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NILPOTENT MATRICES, INTEGRAL CURVES AND JACOBIAN CONJECTURE

JEAN-YVES CHARBONNEL

Abstract. Following the results of L.M. Drużkowski, we are interested in the Jacobian Conjecture.

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1. Introduction

In this note, we are interested in the jacobian conjecture, that is: for n positive integer, an étale polynomial endomorphism of $\mathbb{C}^n$ is a polynomial automorphism. Some results are known for a long time. For example, an étale polynomial endomorphism is an automorphism if it is proper or birational [Bass 82, Theorem 2.1]. According to the results of L.M. Drużkowski, it is sufficient to prove that the following conjecture is true.

Conjecture 1.1. *Let n be a positive integer and $F_1, \dots, F_n$ in the dual of $\mathbb{C}^n$. Denote by Φ the polynomial endomorphism of $\mathbb{C}^n$:*

$$\mathbb{C}^n \xrightarrow{\Phi} \mathbb{C}^n, \qquad x = (x_1, \dots, x_n) \longmapsto (x_1 + F_1(x)^3, \dots, x_n + F_n(x)^3).$$

If Φ is an étale morphism, then Φ is a polynomial automorphism.

As a result, we are interested in the proof of conjecture 1.1.

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1.1. **Sketch of proof.** Let n be a positive integer and $F_1, \dots, F_n, \Phi$ as in Conjecture 1.1. Suppose that Φ is étale. For x in $\mathbb{C}^n$, let X_x be the vector field on $\mathbb{C}^n$,

$$y \longmapsto \Phi'(y)^{-1}(x)$$

with $\Phi'(y)$ the differential of Φ at y. If γ is an integral curve of X_x with initial datum $\gamma(0) = y$, then for all t in the interval of definition of γ,

$$\Phi\circ\gamma(t) = \Phi(y) + tx.$$

The key step of the proof is the following proposition:

Proposition 1.2. *Let x be in $\mathbb{C}^n$ and γ_x the maximal integral curve of X_x with initial datum $\gamma_x(0) = 0$. Then $\mathbb{R}$ is the interval of definition of γ_x.*

For x in $\mathbb{C}^n$, let φ_x be the map from $\mathbb{C}$ to $\mathbb{C}^n$ such that

$$\varphi_x(0) := 0 \quad \text{and} \quad \varphi_x(t) := \gamma_{ux}(|t|) \quad \text{with} \quad u := \frac{t}{|t|},$$

for all t in $\mathbb{C}^*$. The second step is the following result:

Proposition 1.3. *For all x in $\mathbb{C}^n$, φ_x is a polynomial map such that $\Phi\circ\varphi_x(t) = tx$ for all t in $\mathbb{C}$.*

We deduce from this proposition that the map $x \mapsto \varphi_x(1)$ is a polynomial endomorphism. Then Φ is a polynomial automorphism and its inverse is the map $x \mapsto \varphi_x(1)$.

1.2. **Notations and Definition.** • As usual, for k positive integer and for $i = (i_1, \dots, i_k)$ in $\mathbb{N}^k$,

$$|i| := i_1 + \cdots + i_k.$$

For j nonnegative integer, denote by $\mathbb{N}^k_j$, the subset of elements i of $\mathbb{N}^k$ such that $|i| = j$.

• For E subset, $|E|$ is its cardinality when it is finite and $+\infty$ when it is not finite.

• The topology on $\mathbb{C}^n$ is the usual topology and $\|.\|$ is an euclidean norm on $\mathbb{C}^n$. Let d be the distance on $\mathbb{C}^n$ such that $d(x, y) := \|x - y\|$.

• For Φ a holomorphic endomorphism of $\mathbb{C}^n$ and x in $\mathbb{C}^n$, its differential at x is denoted by $\Phi'(x)$.

• For n positive integer and x, y in $\mathbb{C}^n$, set:

$$xy := (x_1y_1, \dots, x_ny_n) \quad \text{with} \quad x = (x_1, \dots, x_n) \quad \text{and} \quad y = (y_1, \dots, y_n).$$

As a result, by induction on k, for $v_1, \dots, v_k$ in $\mathbb{C}^n$, $v_1 \cdots v_k$ is a well defined element of $\mathbb{C}^n$. For F a map from $\mathbb{C}^n$ to $\mathbb{C}^n$ and k positive integer, denote by $F^{(k)}$ the map

$$\mathbb{C}^n \xrightarrow{F^{(k)}} \mathbb{C}^n, \qquad x \longmapsto F(x)^k.$$

• As usual, for n positive integer, $\mathrm{M}_n(\mathbb{C})$ is the algebra of square matrices of order n with complex coefficients. Denote by $\mathbf{1}_n$ the identity element of $\mathrm{M}_n(\mathbb{C})$. For A in $\mathrm{M}_n(\mathbb{C})$ and $1 \leq i, j \leq n$, $A_{i,j}$ is the coefficient of A at its i-row and j-column. The matrix A is identified with the endomorphism of $\mathbb{C}^n$,

$$(x_1, \dots, x_n) \longmapsto (A_{i,1}x_1 + \cdots + A_{i,n}x_n, i = 1, \dots, n).$$

• Let A be in $\mathbb{C}^n$. Denote by F the linear endomorphism of $\mathbb{C}^n$ identified with A and Φ the polynomial endomorphism of $\mathbb{C}^n$,

$$\mathbb{C}^n \xrightarrow{\quad\Phi\quad} \mathbb{C}^n\ , \qquad x \longmapsto x + F^{(3)}(x).$$

For $i = 1, \dots, n$, let F_i be the i-th coordinate of F and for x in $\mathbb{C}^n$, denote by D_x the diagonal matrix whose entries are $F_1(x)^2, \dots, F_n(x)^2$.

Definition 1.4. The matrix A is said AJC (adapted to Jacobian conjecture) if D_xA is a nilpotent matrix for all x in $\mathbb{C}^n$.

By [Dru83, Proposition 2], A is AJC if and only if Φ is an étale morphism.

Lemma 1.5. *Suppose that Φ is étale. Then $\Phi^{-1}(v) = \{v\}$ for all v in the kernel of F.*

Proof. As v is in the kernel of F, $\Phi(v) = v$. Let x be in $\mathbb{C}^n$ such that $\Phi(x) = v$. Then v is the image of x by $\mathbf{1}_n + D_xA$. As Φ is étale, for some positif integer m, $(D_xA)^m = 0$, whence

$$(\mathbf{1}_n + D_xA)^{-1} = \sum_{i=0}^{m-1} (-1)^i (D_xA)^i.$$

Since v is in the kernel of F, for i positive integer, $(D_xA)^i(v) = 0$, whence

$$x = (\mathbf{1}_n + D_xA)^{-1}(v) = v,$$

and the lemma. □

In this note, A is an AJC matrix different from 0. As Φ is an étale morphism, for all x in $\mathbb{C}^n$, there exists an open neighborhood ω of x in $\mathbb{C}^n$ such that the restiction of Φ to ω is a holomorhic isomorphism onto the open neighborhood $\Phi(\omega)$ of $\Phi(x)$ in $\mathbb{C}^n$. Denote by ψ the inverse of the restriction of Φ to ω and call the pair (ω, ψ) *an adapted pair to x.*

2. Semialgebraic sets and maps

We recall some results of semialgebraic geometry (see [Cos02]). Let E and F be finite dimensional real vector spaces. A subset of E is semialgebraic if it is a finite union of subsets defined by a system of a finite number of equalities and inequalities of polynomials. Denoting by $\mathfrak{S}(E)$ the set of semialgebraic subsets of E, $\mathfrak{S}(E)$ is stable under finite union and finite intersection. For S in $\mathfrak{S}(E)$, its closure in E, its interior and $E \setminus S$ are in $\mathfrak{S}(E)$.

For X subset of E and φ a map from X to F, φ is a semialgebraic map if its graph is a semialgebraic subset of $E \times F$. By Tarski-Seidenberg's theorem, X is a semialgebraic subset of E and the image of φ is a semialgebraic subset of F.

Lemma 2.1. *Let X be a subset of E, Y a subset of F and G a real vector space of finite dimension. If φ is a semialgebraic map from X to Y and ψ a semialgebraic map from Y to G then $\psi\circ\varphi$ is a semialgebraic map fom X to G.*

Proof. Let Γ_φ be the graph of φ and Γ_ψ the graph of ψ. Denote by Γ the intersection of $\Gamma_\varphi \times G$ and $E \times \Gamma_\psi$. Then the graph of $\psi\circ\varphi$ is the image of Γ by the projection

$$E \times F \times G \longrightarrow E \times G\ , \qquad (x, y, z) \longmapsto (x, z).$$

By hypothesis, Γ is a semialgebraic subset of $E \times F \times G$ and by Tarski-Seidenberg's theorem, $\psi\circ\varphi$ is a semialgebraic map. □

2.1. **Some results on semialgebraic sets.** According to [Cos02, Secion 2], a semialgebraic subset S of $\mathbb{R}^n$ has a decomposition into a finite disjoint union of semialgebraic subsets semialgebraically homeomorphic to open hypercubes $]0, 1[^d$ (by convention $]0, 1[^0$ is a point). By [Cos02, Proposition 3.15], the dimension of S is the maximum of these d and is independent of the decomposition. Moreover, by [Cos02, Theorem 2.23], S has a finite number of connected components and its connected components are semialgebraic.

Lemma 2.2. *Let S be in $\mathfrak{S}(\mathbb{R}^n)$ and $\overline{S}$ its closure in $\mathbb{R}^n$.*

(i) *The sets S and $\overline{S}$ have the same dimension.*

(ii) *The dimension of $\overline{S} \setminus S$ is smaller than* $\dim S$.

(iii) *The set S has an empty interior if and only if* $\dim S \leq n - 1$.

(iv) *If S is a Zariski closed subset of $\mathbb{R}^n$, then* $\dim S$ *is the dimension of S as an algebraic subset.*

(v) *For m positive integer and φ semialgebraic map from S to $\mathbb{R}^m$,* $\dim \varphi(S) \leq \dim S$.

Proof. Assertions (i) and (ii) are given by [Cos02, Proposition 3.16], Assertion (iv) is given by [Cos02, Theorem 3.20] and Assertion (v) is given by [Cos02, Theorem 3.18].

(iii) By definition, a nonempty semialgebraic open subset of $\mathbb{R}^n$ has dimension n. For T semialgebraic subset T of S, $\dim T \leq \dim S$ by [Cos02, Corollary 3.8]. Then S has an empty interior if $\dim S \leq n - 1$. Conversely if $\dim S = n$, it contains a semialgebraic subset homoeomorphic to $]0, 1[^n$. Hence S has a nonempty interior.

□

Proposition 2.3. *Let Y be a semialgebraic subset of $\mathbb{R}^n$, containing* 0 *and having an empty interior. Then there exists a continuous semialgebraic arc θ,*

$$[0, 1] \xrightarrow{\ \theta\ } \mathbb{R}^n\ , \qquad t \longmapsto \theta(t)$$

such that $\theta(0) = 0$ and $\theta(t) \notin Y$ for all t in $]0, 1]$.

Proof. As Y has an empty interior, $\mathbb{R}^n \setminus Y$ is a dense semialgebraic open subset of $\mathbb{R}^n$, not containing 0, whence the proposition by the Curve selection lemma [Cos02, Theorem 3.13]. □

2.2. **A result on a semialgebraic function.** Let Ω be a connected semialgebraic open subset of $\mathbb{C}^n$ and a a continuous semialgebraic function on Ω with real values.

Lemma 2.4. *Let Ω' be the subset of elements x of Ω such that a is constant in a neighborhood of x. Then Ω' is a semialgebraic open subset of $\mathbb{C}^n$.*

Proof. By definition Ω' is a union of open subsets of $\mathbb{C}^n$. Hence it is open. Let Z be the subset of elements $(x, x_1, x_2, \varepsilon)$ of $\Omega \times \Omega \times \Omega \times \mathbb{R}_+^*$ such that

$$d(x, x_1) \leq \varepsilon, \quad d(x, x_2) \leq \varepsilon, \quad a(x_1) \neq a(x_2).$$

Since Ω, a, d are semialgebraic, Z is a semialgebraic subset of $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{R}$. Denote by $\overline{Z}$ the closure of Z in $\Omega \times \Omega \times \Omega \times \mathbb{R}_+$ and Z_0 the intersection of $\overline{Z}$ and $\Omega \times \Omega \times \Omega \times \{0\}$. Then $\Omega \setminus \Omega'$ is the image of Z_0 by the first projection. As a result, $\Omega \setminus \Omega'$ and Ω' are semialgebraic subsets of $\mathbb{C}^n$, whence the lemma. □

Corollary 2.5. *Suppose that the interior of $\Omega \setminus \Omega'$ is empty. Then the function a is constant.*

Proof. By hypothesis and Lemma 2.4, Ω' is a dense semialgebraic open subset of Ω. Then, by [Cos02, Theorem 2.23], Ω' has a finite number of connected components. Denote them by $O_1, \ldots, O_m$. For $i = 1, \ldots, m$, a is constant on O_i since a is continuous. Denote by a_i its value. As $O \setminus O'$ has an empty interior, for x in $O \setminus O'$, for some i, x is in the closure of O_i so that $a(x) = a_i$. As a result, the image of a is equal to $\{a_1, \ldots, a_m\}$. Hence a is constant since a is continuous and O is connected. □

2.3. **On semialgebraic curves.** In this subsection we give some results on continuous semialgebraic curves.

Lemma 2.6. *Let γ be a continuous semialgebraic function from the interval $]a, b]$ to $\mathbb{R}$.*

(i) *Suppose that γ has a boundary value when t goes to a. Then γ has a continuous extension to $[a, b]$.*

(ii) *If γ is not bounded, then for some positive rational number r and a continuous semialgebraic function γ_0 on $[a, b]$ such that $\gamma_0(a) \neq 0$, $\gamma(t) = (t - a)^{-r}\gamma_0(t)$ for all t in $]a, b]$.*

Proof. (i) Let Γ be the graph of the function γ and $\overline{\Gamma}$ its closure in $[a, b] \times \mathbb{R}$. By hypothesis, $\overline{\Gamma} \setminus \Gamma$ is not empty and contained in $\{a\} \times \mathbb{R}$ since γ is continuous. Let x be in $\mathbb{R}$ such that (a, x) is in $\overline{\Gamma}$ and ε a positive real number. The intersection of $\overline{\Gamma}$ and $[a, b] \times]x - \varepsilon, x + \varepsilon[$ is a semialgebraic subset of $\mathbb{R}^2$. Then it has a finite number of connected components by [Cos02, Theorem 2.23]. Denote them by $C_1, \ldots, C_m$. Since (a, x) is in $\overline{\Gamma}$, for some positive number η and for some i in $\{1, \ldots, m\}$, $]a, a + \eta]$ is contained in the image of C_i by the first projection since a semialgebraic subset of $\mathbb{R}$ is a finite union of intervals and isolated points. As a result, for all t in $]a, a + \eta[$, $|\gamma(t) - x| < \varepsilon$. Hence the extension $\overline{\gamma}$ of γ to $[a, b]$ such that $\overline{\gamma}(a) = x$ is continuous.

(ii) Suppose that γ is not bounded. By (i),

$$\lim_{t \to a} |\gamma(t)| = +\infty.$$

Moreover, the function $t \mapsto |\gamma(t)|$ is a semialgebraic function. Let $P_1, \ldots, P_m$ be the polynomials corresponding to the equalities of a system defining the graph of the function $|\gamma|$. Denote by Q the polynomial

$$Q(t, s) := \prod_{i=1}^{m} P_i(t + a, s).$$

Then, for some prime divisor Q_* of Q, $Q_*(t, \gamma(t+a)) = 0$ for all t, different from 0, in a neighborhood of 0 in $]0, b-a]$. So, by a Puiseux's expansion, for some positive rational number r and positive real number c,

$$\lim_{t\to a}(t-a)^r|\gamma(t)| = c.$$

Setting:

$$\gamma_0(t) = (t-a)^r\gamma(t)$$

for t in $]a, b]$, the function γ_0 on $]a, b]$ is bounded, continuous and semialgebraic. Then, by (i), γ_0 has a continuous expansion to $[a, b]$ such that $\gamma_0(a) \neq 0$. □

Corollary 2.7. *Let E be a finite dimensional real vector space and γ a continuous semialgebraic map from the interval $[a, b]$ to E. For k nonnegative integer, there exist an increasing sequence $r_1 < \cdots < r_{k+1}$ of positive rational numbers and $x_1, \ldots, x_k$ in E such that the map*

$$]a, b] \longrightarrow E\ , \qquad t \longmapsto (t-a)^{-r_{k+1}}(\gamma(t) - (\gamma(a) + (t-a)^{r_1}x_1 + \cdots + (t-a)^{r_k}x_k))$$

is bounded, continuous and semialgebraic.

Proof. Considering the coordinates of γ in a basis of E, it is sufficient to prove the corollary for $E = \mathbb{R}$. The map $t \mapsto \gamma(t) - \gamma(a)$ has a zero at a. Moreover, the set of zeros of this map in $[a, b]$ is a closed semialgebraic subset of $[a, b]$. As a result, it is a union of a finite number of closed intervals and isolated points. Hence $\gamma(t) - \gamma(a) = 0$ for all t in a neighborhood of a in $[a, b]$ or $\gamma(t) - \gamma(a) \neq 0$ for all t different from a in a neighborhood of a in $[a, b]$. In the first case, there is nothing to prove. Suppose that the second case is satisfied. For some positive number η such that $a + \eta \leq b$, the function

$$t \longmapsto \frac{1}{\gamma(t) - \gamma(a)}$$

on $]a, a+\eta]$ is a continuous semialgebraic function and it is not bounded. Hence, by Lemma 2.6(ii), for some positive rational number r_1 and for some continuous semialgebraic function γ_1 on $[a, b]$ such that $\gamma_1(a) \neq 0$,

$$\gamma(t) = \gamma(a) + (t-a)^{r_1}\gamma_1(t)$$

for all t in $[a, b]$, whence the corollary by induction on k. □

Remark 2.8. By Corollary 2.7 and an induction argument, we can find an increasing sequence $r_1 < r_2 < \cdots$ of positive rational numbers and a sequence $x_1, x_2, \ldots$ in $E \setminus \{0\}$ satisfying one of the two following conditions:

(1) if the sequences are finite of cardinality k, then the function

$$t \longmapsto \gamma(t) - (\gamma(a) + (t-a)^{r_1}x_1 + \cdots + (t-a)^{r_k}x_k)$$

is equal to 0 in a neighborhood of a in $[a, b]$,

(2) if the first sequence is not finite so is the second one and for all positive integer k, the map

$$t \longmapsto (t-a)^{-r_{k+1}}(\gamma(t) - (\gamma(a) + (t-a)^{r_1}x_1 + \cdots + (t-a)^{r_k}x_k))$$

is bounded and semialgebraic.

In the second case, by [Duv89], as the rational numbers $r_1, r_2, \ldots$ are the exponents of a Puiseux's serie, for some positive integer m, $mr_1, mr_2, \ldots$ are positive integers so that the sequence $r_1, r_2, \ldots$ is not bounded.

3. Vector fields and integral curves

For x in $\mathbb{C}^n$, denote by X_x the polynomial vector field on $\mathbb{C}^n$,

$$y \longmapsto X_x(y) := \Phi'(y)^{-1}(x).$$

Definition 3.1. An integral curve of the vector field X_x is a derivable map γ from an open interval I of $\mathbb{R}$ to $\mathbb{C}^n$ such that its derative γ' satisfies the equality

$$\gamma'(t) = X_x(\gamma(t))$$

for all t in I.

3.1. Elementary properties.

Let x be in $\mathbb{C}^n$.

Lemma 3.2. *Denote by γ a continuous map from an open interval I of $\mathbb{R}$ to $\mathbb{C}^n$. Then γ is an integral curve of X_x if and only if for some t_0 in I*

$$\Phi\circ\gamma(t) = \Phi\circ\gamma(t_0) + (t - t_0)x$$

for all t in I.

Proof. Suppose that γ is an integral curve of X_x. Then $\Phi\circ\gamma$ is an integral curve of the vector field $z \mapsto \Phi'(z)(X_x(z))$. By definition, $\Phi'(z)(X_x(z)) = x$ for all z in $\mathbb{C}^n$. As a result, by the unicity of the integral curves with a given initial datum,

$$\Phi\circ\gamma(t) = \Phi\circ\gamma(t_0) + (t - t_0)x$$

for all (t, t_0) in $I \times I$. Conversely, suppose that γ is a continuous map from I to $\mathbb{C}^n$ such that for some t_0 in I,

$$\Phi\circ\gamma(t) = \Phi\circ\gamma(t_0) + (t - t_0)x \qquad \forall t \in I.$$

Let s be in I and (ω, ψ) an adapted pair to $\gamma(s)$. Then, for all t in $\gamma^{-1}(\omega)$,

$$\gamma(t) = \psi(\Phi\circ\gamma(t_0) + (t - t_0)x).$$

As a result, γ is a derivable map and $\Phi'(\gamma(t))(\gamma'(t)) = x$ for all t in I, whence the assertion by definition of X_x. □

Denote by $\mathcal{C}_x$ the set of integral curves of X_x whose interval of definition contains 0. For γ and δ in $\mathcal{C}_x$, we say that γ is smaller than δ if the interval of definition of γ is contained in the interval of definition δ and if δ is an expansion of γ. This order is inductive so that any element of $\mathcal{C}_x$ is smaller than a maximal one.

3.2. **On the integral curves.** Denote by Ψ the map

$$\mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^n \xrightarrow{\quad\Psi\quad} \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^n\ , \qquad (x,t,z) \longmapsto (x,t,\Phi(z)).$$

As Φ is étale so is Ψ. Let Δ be the graph of the map

$$\mathbb{C}^n \times \mathbb{C} \longrightarrow \mathbb{C}^n\ , \qquad (x,t) \longmapsto tx.$$

Lemma 3.3. *Denote by Γ the inverse image of Δ by Ψ and for x in $\mathbb{C}^n$ denote by Γ_x the intersection of Γ with $\{x\} \times \mathbb{C} \times \mathbb{C}^n$.*

(i) *The subset Γ of $\mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^n$ is a smooth algebraic variety of dimension $n+1$, containing $\mathbb{C}^n \times \{0,0\}$.*

(ii) *For x in $\mathbb{C}^n$, Γ_x is a smooth algebraic variety of dimension 1, containing $(x,0,0)$.*

Proof. (i) As $\Psi(x,0,0) = (x,0,0)$, $(x,0,0)$ is in Γ for all x in $\mathbb{C}^n$. Let (x,t,y) be in Γ. Since Ψ is étale and Δ is smooth of dimension $n+1$, the tangent space of Γ at (x,t,y) has dimension at most $n+1$. Moreover, the image of some open neighborhood of (x,t,y) in $\mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^n$ by Ψ is an open neighborhood of $(x,t,\Phi(y))$ in $\mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^n$. As a result, Γ has dimension $n+1$. Hence Γ is a smooth algebraic variety, equidimensional of dimension $n+1$.

(ii) Let x be in $\mathbb{C}^n$. By definition, Γ_x is the inverse image by Ψ of the graph Δ_x of the map

$$\{x\} \times \mathbb{C} \times \mathbb{C}^n \longrightarrow \mathbb{C}^n\ , \qquad (x,t) \longmapsto tx.$$

As the restriction of Ψ to $\{x\} \times \mathbb{C} \times \mathbb{C}^n$ is an étale endomorphism of $\{x\} \times \mathbb{C} \times \mathbb{C}^n$, Γ_x is a smooth algebraic variety of dimension 1 since so is Δ_x. Moreover, $(x,0,0)$ is in Γ_x. □

Proposition 3.4. *Let y be in $\mathbb{C}^n$ and γ in $\mathcal{C}_x$ such that $\gamma(0) = y$. Then γ is a semialgebraic map from an interval of $\mathbb{R}$ to $\mathbb{C}^n$.*

Proof. Let I be the interval of definition of γ. According to Lemma 3.2,

$$\Phi\circ\gamma(t) = \Phi(y) + tx$$

for all t in I. Denote by Δ_γ the graph of γ and set:

$$\Delta_I := \{(t,z) \in I \times \mathbb{C}^n \mid \Phi(z) = \Phi(y) + tx\}.$$

Since Φ is a polynomial map, Δ_I is a semialgebraic subset of $\mathbb{R}\times\mathbb{C}^n$. Moreover, Δ_γ is a connected closed subset of Δ_I. For all t in I, denote by (ω_t, ψ_t) an adapted pair to $\gamma(t)$. Let δ_t be the inverse image of ω_t by γ. Then, for (s,z) in the intersection of $\delta_t \times \omega_t$ and Δ_I,

$$\Phi(z) = \Phi(y) + sx \quad \text{and} \quad \Phi\circ\gamma(s) = \Phi(y) + sx,$$

whence $z = \gamma(s)$. As a result, Δ_γ is an open subset of Δ_I. Then Δ_γ is a connected component of Δ_I. So, by [Cos02, Theorem 2.23], Δ_γ is a semialgebraic subset of $\mathbb{R} \times \mathbb{C}^n$ since so is Δ_I. □

Corollary 3.5. *Let y be in $\mathbb{C}^n$ and γ in $\mathcal{C}_x$ such that $\gamma(0) = y$. Suppose that γ is defined over the relatively compact interval $]a,b[$ of $\mathbb{R}$. If $\|\gamma\|$ is bounded then there exists δ in $\mathcal{C}_x$, bigger than γ and such that its domain of definition contains $[a,b]$.*

Proof. By Proposition 3.4 and Lemma 2.6(i), γ has a continuous extension to $[a, b]$. Denote again by γ this extension. For some positive real number η, there exists an integral curve γ_a of X_x defined over $]a-\eta, a+\eta[$ and such that $\gamma_a(a) = \gamma(a)$. Let (ω, ψ) be an adapted pair to $\gamma(a)$. For some positive real number η', smaller than η, for all t in $]a, a+\eta'[$, $\gamma(t)$ is in ω and $\Phi\circ\gamma(t) = \Phi(y) + tx$, whence

$$\Phi\circ\gamma(a) = \Phi(y) + ax \quad \text{and} \quad \Phi\circ\gamma(t) = \Phi\circ\gamma(a) + (t-a)x.$$

By Lemma 3.2, for all t in $[a, a+\eta[$,

$$\Phi\circ\gamma_a(t) = \Phi\circ\gamma(a) + (t-a)x.$$

As a result, for some t in $]a, a+\eta'[$, $\gamma(t) = \gamma_a(t)$. So, by the unicity of the integral curve with initial datum, γ_a and γ have the same restriction to $[a, a+\eta[$. As a result, there exists an integral curve of X_x, extending γ and such that its interval of definition is $]a-\eta, b[$. In the same way, we prove that for some positive real number η'', there exists an integral curve of X_x, extending γ and such that its interval of definition is $]a, b+\eta''[$, whence the corollary. □

4. Special integral curves

For x in $\mathbb{C}^n$, denote by γ_x the maximal element of $\mathcal{C}_x$ such that $\gamma_x(0) = 0$. The goal of the section is the following proposition:

Proposition 4.1. *For all x in $\mathbb{C}^n$, the interval of definition of γ_x is equal to $\mathbb{R}$.*

4.1. **Intervals of definition.** For x in $\mathbb{C}^n$, we denote by I_x the interval of definition of γ_x.

Lemma 4.2. *Let x be in $\mathbb{C}^n$.*

(i) *The interval I_x is invariant under the homeomorphism $t \mapsto -t$. In particular, for some $a(x)$ in $\mathbb{R}_+^* \cup \{+\infty\}$, $I_x =]-a(x), a(x)[$.*

(ii) *For s in $\mathbb{R}^*$, $\gamma_{sx}(t) = \gamma_x(st)$. Moreover, $I_{sx} = s^{-1}I_x$ and $a(sx) = |s|^{-1}a(x)$.*

Proof. (i) By the maximality of γ_x and the unicity of an integral curve with initial datum it is sufficient to prove that the map δ_x,

$$-I \xrightarrow{\delta_x} \mathbb{C}^n, \qquad t \longmapsto -\gamma_x(-t)$$

is an integral curve of X_x since 0 is in I and $\gamma_x(0) = 0$. By definition, for all t in $-I$,

$$\delta_x'(t) = \gamma_x'(-t) = X_x(\gamma_x(-t)) = \Phi'(\gamma_x(-t))^{-1}(x).$$

As Φ is an odd map, $\Phi'(y) = \Phi'(-y)$ for all y in $\mathbb{C}^n$. As a result, δ_x is an integral curve of X_x, whence the assertion.

(ii) Denote by δ the map

$$s^{-1}I_x \xrightarrow{\delta} \mathbb{C}^n, \qquad t \longmapsto \gamma_x(st).$$

Then

$$\delta'(t) = s\gamma_x'(st) = s\Phi'(\gamma_x(st))^{-1}(x) = X_{sx}(\delta(t)).$$

As a result, δ is an integral curve of the vector field X_{sx} such that $\delta(0) = 0$. So by the unicity of the integral curve with initial datum, $s^{-1}I_x$ is contained in I_{sx} and δ is the restriction of γ_{sx} to $s^{-1}I_x$. Considering the map

$$sI_{sx} \xrightarrow{\tau} \mathbb{C}^n\ , \qquad t \longmapsto \gamma_{sx}(s^{-1}t),$$

sI_{sx} is contained in I_x by the same argument, whence $I_{sx} = s^{-1}I_x$ and the assertion. □

According to Lemma 4.2(i), a map a from $\mathbb{C}^n$ to $\mathbb{R}_+^* \cup \{+\infty\}$ is well defined. Let Ω be the subset of elements (x,t) of $\mathbb{C}^n \times \mathbb{R}$ such that $-a(x) < t < a(x)$ and G the graph of the map

$$\Omega \xrightarrow{\gamma} \mathbb{C}^n\ , \qquad (x,t) \longmapsto \gamma(x,t) := \gamma_x(t).$$

Lemma 4.3. *Let (x,t) be in Ω and t' a positive real number such that $|t| < t' < a(x)$. Then for some open neighborhood ω of x in $\mathbb{C}^n$, $\omega \times [-t', t']$ is contained in Ω and the restriction of γ to $\omega \times [-t', t']$ is continuous.*

Proof. Since $[-t', t']$ is a compact interval contained in I_x and the map $(y,z) \mapsto X_z(y)$ is polynomial, for some open neighborhood ω of x in $\mathbb{C}^n$, $\omega \times [-t', t']$ is contained in Ω and the restriction of γ to $\omega \times [-t', t']$ is continuous by [Car67, Théorème 3.6.1]. □

Corollary 4.4. (i) *The subset Ω of $\mathbb{C}^n \times \mathbb{R}$ is open and the map γ is continuous.*

(ii) *The function a is continuous. In particular, $a^{-1}(\{+\infty\})$ is a closed subset of $\mathbb{C}^n$. Moreover $a^{-1}(\{+\infty\})$ contains the kernel of F.*

Proof. (i) By Lemma 4.3, Ω is the union of open subsets of $\mathbb{C}^n \times \mathbb{R}$ at which the restriction of γ is continuous, whence the assertion.

(ii) By (i), the function a is lower continuous. Hence $a^{-1}(\{+\infty\})$ is a closed subset of $\mathbb{C}^n$. By Lemma 4.3, a is upper continuous. Hence a is continuous. By Lemma 1.5 and Lemma 3.2, for x in the kernel of F, $\gamma_x(t) = tx$ for all t in $\mathbb{R}$, whence the assertion. □

4.2. **Properties of the maps γ and a.** Let Γ be as in Subsection 3.2 and $\Gamma_{\mathbb{R}}$ its intersection with $\mathbb{C}^n \times \mathbb{R} \times \mathbb{C}^n$.

Lemma 4.5. *Let Γ_0 be the connected component of $\Gamma_{\mathbb{R}}$ containing $\mathbb{C}^n \times \{0,0\}$.*

(i) *The subset Γ_0 of $\mathbb{C}^n \times \mathbb{R} \times \mathbb{C}^n$ is closed and semialgebraic. Moreover, it contains G.*

(ii) *The subset G of Γ_0 is open.*

Proof. (i) As $\mathbb{C}^n \times \{(0,0)\}$ is contained in Γ, it is contained in a connected component of $\Gamma_{\mathbb{R}}$. Since $\Gamma_{\mathbb{R}}$ is a semialgebraic closed subset of $\mathbb{C}^n \times \mathbb{R} \times \mathbb{C}^n$, Γ_0 is semialgebraic by [Cos02, Theorem 2.23]. The set G is contained in $\Gamma_{\mathbb{R}}$ since for all (x,t) in Ω, $\Phi \circ \gamma_x(t) = tx$ by Lemma 3.2. As Ω is a connected subset of $\mathbb{C}^n \times \mathbb{R}$, G is a connected subset of $\mathbb{C}^n \times \mathbb{R} \times \mathbb{C}^n$ by Corollary 4.4(i). Hence G is contained in Γ_0 since $\mathbb{C}^n \times \{(0,0)\}$ is contained in G.

(ii) Let (x,t,y) be in G and (ω, ψ) an adapted pair to y. As γ is continous, $\gamma^{-1}(\omega) \times \omega$ is an open subset of $\mathbb{C}^n \times \mathbb{R} \times \mathbb{C}^n$. Its intersection with $\Gamma_{\mathbb{R}}$ is the graph of the map

$$\gamma^{-1}(\omega) \longrightarrow \mathbb{C}^n\ , \qquad (z,s) \longmapsto \psi(sz).$$

Since for all (z, s) in $\gamma^{-1}(\omega)$, $\gamma(z, s)$ is in ω and $\Phi\circ\gamma(z, s) = sz$, $\gamma^{-1}(\omega)\times\omega\cap\Gamma_{\mathbb{R}}$ is contained in G, whence the assertion. □

Proposition 4.6. (i) *The graph G is equal to Γ_0.*

(ii) *The map γ is continuous and semialgebraic.*

(iii) *The function a is continuous and semialgebraic. In particular, $a^{-1}(\{+\infty\})$ is a closed semialgebraic subset of $\mathbb{C}^n$.*

Proof. (i) Let $\overline{G}$ be the closure of G in $\mathbb{C}^n\times\mathbb{R}\times\mathbb{C}^n$. Then $\overline{G}$ is contained in Γ_0 since Γ_0 is a closed subset of $\mathbb{C}^n\times\mathbb{R}\times\mathbb{C}^n$. By Lemma 4.5(ii), G is an open subset of Γ_0. Hence it remains to prove $G = \overline{G}$ since Γ_0 is connected by definition.

Suppose that G is strictly contained in $\overline{G}$. A contradiction is expected. Let (x, t, y) be in $\overline{G}\setminus G$. As γ is continuous by Corollary 4.4(i), G is a closed subset of $\Omega\times\mathbb{C}^n$. Hence $t = \pm a(x)$ since a is continous by Corollary 4.4(ii). By Lemma 4.2(i), G and $\overline{G}$ are invariant under the homeomorphism $(x, t, y)\mapsto(x, -t, y)$. So we can suppose $t = a(x)$. For some sequence (x_m, t_m), $m = 1, 2, \ldots$ in Ω,

$$\lim_{m\to+\infty}\gamma(x_m, t_m) = y.$$

Let (ω, ψ) be an adapted pair to y. For z in ω, denote by τ_z the integral curve of the vector field $X_{a(x)^{-1}\Phi(z)}$ such that $\tau_z(a(x)) = z$. In particular, τ_y is the integral curve of the vector field X_x such that $\tau_y(a(x)) = y$ and by Lemma 3.2, for all z in ω and for all t in the interval of definition of τ_z,

$$\Phi\circ\tau_z(t) = \Phi(z) + (t - a(x))a(x)^{-1}\Phi(z) = ta(x)^{-1}\Phi(z).$$

For some positive real number η, $[a(x)-\eta, a(x)+\eta]$ is contained in the interval of definition of τ_y. Then, by [Car67, Théorème 3.6.1], for some open neighborhood ω' of y in ω, $[a(x)-\eta, a(x)+\eta]$ is contained in the interval of definition of τ_z for all z in ω' since the map $(y, z)\mapsto X_{a(x)^{-1}\Phi(z)}(y)$ is polynomial. Moreover, the map

$$\omega'\times[a(x)-\eta, a(x)+\eta]\xrightarrow{\ \tau\ }\mathbb{C}^n\ ,\qquad (z, t)\longmapsto\tau_z(t)$$

is continuous. Hence we can suppose that the image of τ is contained in ω.

For some positive integer M and for m bigger than M, $\gamma(x_m, t_m)$ is in ω' and for s positive real number smaller than $\eta/2$, $t_m - s$ is in $[a(x)-\eta, a(x)+\eta]$. Since $\Phi(\omega')$ is a neighborhood of $\Phi(y)$ and $\Phi\circ\gamma(x_m, t_m) = t_m x_m$, we can suppose that for m bigger than M, $a(x)x_m$ is in $\Phi(\omega')$. Setting $z_m := \psi(a(x)x_m)$, τ_{z_m} is an integral curve of X_{x_m} and $\tau_{z_m}(t_m) = \gamma(z_m, t_m)$ since

$$\Phi\circ\tau_{z_m}(t_m) = \Phi\circ\gamma(z_m, t_m) = t_m z_m.$$

So, by the unicity of the integral curve with initial datum, $\tau_{z_m}(t) = \gamma_{z_m}(t)$ for all t in $[a(x)-\eta, a(x)+\eta]$. As a result, for s positive real number smaller than $\eta/2$,

$$\gamma(x, a(x)-s) = \lim_{m\to+\infty}\gamma(x_m, t_m - s) = \lim_{m\to+\infty}\tau_{z_m}(t_m - s) = \tau_y(a(x)-s)$$

by continuity of γ and τ, whence the contradiction since

$$\lim_{s\to 0}\|\gamma(x, a(x)-s)\| = +\infty$$

by Corollary 3.5.

(ii) By Lemma 4.5(i), Γ_0 is semialgebraic. So, by (i), γ is a semialgebraic map, whence the assertion since γ is continuous.

(iii) By (i), Ω is the image of Γ_0 by the projection

$$\mathbb{C}^n \times \mathbb{R} \times \mathbb{C}^n \longrightarrow \mathbb{C}^n \times \mathbb{R}\,, \qquad (x,t,y) \longmapsto (x,t).$$

Hence Ω and $\overline{\Omega}$ are semialgebraic subsets of $\mathbb{C}^n \times \mathbb{R}$. By Corollary 4.4(ii), a is continuous. Hence, $\overline{\Omega}$ is the disjoint union of Ω and the graphs of the restrictions of a and $-a$ to $\mathbb{C}^n \setminus a^{-1}(\{+\infty\})$ since $a^{-1}(\{+\infty\}) \times \mathbb{R}$ is contained in Ω. In particular, $a^{-1}(\{+\infty\})$ is the complement in $\mathbb{C}^n$ to the image of $\overline{\Omega} \setminus \Omega$ by the projection

$$\mathbb{C}^n \times \mathbb{R} \longrightarrow \mathbb{C}^n\,, \qquad (x,t) \longmapsto x$$

and the graph of the restriction of a to $\mathbb{C}^n \setminus a^{-1}(\{+\infty\})$ is the intersection of $\overline{\Omega} \setminus \Omega$ and $(\mathbb{C}^n \setminus a^{-1}(\{+\infty\})) \times \mathbb{R}_+$. Hence $a^{-1}(\{+\infty\})$ is a semialgebraic subset of $\mathbb{C}^n$ and the restriction of a to $\mathbb{C}^n \setminus a^{-1}(\{+\infty\})$ is semialgebraic. □

4.3. **Etale covering.** Let K be the field of rational fractions $\mathbb{C}(x_1,\dots,x_n)$ and K' the subfield $\mathbb{C}(x_1+F_1^3,\dots,x_n+F_n^3)$ of K. Denote by d the degree of the extension K of K'.

Lemma 4.7. *Let X and Y be two irreducible complex algebraic varieties and ψ a morphism from X to Y. For i nonnegative integer, denote by Y_i the subset of elements y such that $|\psi^{-1}(y)| = i$. Then, for all nonnegative integer i, Y_i is an algebraic constructible subset of Y.*

Proof. Let Y'_n be the subset of elements y of Y such that $|\psi^{-1}(y)| \geq n$. Denote by Z_n the subset of elements $(y,x_1,\dots,x_n)$ of $Y \times X^n$ such that $y = \psi(x_1) = \cdots = \psi(x_n)$. Then Z_n is a closed subset of $Y \times X^n$. Let Z'_n be the open subset of elements $(y,x_1,\dots,x_n)$ of Z_n such that $x_i \neq x_j$ for $i \neq j$. Then Y'_n is an algebraic constructible subset of Y as the image of Z'_n by the projection

$$(y,x_1,\dots,x_n) \longmapsto y.$$

By definition, $Y_n = Y'_n \setminus Y'_{n+1}$. Hence Y_n is an algebraic constructible subset of Y. □

Corollary 4.8. *Suppose that ψ is a quasi-finite dominant morphism. Let j be the degree of the field extension $\mathbb{C}(Y) \subset \mathbb{C}(X)$. Then Y_j contains a dense Zariski open subset of Y.*

Proof. As j is the degree of the separable field extension $\mathbb{C}(Y) \subset \mathbb{C}(X)$, Y_j contains the generic point of Y. So, by Lemma 4.7, Y_j contains a dense Zariski open subset of Y. □

Proposition 4.9. *For some nonempty Zariski open subset U of $\mathbb{C}^n$, the restriction of Φ to $\Phi^{-1}(U)$ is a covering map of degree d. In particular, it is a finite morphism.*

Proof. As Φ is étale, Φ is a quasi finite morphism. So, by Corollary 4.8, for some nonempty Zariski open subset U of $\mathbb{C}^n$, $\Phi^{-1}(\{x\})$ has cardinality d for all x in U. Let x be in U and $x_1,\dots,x_d$ the elements of $\Phi^{-1}(x)$. For $i = 1,\dots,d$, denote by (ω_i,ψ_i) an adapted pair to x_i. Since $x_1,\dots,x_d$ are pairwise different, we can choose $\omega_1,\dots,\omega_d$ pairwise disjoint. Let ω be the intersection of $\Phi(\omega_1),\dots,\Phi(\omega_d)$. Then ω is an open neighborhood of x and $\Phi^{-1}(\omega)$ is the disjoint union of the

open subsets $\omega_1 \cap \Phi^{-1}(\omega), \dots, \omega_d \cap \Phi^{-1}(\omega)$. As a result, the restriction of Φ to $\Phi^{-1}(U)$ is an étale covering of degree d of U, whence the proposition. □

4.4. **At the neighborhood of an element of $\mathbb{C}^n \setminus a^{-1}(\{+\infty\})$.** Let x be in $\mathbb{C}^n \setminus a^{-1}(\{+\infty\}$ and θ a contiuous semialgebraic arc,

$$[0,1] \xrightarrow{\quad\theta\quad} \mathbb{C}^n ,$$

such that $\theta(0) = x$ and $a(\theta(t)) > a(x)$ for all t in $]0,1]$.

Lemma 4.10. *For some positive rational number r and a continuous semialgebraic map ψ from $[0,1]$ to $\mathbb{C}^n$,*

$$\psi(0) \neq 0 \quad \textit{and} \quad \gamma(\theta(t), a(x)) = t^{-r}\psi(t)$$

for all t in $]0,1]$.

Proof. By Proposition 4.6(ii), the map $t \mapsto \gamma(\theta(t), a(x))$ from $]0,1]$ to $\mathbb{C}^n$ is continuous and semialgebraic. Moreover, by Lemma 3.2,

$$\gamma(\theta(t), a(x)) + F^{(3)}(\gamma(\theta(t), a(x))) = a(x)\theta(t) \tag{1}$$

for all t in $]0,1]$. By Lemma 2.6(i), if the map $t \mapsto \|\gamma(\theta(t), a(x))\|$ is bounded, then for some y in $\mathbb{C}^n$,

$$\lim_{t\to 0} \gamma(\theta(t), a(x)) = y$$

so that (x, a, y) is in G and $(x, a(x))$ is in Ω by Proposition 4.6(i). By definition, $(x, a(x))$ is not in Ω. As a result, the map $t \mapsto \|\gamma(\theta(t), a(x))\|$ is not bounded, whence the assertion by Lemma 2.6(ii), □

4.5. **Proof of Proposition 4.1.** Let U be the Zariski open subset of $\mathbb{C}^n$ as in Proposition 4.9 and Y its complement in $\mathbb{C}^n$.

Lemma 4.11. *Let Z be the closure in $\mathbb{C}^n$ of the union of tY, $t \in \mathbb{R}_+^*$ and $O := \mathbb{C}^n \setminus Z$.*

(i) *The subsets Z and O of $\mathbb{C}^n$ are semialgebraic. In particular, the connected components of O are finitely many and semialgebraic.*

(ii) *The open subset O of $\mathbb{C}^n$ is dense in $\mathbb{C}^n$.*

(iii) *Let O_* be a connected component of O and O'_* the subset of elements x of O_* such that a is constant on a neighborhood of x. Then $tO_* = O_*$ and $tO'_* = O'_*$ for all t in $\mathbb{R}_+^*$.*

Proof. (i) Let Z_* be the union of tY, $t \in \mathbb{R}_+^*$. As the map

$$\mathbb{R}_+^* \times \mathbb{C}^n \longrightarrow \mathbb{C}^n , \qquad (t, x) \longmapsto tx$$

is semialgebraic and Y is an algebraic subvariety of $\mathbb{C}^n$, Z_* is a semialgebraic subset of $\mathbb{C}^n$. Hence Z and O are semialgebraic subsets of $\mathbb{C}^n$, whence the assertion by [Cos02, Theorem 2.23].

(ii) By definition, Y is an algebraic subvariety of dimension smaller than n of $\mathbb{C}^n$. Hence its dimension as a semialgebraic subset of $\mathbb{C}^n$ is at most $2n-2$ by Lemma 2.2(iv). As a result, Z_* and Z have dimension at most $2n-1$ by Lemma 2.2, (i) and (v). Hence O is a dense open subset of $\mathbb{C}^n$ by Lemma 2.2(iii).

(iii) By definition, $tO = O$ for all t in $\mathbb{R}_+^*$. Hence the group $\mathbb{R}_+^*$ permutes the connected components of O and the stabilizers are open subgroups of $\mathbb{R}_+^*$. Hence $tO_* = O_*$ for all t in $\mathbb{R}_+^*$. By Lemma 4.2(ii), for x in O_* and t in $\mathbb{R}_+^*$, $a(tx) = t^{-1}a(x)$. Hence O'_* is invariant under the action of $\mathbb{R}_+^*$. □

For O_* a connected component of O, denote by O''_* the interior of the complement to O'_* in O_*. Then O''_* is an open subset of U, invariant under the action of $\mathbb{R}_+^*$ by Lemma 4.11(iii).

Lemma 4.12. *Let O_* be a connected component of O. Then O''_* is empty.*

Proof. Suppose that O''_* is nonempty. A contradiction is expected. Let x be in O''_* and V a semialgebraic compact neighborhood of x in O''_*. As a is continuous, for some y in V, $a(y) \leq a(z)$ for all z in V. Moreover, since a is semialgebraic and V is contained in O''_*, $a^{-1}(\{y\}) \cap V$ is a semialgebraic subset whose interior is empty. Then, by Proposition 2.3, for some continuous semialgebraic arc θ,

$$[0,1] \xrightarrow{\theta} V\ ,$$

$\theta(0) = y$ and $a(\theta(t)) > a(y)$ for all t in $]0,1]$. By Lemma 4.10,

$$\lim_{t\to 0} \|\gamma(\theta(t), a(y))\| = +\infty.$$

Moreover, for all t in $]0,1]$, $\Phi\circ\gamma(\theta(t), a(y)) = a(y)\theta(t)$. As O''_* is invariant under $\mathbb{R}_+^*$, $a(y)\theta(t)$ is in O''_* for all t in $[0,1]$. Let V' be a compact neighborhood of $a(y)y$ in O''_*. As O''_* is contained in U, $\Phi^{-1}(V')$ is compact by Proposition 4.9. Moreover, for some positive real number η, $a(y)\theta(t)$ is in V' and $\gamma(\theta(t), a(y))$ is in $\Phi^{-1}(V')$ for all t in $]0,\eta]$, whence the contradiction. □

By Lemma 4.12 and Corollary 2.5, a is constant on each connected connected of O. By Lemma 4.2(ii), for x in O and t in $\mathbb{R}_+^*$, $a(tx) = t^{-1}a(x)$. Hence O is contained in $a^{-1}(\{+\infty\})$ since the connected components of O are invariant under $\mathbb{R}_+^*$. As a result $a^{-1}(\{+\infty\}) = \mathbb{C}^n$ since a is continuous and O is dense in $\mathbb{C}^n$ by Lemma 4.12(ii), whence Proposition 4.1.

5. Some polynomial maps

Let $p_0, p_1, \ldots$ be the sequence of polynomial endomorphisms of $\mathbb{C}^n$ defined by the induction relations:

$$p_0 = 0, \quad p_1(x) = x, \quad p_j = -\sum_{(i_1,i_2,i_3)\in\mathbb{N}_j^3} F(p_{i_1})F(p_{i_2})F(p_{i_3}),$$

for $j \geq 2$. By induction, for $j = 0, 1, \ldots$, p_j is homogeneous of degree j.

5.1. Some power series and polynomial maps.

For x in $\mathbb{C}^n$, we denote by S_x the formal power serie of the variable t:

$$S_x(t) := p_0(x) + tp_1(x) + t^2p_2(x) + \cdots .$$

Let r_x be the radius of convergence of S_x. As in Section 4, γ_x is the maximal element of $\mathcal{C}_x$ such that $\gamma_x(0) = 0$.

Lemma 5.1. *Let x be in $\mathbb{C}^n$.*

(i) *Let S be in $\mathbb{C}^n[[t]]$. Then $S = S_x$ if and only if $\Phi\circ S(t) = tx$.*

(ii) *The radius of convergence r_x is different from 0 and for t in $]-r_x, r_x[$,*

$$\gamma_x(t) = \sum_{j=0}^{\infty} t^j p_j(x).$$

(iii) *For j even integer, $p_j(x) = 0$.*

Proof. (i) Let $a_0, a_1, \ldots$ the coefficients of S. By Lemma 1.5, $\Phi^{-1}(0) = \{0\}$. Hence $a_0 = 0$ if $\Phi\circ S(t) = tx$. For $j = 0, 1, \ldots$, the coefficient of t^j in the expansion of $F^{(3)}(S(t))$ is equal to

$$\sum_{(i_1,i_2,i_3)\in\mathbb{N}_j^3} F(a_{i_1})F(a_{i_2})F(a_{i_3}).$$

As a result, $\Phi\circ S(t) = tx$ if and only if

$$a_0 = 0, \quad a_1 = x, \quad a_j = -\sum_{(i_1,i_2,i_3)\in\mathbb{N}_j^3} F(a_{i_1})F(a_{i_2})F(a_{i_3}),$$

for $j \geq 2$, whence the assertion.

(ii) Let (ω, ψ) be an adapted pair to 0. Then ψ is a holomorphic map from $\Phi(\omega)$ to $\mathbb{C}^n$. For some positive real number r, for t in $\mathbb{C}$ such that $|t| < r$, tx is in $\Phi(\omega)$ and $\psi(tx)$ is the sum of a power serie

$$\psi(tx) = \sum_{j=0}^{\infty} t^j a_j.$$

By (i), since $\Phi\circ\psi(tx) = tx$, $a_j = p_j(x)$. In particular, $r_x \neq 0$. Moreover, by Lemma 3.2, the map

$$]-r_x, r_x[\longrightarrow \mathbb{C}^n, \qquad t \longmapsto \sum_{j=0}^{\infty} t^j p_j(x)$$

is an integral curve of the vector field X_x. By the unicity of the integral curve with initial datum, it is the restriction of γ_x to $]-r_x, r_x[$, whence the assertion.

(iii) By Lemma 4.2(i), $\gamma_x(t) = -\gamma_x(-t)$ for all t in $]-r_x, r_x[$. Hence by (ii), $p_{2j}(x) = 0$ for all j in $\mathbb{N}$. □

For x in $\mathbb{C}^n$, let φ_x be the map from $\mathbb{C}$ to $\mathbb{C}^n$ defined by the following relations: $\varphi_x(0) = 0$ and for t in $\mathbb{C}^*$,

$$\varphi_x(t) = \gamma_{ux}(|t|) \quad \text{with} \quad u := \frac{t}{|t|}.$$

As the domain of definition of γ_y is $\mathbb{R}$ for all y in $\mathbb{C}^n$, φ_x is well defined.

Lemma 5.2. *Let x be in $\mathbb{C}^n$.*

(i) *For all t in $\mathbb{C}$, $\Phi\circ\varphi_x(t) = tx$.*

(ii) *The map φ_x is continuous and for all t in $\mathbb{C}$ such that $|t| < r_x$, $\varphi_x(t) = S_x(t)$.*

Proof. (i) By definition $\Phi\circ\varphi_x(0) = 0$. By Lemma 3.2, for t in $\mathbb{C}^*$ and $u = t/|t|$,

$$\Phi\circ\varphi_x(t) = \Phi\circ\gamma_{ux}(|t|) = u|t|x = tx,$$

whence the assertion.

(ii) Let $\mathbb{U}$ be the subset of complex numbers of module 1. Since the polynomial maps $p_0, p_1, \dots$ are homogeneous, r_x is the radius of convergence of S_{ux} for all u in $\mathbb{U}$. Moreover, by Lemma 5.1(ii),

$$\varphi_x(t) = \sum_{j=0}^{\infty} t^j p_j(x)$$

for all t in $\mathbb{C}$ such that $|t| < r_x$. In particular, φ_x is continuous in a neighborhood of 0.

Let ψ_x be the map

$$\mathbb{U}\times\mathbb{R}_+^* \xrightarrow{\psi_x} \mathbb{C}^n\ , \qquad (u,t)\longmapsto \varphi_x(tu).$$

With the notations of Subsection 4.1, $\psi_x(u,t) = \gamma(ux,t)$. Hence, by Corollary 4.4(i), the map ψ_x is continuous. As a result, the restriction of φ_x to $\mathbb{C}^*$ is continuous since the map

$$\mathbb{U}\times\mathbb{R}_+^* \longrightarrow \mathbb{C}^*\ , \qquad (u,t)\longmapsto tu$$

is a homeomorphism, whence the assertion. □

Proposition 5.3. *Let x be in $\mathbb{C}^n$. Then φ_x is a polynomial map and for some positive integer j, $p_i(x) = 0$ for all positive integer bigger than j.*

Proof. Let Γ_x be as in Subsection 3.2 and Γ'_x the subset of $\mathbb{C}\times\mathbb{C}^n$ such that $\Gamma_x = \{x\}\times\Gamma'_x$. Denote by G the graph of φ_x. By Lemma 5.2,(i) and (ii), G is a closed connected subset of Γ'_x. Let t be in $\mathbb{C}$ and (ω,ψ) and adapted pair to $\varphi_x(t)$. Then $\varphi_x^{-1}(\omega)\times\omega$ is an open subset of $\mathbb{C}\times\mathbb{C}^n$ and its intersection with Γ'_x is the graph of the map

$$\varphi_x^{-1}(\omega) \longrightarrow \omega\ , \qquad s\longmapsto \psi(sx).$$

As a result, $\varphi_x^{-1}(\omega)\times\omega\cap\Gamma'_x$ is contained in G since $\varphi_x(s)$ is in ω for all s in $\varphi_x^{-1}(\omega)$. Hence G is an irreducible component of Γ'_x by Lemma 3.3(ii). Moreover, the canonical projection from G to $\mathbb{C}$ is birational and injective. So, by Zariski's Main theorem [Mu88, §9], it is an isomorphism of algebraic varieties. Hence φ_x is a polynomial map. As a result, by Lemma 5.2(ii), for some positive integer j, $p_i(x) = 0$ for $j < i$. □

5.2. **Polynomial automorphism.** For i positive integer, set:

$$X_i := \{x\in\mathbb{C}^n \,|\, 0 = p_i(x) = p_{i+1}(x) = \cdots\}.$$

Lemma 5.4. *For some positive integer j, $p_i = 0$ for $i > j$.*

Proof. According to Propostion 5.3, $\mathbb{C}^n$ is the union of $X_1, X_2, \dots$. By definition, for all i, X_i is a Zariski closed subset. By Baire's property, for some positive integer j, X_j contains a nonempty open subset of $\mathbb{C}^n$. Hence, $X_j = \mathbb{C}^n$ and $p_i = 0$ for $i \geq j$. □

Let j be the biggest integer such that $p_j \neq 0$ and set:

$$P := p_1 + \cdots + p_j.$$

Theorem 5.5. *The polynomial endomorphism Φ is a polynomial automorphism and P is its inverse.*

Proof. By Proposition 5.3 and Lemma 5.1(ii), $\varphi_x(1) = P(x)$ for all x in $\mathbb{C}^n$. So, by Lemma 5.1(i), $\Phi \circ P(x) = x$. As a result, P is an étale morphism since so is Φ. In particular, for some open subset ω of $\mathbb{C}^n$, the restriction of P to ω is an isomorphism onto the open subset $P(\omega)$ of $\mathbb{C}^n$. Denoting by ψ the inverse of this isomorphism,

$$\Phi \circ P(x) = x \quad \text{and} \quad P \circ \psi(y) = y$$

for all (x, y) in $\omega \times P(\omega)$, whence

$$\Phi(y) = \psi(y) \quad \text{and} \quad P \circ \Phi(y) = y$$

for all y in $P(\omega)$. Then

$$\Phi \circ P(x) = P \circ \Phi(x) = x$$

for all x in $\mathbb{C}^n$ since P and Φ are polynomial endomorphisms, whence the theorem. □

6. Conclusion

By Theorem 5.5 and [Dru83, Theorem 3], Keller's Jacobian Conjecture is true in any dimension:

Theorem 6.1. *Let n be a positive integer. Then any étale polynomial endomorphism of $\mathbb{C}^n$ is a polynomial automorphism.*

From Theorem 6.1 and [Bass 82, Section I], we deduce the following theorem:

Theorem 6.2. *Let $\Bbbk$ be an integral ring of characteristic 0 and n a positive integer. Then any polynomial endomorphism of $\Bbbk^n$ whose Jacobian matrix has a constant invertible determinant is a polynomial automorphism.*

From Theorem 6.2 and [Ka, Theorem 1], we deduce the following theorem:

Theorem 6.3. *Let $\Bbbk$ be a field of characteristic zero and n a positive integer. Denote by $A_{n,\Bbbk}$ the n-th Weyl algebra over $\Bbbk$. Then any endomorphism of $A_{n,\Bbbk}$ is an automorphism.*

References

[Bass 82] H. Bass, E.H. Connell and D. Wright, *The jacobian conjecture: Reduction of degree and formal expansion of the inverse*, Bulletin of the American Mathematical Society, **7** (1982), p. 287–329.

[Car67] H. Cartan, *Calcul différentiel*, Hermann, Paris, Collection Méthodes.

[Cos02] M. Coste, *An introducion to semialgebraic geometry*, Lecture at University of Rennes1, (2002).

[Duv89] D. Duval, *Rational Puiseux expansions*, Composition Mathematica, **70**, (1989) p. 119–154.

[Dru83] L.M. Drużkowski, *An effective approach to Keller's Jacobian Conjecture*, Mathematisce Annalen, **264** (1983), p. 303–313.

[Dru01] L.M. Drużkowski, *New Reduction in the Jacobian Conjecture*, Universitatis Lagellonicae Acta Mathematica, **39** (2001), p. 203–206.

[Ka] A. Belov-Kanel and M. Kontsevich. *The Jacobian conjecture is stably equivalent to Dixmier's conjecture*, Moscow Mathematical Journal, Vol. 7, **no.2**, (2007), p. 209–218.

[MA86] H. Matsumura, *Commutative ring theory* Cambridge studies in advanced mathematics **n°8** (1986), Cambridge University Press, Cambridge, London, New York, New Rochelle, Melbourne, Sydney.

[Mu88] D. Mumford, *The Red Book of Varieties and Schemes*, Lecture Notes in Mathematics **n°1358** (1988), Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo.

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