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NILPOTENT MATRICES, INTEGRAL CURVES AND JACOBIAN CONJECTURE

JEAN-YVES CHARBONNEL

ABSTRACT. Following the results of L.M. Drużkowski, we are interested in the Jacobian Conjecture.

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1. INTRODUCTION

In this note, we are interested in the jacobian conjecture, that is: for n positive integer, an étale polynomial endomorphism of \mathbb{C}^n is a polynomial automorphism. Some results are known for a long time. For example, an étale polynomial endomorphism is an automorphism if it is proper or birational [Bass 82, Theorem 2.1]. According to the results of L.M. Drużkowski, it is sufficient to prove that the following conjecture is true.

Conjecture 1.1. *Let n be a positive integer and F_1, \dots, F_n in the dual of \mathbb{C}^n . Denote by Φ the polynomial endomorphism of \mathbb{C}^n :*

$$\mathbb{C}^n \xrightarrow{\Phi} \mathbb{C}^n, \quad x = (x_1, \dots, x_n) \mapsto (x_1 + F_1(x)^3, \dots, x_n + F_n(x)^3).$$

If Φ is an étale morphism, then Φ is a polynomial automorphism.

As a result, we are interested in the proof of conjecture 1.1.

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1.1. Sketch of proof. Let n be a positive integer and F_1, \dots, F_n, Φ as in Conjecture 1.1. Suppose that Φ is étale. For x in \mathbb{C}^n , let X_x be the vector field on \mathbb{C}^n ,

$$y \longmapsto \Phi'(y)^{-1}(x)$$

with $\Phi'(y)$ the differential of Φ at y . If γ is an integral curve of X_x with initial datum $\gamma(0) = y$, then for all t in the interval of definition of γ ,

$$\Phi \circ \gamma(t) = \Phi(y) + tx.$$

The key step of the proof is the following proposition:

Proposition 1.2. *Let x be in \mathbb{C}^n and γ_x the maximal integral curve of X_x with initial datum $\gamma_x(0) = 0$. Then \mathbb{R} is the interval of definition of γ_x .*

For x in \mathbb{C}^n , let φ_x be the map from \mathbb{C} to \mathbb{C}^n such that

$$\varphi_x(0) := 0 \quad \text{and} \quad \varphi_x(t) := \gamma_{ux}(|t|) \quad \text{with} \quad u := \frac{t}{|t|},$$

for all t in \mathbb{C}^* . The second step is the following result:

Proposition 1.3. *For all x in \mathbb{C}^n , φ_x is a polynomial map such that $\Phi \circ \varphi_x(t) = tx$ for all t in \mathbb{C} .*

We deduce from this proposition that the map $x \mapsto \varphi_x(1)$ is a polynomial endomorphism. Then Φ is a polynomial automorphism and its inverse is the map $x \mapsto \varphi_x(1)$.

1.2. Notations and Definition. • As usual, for k positive integer and for $i = (i_1, \dots, i_k)$ in \mathbb{N}^k ,

$$|i| := i_1 + \dots + i_k.$$

For j nonnegative integer, denote by \mathbb{N}_j^k , the subset of elements i of \mathbb{N}^k such that $|i| = j$.

- For E subset, $|E|$ is its cardinality when it is finite and $+\infty$ when it is not finite.
- The topology on \mathbb{C}^n is the usual topology and $\|\cdot\|$ is an euclidean norm on \mathbb{C}^n . Let d be the distance on \mathbb{C}^n such that $d(x, y) := \|x - y\|$.
- For Φ a holomorphic endomorphism of \mathbb{C}^n and x in \mathbb{C}^n , its differential at x is denoted by $\Phi'(x)$.
- For n positive integer and x, y in \mathbb{C}^n , set:

$$xy := (x_1y_1, \dots, x_ny_n) \quad \text{with} \quad x = (x_1, \dots, x_n) \quad \text{and} \quad y = (y_1, \dots, y_n).$$

As a result, by induction on k , for v_1, \dots, v_k in \mathbb{C}^n , $v_1 \cdots v_k$ is a well defined element of \mathbb{C}^n . For F a map from \mathbb{C}^n to \mathbb{C}^n and k positive integer, denote by $F^{(k)}$ the map

$$\mathbb{C}^n \xrightarrow{F^{(k)}} \mathbb{C}^n, \quad x \longmapsto F(x)^k.$$

• As usual, for n positive integer, $M_n(\mathbb{C})$ is the algebra of square matrices of order n with complex coefficients. Denote by $\mathbf{1}_n$ the identity element of $M_n(\mathbb{C})$. For A in $M_n(\mathbb{C})$ and $1 \leq i, j \leq n$, $A_{i,j}$ is the coefficient of A at its i -row and j -column. The matrix A is identified with the endomorphism of \mathbb{C}^n ,

$$(x_1, \dots, x_n) \longmapsto (A_{i,1}x_1 + \dots + A_{i,n}x_n, i = 1, \dots, n).$$

• Let A be in \mathbb{C}^n . Denote by F the linear endomorphism of \mathbb{C}^n identified with A and Φ the polynomial endomorphism of \mathbb{C}^n ,

$$\mathbb{C}^n \xrightarrow{\Phi} \mathbb{C}^n, \quad x \mapsto x + F^{(3)}(x).$$

For $i = 1, \dots, n$, let F_i be the i -th coordinate of F and for x in \mathbb{C}^n , denote by D_x the diagonal matrix whose entries are $F_1(x)^2, \dots, F_n(x)^2$.

Definition 1.4. The matrix A is said AJC (adapted to Jacobian conjecture) if $D_x A$ is a nilpotent matrix for all x in \mathbb{C}^n .

By [Dru83, Proposition 2], A is AJC if and only if Φ is an étale morphism.

Lemma 1.5. *Suppose that Φ is étale. Then $\Phi^{-1}(v) = \{v\}$ for all v in the kernel of F .*

Proof. As v is in the kernel of F , $\Phi(v) = v$. Let x be in \mathbb{C}^n such that $\Phi(x) = v$. Then v is the image of x by $\mathbf{1}_n + D_x A$. As Φ is étale, for some positive integer m , $(D_x A)^m = 0$, whence

$$(\mathbf{1}_n + D_x A)^{-1} = \sum_{i=0}^{m-1} (-1)^i (D_x A)^i.$$

Since v is in the kernel of F , for i positive integer, $(D_x A)^i(v) = 0$, whence

$$x = (\mathbf{1}_n + D_x A)^{-1}(v) = v,$$

and the lemma. □

In this note, A is an AJC matrix different from 0. As Φ is an étale morphism, for all x in \mathbb{C}^n , there exists an open neighborhood ω of x in \mathbb{C}^n such that the restriction of Φ to ω is a holomorphic isomorphism onto the open neighborhood $\Phi(\omega)$ of $\Phi(x)$ in \mathbb{C}^n . Denote by ψ the inverse of the restriction of Φ to ω and call the pair (ω, ψ) an adapted pair to x .

2. SEMIALGEBRAIC SETS AND MAPS

We recall some results of semialgebraic geometry (see [Cos02]). Let E and F be finite dimensional real vector spaces. A subset of E is semialgebraic if it is a finite union of subsets defined by a system of a finite number of equalities and inequalities of polynomials. Denoting by $\mathfrak{S}(E)$ the set of semialgebraic subsets of E , $\mathfrak{S}(E)$ is stable under finite union and finite intersection. For S in $\mathfrak{S}(E)$, its closure in E , its interior and $E \setminus S$ are in $\mathfrak{S}(E)$.

For X subset of E and φ a map from X to F , φ is a semialgebraic map if its graph is a semialgebraic subset of $E \times F$. By Tarski-Seidenberg's theorem, X is a semialgebraic subset of E and the image of φ is a semialgebraic subset of F .

Lemma 2.1. *Let X be a subset of E , Y a subset of F and G a real vector space of finite dimension. If φ is a semialgebraic map from X to Y and ψ a semialgebraic map from Y to G then $\psi \circ \varphi$ is a semialgebraic map from X to G .*

Proof. Let Γ_φ be the graph of φ and Γ_ψ the graph of ψ . Denote by Γ the intersection of $\Gamma_\varphi \times G$ and $E \times \Gamma_\psi$. Then the graph of $\psi \circ \varphi$ is the image of Γ by the projection

$$E \times F \times G \longrightarrow E \times G, \quad (x, y, z) \mapsto (x, z).$$

By hypothesis, Γ is a semialgebraic subset of $E \times F \times G$ and by Tarski-Seidenberg's theorem, $\psi \circ \varphi$ is a semialgebraic map. \square

2.1. Some results on semialgebraic sets. According to [Cos02, Section 2], a semialgebraic subset S of \mathbb{R}^n has a decomposition into a finite disjoint union of semialgebraic subsets semialgebraically homeomorphic to open hypercubes $]0, 1[^d$ (by convention $]0, 1[^0$ is a point). By [Cos02, Proposition 3.15], the dimension of S is the maximum of these d and is independent of the decomposition. Moreover, by [Cos02, Theorem 2.23], S has a finite number of connected components and its connected components are semialgebraic.

Lemma 2.2. *Let S be in $\mathfrak{S}(\mathbb{R}^n)$ and \overline{S} its closure in \mathbb{R}^n .*

(i) *The sets S and \overline{S} have the same dimension.*

(ii) *The dimension of $\overline{S} \setminus S$ is smaller than $\dim S$.*

(iii) *The set S has an empty interior if and only if $\dim S \leq n - 1$.*

(iv) *If S is a Zariski closed subset of \mathbb{R}^n , then $\dim S$ is the dimension of S as an algebraic subset.*

(v) *For m positive integer and φ semialgebraic map from S to \mathbb{R}^m , $\dim \varphi(S) \leq \dim S$.*

Proof. Assertions (i) and (ii) are given by [Cos02, Proposition 3.16], Assertion (iv) is given by [Cos02, Theorem 3.20] and Assertion (v) is given by [Cos02, Theorem 3.18].

(iii) By definition, a nonempty semialgebraic open subset of \mathbb{R}^n has dimension n . For T semialgebraic subset T of S , $\dim T \leq \dim S$ by [Cos02, Corollary 3.8]. Then S has an empty interior if $\dim S \leq n - 1$. Conversely if $\dim S = n$, it contains a semialgebraic subset homeomorphic to $]0, 1[^n$. Hence S has a nonempty interior. \square

Proposition 2.3. *Let Y be a semialgebraic subset of \mathbb{R}^n , containing 0 and having an empty interior. Then there exists a continuous semialgebraic arc θ ,*

$$[0, 1] \xrightarrow{\theta} \mathbb{R}^n, \quad t \mapsto \theta(t)$$

such that $\theta(0) = 0$ and $\theta(t) \notin Y$ for all t in $]0, 1]$.

Proof. As Y has an empty interior, $\mathbb{R}^n \setminus Y$ is a dense semialgebraic open subset of \mathbb{R}^n , not containing 0, whence the proposition by the Curve selection lemma [Cos02, Theorem 3.13]. \square

2.2. A result on a semialgebraic function. Let Ω be a connected semialgebraic open subset of \mathbb{C}^n and a a continuous semialgebraic function on Ω with real values.

Lemma 2.4. *Let Ω' be the subset of elements x of Ω such that a is constant in a neighborhood of x . Then Ω' is a semialgebraic open subset of \mathbb{C}^n .*

Proof. By definition Ω' is a union of open subsets of \mathbb{C}^n . Hence it is open. Let Z be the subset of elements $(x, x_1, x_2, \varepsilon)$ of $\Omega \times \Omega \times \Omega \times \mathbb{R}_+^*$ such that

$$d(x, x_1) \leq \varepsilon, \quad d(x, x_2) \leq \varepsilon, \quad a(x_1) \neq a(x_2).$$

Since Ω, a, d are semialgebraic, Z is a semialgebraic subset of $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{R}$. Denote by \bar{Z} the closure of Z in $\Omega \times \Omega \times \Omega \times \mathbb{R}_+$ and Z_0 the intersection of \bar{Z} and $\Omega \times \Omega \times \Omega \times \{0\}$. Then $\Omega \setminus \Omega'$ is the image of Z_0 by the first projection. As a result, $\Omega \setminus \Omega'$ and Ω' are semialgebraic subsets of \mathbb{C}^n , whence the lemma. \square

Corollary 2.5. *Suppose that the interior of $\Omega \setminus \Omega'$ is empty. Then the function a is constant.*

Proof. By hypothesis and Lemma 2.4, Ω' is a dense semialgebraic open subset of Ω . Then, by [Cos02, Theorem 2.23], Ω' has a finite number of connected components. Denote them by O_1, \dots, O_m . For $i = 1, \dots, m$, a is constant on O_i since a is continuous. Denote by a_i its value. As $O \setminus O'$ has an empty interior, for x in $O \setminus O'$, for some i , x is in the closure of O_i so that $a(x) = a_i$. As a result, the image of a is equal to $\{a_1, \dots, a_m\}$. Hence a is constant since a is continuous and O is connected. \square

2.3. On semialgebraic curves. In this subsection we give some results on continuous semialgebraic curves.

Lemma 2.6. *Let γ be a continuous semialgebraic function from the interval $]a, b[$ to \mathbb{R} .*

(i) *Suppose that γ has a boundary value when t goes to a . Then γ has a continuous extension to $[a, b]$.*

(ii) *If γ is not bounded, then for some positive rational number r and a continuous semialgebraic function γ_0 on $[a, b]$ such that $\gamma_0(a) \neq 0$, $\gamma(t) = (t - a)^{-r} \gamma_0(t)$ for all t in $]a, b[$.*

Proof. (i) Let Γ be the graph of the function γ and $\bar{\Gamma}$ its closure in $[a, b] \times \mathbb{R}$. By hypothesis, $\bar{\Gamma} \setminus \Gamma$ is not empty and contained in $\{a\} \times \mathbb{R}$ since γ is continuous. Let x be in \mathbb{R} such that (a, x) is in $\bar{\Gamma}$ and ε a positive real number. The intersection of $\bar{\Gamma}$ and $[a, b] \times]x - \varepsilon, x + \varepsilon[$ is a semialgebraic subset of \mathbb{R}^2 . Then it has a finite number of connected components by [Cos02, Theorem 2.23]. Denote them by C_1, \dots, C_m . Since (a, x) is in $\bar{\Gamma}$, for some positive number η and for some i in $\{1, \dots, m\}$, $]a, a + \eta[$ is contained in the image of C_i by the first projection since a semialgebraic subset of \mathbb{R} is a finite union of intervals and isolated points. As a result, for all t in $]a, a + \eta[$, $|\gamma(t) - x| < \varepsilon$. Hence the extension $\bar{\gamma}$ of γ to $[a, b]$ such that $\bar{\gamma}(a) = x$ is continuous.

(ii) Suppose that γ is not bounded. By (i),

$$\lim_{t \rightarrow a} |\gamma(t)| = +\infty.$$

Moreover, the function $t \mapsto |\gamma(t)|$ is a semialgebraic function. Let P_1, \dots, P_m be the polynomials corresponding to the equalities of a system defining the graph of the function $|\gamma|$. Denote by Q the polynomial

$$Q(t, s) := \prod_{i=1}^m P_i(t + a, s).$$

Then, for some prime divisor Q_* of Q , $Q_*(t, \gamma(t+a)) = 0$ for all t , different from 0, in a neighborhood of 0 in $]0, b-a]$. So, by a Puiseux's expansion, for some positive rational number r and positive real number c ,

$$\lim_{t \rightarrow a} (t-a)^r |\gamma(t)| = c.$$

Setting:

$$\gamma_0(t) = (t-a)^r \gamma(t)$$

for t in $]a, b]$, the function γ_0 on $]a, b]$ is bounded, continuous and semialgebraic. Then, by (i), γ_0 has a continuous expansion to $[a, b]$ such that $\gamma_0(a) \neq 0$. \square

Corollary 2.7. *Let E be a finite dimensional real vector space and γ a continuous semialgebraic map from the interval $[a, b]$ to E . For k nonnegative integer, there exist an increasing sequence $r_1 < \dots < r_{k+1}$ of positive rational numbers and x_1, \dots, x_k in E such that the map*

$$[a, b] \longrightarrow E, \quad t \longmapsto (t-a)^{-r_{k+1}} (\gamma(t) - (\gamma(a) + (t-a)^{r_1} x_1 + \dots + (t-a)^{r_k} x_k))$$

is bounded, continuous and semialgebraic.

Proof. Considering the coordinates of γ in a basis of E , it is sufficient to prove the corollary for $E = \mathbb{R}$. The map $t \mapsto \gamma(t) - \gamma(a)$ has a zero at a . Moreover, the set of zeros of this map in $[a, b]$ is a closed semialgebraic subset of $[a, b]$. As a result, it is a union of a finite number of closed intervals and isolated points. Hence $\gamma(t) - \gamma(a) = 0$ for all t in a neighborhood of a in $[a, b]$ or $\gamma(t) - \gamma(a) \neq 0$ for all t different from a in a neighborhood of a in $[a, b]$. In the first case, there is nothing to prove. Suppose that the second case is satisfied. For some positive number η such that $a + \eta \leq b$, the function

$$t \longmapsto \frac{1}{\gamma(t) - \gamma(a)}$$

on $]a, a+\eta]$ is a continuous semialgebraic function and it is not bounded. Hence, by Lemma 2.6(ii), for some positive rational number r_1 and for some continuous semialgebraic function γ_1 on $[a, b]$ such that $\gamma_1(a) \neq 0$,

$$\gamma(t) = \gamma(a) + (t-a)^{r_1} \gamma_1(t)$$

for all t in $[a, b]$, whence the corollary by induction on k . \square

Remark 2.8. By Corollary 2.7 and an induction argument, we can find an increasing sequence $r_1 < r_2 < \dots$ of positive rational numbers and a sequence x_1, x_2, \dots in $E \setminus \{0\}$ satisfying one of the two following conditions:

- (1) if the sequences are finite of cardinality k , then the function

$$t \longmapsto \gamma(t) - (\gamma(a) + (t-a)^{r_1} x_1 + \dots + (t-a)^{r_k} x_k)$$

is equal to 0 in a neighborhood of a in $[a, b]$,

- (2) if the first sequence is not finite so is the second one and for all positive integer k , the map

$$t \longmapsto (t-a)^{-r_{k+1}} (\gamma(t) - (\gamma(a) + (t-a)^{r_1} x_1 + \dots + (t-a)^{r_k} x_k))$$

is bounded and semialgebraic.

In the second case, by [Duv89], as the rational numbers r_1, r_2, \dots are the exponents of a Puiseux's serie, for some positive integer m , mr_1, mr_2, \dots are positive integers so that the sequence r_1, r_2, \dots is not bounded.

3. VECTOR FIELDS AND INTEGRAL CURVES

For x in \mathbb{C}^n , denote by X_x the polynomial vector field on \mathbb{C}^n ,

$$y \mapsto X_x(y) := \Phi'(y)^{-1}(x).$$

Definition 3.1. An integral curve of the vector field X_x is a derivable map γ from an open interval I of \mathbb{R} to \mathbb{C}^n such that its derative γ' satisfies the equality

$$\gamma'(t) = X_x(\gamma(t))$$

for all t in I .

3.1. Elementary properties. Let x be in \mathbb{C}^n .

Lemma 3.2. Denote by γ a continuous map from an open interval I of \mathbb{R} to \mathbb{C}^n . Then γ is an integral curve of X_x if and only if for some t_0 in I

$$\Phi \circ \gamma(t) = \Phi \circ \gamma(t_0) + (t - t_0)x$$

for all t in I .

Proof. Suppose that γ is an integral curve of X_x . Then $\Phi \circ \gamma$ is an integral curve of the vector field $z \mapsto \Phi'(z)(X_x(z))$. By definition, $\Phi'(z)(X_x(z)) = x$ for all z in \mathbb{C}^n . As a result, by the unicity of the integral curves with a given initial datum,

$$\Phi \circ \gamma(t) = \Phi \circ \gamma(t_0) + (t - t_0)x$$

for all (t, t_0) in $I \times I$. Conversely, suppose that γ is a continuous map from I to \mathbb{C}^n such that for some t_0 in I ,

$$\Phi \circ \gamma(t) = \Phi \circ \gamma(t_0) + (t - t_0)x \quad \forall t \in I.$$

Let s be in I and (ω, ψ) an adapted pair to $\gamma(s)$. Then, for all t in $\gamma^{-1}(\omega)$,

$$\gamma(t) = \psi(\Phi \circ \gamma(t_0) + (t - t_0)x).$$

As a result, γ is a derivable map and $\Phi'(\gamma(t))(\gamma'(t)) = x$ for all t in I , whence the assertion by definition of X_x . □

Denote by \mathcal{C}_x the set of integral curves of X_x whose interval of definition contains 0. For γ and δ in \mathcal{C}_x , we say that γ is smaller than δ if the interval of definition of γ is contained in the interval of definition δ and if δ is an expansion of γ . This order is inductive so that any element of \mathcal{C}_x is smaller than a maximal one.

3.2. On the integral curves. Denote by Ψ the map

$$\mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^n \xrightarrow{\Psi} \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^n, \quad (x, t, z) \mapsto (x, t, \Phi(z)).$$

As Φ is étale so is Ψ . Let Δ be the graph of the map

$$\mathbb{C}^n \times \mathbb{C} \longrightarrow \mathbb{C}^n, \quad (x, t) \mapsto tx.$$

Lemma 3.3. *Denote by Γ the inverse image of Δ by Ψ and for x in \mathbb{C}^n denote by Γ_x the intersection of Γ with $\{x\} \times \mathbb{C} \times \mathbb{C}^n$.*

(i) *The subset Γ of $\mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^n$ is a smooth algebraic variety of dimension $n + 1$, containing $\mathbb{C}^n \times \{0, 0\}$.*

(ii) *For x in \mathbb{C}^n , Γ_x is a smooth algebraic variety of dimension 1, containing $(x, 0, 0)$.*

Proof. (i) As $\Psi(x, 0, 0) = (x, 0, 0)$, $(x, 0, 0)$ is in Γ for all x in \mathbb{C}^n . Let (x, t, y) be in Γ . Since Ψ is étale and Δ is smooth of dimension $n + 1$, the tangent space of Γ at (x, t, y) has dimension at most $n + 1$. Moreover, the image of some open neighborhood of (x, t, y) in $\mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^n$ by Ψ is an open neighborhood of $(x, t, \Phi(y))$ in $\mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^n$. As a result, Γ has dimension $n + 1$. Hence Γ is a smooth algebraic variety, equidimensional of dimension $n + 1$.

(ii) Let x be in \mathbb{C}^n . By definition, Γ_x is the inverse image by Ψ of the graph Δ_x of the map

$$\{x\} \times \mathbb{C} \times \mathbb{C}^n \longrightarrow \mathbb{C}^n, \quad (x, t) \mapsto tx.$$

As the restriction of Ψ to $\{x\} \times \mathbb{C} \times \mathbb{C}^n$ is an étale endomorphism of $\{x\} \times \mathbb{C} \times \mathbb{C}^n$, Γ_x is a smooth algebraic variety of dimension 1 since so is Δ_x . Moreover, $(x, 0, 0)$ is in Γ_x . \square

Proposition 3.4. *Let y be in \mathbb{C}^n and γ in \mathcal{C}_x such that $\gamma(0) = y$. Then γ is a semialgebraic map from an interval of \mathbb{R} to \mathbb{C}^n .*

Proof. Let I be the interval of definition of γ . According to Lemma 3.2,

$$\Phi \circ \gamma(t) = \Phi(y) + tx$$

for all t in I . Denote by Δ_γ the graph of γ and set:

$$\Delta_I := \{(t, z) \in I \times \mathbb{C}^n \mid \Phi(z) = \Phi(y) + tx\}.$$

Since Φ is a polynomial map, Δ_I is a semialgebraic subset of $\mathbb{R} \times \mathbb{C}^n$. Moreover, Δ_γ is a connected closed subset of Δ_I . For all t in I , denote by (ω_t, ψ_t) an adapted pair to $\gamma(t)$. Let δ_t be the inverse image of ω_t by γ . Then, for (s, z) in the intersection of $\delta_t \times \omega_t$ and Δ_I ,

$$\Phi(z) = \Phi(y) + sx \quad \text{and} \quad \Phi \circ \gamma(s) = \Phi(y) + sx,$$

whence $z = \gamma(s)$. As a result, Δ_γ is an open subset of Δ_I . Then Δ_γ is a connected component of Δ_I . So, by [Cos02, Theorem 2.23], Δ_γ is a semialgebraic subset of $\mathbb{R} \times \mathbb{C}^n$ since so is Δ_I . \square

Corollary 3.5. *Let y be in \mathbb{C}^n and γ in \mathcal{C}_x such that $\gamma(0) = y$. Suppose that γ is defined over the relatively compact interval $]a, b[$ of \mathbb{R} . If $\|\gamma\|$ is bounded then there exists δ in \mathcal{C}_x , bigger than γ and such that its domain of definition contains $[a, b]$.*

Proof. By Proposition 3.4 and Lemma 2.6(i), γ has a continuous extension to $[a, b]$. Denote again by γ this extension. For some positive real number η , there exists an integral curve γ_a of X_x defined over $]a - \eta, a + \eta[$ and such that $\gamma_a(a) = \gamma(a)$. Let (ω, ψ) be an adapted pair to $\gamma(a)$. For some positive real number η' , smaller than η , for all t in $]a, a + \eta'[$, $\gamma(t)$ is in ω and $\Phi \circ \gamma(t) = \Phi(y) + tx$, whence

$$\Phi \circ \gamma(a) = \Phi(y) + ax \quad \text{and} \quad \Phi \circ \gamma(t) = \Phi \circ \gamma(a) + (t - a)x.$$

By Lemma 3.2, for all t in $[a, a + \eta[$,

$$\Phi \circ \gamma_a(t) = \Phi \circ \gamma(a) + (t - a)x.$$

As a result, for some t in $]a, a + \eta'[$, $\gamma(t) = \gamma_a(t)$. So, by the unicity of the integral curve with initial datum, γ_a and γ have the same restriction to $[a, a + \eta[$. As a result, there exists an integral curve of X_x , extending γ and such that its interval of definition is $]a - \eta, b[$. In the same way, we prove that for some positive real number η'' , there exists an integral curve of X_x , extending γ and such that its interval of definition is $]a, b + \eta''[$, whence the corollary. \square

4. SPECIAL INTEGRAL CURVES

For x in \mathbb{C}^n , denote by γ_x the maximal element of \mathcal{C}_x such that $\gamma_x(0) = 0$. The goal of the section is the following proposition:

Proposition 4.1. *For all x in \mathbb{C}^n , the interval of definition of γ_x is equal to \mathbb{R} .*

4.1. Intervals of definition. For x in \mathbb{C}^n , we denote by I_x the interval of definition of γ_x .

Lemma 4.2. *Let x be in \mathbb{C}^n .*

(i) *The interval I_x is invariant under the homeomorphism $t \mapsto -t$. In particular, for some $a(x)$ in $\mathbb{R}_+^* \cup \{+\infty\}$, $I_x =]-a(x), a(x)[$.*

(ii) *For s in \mathbb{R}^* , $\gamma_{sx}(t) = \gamma_x(st)$. Moreover, $I_{sx} = s^{-1}I_x$ and $a(sx) = |s|^{-1}a(x)$.*

Proof. (i) By the maximality of γ_x and the unicity of an integral curve with initial datum it is sufficient to prove that the map δ_x ,

$$-I \xrightarrow{\delta_x} \mathbb{C}^n, \quad t \mapsto -\gamma_x(-t)$$

is an integral curve of X_x since 0 is in I and $\gamma_x(0) = 0$. By definition, for all t in $-I$,

$$\delta'_x(t) = \gamma'_x(-t) = X_x(\gamma_x(-t)) = \Phi'(\gamma_x(-t))^{-1}(x).$$

As Φ is an odd map, $\Phi'(y) = \Phi'(-y)$ for all y in \mathbb{C}^n . As a result, δ_x is an integral curve of X_x , whence the assertion.

(ii) Denote by δ the map

$$s^{-1}I_x \xrightarrow{\delta} \mathbb{C}^n, \quad t \mapsto \gamma_x(st).$$

Then

$$\delta'(t) = s\gamma'_x(st) = s\Phi'(\gamma_x(st))^{-1}(x) = X_{sx}(\delta(t)).$$

As a result, δ is an integral curve of the vector field X_{sx} such that $\delta(0) = 0$. So by the unicity of the integral curve with initial datum, $s^{-1}I_x$ is contained in I_{sx} and δ is the restriction of γ_{sx} to $s^{-1}I_x$. Considering the map

$$sI_{sx} \xrightarrow{\tau} \mathbb{C}^n, \quad t \mapsto \gamma_{sx}(s^{-1}t),$$

sI_{sx} is contained in I_x by the same argument, whence $I_{sx} = s^{-1}I_x$ and the assertion. \square

According to Lemma 4.2(i), a map a from \mathbb{C}^n to $\mathbb{R}_+^* \cup \{+\infty\}$ is well defined. Let Ω be the subset of elements (x, t) of $\mathbb{C}^n \times \mathbb{R}$ such that $-a(x) < t < a(x)$ and G the graph of the map

$$\Omega \xrightarrow{\gamma} \mathbb{C}^n, \quad (x, t) \mapsto \gamma(x, t) := \gamma_x(t).$$

Lemma 4.3. *Let (x, t) be in Ω and t' a positive real number such that $|t| < t' < a(x)$. Then for some open neighborhood ω of x in \mathbb{C}^n , $\omega \times [-t', t']$ is contained in Ω and the restriction of γ to $\omega \times [-t', t']$ is continuous.*

Proof. Since $[-t', t']$ is a compact interval contained in I_x and the map $(y, z) \mapsto X_z(y)$ is polynomial, for some open neighborhood ω of x in \mathbb{C}^n , $\omega \times [-t', t']$ is contained in Ω and the restriction of γ to $\omega \times [-t', t']$ is continuous by [Car67, Théorème 3.6.1]. \square

Corollary 4.4. (i) *The subset Ω of $\mathbb{C}^n \times \mathbb{R}$ is open and the map γ is continuous.*

(ii) *The function a is continuous. In particular, $a^{-1}(\{+\infty\})$ is a closed subset of \mathbb{C}^n . Moreover $a^{-1}(\{+\infty\})$ contains the kernel of F .*

Proof. (i) By Lemma 4.3, Ω is the union of open subsets of $\mathbb{C}^n \times \mathbb{R}$ at which the restriction of γ is continuous, whence the assertion.

(ii) By (i), the function a is lower continuous. Hence $a^{-1}(\{+\infty\})$ is a closed subset of \mathbb{C}^n . By Lemma 4.3, a is upper continuous. Hence a is continuous. By Lemma 1.5 and Lemma 3.2, for x in the kernel of F , $\gamma_x(t) = tx$ for all t in \mathbb{R} , whence the assertion. \square

4.2. Properties of the maps γ and a . Let Γ be as in Subsection 3.2 and $\Gamma_{\mathbb{R}}$ its intersection with $\mathbb{C}^n \times \mathbb{R} \times \mathbb{C}^n$.

Lemma 4.5. *Let Γ_0 be the connected component of $\Gamma_{\mathbb{R}}$ containing $\mathbb{C}^n \times \{0, 0\}$.*

(i) *The subset Γ_0 of $\mathbb{C}^n \times \mathbb{R} \times \mathbb{C}^n$ is closed and semialgebraic. Moreover, it contains G .*

(ii) *The subset G of Γ_0 is open.*

Proof. (i) As $\mathbb{C}^n \times \{(0, 0)\}$ is contained in Γ , it is contained in a connected component of $\Gamma_{\mathbb{R}}$. Since $\Gamma_{\mathbb{R}}$ is a semialgebraic closed subset of $\mathbb{C}^n \times \mathbb{R} \times \mathbb{C}^n$, Γ_0 is semialgebraic by [Cos02, Theorem 2.23]. The set G is contained in $\Gamma_{\mathbb{R}}$ since for all (x, t) in Ω , $\Phi \circ \gamma_x(t) = tx$ by Lemma 3.2. As Ω is a connected subset of $\mathbb{C}^n \times \mathbb{R}$, G is a connected subset of $\mathbb{C}^n \times \mathbb{R} \times \mathbb{C}^n$ by Corollary 4.4(i). Hence G is contained in Γ_0 since $\mathbb{C}^n \times \{(0, 0)\}$ is contained in G .

(ii) Let (x, t, y) be in G and (ω, ψ) an adapted pair to y . As γ is continuous, $\gamma^{-1}(\omega) \times \omega$ is an open subset of $\mathbb{C}^n \times \mathbb{R} \times \mathbb{C}^n$. Its intersection with $\Gamma_{\mathbb{R}}$ is the graph of the map

$$\gamma^{-1}(\omega) \longrightarrow \mathbb{C}^n, \quad (z, s) \mapsto \psi(sz).$$

Since for all (z, s) in $\gamma^{-1}(\omega)$, $\gamma(z, s)$ is in ω and $\Phi \circ \gamma(z, s) = sz$, $\gamma^{-1}(\omega) \times \omega \cap \Gamma_{\mathbb{R}}$ is contained in G , whence the assertion. \square

Proposition 4.6. (i) *The graph G is equal to Γ_0 .*

(ii) *The map γ is continuous and semialgebraic.*

(iii) *The function a is continuous and semialgebraic. In particular, $a^{-1}(\{+\infty\})$ is a closed semialgebraic subset of \mathbb{C}^n .*

Proof. (i) Let \overline{G} be the closure of G in $\mathbb{C}^n \times \mathbb{R} \times \mathbb{C}^n$. Then \overline{G} is contained in Γ_0 since Γ_0 is a closed subset of $\mathbb{C}^n \times \mathbb{R} \times \mathbb{C}^n$. By Lemma 4.5(ii), G is an open subset of Γ_0 . Hence it remains to prove $G = \overline{G}$ since Γ_0 is connected by definition.

Suppose that G is strictly contained in \overline{G} . A contradiction is expected. Let (x, t, y) be in $\overline{G} \setminus G$. As γ is continuous by Corollary 4.4(i), G is a closed subset of $\Omega \times \mathbb{C}^n$. Hence $t = \pm a(x)$ since a is continuous by Corollary 4.4(ii). By Lemma 4.2(i), G and \overline{G} are invariant under the homeomorphism $(x, t, y) \mapsto (x, -t, y)$. So we can suppose $t = a(x)$. For some sequence (x_m, t_m) , $m = 1, 2, \dots$ in Ω ,

$$\lim_{m \rightarrow +\infty} \gamma(x_m, t_m) = y.$$

Let (ω, ψ) be an adapted pair to y . For z in ω , denote by τ_z the integral curve of the vector field $X_{a(x)^{-1}\Phi(z)}$ such that $\tau_z(a(x)) = z$. In particular, τ_y is the integral curve of the vector field X_x such that $\tau_y(a(x)) = y$ and by Lemma 3.2, for all z in ω and for all t in the interval of definition of τ_z ,

$$\Phi \circ \tau_z(t) = \Phi(z) + (t - a(x))a(x)^{-1}\Phi(z) = ta(x)^{-1}\Phi(z).$$

For some positive real number η , $[a(x) - \eta, a(x) + \eta]$ is contained in the interval of definition of τ_y . Then, by [Car67, Théorème 3.6.1], for some open neighborhood ω' of y in ω , $[a(x) - \eta, a(x) + \eta]$ is contained in the interval of definition of τ_z for all z in ω' since the map $(y, z) \mapsto X_{a(x)^{-1}\Phi(z)}(y)$ is polynomial. Moreover, the map

$$\omega' \times [a(x) - \eta, a(x) + \eta] \xrightarrow{\tau} \mathbb{C}^n, \quad (z, t) \mapsto \tau_z(t)$$

is continuous. Hence we can suppose that the image of τ is contained in ω .

For some positive integer M and for m bigger than M , $\gamma(x_m, t_m)$ is in ω' and for s positive real number smaller than $\eta/2$, $t_m - s$ is in $[a(x) - \eta, a(x) + \eta]$. Since $\Phi(\omega')$ is a neighborhood of $\Phi(y)$ and $\Phi \circ \gamma(x_m, t_m) = t_m x_m$, we can suppose that for m bigger than M , $a(x)x_m$ is in $\Phi(\omega')$. Setting $z_m := \psi(a(x)x_m)$, τ_{z_m} is an integral curve of X_{x_m} and $\tau_{z_m}(t_m) = \gamma(z_m, t_m)$ since

$$\Phi \circ \tau_{z_m}(t_m) = \Phi \circ \gamma(z_m, t_m) = t_m z_m.$$

So, by the unicity of the integral curve with initial datum, $\tau_{z_m}(t) = \gamma_{z_m}(t)$ for all t in $[a(x) - \eta, a(x) + \eta]$. As a result, for s positive real number smaller than $\eta/2$,

$$\gamma(x, a(x) - s) = \lim_{m \rightarrow +\infty} \gamma(x_m, t_m - s) = \lim_{m \rightarrow +\infty} \tau_{z_m}(t_m - s) = \tau_y(a(x) - s)$$

by continuity of γ and τ , whence the contradiction since

$$\lim_{s \rightarrow 0} \|\gamma(x, a(x) - s)\| = +\infty$$

by Corollary 3.5.

(ii) By Lemma 4.5(i), Γ_0 is semialgebraic. So, by (i), γ is a semialgebraic map, whence the assertion since γ is continuous.

(iii) By (i), Ω is the image of Γ_0 by the projection

$$\mathbb{C}^n \times \mathbb{R} \times \mathbb{C}^n \longrightarrow \mathbb{C}^n \times \mathbb{R}, \quad (x, t, y) \mapsto (x, t).$$

Hence Ω and $\overline{\Omega}$ are semialgebraic subsets of $\mathbb{C}^n \times \mathbb{R}$. By Corollary 4.4(ii), a is continuous. Hence, $\overline{\Omega}$ is the disjoint union of Ω and the graphs of the restrictions of a and $-a$ to $\mathbb{C}^n \setminus a^{-1}(+\infty)$ since $a^{-1}(+\infty) \times \mathbb{R}$ is contained in Ω . In particular, $a^{-1}(+\infty)$ is the complement in \mathbb{C}^n to the image of $\overline{\Omega} \setminus \Omega$ by the projection

$$\mathbb{C}^n \times \mathbb{R} \longrightarrow \mathbb{C}^n, \quad (x, t) \mapsto x$$

and the graph of the restriction of a to $\mathbb{C}^n \setminus a^{-1}(+\infty)$ is the intersection of $\overline{\Omega} \setminus \Omega$ and $(\mathbb{C}^n \setminus a^{-1}(+\infty)) \times \mathbb{R}_+$. Hence $a^{-1}(+\infty)$ is a semialgebraic subset of \mathbb{C}^n and the restriction of a to $\mathbb{C}^n \setminus a^{-1}(+\infty)$ is semialgebraic. \square

4.3. Étale covering. Let K be the field of rational fractions $\mathbb{C}(x_1, \dots, x_n)$ and K' the subfield $\mathbb{C}(x_1 + F_1^3, \dots, x_n + F_n^3)$ of K . Denote by d the degree of the extension K of K' .

Lemma 4.7. *Let X and Y be two irreducible complex algebraic varieties and ψ a morphism from X to Y . For i nonnegative integer, denote by Y_i the subset of elements y such that $|\psi^{-1}(y)| = i$. Then, for all nonnegative integer i , Y_i is an algebraic constructible subset of Y .*

Proof. Let Y'_n be the subset of elements y of Y such that $|\psi^{-1}(y)| \geq n$. Denote by Z_n the subset of elements (y, x_1, \dots, x_n) of $Y \times X^n$ such that $y = \psi(x_1) = \dots = \psi(x_n)$. Then Z_n is a closed subset of $Y \times X^n$. Let Z'_n be the open subset of elements (y, x_1, \dots, x_n) of Z_n such that $x_i \neq x_j$ for $i \neq j$. Then Y'_n is an algebraic constructible subset of Y as the image of Z'_n by the projection

$$(y, x_1, \dots, x_n) \mapsto y.$$

By definition, $Y_n = Y'_n \setminus Y'_{n+1}$. Hence Y_n is an algebraic constructible subset of Y . \square

Corollary 4.8. *Suppose that ψ is a quasi-finite dominant morphism. Let j be the degree of the field extension $\mathbb{C}(Y) \subset \mathbb{C}(X)$. Then Y_j contains a dense Zariski open subset of Y .*

Proof. As j is the degree of the separable field extension $\mathbb{C}(Y) \subset \mathbb{C}(X)$, Y_j contains the generic point of Y . So, by Lemma 4.7, Y_j contains a dense Zariski open subset of Y . \square

Proposition 4.9. *For some nonempty Zariski open subset U of \mathbb{C}^n , the restriction of Φ to $\Phi^{-1}(U)$ is a covering map of degree d . In particular, it is a finite morphism.*

Proof. As Φ is étale, Φ is a quasi finite morphism. So, by Corollary 4.8, for some nonempty Zariski open subset U of \mathbb{C}^n , $\Phi^{-1}(U)$ has cardinality d for all x in U . Let x be in U and x_1, \dots, x_d the elements of $\Phi^{-1}(x)$. For $i = 1, \dots, d$, denote by (ω_i, ψ_i) an adapted pair to x_i . Since x_1, \dots, x_d are pairwise different, we can choose $\omega_1, \dots, \omega_d$ pairwise disjoint. Let ω be the intersection of $\Phi(\omega_1), \dots, \Phi(\omega_d)$. Then ω is an open neighborhood of x and $\Phi^{-1}(\omega)$ is the disjoint union of the

open subsets $\omega_1 \cap \Phi^{-1}(\omega), \dots, \omega_d \cap \Phi^{-1}(\omega)$. As a result, the restriction of Φ to $\Phi^{-1}(U)$ is an étale covering of degree d of U , whence the proposition. \square

4.4. At the neighborhood of an element of $\mathbb{C}^n \setminus a^{-1}(\{+\infty\})$. Let x be in $\mathbb{C}^n \setminus a^{-1}(\{+\infty\})$ and θ a continuous semialgebraic arc,

$$[0, 1] \xrightarrow{\theta} \mathbb{C}^n,$$

such that $\theta(0) = x$ and $a(\theta(t)) > a(x)$ for all t in $]0, 1]$.

Lemma 4.10. *For some positive rational number r and a continuous semialgebraic map ψ from $[0, 1]$ to \mathbb{C}^n ,*

$$\psi(0) \neq 0 \quad \text{and} \quad \gamma(\theta(t), a(x)) = t^{-r} \psi(t)$$

for all t in $]0, 1]$.

Proof. By Proposition 4.6(ii), the map $t \mapsto \gamma(\theta(t), a(x))$ from $]0, 1]$ to \mathbb{C}^n is continuous and semialgebraic. Moreover, by Lemma 3.2,

$$(1) \quad \gamma(\theta(t), a(x)) + F^{(3)}(\gamma(\theta(t), a(x))) = a(x)\theta(t)$$

for all t in $]0, 1]$. By Lemma 2.6(i), if the map $t \mapsto \|\gamma(\theta(t), a(x))\|$ is bounded, then for some y in \mathbb{C}^n ,

$$\lim_{t \rightarrow 0} \gamma(\theta(t), a(x)) = y$$

so that (x, a, y) is in G and $(x, a(x))$ is in Ω by Proposition 4.6(i). By definition, $(x, a(x))$ is not in Ω . As a result, the map $t \mapsto \|\gamma(\theta(t), a(x))\|$ is not bounded, whence the assertion by Lemma 2.6(ii), \square

4.5. Proof of Proposition 4.1. Let U be the Zariski open subset of \mathbb{C}^n as in Proposition 4.9 and Y its complement in \mathbb{C}^n .

Lemma 4.11. *Let Z be the closure in \mathbb{C}^n of the union of tY , $t \in \mathbb{R}_+^*$ and $O := \mathbb{C}^n \setminus Z$.*

(i) *The subsets Z and O of \mathbb{C}^n are semialgebraic. In particular, the connected components of O are finitely many and semialgebraic.*

(ii) *The open subset O of \mathbb{C}^n is dense in \mathbb{C}^n .*

(iii) *Let O_* be a connected component of O and O'_* the subset of elements x of O_* such that a is constant on a neighborhood of x . Then $tO_* = O_*$ and $tO'_* = O'_*$ for all t in \mathbb{R}_+^* .*

Proof. (i) Let Z_* be the union of tY , $t \in \mathbb{R}_+^*$. As the map

$$\mathbb{R}_+^* \times \mathbb{C}^n \longrightarrow \mathbb{C}^n, \quad (t, x) \longmapsto tx$$

is semialgebraic and Y is an algebraic subvariety of \mathbb{C}^n , Z_* is a semialgebraic subset of \mathbb{C}^n . Hence Z and O are semialgebraic subsets of \mathbb{C}^n , whence the assertion by [Cos02, Theorem 2.23].

(ii) By definition, Y is an algebraic subvariety of dimension smaller than n of \mathbb{C}^n . Hence its dimension as a semialgebraic subset of \mathbb{C}^n is at most $2n - 2$ by Lemma 2.2(iv). As a result, Z_* and Z have dimension at most $2n - 1$ by Lemma 2.2, (i) and (v). Hence O is a dense open subset of \mathbb{C}^n by Lemma 2.2(iii).

(iii) By definition, $tO = O$ for all t in \mathbb{R}_+^* . Hence the group \mathbb{R}_+^* permutes the connected components of O and the stabilizers are open subgroups of \mathbb{R}_+^* . Hence $tO_* = O_*$ for all t in \mathbb{R}_+^* . By Lemma 4.2(ii), for x in O_* and t in \mathbb{R}_+^* , $a(tx) = t^{-1}a(x)$. Hence O'_* is invariant under the action of \mathbb{R}_+^* . \square

For O_* a connected component of O , denote by O''_* the interior of the complement to O'_* in O_* . Then O''_* is an open subset of U , invariant under the action of \mathbb{R}_+^* by Lemma 4.11(iii).

Lemma 4.12. *Let O_* be a connected component of O . Then O''_* is empty.*

Proof. Suppose that O''_* is nonempty. A contradiction is expected. Let x be in O''_* and V a semialgebraic compact neighborhood of x in O''_* . As a is continuous, for some y in V , $a(y) \leq a(z)$ for all z in V . Moreover, since a is semialgebraic and V is contained in O''_* , $a^{-1}(\{y\}) \cap V$ is a semialgebraic subset whose interior is empty. Then, by Proposition 2.3, for some continuous semialgebraic arc θ ,

$$[0, 1] \xrightarrow{\theta} V,$$

$\theta(0) = y$ and $a(\theta(t)) > a(y)$ for all t in $]0, 1]$. By Lemma 4.10,

$$\lim_{t \rightarrow 0} \|\gamma(\theta(t), a(y))\| = +\infty.$$

Moreover, for all t in $]0, 1]$, $\Phi \circ \gamma(\theta(t), a(y)) = a(y)\theta(t)$. As O''_* is invariant under \mathbb{R}_+^* , $a(y)\theta(t)$ is in O''_* for all t in $]0, 1]$. Let V' be a compact neighborhood of $a(y)y$ in O''_* . As O''_* is contained in U , $\Phi^{-1}(V')$ is compact by Proposition 4.9. Moreover, for some positive real number η , $a(y)\theta(t)$ is in V' and $\gamma(\theta(t), a(y))$ is in $\Phi^{-1}(V')$ for all t in $]0, \eta]$, whence the contradiction. \square

By Lemma 4.12 and Corollary 2.5, a is constant on each connected component of O . By Lemma 4.2(ii), for x in O and t in \mathbb{R}_+^* , $a(tx) = t^{-1}a(x)$. Hence O is contained in $a^{-1}(\{+\infty\})$ since the connected components of O are invariant under \mathbb{R}_+^* . As a result $a^{-1}(\{+\infty\}) = \mathbb{C}^n$ since a is continuous and O is dense in \mathbb{C}^n by Lemma 4.12(ii), whence Proposition 4.1.

5. SOME POLYNOMIAL MAPS

Let p_0, p_1, \dots be the sequence of polynomial endomorphisms of \mathbb{C}^n defined by the induction relations:

$$p_0 = 0, \quad p_1(x) = x, \quad p_j = - \sum_{(i_1, i_2, i_3) \in \mathbb{N}_j^3} F(p_{i_1})F(p_{i_2})F(p_{i_3}),$$

for $j \geq 2$. By induction, for $j = 0, 1, \dots$, p_j is homogeneous of degree j .

5.1. Some power series and polynomial maps. For x in \mathbb{C}^n , we denote by S_x the formal power series of the variable t :

$$S_x(t) := p_0(x) + tp_1(x) + t^2p_2(x) + \dots$$

Let r_x be the radius of convergence of S_x . As in Section 4, γ_x is the maximal element of \mathcal{C}_x such that $\gamma_x(0) = 0$.

Lemma 5.1. *Let x be in \mathbb{C}^n .*

(i) *Let S be in $\mathbb{C}^n[[t]]$. Then $S = S_x$ if and only if $\Phi \circ S(t) = tx$.*

(ii) *The radius of convergence r_x is different from 0 and for t in $] - r_x, r_x[$,*

$$\gamma_x(t) = \sum_{j=0}^{\infty} t^j p_j(x).$$

(iii) *For j even integer, $p_j(x) = 0$.*

Proof. (i) Let a_0, a_1, \dots the coefficients of S . By Lemma 1.5, $\Phi^{-1}(0) = \{0\}$. Hence $a_0 = 0$ if $\Phi \circ S(t) = tx$. For $j = 0, 1, \dots$, the coefficient of t^j in the expansion of $F^{(3)}(S(t))$ is equal to

$$\sum_{(i_1, i_2, i_3) \in \mathbb{N}_j^3} F(a_{i_1})F(a_{i_2})F(a_{i_3}).$$

As a result, $\Phi \circ S(t) = tx$ if and only if

$$a_0 = 0, \quad a_1 = x, \quad a_j = - \sum_{(i_1, i_2, i_3) \in \mathbb{N}_j^3} F(a_{i_1})F(a_{i_2})F(a_{i_3}),$$

for $j \geq 2$, whence the assertion.

(ii) Let (ω, ψ) be an adapted pair to 0. Then ψ is a holomorphic map from $\Phi(\omega)$ to \mathbb{C}^n . For some positive real number r , for t in \mathbb{C} such that $|t| < r$, tx is in $\Phi(\omega)$ and $\psi(tx)$ is the sum of a power serie

$$\psi(tx) = \sum_{j=0}^{\infty} t^j a_j.$$

By (i), since $\Phi \circ \psi(tx) = tx$, $a_j = p_j(x)$. In particular, $r_x \neq 0$. Moreover, by Lemma 3.2, the map

$$] - r_x, r_x[\longrightarrow \mathbb{C}^n, \quad t \longmapsto \sum_{j=0}^{\infty} t^j p_j(x)$$

is an integral curve of the vector field X_x . By the unicity of the integral curve with initial datum, it is the restriction of γ_x to $] - r_x, r_x[$, whence the assertion.

(iii) By Lemma 4.2(i), $\gamma_x(t) = -\gamma_x(-t)$ for all t in $] - r_x, r_x[$. Hence by (ii), $p_{2j}(x) = 0$ for all j in \mathbb{N} . □

For x in \mathbb{C}^n , let φ_x be the map from \mathbb{C} to \mathbb{C}^n defined by the following relations: $\varphi_x(0) = 0$ and for t in \mathbb{C}^* ,

$$\varphi_x(t) = \gamma_{ux}(|t|) \quad \text{with} \quad u := \frac{t}{|t|}.$$

As the domain of definition of γ_y is \mathbb{R} for all y in \mathbb{C}^n , φ_x is well defined.

Lemma 5.2. *Let x be in \mathbb{C}^n .*

(i) *For all t in \mathbb{C} , $\Phi \circ \varphi_x(t) = tx$.*

(ii) *The map φ_x is continuous and for all t in \mathbb{C} such that $|t| < r_x$, $\varphi_x(t) = S_x(t)$.*

Proof. (i) By definition $\Phi \circ \varphi_x(0) = 0$. By Lemma 3.2, for t in \mathbb{C}^* and $u = t/|t|$,

$$\Phi \circ \varphi_x(t) = \Phi \circ \gamma_{ux}(|t|) = u|t|x = tx,$$

whence the assertion.

(ii) Let \mathbb{U} be the subset of complex numbers of module 1. Since the polynomial maps p_0, p_1, \dots are homogeneous, r_x is the radius of convergence of S_{ux} for all u in \mathbb{U} . Moreover, by Lemma 5.1(ii),

$$\varphi_x(t) = \sum_{j=0}^{\infty} t^j p_j(x)$$

for all t in \mathbb{C} such that $|t| < r_x$. In particular, φ_x is continuous in a neighborhood of 0.

Let ψ_x be the map

$$\mathbb{U} \times \mathbb{R}_+^* \xrightarrow{\psi_x} \mathbb{C}^n, \quad (u, t) \mapsto \varphi_x(tu).$$

With the notations of Subsection 4.1, $\psi_x(u, t) = \gamma(ux, t)$. Hence, by Corollary 4.4(i), the map ψ_x is continuous. As a result, the restriction of φ_x to \mathbb{C}^* is continuous since the map

$$\mathbb{U} \times \mathbb{R}_+^* \longrightarrow \mathbb{C}^*, \quad (u, t) \mapsto tu$$

is a homeomorphism, whence the assertion. \square

Proposition 5.3. *Let x be in \mathbb{C}^n . Then φ_x is a polynomial map and for some positive integer j , $p_i(x) = 0$ for all positive integer bigger than j .*

Proof. Let Γ_x be as in Subsection 3.2 and Γ'_x the subset of $\mathbb{C} \times \mathbb{C}^n$ such that $\Gamma_x = \{x\} \times \Gamma'_x$. Denote by G the graph of φ_x . By Lemma 5.2, (i) and (ii), G is a closed connected subset of Γ'_x . Let t be in \mathbb{C} and (ω, ψ) an adapted pair to $\varphi_x(t)$. Then $\varphi_x^{-1}(\omega) \times \omega$ is an open subset of $\mathbb{C} \times \mathbb{C}^n$ and its intersection with Γ'_x is the graph of the map

$$\varphi_x^{-1}(\omega) \longrightarrow \omega, \quad s \mapsto \psi(sx).$$

As a result, $\varphi_x^{-1}(\omega) \times \omega \cap \Gamma'_x$ is contained in G since $\varphi_x(s)$ is in ω for all s in $\varphi_x^{-1}(\omega)$. Hence G is an irreducible component of Γ'_x by Lemma 3.3(ii). Moreover, the canonical projection from G to \mathbb{C} is birational and injective. So, by Zariski's Main theorem [Mu88, §9], it is an isomorphism of algebraic varieties. Hence φ_x is a polynomial map. As a result, by Lemma 5.2(ii), for some positive integer j , $p_i(x) = 0$ for $j < i$. \square

5.2. Polynomial automorphism. For i positive integer, set:

$$X_i := \{x \in \mathbb{C}^n \mid 0 = p_i(x) = p_{i+1}(x) = \dots\}.$$

Lemma 5.4. *For some positive integer j , $p_i = 0$ for $i > j$.*

Proof. According to Proposition 5.3, \mathbb{C}^n is the union of X_1, X_2, \dots . By definition, for all i , X_i is a Zariski closed subset. By Baire's property, for some positive integer j , X_j contains a nonempty open subset of \mathbb{C}^n . Hence, $X_j = \mathbb{C}^n$ and $p_i = 0$ for $i \geq j$. \square

Let j be the biggest integer such that $p_j \neq 0$ and set:

$$P := p_1 + \cdots + p_j.$$

Theorem 5.5. *The polynomial endomorphism Φ is a polynomial automorphism and P is its inverse.*

Proof. By Proposition 5.3 and Lemma 5.1(ii), $\varphi_x(1) = P(x)$ for all x in \mathbb{C}^n . So, by Lemma 5.1(i), $\Phi \circ P(x) = x$. As a result, P is an étale morphism since so is Φ . In particular, for some open subset ω of \mathbb{C}^n , the restriction of P to ω is an isomorphism onto the open subset $P(\omega)$ of \mathbb{C}^n . Denoting by ψ the inverse of this isomorphism,

$$\Phi \circ P(x) = x \quad \text{and} \quad P \circ \psi(y) = y$$

for all (x, y) in $\omega \times P(\omega)$, whence

$$\Phi(y) = \psi(y) \quad \text{and} \quad P \circ \Phi(y) = y$$

for all y in $P(\omega)$. Then

$$\Phi \circ P(x) = P \circ \Phi(x) = x$$

for all x in \mathbb{C}^n since P and Φ are polynomial endomorphisms, whence the theorem. □

6. CONCLUSION

By Theorem 5.5 and [Dru83, Theorem 3], Keller's Jacobian Conjecture is true in any dimension:

Theorem 6.1. *Let n be a positive integer. Then any étale polynomial endomorphism of \mathbb{C}^n is a polynomial automorphism.*

From Theorem 6.1 and [Bass 82, Section I], we deduce the following theorem:

Theorem 6.2. *Let \mathbb{k} be an integral ring of characteristic 0 and n a positive integer. Then any polynomial endomorphism of \mathbb{k}^n whose Jacobian matrix has a constant invertible determinant is a polynomial automorphism.*

From Theorem 6.2 and [Ka, Theorem 1], we deduce the following theorem:

Theorem 6.3. *Let \mathbb{k} be a field of characteristic zero and n a positive integer. Denote by $A_{n,\mathbb{k}}$ the n -th Weyl algebra over \mathbb{k} . Then any endomorphism of $A_{n,\mathbb{k}}$ is an automorphism.*

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