau$ , whence the contradiction since

$$\lim_{s \rightarrow 0} \|\gamma(x, a(x) - s)\| = +\infty$$

by Corollary 3.5.

(ii) By Lemma 4.5(i),  $\Gamma_0$  is semialgebraic. So, by (i),  $\gamma$  is a semialgebraic map, whence the assertion since  $\gamma$  is continuous.

(iii) By (i),  $\Omega$  is the image of  $\Gamma_0$  by the projection

$$\mathbb{C}^n \times \mathbb{R} \times \mathbb{C}^n \longrightarrow \mathbb{C}^n \times \mathbb{R}, \quad (x, t, y) \mapsto (x, t).$$

Hence  $\Omega$  and  $\overline{\Omega}$  are semialgebraic subsets of  $\mathbb{C}^n \times \mathbb{R}$ . By Corollary 4.4(ii),  $a$  is continuous. Hence,  $\overline{\Omega}$  is the disjoint union of  $\Omega$  and the graphs of the restrictions of  $a$  and  $-a$  to  $\mathbb{C}^n \setminus a^{-1}(+\infty)$  since  $a^{-1}(+\infty) \times \mathbb{R}$  is contained in  $\Omega$ . In particular,  $a^{-1}(+\infty)$  is the complement in  $\mathbb{C}^n$  to the image of  $\overline{\Omega} \setminus \Omega$  by the projection

$$\mathbb{C}^n \times \mathbb{R} \longrightarrow \mathbb{C}^n, \quad (x, t) \mapsto x$$

and the graph of the restriction of  $a$  to  $\mathbb{C}^n \setminus a^{-1}(+\infty)$  is the intersection of  $\overline{\Omega} \setminus \Omega$  and  $(\mathbb{C}^n \setminus a^{-1}(+\infty)) \times \mathbb{R}_+$ . Hence  $a^{-1}(+\infty)$  is a semialgebraic subset of  $\mathbb{C}^n$  and the restriction of  $a$  to  $\mathbb{C}^n \setminus a^{-1}(+\infty)$  is semialgebraic.  $\square$

**4.3. Étale covering.** Let  $K$  be the field of rational fractions  $\mathbb{C}(x_1, \dots, x_n)$  and  $K'$  the subfield  $\mathbb{C}(x_1 + F_1^3, \dots, x_n + F_n^3)$  of  $K$ . Denote by  $d$  the degree of the extension  $K$  of  $K'$ .

**Lemma 4.7.** *Let  $X$  and  $Y$  be two irreducible complex algebraic varieties and  $\psi$  a morphism from  $X$  to  $Y$ . For  $i$  nonnegative integer, denote by  $Y_i$  the subset of elements  $y$  such that  $|\psi^{-1}(y)| = i$ . Then, for all nonnegative integer  $i$ ,  $Y_i$  is an algebraic constructible subset of  $Y$ .*

*Proof.* Let  $Y'_n$  be the subset of elements  $y$  of  $Y$  such that  $|\psi^{-1}(y)| \geq n$ . Denote by  $Z_n$  the subset of elements  $(y, x_1, \dots, x_n)$  of  $Y \times X^n$  such that  $y = \psi(x_1) = \dots = \psi(x_n)$ . Then  $Z_n$  is a closed subset of  $Y \times X^n$ . Let  $Z'_n$  be the open subset of elements  $(y, x_1, \dots, x_n)$  of  $Z_n$  such that  $x_i \neq x_j$  for  $i \neq j$ . Then  $Y'_n$  is an algebraic constructible subset of  $Y$  as the image of  $Z'_n$  by the projection

$$(y, x_1, \dots, x_n) \mapsto y.$$

By definition,  $Y_n = Y'_n \setminus Y'_{n+1}$ . Hence  $Y_n$  is an algebraic constructible subset of  $Y$ .  $\square$

**Corollary 4.8.** *Suppose that  $\psi$  is a quasi-finite dominant morphism. Let  $j$  be the degree of the field extension  $\mathbb{C}(Y) \subset \mathbb{C}(X)$ . Then  $Y_j$  contains a dense Zariski open subset of  $Y$ .*

*Proof.* As  $j$  is the degree of the separable field extension  $\mathbb{C}(Y) \subset \mathbb{C}(X)$ ,  $Y_j$  contains the generic point of  $Y$ . So, by Lemma 4.7,  $Y_j$  contains a dense Zariski open subset of  $Y$ .  $\square$

**Proposition 4.9.** *For some nonempty Zariski open subset  $U$  of  $\mathbb{C}^n$ , the restriction of  $\Phi$  to  $\Phi^{-1}(U)$  is a covering map of degree  $d$ . In particular, it is a finite morphism.*

*Proof.* As  $\Phi$  is étale,  $\Phi$  is a quasi finite morphism. So, by Corollary 4.8, for some nonempty Zariski open subset  $U$  of  $\mathbb{C}^n$ ,  $\Phi^{-1}(U)$  has cardinality  $d$  for all  $x$  in  $U$ . Let  $x$  be in  $U$  and  $x_1, \dots, x_d$  the elements of  $\Phi^{-1}(x)$ . For  $i = 1, \dots, d$ , denote by  $(\omega_i, \psi_i)$  an adapted pair to  $x_i$ . Since  $x_1, \dots, x_d$  are pairwise different, we can choose  $\omega_1, \dots, \omega_d$  pairwise disjoint. Let  $\omega$  be the intersection of  $\Phi(\omega_1), \dots, \Phi(\omega_d)$ . Then  $\omega$  is an open neighborhood of  $x$  and  $\Phi^{-1}(\omega)$  is the disjoint union of the

open subsets  $\omega_1 \cap \Phi^{-1}(\omega), \dots, \omega_d \cap \Phi^{-1}(\omega)$ . As a result, the restriction of  $\Phi$  to  $\Phi^{-1}(U)$  is an étale covering of degree  $d$  of  $U$ , whence the proposition.  $\square$

**4.4. At the neighborhood of an element of  $\mathbb{C}^n \setminus a^{-1}(\{+\infty\})$ .** Let  $x$  be in  $\mathbb{C}^n \setminus a^{-1}(\{+\infty\})$  and  $\theta$  a continuous semialgebraic arc,

$$[0, 1] \xrightarrow{\theta} \mathbb{C}^n,$$

such that  $\theta(0) = x$  and  $a(\theta(t)) > a(x)$  for all  $t$  in  $]0, 1]$ .

**Lemma 4.10.** *For some positive rational number  $r$  and a continuous semialgebraic map  $\psi$  from  $[0, 1]$  to  $\mathbb{C}^n$ ,*

$$\psi(0) \neq 0 \quad \text{and} \quad \gamma(\theta(t), a(x)) = t^{-r} \psi(t)$$

for all  $t$  in  $]0, 1]$ .

*Proof.* By Proposition 4.6(ii), the map  $t \mapsto \gamma(\theta(t), a(x))$  from  $]0, 1]$  to  $\mathbb{C}^n$  is continuous and semialgebraic. Moreover, by Lemma 3.2,

$$(1) \quad \gamma(\theta(t), a(x)) + F^{(3)}(\gamma(\theta(t), a(x))) = a(x)\theta(t)$$

for all  $t$  in  $]0, 1]$ . By Lemma 2.6(i), if the map  $t \mapsto \|\gamma(\theta(t), a(x))\|$  is bounded, then for some  $y$  in  $\mathbb{C}^n$ ,

$$\lim_{t \rightarrow 0} \gamma(\theta(t), a(x)) = y$$

so that  $(x, a, y)$  is in  $G$  and  $(x, a(x))$  is in  $\Omega$  by Proposition 4.6(i). By definition,  $(x, a(x))$  is not in  $\Omega$ . As a result, the map  $t \mapsto \|\gamma(\theta(t), a(x))\|$  is not bounded, whence the assertion by Lemma 2.6(ii),  $\square$

**4.5. Proof of Proposition 4.1.** Let  $U$  be the Zariski open subset of  $\mathbb{C}^n$  as in Proposition 4.9 and  $Y$  its complement in  $\mathbb{C}^n$ .

**Lemma 4.11.** *Let  $Z$  be the closure in  $\mathbb{C}^n$  of the union of  $tY$ ,  $t \in \mathbb{R}_+^*$  and  $O := \mathbb{C}^n \setminus Z$ .*

(i) *The subsets  $Z$  and  $O$  of  $\mathbb{C}^n$  are semialgebraic. In particular, the connected components of  $O$  are finitely many and semialgebraic.*

(ii) *The open subset  $O$  of  $\mathbb{C}^n$  is dense in  $\mathbb{C}^n$ .*

(iii) *Let  $O_*$  be a connected component of  $O$  and  $O'_*$  the subset of elements  $x$  of  $O_*$  such that  $a$  is constant on a neighborhood of  $x$ . Then  $tO_* = O_*$  and  $tO'_* = O'_*$  for all  $t$  in  $\mathbb{R}_+^*$ .*

*Proof.* (i) Let  $Z_*$  be the union of  $tY$ ,  $t \in \mathbb{R}_+^*$ . As the map

$$\mathbb{R}_+^* \times \mathbb{C}^n \longrightarrow \mathbb{C}^n, \quad (t, x) \longmapsto tx$$

is semialgebraic and  $Y$  is an algebraic subvariety of  $\mathbb{C}^n$ ,  $Z_*$  is a semialgebraic subset of  $\mathbb{C}^n$ . Hence  $Z$  and  $O$  are semialgebraic subsets of  $\mathbb{C}^n$ , whence the assertion by [Cos02, Theorem 2.23].

(ii) By definition,  $Y$  is an algebraic subvariety of dimension smaller than  $n$  of  $\mathbb{C}^n$ . Hence its dimension as a semialgebraic subset of  $\mathbb{C}^n$  is at most  $2n - 2$  by Lemma 2.2(iv). As a result,  $Z_*$  and  $Z$  have dimension at most  $2n - 1$  by Lemma 2.2, (i) and (v). Hence  $O$  is a dense open subset of  $\mathbb{C}^n$  by Lemma 2.2(iii).

(iii) By definition,  $tO = O$  for all  $t$  in  $\mathbb{R}_+^*$ . Hence the group  $\mathbb{R}_+^*$  permutes the connected components of  $O$  and the stabilizers are open subgroups of  $\mathbb{R}_+^*$ . Hence  $tO_* = O_*$  for all  $t$  in  $\mathbb{R}_+^*$ . By Lemma 4.2(ii), for  $x$  in  $O_*$  and  $t$  in  $\mathbb{R}_+^*$ ,  $a(tx) = t^{-1}a(x)$ . Hence  $O'_*$  is invariant under the action of  $\mathbb{R}_+^*$ .  $\square$

For  $O_*$  a connected component of  $O$ , denote by  $O''_*$  the interior of the complement to  $O'_*$  in  $O_*$ . Then  $O''_*$  is an open subset of  $U$ , invariant under the action of  $\mathbb{R}_+^*$  by Lemma 4.11(iii).

**Lemma 4.12.** *Let  $O_*$  be a connected component of  $O$ . Then  $O''_*$  is empty.*

*Proof.* Suppose that  $O''_*$  is nonempty. A contradiction is expected. Let  $x$  be in  $O''_*$  and  $V$  a semialgebraic compact neighborhood of  $x$  in  $O''_*$ . As  $a$  is continuous, for some  $y$  in  $V$ ,  $a(y) \leq a(z)$  for all  $z$  in  $V$ . Moreover, since  $a$  is semialgebraic and  $V$  is contained in  $O''_*$ ,  $a^{-1}(\{y\}) \cap V$  is a semialgebraic subset whose interior is empty. Then, by Proposition 2.3, for some continuous semialgebraic arc  $\theta$ ,

$$[0, 1] \xrightarrow{\theta} V,$$

$\theta(0) = y$  and  $a(\theta(t)) > a(y)$  for all  $t$  in  $]0, 1]$ . By Lemma 4.10,

$$\lim_{t \rightarrow 0} \|\gamma(\theta(t), a(y))\| = +\infty.$$

Moreover, for all  $t$  in  $]0, 1]$ ,  $\Phi \circ \gamma(\theta(t), a(y)) = a(y)\theta(t)$ . As  $O''_*$  is invariant under  $\mathbb{R}_+^*$ ,  $a(y)\theta(t)$  is in  $O''_*$  for all  $t$  in  $[0, 1]$ . Let  $V'$  be a compact neighborhood of  $a(y)y$  in  $O''_*$ . As  $O''_*$  is contained in  $U$ ,  $\Phi^{-1}(V')$  is compact by Proposition 4.9. Moreover, for some positive real number  $\eta$ ,  $a(y)\theta(t)$  is in  $V'$  and  $\gamma(\theta(t), a(y))$  is in  $\Phi^{-1}(V')$  for all  $t$  in  $]0, \eta]$ , whence the contradiction.  $\square$

By Lemma 4.12 and Corollary 2.5,  $a$  is constant on each connected component of  $O$ . By Lemma 4.2(ii), for  $x$  in  $O$  and  $t$  in  $\mathbb{R}_+^*$ ,  $a(tx) = t^{-1}a(x)$ . Hence  $O$  is contained in  $a^{-1}(\{+\infty\})$  since the connected components of  $O$  are invariant under  $\mathbb{R}_+^*$ . As a result  $a^{-1}(\{+\infty\}) = \mathbb{C}^n$  since  $a$  is continuous and  $O$  is dense in  $\mathbb{C}^n$  by Lemma 4.12(ii), whence Proposition 4.1.

## 5. SOME POLYNOMIAL MAPS

Let  $p_0, p_1, \dots$  be the sequence of polynomial endomorphisms of  $\mathbb{C}^n$  defined by the induction relations:

$$p_0 = 0, \quad p_1(x) = x, \quad p_j = - \sum_{(i_1, i_2, i_3) \in \mathbb{N}_j^3} F(p_{i_1})F(p_{i_2})F(p_{i_3}),$$

for  $j \geq 2$ . By induction, for  $j = 0, 1, \dots$ ,  $p_j$  is homogeneous of degree  $j$ .

**5.1. Some power series and polynomial maps.** For  $x$  in  $\mathbb{C}^n$ , we denote by  $S_x$  the formal power series of the variable  $t$ :

$$S_x(t) := p_0(x) + tp_1(x) + t^2p_2(x) + \dots$$

Let  $r_x$  be the radius of convergence of  $S_x$ . As in Section 4,  $\gamma_x$  is the maximal element of  $\mathcal{C}_x$  such that  $\gamma_x(0) = 0$ .

**Lemma 5.1.** *Let  $x$  be in  $\mathbb{C}^n$ .*

(i) *Let  $S$  be in  $\mathbb{C}^n[[t]]$ . Then  $S = S_x$  if and only if  $\Phi \circ S(t) = tx$ .*

(ii) *The radius of convergence  $r_x$  is different from 0 and for  $t$  in  $] - r_x, r_x[$ ,*

$$\gamma_x(t) = \sum_{j=0}^{\infty} t^j p_j(x).$$

(iii) *For  $j$  even integer,  $p_j(x) = 0$ .*

*Proof.* (i) Let  $a_0, a_1, \dots$  the coefficients of  $S$ . By Lemma 1.5,  $\Phi^{-1}(0) = \{0\}$ . Hence  $a_0 = 0$  if  $\Phi \circ S(t) = tx$ . For  $j = 0, 1, \dots$ , the coefficient of  $t^j$  in the expansion of  $F^{(3)}(S(t))$  is equal to

$$\sum_{(i_1, i_2, i_3) \in \mathbb{N}_j^3} F(a_{i_1})F(a_{i_2})F(a_{i_3}).$$

As a result,  $\Phi \circ S(t) = tx$  if and only if

$$a_0 = 0, \quad a_1 = x, \quad a_j = - \sum_{(i_1, i_2, i_3) \in \mathbb{N}_j^3} F(a_{i_1})F(a_{i_2})F(a_{i_3}),$$

for  $j \geq 2$ , whence the assertion.

(ii) Let  $(\omega, \psi)$  be an adapted pair to 0. Then  $\psi$  is a holomorphic map from  $\Phi(\omega)$  to  $\mathbb{C}^n$ . For some positive real number  $r$ , for  $t$  in  $\mathbb{C}$  such that  $|t| < r$ ,  $tx$  is in  $\Phi(\omega)$  and  $\psi(tx)$  is the sum of a power serie

$$\psi(tx) = \sum_{j=0}^{\infty} t^j a_j.$$

By (i), since  $\Phi \circ \psi(tx) = tx$ ,  $a_j = p_j(x)$ . In particular,  $r_x \neq 0$ . Moreover, by Lemma 3.2, the map

$$] - r_x, r_x[ \longrightarrow \mathbb{C}^n, \quad t \longmapsto \sum_{j=0}^{\infty} t^j p_j(x)$$

is an integral curve of the vector field  $X_x$ . By the unicity of the integral curve with initial datum, it is the restriction of  $\gamma_x$  to  $] - r_x, r_x[$ , whence the assertion.

(iii) By Lemma 4.2(i),  $\gamma_x(t) = -\gamma_x(-t)$  for all  $t$  in  $] - r_x, r_x[$ . Hence by (ii),  $p_{2j}(x) = 0$  for all  $j$  in  $\mathbb{N}$ . □

For  $x$  in  $\mathbb{C}^n$ , let  $\varphi_x$  be the map from  $\mathbb{C}$  to  $\mathbb{C}^n$  defined by the following relations:  $\varphi_x(0) = 0$  and for  $t$  in  $\mathbb{C}^*$ ,

$$\varphi_x(t) = \gamma_{ux}(|t|) \quad \text{with} \quad u := \frac{t}{|t|}.$$

As the domain of definition of  $\gamma_y$  is  $\mathbb{R}$  for all  $y$  in  $\mathbb{C}^n$ ,  $\varphi_x$  is well defined.

**Lemma 5.2.** *Let  $x$  be in  $\mathbb{C}^n$ .*

(i) *For all  $t$  in  $\mathbb{C}$ ,  $\Phi \circ \varphi_x(t) = tx$ .*

(ii) *The map  $\varphi_x$  is continuous and for all  $t$  in  $\mathbb{C}$  such that  $|t| < r_x$ ,  $\varphi_x(t) = S_x(t)$ .*



*Proof.* (i) By definition  $\Phi \circ \varphi_x(0) = 0$ . By Lemma 3.2, for  $t$  in  $\mathbb{C}^*$  and  $u = t/|t|$ ,

$$\Phi \circ \varphi_x(t) = \Phi \circ \gamma_{ux}(|t|) = u|t|x = tx,$$

whence the assertion.

(ii) Let  $\mathbb{U}$  be the subset of complex numbers of module 1. Since the polynomial maps  $p_0, p_1, \dots$  are homogeneous,  $r_x$  is the radius of convergence of  $S_{ux}$  for all  $u$  in  $\mathbb{U}$ . Moreover, by Lemma 5.1(ii),

$$\varphi_x(t) = \sum_{j=0}^{\infty} t^j p_j(x)$$

for all  $t$  in  $\mathbb{C}$  such that  $|t| < r_x$ . In particular,  $\varphi_x$  is continuous in a neighborhood of 0.

Let  $\psi_x$  be the map

$$\mathbb{U} \times \mathbb{R}_+^* \xrightarrow{\psi_x} \mathbb{C}^n, \quad (u, t) \mapsto \varphi_x(tu).$$

With the notations of Subsection 4.1,  $\psi_x(u, t) = \gamma(ux, t)$ . Hence, by Corollary 4.4(i), the map  $\psi_x$  is continuous. As a result, the restriction of  $\varphi_x$  to  $\mathbb{C}^*$  is continuous since the map

$$\mathbb{U} \times \mathbb{R}_+^* \longrightarrow \mathbb{C}^*, \quad (u, t) \mapsto tu$$

is a homeomorphism, whence the assertion.  $\square$

**Proposition 5.3.** *Let  $x$  be in  $\mathbb{C}^n$ . Then  $\varphi_x$  is a polynomial map and for some positive integer  $j$ ,  $p_i(x) = 0$  for all positive integer bigger than  $j$ .*

*Proof.* Let  $\Gamma_x$  be as in Subsection 3.2 and  $\Gamma'_x$  the subset of  $\mathbb{C} \times \mathbb{C}^n$  such that  $\Gamma_x = \{x\} \times \Gamma'_x$ . Denote by  $G$  the graph of  $\varphi_x$ . By Lemma 5.2, (i) and (ii),  $G$  is a closed connected subset of  $\Gamma'_x$ . Let  $t$  be in  $\mathbb{C}$  and  $(\omega, \psi)$  an adapted pair to  $\varphi_x(t)$ . Then  $\varphi_x^{-1}(\omega) \times \omega$  is an open subset of  $\mathbb{C} \times \mathbb{C}^n$  and its intersection with  $\Gamma'_x$  is the graph of the map

$$\varphi_x^{-1}(\omega) \longrightarrow \omega, \quad s \mapsto \psi(sx).$$

As a result,  $\varphi_x^{-1}(\omega) \times \omega \cap \Gamma'_x$  is contained in  $G$  since  $\varphi_x(s)$  is in  $\omega$  for all  $s$  in  $\varphi_x^{-1}(\omega)$ . Hence  $G$  is an irreducible component of  $\Gamma'_x$  by Lemma 3.3(ii). Moreover, the canonical projection from  $G$  to  $\mathbb{C}$  is birational and injective. So, by Zariski's Main theorem [Mu88, §9], it is an isomorphism of algebraic varieties. Hence  $\varphi_x$  is a polynomial map. As a result, by Lemma 5.2(ii), for some positive integer  $j$ ,  $p_i(x) = 0$  for  $j < i$ .  $\square$

**5.2. Polynomial automorphism.** For  $i$  positive integer, set:

$$X_i := \{x \in \mathbb{C}^n \mid 0 = p_i(x) = p_{i+1}(x) = \dots\}.$$

**Lemma 5.4.** *For some positive integer  $j$ ,  $p_i = 0$  for  $i > j$ .*

*Proof.* According to Proposition 5.3,  $\mathbb{C}^n$  is the union of  $X_1, X_2, \dots$ . By definition, for all  $i$ ,  $X_i$  is a Zariski closed subset. By Baire's property, for some positive integer  $j$ ,  $X_j$  contains a nonempty open subset of  $\mathbb{C}^n$ . Hence,  $X_j = \mathbb{C}^n$  and  $p_i = 0$  for  $i \geq j$ .  $\square$

Let  $j$  be the biggest integer such that  $p_j \neq 0$  and set:

$$P := p_1 + \cdots + p_j.$$

**Theorem 5.5.** *The polynomial endomorphism  $\Phi$  is a polynomial automorphism and  $P$  is its inverse.*

*Proof.* By Proposition 5.3 and Lemma 5.1(ii),  $\varphi_x(1) = P(x)$  for all  $x$  in  $\mathbb{C}^n$ . So, by Lemma 5.1(i),  $\Phi \circ P(x) = x$ . As a result,  $P$  is an étale morphism since so is  $\Phi$ . In particular, for some open subset  $\omega$  of  $\mathbb{C}^n$ , the restriction of  $P$  to  $\omega$  is an isomorphism onto the open subset  $P(\omega)$  of  $\mathbb{C}^n$ . Denoting by  $\psi$  the inverse of this isomorphism,

$$\Phi \circ P(x) = x \quad \text{and} \quad P \circ \psi(y) = y$$

for all  $(x, y)$  in  $\omega \times P(\omega)$ , whence

$$\Phi(y) = \psi(y) \quad \text{and} \quad P \circ \Phi(y) = y$$

for all  $y$  in  $P(\omega)$ . Then

$$\Phi \circ P(x) = P \circ \Phi(x) = x$$

for all  $x$  in  $\mathbb{C}^n$  since  $P$  and  $\Phi$  are polynomial endomorphisms, whence the theorem.  $\square$

## 6. CONCLUSION

By Theorem 5.5 and [Dru83, Theorem 3], Keller's Jacobian Conjecture is true in any dimension:

**Theorem 6.1.** *Let  $n$  be a positive integer. Then any étale polynomial endomorphism of  $\mathbb{C}^n$  is a polynomial automorphism.*

From Theorem 6.1 and [Bass 82, Section I], we deduce the following theorem:

**Theorem 6.2.** *Let  $\mathbb{k}$  be an integral ring of characteristic 0 and  $n$  a positive integer. Then any polynomial endomorphism of  $\mathbb{k}^n$  whose Jacobian matrix has a constant invertible determinant is a polynomial automorphism.*

From Theorem 6.2 and [Ka, Theorem 1], we deduce the following theorem:

**Theorem 6.3.** *Let  $\mathbb{k}$  be a field of characteristic zero and  $n$  a positive integer. Denote by  $A_{n,\mathbb{k}}$  the  $n$ -th Weyl algebra over  $\mathbb{k}$ . Then any endomorphism of  $A_{n,\mathbb{k}}$  is an automorphism.*

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