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A PSEUDODIFFERENTIAL ANALYTIC PERSPECTIVE ON GETZLER'S RESCALING

GEORGES HABIB AND SYLVIE PAYCHA

ABSTRACT. Inspired by Gilkey's invariance theory, Getzler's rescaling method and Scott's approach to the index via Wodzicki residues, we give a localisation formula for the \mathbb{Z}_2 -graded Wodzicki residue of the logarithm of a class of differential operators acting on sections of a spinor bundle over an even dimensional manifold. This formula is expressed in terms of another local density built from the symbol of the logarithm of a limit of rescaled differential operators acting on differential forms. When applied to complex powers of the square of a Dirac operator, it amounts to expressing the index of a Dirac operator in terms of a local density involving the logarithm of the Getzler rescaled limit of its square.

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INTRODUCTION

On a closed Riemannian manifold (M^n, g) , the algebra $\Psi_{\text{cl}}(M, E)$ of classical pseudodifferential operators acting on the smooth sections of a finite rank vector bundle E over M , admits a unique (up to a multiplicative factor) trace, called the Wodzicki [W] or the noncommutative residue, built from a residue density defined as follows. Given Q in $\Psi_{\text{cl}}(M, E)$, the residue of Q is defined as the integral over M of the residue density $\omega_Q^{\text{Res}}(x) := \text{res}(\sigma(Q)(x, \cdot)) dx^1 \wedge \dots \wedge dx^n$ defined in (4) with

$$\text{res}(\sigma(Q)(x, \cdot)) := \frac{1}{(2\pi)^n} \int_{|\xi|=1} \text{tr}^E(\sigma_{-n}(Q)(x, \xi)) d_S \xi.$$

Here, n is the dimension of M , tr^E stands for the fibrewise trace on $\text{End}(E)$, (x, ξ) is an element in T^*M , and $\sigma_{-n}(Q)(x, \xi)$ is the $(-n)$ -th homogeneous part of the symbol at (x, ξ) . The Wodzicki residue extends beyond classical pseudodifferential operators to the logarithm $\log_\theta Q$ of a pseudodifferential operator Q with Agmon angle θ (see (17)), giving rise to the logarithmic residue $\omega_{\log_\theta Q}^{\text{Res}}(x)$. Here the logarithm $\log_\theta Q$ is defined as the derivative with respect to z of the complex power Q_θ^z [Se] at the point $z = 0$ (see §1.4 for the precise definition) and the Agmon angle θ gives a determination of the logarithm.

The Wodzicki residue is local in so far as it is expressed as the integral on M of a volume form involving the $(-n)$ -homogeneous component of the symbol. So it comes as no surprise that the index of the Dirac operator can be expressed in terms of the residue. For a \mathbb{Z}_2 -graded vector bundle $E = E^+ \oplus E^-$, the index of an elliptic odd operator $\mathcal{D}^+ : \Psi_{\text{cl}}(M, E^+) \rightarrow \Psi_{\text{cl}}(M, E^-)$ with formal adjoint $\mathcal{D}^- = (\mathcal{D}^+)^*$ can be written [Sc]

$$\text{Index}(\mathcal{D}^+) = -\frac{1}{2} \text{sres}(\log_\theta(\mathcal{D}^2)) = -\frac{1}{2} \int_M \omega_{\log_\theta \mathcal{D}^2}^{\text{sRes}}(x) dx,$$

where $\mathcal{D} := \mathcal{D}^+ \oplus \mathcal{D}^-$ so that $\mathcal{D}^2 = \mathcal{D}^- \mathcal{D}^+ + \mathcal{D}^+ \mathcal{D}^-$ and $\theta = \pi$. The graded residue “sres” is defined in the same way as the residue with the fibrewise trace on $\text{End}(E)$ replaced by the \mathbb{Z}_2 -graded trace and $\omega_{\log_\theta(\mathcal{D}^2)}^{\text{sRes}}(x) := \text{sres}(\sigma(\log_\theta(\mathcal{D}^2))(x, \cdot)) dx^1 \wedge \dots \wedge dx^n$.

Inspired by the approach adopted in [Sc], we revisit Getzler’s rescaling in the context of index theory in the light of the logarithmic Wodzicki residue. For a class of differential operators acting on spinors which includes \mathcal{D}^2 , we express the logarithmic residue density evaluated at a point p in M in terms of another local density $\tilde{\omega}_{\log_\theta \mathbb{P}^{\text{lim}}}^{\text{sres}}(x)$ (see formula (1)) involving a limit \mathbb{P}^{lim} as the parameter λ goes to zero of a family of operators $\tilde{\mathbb{P}}_\lambda^{\text{Ge}}$ built from the original one by rescaling it at the point p (see Formula (54)). This new local density is defined on operators acting on differential forms and the rescaling defined in terms of a local diffeomorphism \mathfrak{f}_λ along geodesics (see Formula (6)) composed with a dilation/contraction map U_λ acting on tensors (see Formula (27)), amounts to pulling back the tensor from the manifold to its deformation to the normal cone at the point p (see Formula (6)). In this sense, Formula (3) at the limit as λ tends to zero, can be viewed as a localisation formula of the logarithmic residue at point p .

For this purpose, we single out a class of differential operators acting on smooth sections of a vector bundle E , which we call **geometric** w.r. to a metric g , that send geometric sections of the bundle to geometric sections (Definition 4.10). Stalks of geometric sections at a point p are polynomials in the jets of the vielbeins for the metric g (Definition 4.5). Inspired by Gilkey [G], we define the **Gilkey order** (at a point p) of a geometric section (see (48)) as being the order of those jets. Then a geometric operator is required to transform a geometric section to one whose Gilkey order is obtained by adding the order of the operator to the Gilkey order of the initial section. Geometric differential operators enjoy nice transformation properties under local dilations $(f_\lambda)_{\lambda \in [0,1]}$ along local geodesics defined by means of exponential geodesic normal coordinates (see Equation (6) for the definition). Indeed a geometric differential operator w.r. to g transforms to one w.r. to g_λ (Proposition 4.15), where $g_\lambda = \lambda^{-2} f_\lambda^* g$. This transformed metric can be viewed as the pull-back metric under the canonical projection $\hat{\pi} : \mathbb{M} \rightarrow M$ of the deformed manifold \mathbb{M} via a deformation to the normal cone to p , see (41).

We consider the bundle $E = \Lambda T^*M$. From a differential operator P in $\Psi_{\text{cl}}(M, \Lambda T^*M)$ acting on differential forms, we define a family of operators $\tilde{\mathbb{P}}_\lambda^{\text{Ge}} := \lambda^{\text{ord}(P)} U_\lambda^\# f_\lambda^\# P$ (see (54)) using notations borrowed from [vEY], which are obtained under the combined action of the dilations f_λ mentioned previously and the so-called **Getzler map** U_λ that acts on tensors, see Definition 2.2. We call a geometric differential operator P **rescalable** if $\tilde{\mathbb{P}}_\lambda^{\text{Ge}}$ admits a limit $\tilde{\mathbb{P}}^{\text{lim}}$ when $\lambda \rightarrow 0$ (Definition 5.1). In Proposition 5.6, we give a necessary and sufficient condition for the rescalability of a geometric differential operator in $\Psi_{\text{cl}}(M, \Lambda T^*M)$ and show that the coefficients of the limit $\tilde{\mathbb{P}}^{\text{lim}}$ are polynomial expressions in the jets of the Riemannian curvature tensor.

A first result is the localisation formula (2) for a differential operator P acting on differential forms. It involves a local n -degree form $\tilde{\omega}_Q^{\text{Res}}(x)$, inspired by Scott's proof of the index theorem [Sc, §3.5.3]. This local form is actually defined for more general operators Q in $\Psi_{\text{cl}}(M, \Lambda T^*M)$ (see Section 2 for the definitions), whose homogeneous symbol $\sigma_{-n}(Q)(x, \xi)$ of degree $-n$ at (x, ξ) in T^*M lies in $\text{End}(\Lambda T_x^*M)$ so that $\sigma_{-n}(Q)(x, \xi) \mathbf{1}_x$ lies in ΛT_x^*M . It is defined as (see Equation (31)):

$$(1) \quad \tilde{\omega}_Q^{\text{Res}}(x) := \frac{1}{(2\pi)^n} \int_{S_x^* U_p} [\sigma_{-n}(Q)(x, \xi) \mathbf{1}_x]_{[n]} d_S \xi,$$

where U_p is a local exponential neighborhood of a point p in M and $S_x^* U_p$ is the unit sphere in the cotangent space $T_x^* U_p$ at point x and $\alpha_{[n]}$ is the n -degree part of α . This local form differs from the Wodzicki residue density $\omega_Q^{\text{Res}}(x)$, which involves the fibrewise trace $\text{tr}^E(\sigma_{-n}(Q)(x, \xi))$. In general it does not induce a global form. Its relevance lies in the fact that it does define a global form on a subclass of operators in the range of a Clifford map in so far as there, it is proportional to the Wodzicki residue density (Corollary 2.11). The local form $\tilde{\omega}_Q^{\text{Res}}$ extends to logarithmic pseudodifferential operators $Q = \log_\theta P$ for a differential operator P with Agmon angle θ . In Proposition 2.7 we show that the local n -form $\tilde{\omega}_{\log_\theta P}^{\text{Res}}(\cdot)$ at the point $f_\lambda(x)$ is the local n -form associated to the pull-back operator $U_\lambda^\# f_\lambda^\# (\log_\theta P)$

at the point x . This allows us to show that for small enough positive λ

$$\tilde{\omega}_{\log_\theta(P)}^{\text{Res}}(f_\lambda(x)) = \tilde{\omega}_{\log_\theta(\tilde{\mathbb{P}}_\lambda^{\text{Ge}})}^{\text{Res}}(x), \quad \forall x \in U_p.$$

If moreover P is rescalable, taking the limit as λ tends to zero yields the **localisation formula**

$$(2) \quad \tilde{\omega}_{\log_\theta(P)}^{\text{Res}}(p) = \tilde{\omega}_{\log_\theta(\tilde{\mathbb{P}}^{\text{lim}})}^{\text{Res}}(x), \quad \forall x \in U_p.$$

This formula can be applied to the Hodge Laplacian which is a geometric and rescalable operator (see Example 5.8).

We then consider the case of a spinor module $E = \Sigma M$ when M is a spin manifold of even dimension. To define rescalability of geometric differential operators in $\Psi_{\text{cl}}(M, \Sigma M)$ we use the identification $C\ell(TM) \otimes \mathbb{C} \simeq \text{End}(\Sigma M)$, and the **Clifford map** $c^\theta : C\ell(TM) \rightarrow \text{End}(\Lambda T^*M)$ (see (25)) which sends an element of the Clifford algebra $C\ell(TM)$ on the tangent bundle to an endomorphism of ΛT^*M . We call a geometric differential operator P in $\Psi_{\text{cl}}(M, \Sigma M)$ rescalable if $c^\theta(P)$ is rescalable in $\Psi_{\text{cl}}(M, \Lambda T^*M)$. We then give a necessary and sufficient condition for the rescalability of geometric differential operators in $\Psi_{\text{cl}}(M, \Sigma M)$, see Proposition 5.9. In this part of the work, we consider even dimensional spin manifolds, however our study extends to manifolds with a spin^c structure. Indeed the construction relies on the identification $C\ell(TM) \otimes \mathbb{C} \simeq \text{End}(\Sigma M)$ which can be extended to manifolds with a spin^c structure, in which case we have the identification $C\ell(TM) \otimes \mathbb{C} \simeq \text{End}(E)$ [BHMMM, Thm. 2.13] with E a vector bundle isomorphic to the spinor bundle of the spin^c bundle. For simplicity, we restrict ourselves to spin manifolds.

It follows from Proposition 2.9 that for a differential operator $P \in \Psi_{\text{cl}}(M, \Sigma M)$ with Agmon angle θ , the form $\tilde{\omega}_{\log_\theta c^\theta(P)}^{\text{Res}}$ defines a global density. In Corollary 5.10, we infer from the above localisation formula (2) a second **localisation formula** for operators in $\text{Diff}(M, \Sigma M)$:

$$(3) \quad \omega_{\log_\theta P}^{\text{sRes}}(p) = (-2i)^{n/2} \tilde{\omega}_{\log_\theta(\tilde{\mathbb{P}}^{\text{lim}})}^{\text{Res}}(x),$$

where P is a rescalable geometric differential operator in $\text{Diff}(M, \Sigma M)$ of Agmon angle θ which is even for the \mathbb{Z}_2 -grading $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$. This formula expresses the residue density $\omega_{\log_\theta P}^{\text{sRes}}(p)$ at a point p in terms of a local density $\tilde{\omega}_{\log_\theta(\tilde{\mathbb{P}}^{\text{lim}})}^{\text{Res}}(x)$ of the limit $\tilde{\mathbb{P}}^{\text{lim}}$ of the rescaled operators $\tilde{\mathbb{P}}_\lambda^{\text{Ge}}$.

The localisation formula (3) applied to the square of the Dirac operator (Proposition 6.1), which is proven to be a rescalable geometric differential operator, confirms the results of [Sc, §3.5.3.3] (identification of (3.5.3.12) and (3.5.3.40)). Although the limit operator $\tilde{\mathbb{P}}^{\text{lim}}$ is expected to have a simpler form than the original operator as in the case of the Dirac operator, computing $\tilde{\omega}_{\log_\theta(\tilde{\mathbb{P}}^{\text{lim}})}^{\text{Res}}(x)$ nevertheless remains a challenge since it involves its $(-n)$ -th homogeneous symbol.

1. THE WODZICKI RESIDUE DENSITY FOR CLASSICAL PSEUDODIFFERENTIAL OPERATORS

In this section, we review the definition of the Wodzicki residue for classical pseudodifferential operators acting on sections of a given vector bundle. We recall the

covariance property of the Wodzicki residue under local dilations (see Proposition 1.3). We also recall how the Wodzicki residue extends to logarithms of classical pseudodifferential operators with appropriate spectral properties and refer to this extension as logarithmic Wodzicki residue. Specialising to the trivial vector bundle, we show a localisation formula for the logarithmic residue of scalar differential operators. It identifies the logarithmic residue density at the point p of a differential operator P with the logarithmic residue density at any point x in a small neighborhood of p of the same operator localised at p (see Proposition 1.5).

1.1. The Wodzicki residue for classical pseudodifferential operators. Let (E, π, M) be a vector bundle over M of rank k and let $\Psi_{\text{cl}}(M, E)$ denote the algebra of classical polyhomogeneous pseudodifferential operators acting on the space $C^\infty(M, E)$ of smooth sections of a vector bundle E modelled on a vector space V , based on an n -dimensional smooth manifold M . These are linear maps $Q : C^\infty(M, E) \rightarrow C^\infty(M, E)$, which read $Q = \sum_{i \in I} Q_{U_i} + S_Q$, where given a partition of unity $\chi_i, i \in I$ of M subordinated to a finite covering $(U_i, i \in I)$ of M with open subsets, $Q_{U_i} := \chi_i Q|_{U_i}, i \in I$ are localisations of Q in open subsets U_i of M , and S_Q is a smoothing operator—it maps any Sobolev section to a smooth section. Since we are interested in singular linear forms which vanish on smoothing operators, we reduce our study to localised operators Q_U . To simplify notations, we drop the explicit mention of the localisation and simply write Q . A pseudodifferential operator (localised on some open subset U of M) acting on $C^\infty(U, E)$ is called **classical** or **polyhomogeneous** if it is a linear combination of pseudodifferential operators Q whose (local) symbol $\sigma(Q)$ —which lies in $C^\infty(T^*U \setminus U \times \{0\}, \text{End}(V))$, in any local trivialisation of E over U —has a polyhomogeneous expansion of the form

$$\sigma(Q) \sim \sum_{j=0}^{\infty} \sigma_{m-j}(Q),$$

with m in \mathbb{C} , the order of Q . Explicitly, for any N in \mathbb{N} , the difference $\sigma(Q) - \sum_{j=0}^N \chi \sigma_{m-j}(Q)$ is a smooth pseudodifferential symbol of order $\leq \Re(m) - N$, with χ a smooth function which vanishes in a neighborhood of zero, and $\sigma_\alpha(Q)$ positively homogeneous of degree $\alpha \in \mathbb{C}$, that is,

$$\sigma_\alpha(Q)(x, \lambda \xi) = \lambda^\alpha \sigma_\alpha(Q)(x, \xi)$$

for any $(x, \xi) \in TU^* \setminus U \times \{0\}$ and $\lambda > 0$. For further details, we refer to classical books on the subject such as [Sh], see also [Sc, Example 1.1.8]. We also consider the class of logarithmic pseudodifferential operators, namely those whose symbols have a log polyhomogeneous expansion of the form

$$\sigma(Q)(x, \xi) = m \log(|\xi|) \text{Id} + \sigma_0(Q)(x, \xi),$$

where $\sigma_0(Q)$ is a classical symbol of nonpositive order. We define the local **residue density**¹

¹ s -densities on an n -dimensional real vector space V are functions $\mu : V^n \rightarrow \mathbb{R}$ such that $\mu(Av_1, Av_2, \dots, Av_n) = |\det(A)|^s \mu(v_1, \dots, v_n)$ for any linear isomorphism A of V and form a one dimensional vector space $|\Lambda|^s(V)$. An s -density on a manifold M is a section of the s -density bundle $|\Lambda|^s(TM)$ over M whose fibre over x consists of s -densities on the tangent space $T_x M$. On an n -dimensional manifold M , 1-densities, also called densities, can be canonically identified with the n -forms on M .

$$(4) \quad \omega_Q^{\text{Res}}(x) := \text{res}(\sigma(Q)(x, \cdot)) dx^1 \wedge \dots \wedge dx^n$$

where $dx^1 \wedge \dots \wedge dx^n$ is the flat volume form in local coordinates on the (oriented) n -dimensional manifold M and

$$\text{res}(\sigma(Q)(x, \cdot)) := \frac{1}{(2\pi)^n} \int_{|\xi|=1} \text{tr}^E(\sigma_{-n}(Q)(x, \xi)) d_S \xi,$$

where tr^E stands for the fibrewise trace on $\text{End}(E)$, $d_S \xi$ for the standard density on the unit sphere \mathbb{S}^{n-1} obtained as the interior product of the flat volume form $d\xi^1 \wedge \dots \wedge \widehat{d\xi^j} \wedge \dots \wedge d\xi^n$ by the radial vector field $\mathcal{R} := \sum_{i=1}^n \xi^i \frac{\partial}{\partial \xi^i}$

$$d_S \xi := \mathcal{R} \lrcorner (d\xi^1 \wedge \dots \wedge \dots \wedge d\xi^n) = \sum_{j=1}^{n-1} (-1)^{n-1} \xi^j d\xi^1 \wedge \dots \wedge \widehat{d\xi^j} \wedge \dots \wedge d\xi^n.$$

A priori, $\omega_Q^{\text{Res}}(x)$, which is defined using a localisation of the operator Q around x , depends on the choice of local coordinates in a neighborhood of x . M. Wodzicki [W] showed that it actually defines a global n -form, which can be integrated to define the linear form Res on $\Psi_{\text{cl}}(M, E)$, called the Wodzicki or noncommutative residue:

$$\text{Res}(Q) := \int_M \omega_Q^{\text{Res}}(x).$$

Remark 1.1. (1) If (M^n, g) is a Riemannian manifold, we can equivalently define

$$\text{res}(\sigma(Q)(x, \cdot)) := \frac{1}{(2\pi)^n} \int_{S_x^* M} \text{tr}^E(\sigma_{-n}(Q)(x, \xi)) \nu_x(\xi),$$

as an integral over the cotangent unit sphere $S_x^* M := \{\xi \in T_x^* M, |\xi| = 1\}$ endowed with the induced Riemannian volume form ν_x . As before, n stands for the dimension of M .

(2) The Wodzicki residue easily extends to a \mathbb{Z}_2 -graded vector bundle $E = E^+ \oplus E^-$ replacing the fibrewise trace tr^E by a graded trace $\text{str}^E := \text{tr}^{E^+} - \text{tr}^{E^-}$, in which case we set

$$\text{sres}(\sigma(Q)(x, \cdot)) := \frac{1}{(2\pi)^n} \int_{|\xi|=1} \text{str}^E(\sigma_{-n}(Q)(x, \xi)) d_S \xi,$$

and

$$\omega_Q^{\text{sRes}}(x) := \text{sres}(\sigma(Q)(x, \cdot)) dx^1 \wedge \dots \wedge dx^n.$$

1.2. Local dilations. Throughout the paper, (M^n, g) denotes a smooth Riemannian manifold and p a point in M . The local identification uses the exponential map

$$(5) \quad \exp_p : T_p M \supset B_r \longrightarrow U_p \subset M$$

around p which yields a local diffeomorphism from a ball B_r of radius $r > 0$ centered at 0 to a local geodesic neighborhood U_p of p . This exponential map is combined with a rescaling leading to the map (This is the map $\exp \circ T_\epsilon$ in [F, Formula (4.4.7)])

$$\exp_p \circ h_\lambda : T_p M \supset B_{r/\lambda} \longrightarrow U_p \subset M,$$

where

$$\begin{aligned} h_\lambda : B_{r/\lambda} \subset T_p M &\longrightarrow B_r \subset T_p M \\ \mathbf{x} &\longmapsto \lambda \mathbf{x}. \end{aligned}$$

For $1 > \lambda > 0$ and $r > 0$ we set $U_p^\lambda := \exp_p(B_{\frac{r}{|\lambda|}})$, so that $U_p \subset U_p^\lambda$ and

$$V_p^\lambda := \exp_p(B_{\frac{r}{|\lambda|}}) \times \{\lambda\} \subset M^\lambda := M \times \{\lambda\} \simeq M.$$

The map (5) induces local transformations

$$(6) \quad \begin{aligned} \mathfrak{f}_{p,\lambda} : U_p^\lambda &\longrightarrow U_p \\ \exp_p(\mathbf{x}) &\longmapsto \exp_p(\lambda \mathbf{x}) \end{aligned}$$

which we shall denote by \mathfrak{f}_λ to simplify notations. As the map $\lambda \rightarrow \exp_p(\lambda \mathbf{x})$ is continuous, we define $\mathfrak{f}_0 = \lim_{\lambda \rightarrow 0} \mathfrak{f}_\lambda|_{U_p}$. Hence, $\mathfrak{f}_0(\exp_p(\mathbf{x})) = p$ for any $\mathbf{x} \in B_r$.

Throughout the paper, we use the following notations. From a given orthonormal basis $e_1(p), \dots, e_n(p)$ of $T_p M$ at $p \in M$, we build:

- a local orthonormal frame

$$(7) \quad O_p(x, g) := \{e_1(x, g), \dots, e_n(x, g)\}, x \in U_p$$

of $T_x M$ by the parallel transport $\tau_c : T_p M \rightarrow T_x M$ along the geodesic $c(t) = \exp_x(t \mathbf{x})$, with \mathbf{x} in $B_r \subset T_p M$, which takes p to $x = c(1)$ so that $e_j(x, g) = \tau_c(e_j(p))$;

- normal geodesic coordinates at any point $x \in U_p$, as follows: Let \tilde{B}_r be the open ball in \mathbb{R}^n of radius r centered at 0 and consider the map $\tilde{B}_r \subset \mathbb{R}^n \rightarrow U_p$; $(x^1, \dots, x^n) \mapsto x$, which is defined via the local exponential map \exp_p in (5) by

$$(8) \quad x = \exp_p \left(\sum_{i=1}^n x^i e_i(p) \right) \in \exp_p(B_r).$$

Unless specified otherwise, throughout the paper, we use normal geodesic coordinates. As usual, we identify any point $x \in U_p$ with its coordinates $X := (x^1, \dots, x^n)$.

Composing the map $\mathfrak{f}_\lambda^{-1}$ with the normal geodesic coordinates $\tilde{B}_r \subset \mathbb{R}^n \rightarrow U_p$ gives rise to a new coordinate system on U_p^λ

$$\tilde{B}_r \subset \mathbb{R}^n \rightarrow U_p^\lambda; (y^1, \dots, y^n) \mapsto \exp_p \left(\sum_{i=1}^n \lambda^{-1} y^i e_i(p) \right).$$

Since $U_p \subset U_p^\lambda$ for any $\lambda > 0$ small enough, there are two coordinate systems $X := (x^1, \dots, x^n)$ and

$$Y := (y^1 := \lambda x^1, \dots, y^n := \lambda x^n)$$

at any point in U_p . A direct computation shows that the coordinate system Y is indeed the normal geodesic coordinates associated to the metric $\mathfrak{f}_\lambda^* g$ on U_p^λ since $(\mathfrak{f}_\lambda^* g)_p = \lambda^2 g|_p$. Hence, for any $\lambda > 0$ small enough, we have

$$(9) \quad \mathfrak{f}_\lambda^* \left(\frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial y^i} = \lambda^{-1} \frac{\partial}{\partial x^i} \circ \mathfrak{f}_\lambda \quad \text{and} \quad \mathfrak{f}_\lambda^* dx^i = dy^i = \lambda dx^i \circ \mathfrak{f}_\lambda.$$

In the following, we often identify a point y in U_p^λ with its coordinates $Y = (y^1, \dots, y^n)$.

1.3. The behaviour of the Wodzicki residue under local dilations. Let us now recall the general fact on pull-back of operators. Any local diffeomorphism $f : U \rightarrow V$ induces a local transformation on a localised pseudodifferential operators as follows: Given any Q in $\Psi_{\text{cl}}(V, E)$ where (E, π, M) is a vector bundle over M , we define $f^\sharp Q \in \Psi_{\text{cl}}(U, f^*E)$ by

$$(10) \quad (f^\sharp Q) s := f^* \left(Q \left((f^*)^{-1}(s) \right) \right) = Q (s \circ f^{-1}) \circ f,$$

where s is any local section in f^*E above U . Here, f^*E is the pull-back bundle over U of the bundle E given by

$$f^*E = \{(x, y) \in U \times E \mid f(x) = \pi(y)\}.$$

The following lemma is an easy consequence of the transformation property of symbols under the local diffeomorphism f_λ defined in (6). We nevertheless provide an explicit proof.

Lemma 1.2. *Given any Q in $\Psi_{\text{cl}}(M, E)$, we have for small enough positive λ ,*

$$(11) \quad \sigma(f_\lambda^\sharp Q)(x, \xi) = \sigma(Q) (f_\lambda(x), (f_\lambda^*)^t(\xi)),$$

at any given point x in $U_p \subset U_p^\lambda$.

Proof. Under a local diffeomorphism f , the symbol $\sigma(Q)$ of Q transforms as [Sc, Equation (3.5.3.31)]

$$(12) \quad \sigma(f^\sharp Q)(x, \xi) = \sigma(Q) (f(x), (f^*)^t(\xi)) + \text{lower order terms}.$$

Here ‘‘lower order terms’’ stands for the push forward by f of the sum $\sum_{|\alpha|>0} \frac{1}{\alpha!} \Phi_\alpha(x, \xi) \partial_\xi^\alpha \sigma(x, \xi)$ with

$$\Phi_\alpha(x, \xi) := D_z^\alpha e^{i(\varphi_x^f(z), \xi)}|_{z=x}, \quad \text{and} \quad \varphi_x^f(z) := f(z) - f(x) - df(x)(z - x),$$

which is a polynomial in ξ of degree $\leq \frac{|\alpha|}{2}$ whose coefficients are linear combinations of products of derivatives $\prod_\gamma \partial^\gamma f(x)$ of f at x with $\sum |\gamma| = |\alpha|$ and $|\gamma| \geq 2$. In the case of $f := f_\lambda$, we have $\partial_i \partial_j f \equiv 0$ for any indices i, j running from 1 to n , so that the lower order terms vanish leading to (11). \square

Proposition 1.3. *Let E be a vector bundle over M . For any given Q in $\Psi_{\text{cl}}(M, E)$, the n -form ω_Q^{Res} transforms covariantly under dilations f_λ as*

$$\omega_Q^{\text{Res}} \circ f_\lambda = \omega_{f_\lambda^\sharp Q}^{\text{Res}}.$$

Proof. Applying the local residue density (4) at the point $f_\lambda(x)$, we have

$$\begin{aligned}
(2\pi)^n \omega_Q^{\text{Res}}(f_\lambda(x)) &= \left(\int_{|\xi|=1} \text{tr}^E(\sigma_{-n}(Q)(f_\lambda(x), \xi)) d_S \xi \right) dy^1 \wedge \dots \wedge dy^n \\
&= \left(\int_{|\xi|=1} \text{tr}^E(\sigma_{-n}(Q)(f_\lambda(x), \lambda^{-1} \xi)) d_S \xi \right) dx^1 \wedge \dots \wedge dx^n \\
&= \left(\int_{|\xi|=1} \text{tr}^E(\sigma_{-n}(Q)(f_\lambda(x), (f_\lambda^*)^t(\xi))) d_S \xi \right) dx^1 \wedge \dots \wedge dx^n \\
&\stackrel{(11)}{=} (2\pi)^n \omega_{f_\lambda^*(Q)}^{\text{Res}}(x).
\end{aligned}$$

This finishes the proof of the lemma. \square

1.4. The Wodzicki residue for logarithmic pseudodifferential operators.

For later purposes, we review here how the Wodzicki residue can be extended to a class of logarithmic pseudodifferential operators, a notion which we first briefly recall (we refer the reader to [Sc, §2.6.1.3] for further details). As before, we consider a vector bundle E over M of rank k . We say that an operator Q in $\Psi_{\text{cl}}(M, E)$ of order m has a principal angle $\theta \in [0, 2\pi)$ (see [Sc, §1.5.7.1]) if there is a conical neighbourhood of the ray $L_\theta := \{r e^{i\theta}, r \geq 0\}$ such that its leading symbol matrix $\sigma_L(Q)(x, \xi) := \sigma_m(Q)(x, \xi)$ has no eigenvalues in this neighbourhood for every $(x, \xi) \in T^*U \setminus U \times \{0\}$. In particular, the operator is elliptic and, therefore, has a purely discrete spectrum. A principal angle θ of an operator Q is said to be an Agmon angle if there are no eigenvalues of Q on the ray L_θ . In this case, Q is invertible and there exists a solid angle of the ray

$$\Lambda_{\epsilon, \theta} = \{r e^{i\alpha}, r \geq 0, \theta - \epsilon \leq \alpha \leq \theta + \epsilon\},$$

for some $\epsilon > 0$, that contains no eigenvalue of Q . Now, for any operator Q with Agmon angle θ and for a holomorphic function $\varphi_\theta : \mathbb{C} \setminus L_\theta \rightarrow \mathbb{C}$, we define the Cauchy integral

$$\varphi_\theta(Q) := \frac{i}{2\pi} \int_{\Gamma_\theta} \varphi_\theta(\lambda) (Q - \lambda)^{-1} d\lambda,$$

where Γ_θ is a closed contour along the ray L_θ around the spectrum of Q . For example, this contour can be taken as follows: Fix $0 < \delta < R$, we define

$$\Gamma_\theta = \Gamma_\theta^1 \cup \Gamma_\theta^2 \cup \Gamma_\theta^3 \cup \Gamma_\theta^4,$$

where

$$\Gamma_\theta^1 = \{r e^{i\theta}, \delta \leq r \leq R\}, \quad \Gamma_\theta^2 = \{r e^{i(\theta-2\pi)}, \delta \leq r \leq R\}$$

and

$$\Gamma_\theta^3 = \{\delta e^{it}, \theta - 2\pi \leq t \leq \theta\}, \quad \Gamma_\theta^4 = \{R e^{it}, \theta - 2\pi \leq t \leq \theta\}.$$

One can easily check that $Q\varphi_\theta(Q) = \varphi_\theta(Q)Q$. The symbol of $\varphi_\theta(Q)$ reads as

$$(13) \quad \sigma(\varphi_\theta(Q)) \sim \frac{i}{2\pi} \int_{\Gamma_\theta} \varphi_\theta(\lambda) (\sigma(Q) - \lambda)^{* -1} d\lambda,$$

where $\sigma(Q)$ is the symbol of Q , the star in $^{*-1}$ stands for the inverse in the symbol algebra and

$$\begin{aligned} (\sigma(Q) - \lambda)^{*-1} &= \sigma_L(Q)^{*-1} (\sigma(Q)\sigma_L(Q)^{*-1} - \lambda\sigma_L(Q)^{*-1})^{*-1} \\ &= \sigma_L(Q)^{-1} (1 + \text{symbol of order } < 0)^{*-1} \end{aligned}$$

is obtained by means of an expansion in ξ . Given a local diffeomorphism $\mathfrak{f} : U \rightarrow V$ and an operator $Q \in \Psi_{\text{cl}}(M, E)$ with Agmon angle θ and order m , the operator $\mathfrak{f}^\#Q$ defined in (10) is in $\Psi_{\text{cl}}(M, \mathfrak{f}^*E)$ with the same Agmon angle θ , since from Equation (12), we have that

$$\sigma_L(\mathfrak{f}^\#Q)(x, \xi) = \sigma_L(Q) (\mathfrak{f}(x), (\mathfrak{f}^*)^t(\xi)).$$

Also the relation $\mathfrak{f}^\#(Q - \lambda)^{-1} = (\mathfrak{f}^\#Q - \lambda)^{-1}$ gives that $\mathfrak{f}^\#(\varphi_\theta(Q)) = \varphi_\theta(\mathfrak{f}^\#Q)$. Now, we consider the particular case where $\varphi_\theta(\lambda) = \lambda_\theta^z = |\lambda|^z e^{iz\arg(\lambda)}$ for $\Re(z) < 0$ and $\theta \leq \arg(\lambda) < \theta + 2\pi$ and define the complex power

$$Q_\theta^z = \varphi_\theta(Q) = \frac{i}{2\pi} \int_{\Gamma_\theta} \lambda_\theta^z (Q - \lambda)^{-1} d\lambda.$$

In this case, the symbol of Q_θ^z has the expansion

$$\sigma(Q_\theta^z)(x, \xi) \sim \sum_{j=0}^{\infty} \sigma_{mz-j}(Q_\theta^z)(x, \xi),$$

where $\sigma_{mz-j}(Q_\theta^z)(x, \xi)$ are the positive homogenous of degree $mz - j$ given by [Sh, §11.1]

$$(14) \quad \sigma_{mz-j}(Q_\theta^z)(x, \xi) = \frac{i}{2\pi} \int_{\Gamma_\theta} \lambda_\theta^z \sigma_{-m-j}((Q - \lambda)^{-1})(x, \xi) d\lambda.$$

The map $z \mapsto Q_\theta^z$ is holomorphic on the complex plane with values in the space $B(H^s(M, E), H^{s-m\Re(z)}(M, E))$ for all $s \in \mathbb{R}$. Therefore, we define the logarithm of Q as:

$$\begin{aligned} \log_\theta Q &:= (\partial_z Q_\theta^z)_{z=0} \\ &= \frac{i}{2\pi} \left(\partial_z \int_{\Gamma_\theta} \lambda_\theta^z (Q - \lambda)^{-1} d\lambda \right)_{z=0} \\ &= \frac{i}{2\pi} \int_{\Gamma_\theta} \log_\theta \lambda (Q - \lambda)^{-1} d\lambda. \end{aligned}$$

By construction, one clearly has that $(\log_\theta Q) Q_\theta^z = Q_\theta^z (\log_\theta Q)$. We also have that $\mathfrak{f}^\#(\log_\theta Q) = \log_\theta(\mathfrak{f}^\#Q)$, since $\mathfrak{f}^\#$ and Q_θ^z both commute. Note that the logarithm $\log_\theta Q$ of a classical pseudodifferential operator is not classical, it is “nearly” classical in so far as its local symbol differs from a classical symbol by a logarithm term. Indeed, it is shown in [Sc, p. 219] that the symbol of the logarithm reads as

$$(15) \quad \sigma(\log_\theta Q)(x, \xi) = m \log(|\xi|) \text{Id} + \sigma_{\text{cl}}(\log_\theta Q)(x, \xi),$$

where $\sigma_{\text{cl}}(\log_\theta Q)$ is a classical symbol of order zero with homogeneous components $\sigma_{-j}(\log_\theta Q)$ of degree $-j$, $j \geq 0$ given by

$$(16) \quad \sigma_{-j}(\log_\theta Q)(x, \xi) = |\xi|^{-j} \left(\partial_z \left(\sigma_{mz-j}(Q_\theta^z)(x, \frac{\xi}{|\xi|}) \right) \right)_{z=0}.$$

The fact that the logarithmic part of the symbol vanishes on the cotangent unit sphere underlies the extendibility of the Wodzicki residue to logarithmic pseudodifferential operators (for a detailed discussion, we refer the reader to [Sc, §2.7.1]). We shall henceforth apply without any further ado, the above constructions to $\log_\theta Q$ for an operator Q in $\Psi_{\text{cl}}(M, E)$ with Agmon angle θ . In analogy with (4), we set

$$\begin{aligned} \text{res}(\sigma(\log_\theta Q)(x, \cdot)) &:= \frac{1}{(2\pi)^n} \int_{S_x^* M} \text{tr}^E(\sigma_{-n}(\log_\theta Q)(x, \xi)) d_S \xi, \\ (17) \quad \omega_{\log_\theta Q}^{\text{Res}}(x) &:= \text{res}(\sigma(\log_\theta Q)(x, \cdot)) dx^1 \wedge \dots \wedge dx^n, \end{aligned}$$

which we call the **logarithmic residue density of Q** . On the grounds of (15), $\sigma_{-n}(\log_\theta Q) = (\sigma_{\text{cl}})_{-n}(\log_\theta Q)$ so that one can easily adapt the proof of Proposition 1.3 to show the covariance of the logarithmic residue:

$$(18) \quad \omega_{\log_\theta Q}^{\text{Res}} \circ \mathfrak{f}_\lambda = \omega_{\log_\theta \mathfrak{f}_\lambda^\# Q}^{\text{Res}}$$

where we have used the fact that $\mathfrak{f}_\lambda^\#$ and \log_θ commute.

Remark 1.4. We shall drop the explicit mention of the principal angle when we can choose $\theta = \pi$.

1.5. A localisation formula for the logarithmic residue density of scalar differential operators. We now focus on logarithms of scalar differential operators, for which we prove a localisation formula for the Wodzicki residue density. In the sequel, we use the following notations. For any multiindex $\gamma = \{i_1, \dots, i_s\}$, we set

$$(19) \quad D_X^\gamma := \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_s}},$$

in the local normal geodesic coordinates $X = (x^1, \dots, x^n)$ at point x with the usual identification $x \leftrightarrow X$.

To simplify notations, unless this gives rise to an ambiguity, we henceforth write D^γ instead of D_X^γ .

Given a vector bundle $E \rightarrow M$ of rank k , trivialised over an open subset U of M , D^γ acts on a local section $s|_U = \sum_{i=1}^k \alpha_i s_i|_U$ on U by

$$(20) \quad D^\gamma s := \sum_{i=1}^k D^\gamma(\alpha_i) s_i.$$

Here $\{s_i\}_{i=1, \dots, k}$ is a basis of the bundle $E|_U$ in the local trivialisation $E|_U \simeq U \times \mathbb{R}^k$. A differential operator of order $m \in \mathbb{Z}_{\geq 0}$ reads $P = \sum_{|\gamma| \leq m} P_\gamma D^\gamma$, which means that in the local trivialisation $E|_U \simeq U \times \mathbb{R}^k$ of E , it acts as

$$(21) \quad P \left(\sum_{j=1}^k \alpha_j s_j \right) = \sum_{|\gamma| \leq m} \sum_{i,j=1}^k (P_\gamma)_{ij} D^\gamma(\alpha_j) s_i,$$

where we have used Equation (20). Differential operators form an algebra $\text{Diff}(M, E)$ and we have the following isomorphism of C^∞ -modules:

$$\text{Diff}(M, E) \simeq \text{Diff}(M) \otimes_{C^\infty(M)} C^\infty(M, \text{End}(E)),$$

where we have set $\text{Diff}(M) := \text{Diff}(M, M \times \mathbb{R})$. Following [vEY], we define a family of rescaled differential operators for any $P \in \text{Diff}(M, M \times \mathbb{R})$ by

$$(22) \quad \mathbb{P} := \lambda^m P, \quad \lambda > 0,$$

and set for any small positive λ :

$$(23) \quad \tilde{\mathbb{P}}_\lambda := \lambda^m \mathfrak{f}_\lambda^\sharp P.$$

In local normal geodesic coordinates, we have $\mathfrak{f}_\lambda^\sharp D^\gamma = \lambda^{-|\gamma|} D^\gamma$ so that the family of rescaled operators built from a differential operator $P = \sum_{|\gamma| \leq m} P_\gamma D^\gamma$, locally reads (these and the above notations \mathbb{P}_λ are borrowed from [vEY])

$$\tilde{\mathbb{P}}_\lambda = \sum_{\gamma} \lambda^{m-|\gamma|} (P_\gamma \circ \mathfrak{f}_\lambda) D^\gamma.$$

As λ tends to zero, $\tilde{\mathbb{P}}_\lambda$ converges to the operator P evaluated at the limit point p :

$$(24) \quad \lim_{\lambda \rightarrow 0} \tilde{\mathbb{P}}_\lambda|_{U_p} = \sum_{|\gamma|=m} P_\gamma(p) D^\gamma|_p = P|_p,$$

where $P_\gamma(p)$ corresponds to $P_\gamma(x)$ evaluated at the reference point p . In the following, we state a localisation formula for the residue of the logarithm of a differential operator (see [Sc, (3.5.3.33) p.382] for a similar formula)

Proposition 1.5. *For any differential operator P in $\text{Diff}(M, M \times \mathbb{R})$ with Agmon angle θ , we have the following localisation formula:*

$$\omega_{\log_\theta P}^{\text{Res}}(p) = \omega_{\log_\theta P|_p}^{\text{Res}}(x),$$

for all $x \in U_p$.

Proof. We first observe that for small positive λ

$$\log_\theta \tilde{\mathbb{P}}_\lambda = \log_\theta(\lambda^m \mathfrak{f}_\lambda^\sharp P) = (m \log \lambda) \text{Id} + \log_\theta \left(\mathfrak{f}_\lambda^\sharp P \right).$$

Since the residue density vanishes on differential operators and hence on Id, we have $\omega_{\log_\theta \tilde{\mathbb{P}}_\lambda}^{\text{Res}} = \omega_{\log_\theta (\mathfrak{f}_\lambda^\sharp P)}^{\text{Res}}$. Equation (18) implies that $\omega_{\log_\theta \tilde{\mathbb{P}}_\lambda}^{\text{Res}} = \omega_{\log_\theta P}^{\text{Res}} \circ \mathfrak{f}_\lambda$. We then take the limit as $\lambda \rightarrow 0$, by which $\tilde{\mathbb{P}}_\lambda$ tends to $P|_p$ by (24). The continuity of the logarithm combined with the continuity of the Wodzicki residue for the Fréchet topology of (log-)classical operators of constant order then yields the statement of the proposition. \square

2. A LOCAL BEREZIN TYPE n -FORM ON $\Psi_{\text{cl}}(M, \Lambda T^*M)$

In this section, we define a local n -form $\tilde{\omega}^{\text{Res}}$ on $\Psi_{\text{cl}}(M, \Lambda T^*M)$ (see Eq. (31)), which unlike the Wodzicki density, is not covariant under dilations defined in the previous section. We give in Proposition 2.7 the behaviour of this local n -form $\tilde{\omega}^{\text{Res}}$ under Getzler rescaling map (see Definition 2.2) combined with the local dilations. When the manifold M is spin and for a differential operator P acting on smooth sections of its spinor bundle, we use the expression of the super trace in terms of a Berezin integral (see (26)) to relate the local n -form $\tilde{\omega}_{\log_\theta c^g(P)}^{\text{Res}}$ of the logarithm (with spectral cut θ) of $c^g(P)$ (defined in Equation (37)) to its (super-)Wodzicki residue $\omega_{\log_\theta c^g(P)}^{\text{sRes}}$. Much of this section is inspired from Simon Scott's approach

to the local Atiyah-Singer index theorem by means of the Wodzicki residue [Sc, §3.5.3].

2.1. Prerequisites: The supertrace versus the Berezin integral and the Getzler rescaling. In the sequel, following [BGV, Chap. 3] we review the construction and properties of the Berezin integral together with its relation with the supertrace as well as Getzler's rescaling on differential forms. Let V be a real vector space. A linear map $T : \Lambda V \rightarrow \mathbb{R}$ is called a Berezin integral if T vanishes on $\Lambda^k V$ for $k < n = \dim(V)$. If V is an oriented Euclidean vector space equipped with a scalar product g , there exist a canonical Berezin integral defined as the projection of any element in ΛV onto its component on the n -form $e^1 \wedge \dots \wedge e^n$ where $\{e_1, \dots, e_n\}$ is an orthonormal basis of (V, g) . We will denote this Berezin integral by T :

$$T(e^1 \wedge \dots \wedge e^n) = 1, \quad T(e^I) = 0, \quad \text{if } |I| < n.$$

Let us now consider the Clifford algebra $\text{Cl}(V)$ of (V, g) . The isometry $-\text{Id} \in O(V, g)$ gives rise to the map

$$\begin{aligned} \Phi : \text{Cl}(V) &\longrightarrow \text{Cl}(V) \\ e_{i_1} \cdot_g \dots \cdot_g e_{i_k} &\longmapsto (-1)^k e_{i_1} \cdot_g \dots \cdot_g e_{i_k} \end{aligned}$$

with $i_1 < \dots < i_k$. Here " \cdot_g " denotes the Clifford multiplication with respect to the metric g . The map Φ clearly satisfies $\Phi^2 = \text{Id}$. Therefore, we get a splitting of $\text{Cl}(V)$ into

$$\text{Cl}(V) = \text{Cl}(V)^+ \oplus \text{Cl}(V)^-,$$

where $\text{Cl}(V)^\pm := \{a \in \text{Cl}(V) \mid \Phi(a) = \pm a\}$. Now we have the proposition [BGV, Prop. 3.19]

Proposition 2.1. *Let V be an oriented Euclidean vector space of even dimension n . There exists a unique \mathbb{Z}_2 -graded Clifford module $S = S^+ \oplus S^-$, such that*

$$\text{Cl}(V) \otimes \mathbb{C} \simeq \text{End}(S).$$

In particular, $\dim S^\pm = 2^{\frac{n}{2}-1}$. Also, we have that $\text{Cl}(V)^+ \cdot S^\pm \subset S^\pm$ and $\text{Cl}(V)^- \cdot S^\pm \subset S^\mp$. Therefore, we have the isomorphism

$$\text{Cl}(V)^+ \otimes \mathbb{C} \simeq \text{End}(S^\pm).$$

Notice that S^\pm are defined as the eigenspaces of S associated with the eigenvalues ± 1 of the complex volume form $\omega_{\mathbb{C}} = i^{\frac{n}{2}} e_1 \cdot_g \dots \cdot_g e_n$ with n even. Now, there is a natural super trace on $\text{Cl}(V) \otimes \mathbb{C}$, defined by

$$\text{str}(a) := \begin{cases} \text{tr}^{S^+}(a) - \text{tr}^{S^-}(a), & \text{if } a \in \text{Cl}(V)^+, \\ 0, & \text{if } a \in \text{Cl}(V)^-, \end{cases}$$

where, as before, tr^E stands for the fibrewise trace on $\text{End}(E)$. In order to relate this supertrace with the Berezin integral, we assign to any vector $v \in V^*$ the endomorphism $c^g(v) \in \text{End}(\Lambda V)$ that uniquely extends to a morphism of algebra bundles

$$(25) \quad c^g : \text{Cl}(V) \longrightarrow \text{End}(\Lambda V)$$

defined by

$$c^g(v) \bullet = v \wedge \bullet - v^{\sharp_g} \lrcorner \bullet,$$

where v^{\sharp_g} is the vector in V associated to v by the musical isomorphism. The symbol map $\mathbf{s}^g : \text{Cl}(V) \rightarrow \Lambda V$ is the isomorphism given by $\mathbf{s}^g(a) := c^g(a)1$. Indeed,

to show \mathbf{s}^g (as well as c^g) is injective, we assume that $\mathbf{s}^g(a) = 0$ for some $a = \sum_{i_1 < \dots < i_k} a_{i_1 \dots i_k} e_{i_1} \cdot g \dots \cdot g e_{i_k}$. Then $c^g(a)1 = \sum_{i_1 < \dots < i_k} a_{i_1 \dots i_k} e_{i_1} \wedge \dots \wedge e_{i_k} = 0$. Hence, $a_{i_1 \dots i_k} = 0$ and, thus, $a = 0$. The bijectivity of \mathbf{s}^g comes from the equality of the dimensions. In [BGV, Prop. 3.21], it is shown that there is a unique supertrace which is related to the Berezin integral by the following

$$(26) \quad \text{str}(a) = (-2i)^{\frac{r}{2}} (T \circ \mathbf{s}^g)(a).$$

for any $a \in \text{Cl}(V) \otimes \mathbb{C}$.

Definition 2.2. *The Getzler rescaling map is the tensor bundle morphism defined for any $\lambda > 0$, by*

$$(27) \quad \begin{aligned} U_\lambda : \otimes_r^q V &\longrightarrow \otimes_r^q V \\ t &\longmapsto \lambda^{q-r} t. \end{aligned}$$

The Getzler rescaling map U_λ restricted to ΛV induces a map

$$(28) \quad \begin{aligned} U_\lambda^\sharp : \text{End}(\Lambda V) &\longrightarrow \text{End}(\Lambda V) \\ Q &\longmapsto U_\lambda^\sharp Q : \omega \mapsto U_\lambda Q U_\lambda^{-1} \omega, \end{aligned}$$

which satisfies:

$$(29) \quad U_\lambda^\sharp(v \wedge \bullet) = \lambda^{-1} v \wedge \bullet \quad \text{and} \quad U_\lambda^\sharp(v \lrcorner \bullet) = \lambda v \lrcorner \bullet.$$

for any $v \in V^*$. Combining (29) with $c^g(v) \bullet = v \wedge \bullet - v \lrcorner \bullet$ yields the map

$$U_\lambda^\sharp \circ c^g : \text{Cl}(V) \longrightarrow \text{End}(\Lambda V)$$

given by

$$\left(U_\lambda^\sharp \circ c^g \right) (v) \bullet = \lambda^{-1} v \wedge \bullet - \lambda v \lrcorner \bullet.$$

We have the following straightforward lemma that we will use later

Lemma 2.3. *Let $e^I := e^{i_1} \cdot g e^{i_2} \dots \cdot g e^{i_k}$ for $i_1 < i_2 < \dots < i_k$ with $|I| = i_1 + \dots + i_k$, it follows that*

$$(30) \quad \lim_{\lambda \rightarrow 0} \lambda^{|I|} \left(U_\lambda^\sharp \circ c^g \right) (e^I) = e^I \wedge$$

where $e^I := e^{i_1} \wedge \dots \wedge e^{i_k}$.

2.2. A local n -form on $\Psi_{\text{cl}}(M, \Lambda T^* M)$ and Getzler rescaling. In order to define the local n -form, we fix a normal geodesic neighborhood U_p around a point p in M and for $(x, \xi) \in T^* U_p$ and consider the symbol $\sigma(Q)(x, \xi) \in \text{End}(\Lambda T_x^* U_p)$ of an operator Q in $\Psi_{\text{cl}}(M, \Lambda T^* M)$ in the corresponding coordinate chart. Its homogeneous component $\sigma_{-n}(Q)(x, \xi)$ of degree $-n$ evaluated in $\mathbf{1}_x$ yields a differential form $\sigma_{-n}(Q)(x, \xi) \mathbf{1}_x \in \Lambda T_x^* U_p$. Hence we define

$$(31) \quad \tilde{\omega}_Q^{\text{Res}}(x) := \frac{1}{(2\pi)^n} \int_{|\xi|=1} [\sigma_{-n}(Q)(x, \xi) \mathbf{1}_x]_{[n]} d_S \xi,$$

where $\alpha_{[n]}$ stands for the part of degree n of a form α in $\Lambda T_x^* M$.

Remark 2.4. • Note that this differs from the Wodzicki residue density. Contrarily to ω_Q^{Res} which is covariant w.r. to the action of \mathfrak{f}_λ , as we shall see shortly, $\tilde{\omega}_Q^{\text{Res}}$ is not. Getzler's rescaling map will enable us to compensate this lack of covariance.

- The above constructions generalise beyond classical pseudodifferential operators, to logarithmic pseudodifferential operators. For a differential operator P in $\text{Diff}(M, \Lambda T^*M)$ with Agmon angle θ , similarly to (31), we define:

$$\tilde{\omega}_{\log_\theta P}^{\text{Res}}(x) := \frac{1}{(2\pi)^n} \int_{|\xi|=1} [\sigma_{-n}(\log_\theta P)(x, \xi) \mathbf{1}_x]_{[n]} d_S \xi.$$

The maps U_λ^\sharp defined in (28) induces a transformation on differential operators as follows: for any $P = \sum_{|\gamma| \leq m} P_\gamma D^\gamma$ in $\text{Diff}(M, \Lambda T^*M)$ of order m , we define

$$(32) \quad U_\lambda^\sharp P := \sum_{|\gamma| \leq m} U_\lambda^\sharp(P_\gamma) D^\gamma \in \text{Diff}(M, \Lambda T^*M),$$

A first direct consequence of (32) is that the operator $U_\lambda^\sharp P$ is also of order m and of Agmon angle θ . For any $(x, \xi) \in T^*U$, the symbols of $U_\lambda^\sharp P$ and P are related by

$$\sigma_{m-j}(U_\lambda^\sharp P)(x, \xi) = U_\lambda^\sharp(\sigma_{m-j}(P)(x, \xi)),$$

for all $j \geq 0$. To check that the Agmon angle remains unchanged, we observe that the eigenvalues of $\sigma_L(U_\lambda^\sharp P)(x, \xi)$ are the same as those of $\sigma_L(P)(x, \xi)$. The eigenvectors of $\sigma_L(U_\lambda^\sharp P)(x, \xi)$ are derived from those of $\sigma_L(P)(x, \xi)$ by applying U_λ . Also the eigenvalues of $U_\lambda^\sharp P$ are the same as the ones of P , since $U_\lambda^\sharp(P - \beta \text{Id}) = U_\lambda^\sharp P - \beta \text{Id}$. We then define

$$(33) \quad U_\lambda^\sharp(\varphi_\theta(P)) := \varphi_\theta(U_\lambda^\sharp P).$$

where we recall that $\varphi_\theta : \mathbb{C} \setminus L_\theta \rightarrow \mathbb{C}$ is a holomorphic function. As a consequence, we can prove that

Lemma 2.5. *Given any $P \in \text{Diff}(M, \Lambda T^*M)$ of Agmon angle θ and $Q := \varphi_\theta(P)$ we have for any $\lambda > 0$*

$$(U_\lambda^\sharp \circ \mathfrak{f}^\sharp)(Q) = (\mathfrak{f}^\sharp \circ U_\lambda^\sharp)(Q)$$

for any local diffeomorphism $\mathfrak{f} : U \rightarrow V$.

Proof. First, we show that $U_\lambda \circ \mathfrak{f}^* = \mathfrak{f}^* \circ U_\lambda$, where by definition $\mathfrak{f}^* \omega = \omega \circ \mathfrak{f}$ for any differential form ω . Indeed, we compute

$$\begin{aligned} (U_\lambda \circ \mathfrak{f}^*) \omega &= U_\lambda(\omega \circ \mathfrak{f}) \\ &= \sum_{i=1}^n \lambda^{-i} (\omega \circ \mathfrak{f})_{[i]} \\ &= \sum_{i=1}^n \lambda^{-i} \omega_{[i]} \circ \mathfrak{f} \\ &= (\mathfrak{f}^* \circ U_\lambda) \omega. \end{aligned}$$

Hence, we get for $P \in \text{Diff}(M, \Lambda T^*M)$

$$(U_\lambda^\sharp \circ \mathfrak{f}^\sharp)(P) = U_\lambda \circ \mathfrak{f}^* \circ P \circ \mathfrak{f}_* \circ U_\lambda^{-1} = \mathfrak{f}^* \circ U_\lambda \circ P \circ U_\lambda^{-1} \circ \mathfrak{f}_* = (\mathfrak{f}^\sharp \circ U_\lambda^\sharp)(P).$$

Now, for $Q = \varphi_\theta(P)$, we deduce the statement by using the fact that \mathfrak{f}^\sharp and U_λ^\sharp both commute with φ_θ . This finishes the proof. \square

We now have:

Lemma 2.6. *For any differential operator $P \in \text{Diff}(M, \Lambda T^*M)$ of Agmon angle θ and order m , we have for $Q = \varphi_\theta(P)$ that*

$$(34) \quad \sigma_{m-j}(U_\lambda^\sharp Q)(x, \xi) = U_\lambda^\sharp(\sigma_{m-j}(Q)(x, \xi)),$$

for any $(x, \xi) \in T^*U$ and $j \geq 0$.

Proof. Using Equation (13) for the symbol, we have that

$$\begin{aligned} \sigma_{m-j}(U_\lambda^\sharp \varphi_\theta(P)) &= \sigma_{m-j}(\varphi_\theta(U_\lambda^\sharp P))(x, \xi) \\ &= \frac{i}{2\pi} \int_{\Gamma_\theta} \varphi_\theta(\lambda) (\sigma_{m-j}(U_\lambda^\sharp P) - \lambda)^{* -1} d\lambda \\ &= \frac{i}{2\pi} \int_{\Gamma_\theta} \varphi_\theta(\lambda) (U_\lambda^\sharp \sigma_{m-j}(P) - \lambda)^{* -1} d\lambda \\ &= U_\lambda^\sharp(\sigma_{m-j}(\varphi_\theta(P))). \end{aligned}$$

This finishes the proof. \square

As a direct consequence of Equation (34), we get the following

Proposition 2.7. *For any differential operator $P \in \text{Diff}(M, \Lambda T^*M)$ of Agmon angle θ , we have for $Q = \varphi_\theta(P)$ that for any $\lambda > 0$.*

$$(35) \quad \omega_{U_\lambda^\sharp Q}^{\text{Res}} = \omega_Q^{\text{Res}}, \quad \tilde{\omega}_{U_\lambda^\sharp Q}^{\text{Res}} = \lambda^{-n} \tilde{\omega}_Q^{\text{Res}} \quad \text{and} \quad \tilde{\omega}_{\mathfrak{f}_\lambda^\sharp Q}^{\text{Res}} = \lambda^n \tilde{\omega}_Q^{\text{Res}} \circ \mathfrak{f}_\lambda.$$

In particular, we get

$$(36) \quad \tilde{\omega}_{U_\lambda^\sharp \mathfrak{f}_\lambda^\sharp Q}^{\text{Res}} = \tilde{\omega}_Q^{\text{Res}} \circ \mathfrak{f}_\lambda.$$

Proof. For any $\lambda > 0$, we write

$$\begin{aligned} (2\pi)^n \omega_{U_\lambda^\sharp Q}^{\text{Res}}(x) &= \left(\int_{|\xi|=1} \text{tr}^{\Lambda T^*M} \left(\sigma_{-n}(U_\lambda^\sharp Q)(x, \xi) \right) d_S \xi \right) dx^1 \wedge \dots \wedge dx^n \\ &\stackrel{(34)}{=} \left(\int_{|\xi|=1} \text{tr}^{\Lambda T^*M} \left(U_\lambda^\sharp (\sigma_{-n}(Q)(x, \xi)) \right) d_S \xi \right) dx^1 \wedge \dots \wedge dx^n \\ &= \left(\int_{|\xi|=1} \text{tr}^{\Lambda T^*M} \left(U_\lambda \sigma_{-n}(Q)(x, \xi) U_\lambda^{-1} \right) d_S \xi \right) dx^1 \wedge \dots \wedge dx^n \\ &= \left(\int_{|\xi|=1} \text{tr}^{\Lambda T^*M} \left(\sigma_{-n}(Q)(x, \xi) \right) d_S \xi \right) dx^1 \wedge \dots \wedge dx^n \\ &= (2\pi)^n \omega_Q^{\text{Res}}(x). \end{aligned}$$

To prove the two other equalities, we also compute

$$\begin{aligned}
(2\pi)^n \tilde{\omega}_{U_\lambda^\sharp Q}^{\text{Res}}(x) &= \int_{|\xi|=1} \left[\left(\sigma_{-n}(U_\lambda^\sharp Q)(x, \xi) \right) \mathbf{1}_x \right]_{[n]} d_S \xi \\
&\stackrel{(34)}{=} \int_{|\xi|=1} \left[\left(U_\lambda^\sharp(\sigma_{-n}(Q))(x, \xi) \right) \mathbf{1}_x \right]_{[n]} d_S \xi \\
&= \int_{|\xi|=1} [U_\lambda(\sigma_{-n}(Q))(x, \xi) \mathbf{1}_x]_{[n]} d_S \xi \\
&= \int_{|\xi|=1} \sum_{i=0}^n [U_\lambda [\sigma_{-n}(Q)(x, \cdot) \mathbf{1}_x]_{[i]}]_{[n]} d_S \xi \\
&= \int_{|\xi|=1} \sum_{i=0}^n [\lambda^{-i} [\sigma_{-n}(Q)(x, \cdot) \mathbf{1}_x]_{[i]}]_{[n]} d_S \xi \\
&= \lambda^{-n} \int_{|\xi|=1} [\sigma_{-n}(Q)(x, \cdot) \mathbf{1}_x]_{[n]} d_S \xi \\
&= \lambda^{-n} (2\pi)^n \tilde{\omega}_Q^{\text{Res}}(x).
\end{aligned}$$

To prove the last equality in (35), we use Equality (11) to write

$$\begin{aligned}
(2\pi)^n \tilde{\omega}_{f_\lambda^\sharp Q}^{\text{Res}}(x) &= \int_{|\xi|=1} \left[\sigma_{-n}(f_\lambda^\sharp Q)(x, \xi) \mathbf{1}_x \right]_{[n]} d_S \xi \\
&= \int_{|\xi|=1} \left[(\sigma_{-n}(Q)(f_\lambda(x), (f_\lambda^*)^t(\xi))) \mathbf{1}_x \right]_{[n]} d_S \xi \\
&= \int_{|\xi|=1} \left[(\sigma_{-n}(Q)(f_\lambda(x), \lambda^{-1}\xi)) \mathbf{1}_x \right]_{[n]} d_S \xi \\
&= \lambda^n (2\pi)^n \tilde{\omega}_Q^{\text{Res}}(x).
\end{aligned}$$

Finally, Equality (36) is obtained by combining the last two identities in (35). This gives the statement. \square

2.3. The Wodzicki residue density versus a local Berezin type density. In this paragraph, we enhance the well-known algebraic identity (26) to a lesser known identity of local densities on spin manifolds. Let now (M^n, g) be a spin manifold of even dimension and let ΣM be its spinor bundle. The morphism c^g defined in (25) induces on a differential operator $P = \sum_{|\gamma| \leq m} P_\gamma D^\gamma$ in $\text{Diff}(M, \Sigma M)$ of order m , the operator $c^g(P)$ given by

$$(37) \quad c^g(P) := \sum_{|\gamma| \leq m} c^g(P_\gamma) D^\gamma \in \text{Diff}(M, \Lambda T^* M).$$

Clearly, the operator $c^g(P)$ has the same order as P and,

$$(38) \quad \sigma_{m-j}(c^g(P))(x, \xi) = c^g(\sigma_{m-j}(P))(x, \xi),$$

for any $(x, \xi) \in T^*U \setminus \{0\}$. In order to find the relation between $\tilde{\omega}^{\text{Res}}$ and ω^{Res} , we need the following lemma:

Lemma 2.8. *For any differential operator $P \in \text{Diff}(M, \Sigma M)$ of Agmon angle θ and order m , the operator $c^g(P)$ has also an Agmon angle θ . Also, we have that*

$$\sigma_{-j}(\log_\theta c^g(P))(x, \xi) = c^g(\sigma_{-j}(\log_\theta P))(x, \xi),$$

for $j \geq 0$.

Proof. We shall prove that the set of eigenvalues of $c^g(\sigma_L(P)(x, \xi))$ is a subset of the one of $\sigma_L(P)(x, \xi)$. Thus, as the eigenvalues of $\sigma_L(P)(x, \xi)$ do not meet a neighborhood of the ray L_θ , then neither do the eigenvalues of $c^g(\sigma_L(P)(x, \xi))$. Indeed, if λ is an eigenvalue of $c^g(\sigma_L(P)(x, \xi))$, then $c^g(\sigma_L(P)(x, \xi)) - \lambda \text{Id}$ is non injective. Hence $c^g(\sigma_L(P)(x, \xi) - \lambda \text{Id})$ is non injective and, therefore, $\sigma_L(P)(x, \xi) - \lambda \text{Id}$ is non injective since c^g is injective, meaning that λ is an eigenvalue of $\sigma_L(P)(x, \xi)$. In the same way, we can prove that the set of eigenvalues of $c^g(P)$ is a subset of the one of P . To prove the second part, we first have for all j that $\sigma_{mz-j}(c^g(P)_\theta^z)(x, \xi) = c^g(\sigma_{mz-j}(P_\theta^z)(x, \xi))$. Indeed, using Equation (14), we compute

$$\begin{aligned} \sigma_{mz-j}(c^g(P)_\theta^z)(x, \xi) &= \frac{i}{2\pi} \int_{\Gamma_\theta} \lambda^z \sigma_{-m-j}((c^g(P) - \lambda)^{-1})(x, \xi) d\lambda \\ &= \frac{i}{2\pi} \int_{\Gamma_\theta} \lambda^z \sigma_{-m-j}(c^g((P - \lambda \text{Id})^{-1}))(x, \xi) d\lambda \\ &\stackrel{(38)}{=} \frac{i}{2\pi} \int_{\Gamma_\theta} \lambda^z c^g(\sigma_{-m-j}((P - \lambda \text{Id})^{-1}))(x, \xi) d\lambda \\ &\stackrel{(14)}{=} c^g(\sigma_{mz-j}(P_\theta^z)(x, \xi)). \end{aligned}$$

Now, using Equation (16), we write

$$\begin{aligned} \sigma_{-j}(\log_\theta c^g(P))(x, \xi) &= |\xi|^{-j} \left(\partial_z \left(\sigma_{mz-j}(c^g(P)_\theta^z)(x, \frac{\xi}{|\xi|}) \right) \right)_{z=0} \\ &= |\xi|^{-j} \left(\partial_z \left(c^g \left(\sigma_{mz-j}(P_\theta^z)(x, \frac{\xi}{|\xi|}) \right) \right) \right)_{z=0} \\ &\stackrel{(16)}{=} c^g(\sigma_{-j}(\log_\theta P))(x, \xi). \end{aligned}$$

This ends the proof. \square

By choosing $a = \sigma_{-n}(\log_\theta P)(x, \xi)$ in Equation (26), for any differential operator $P \in \text{Diff}(M, \Sigma M)$ which is \mathbb{Z}_2 -grading, we get that

Proposition 2.9. *For any differential operator $P \in \text{Diff}(M, \Sigma M)$ with Agmon angle θ , which is even for the \mathbb{Z}_2 -grading $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$, we have*

$$\tilde{\omega}_{\log_\theta c^g(P)}^{\text{Res}}(x) = j_g(x) (-2i)^{-n/2} \omega_{\log_\theta P}^{\text{sRes}}(x),$$

where $j_g(x) = \sqrt{\det(g_{ij}(x))}$.

Proof. Using Lemma 2.8 for $j = n$, we compute

$$\begin{aligned}
\tilde{\omega}_{\log_\theta c^g(P)}^{\text{Res}}(x) &= \frac{1}{(2\pi)^n} \int_{|\xi|=1} [\sigma_{-n}(\log_\theta c^g(P))(x, \xi) \mathbf{1}_x]_{[n]} d_S \xi \\
&= \frac{1}{(2\pi)^n} \int_{|\xi|=1} [c^g(\sigma_{-n}(\log_\theta P)(x, \xi)) \mathbf{1}_x]_{[n]} d_S \xi \\
&= \frac{1}{(2\pi)^n} \int_{|\xi|=1} [\mathbf{s}^g(\sigma_{-n}(\log_\theta P)(x, \xi))]_{[n]} d_S \xi \\
&= \frac{1}{(2\pi)^n} \int_{|\xi|=1} (T \circ \mathbf{s}^g)(\sigma_{-n}(\log_\theta P)(x, \xi)) e^1 \wedge \dots \wedge e^n d_S \xi \\
&\stackrel{(26)}{=} \frac{j_g(x)(-2i)^{-n/2}}{(2\pi)^n} \left(\int_{|\xi|=1} \text{str}(\sigma_{-n}(\log_\theta P)(x, \xi)) d_S \xi \right) dx^1 \wedge \dots \wedge dx^n \\
&= j_g(x)(-2i)^{-n/2} \omega_{\log_\theta P}^{\text{sRes}}(x).
\end{aligned}$$

Here, we use the fact that P is \mathbb{Z}_2 -graded, meaning that $\sigma_{-j}(\log_\theta P)(x, \xi)$ is in $\text{End}(\Sigma_x^\pm M) \simeq \mathcal{C}\ell(T_x M)^+ \otimes \mathbb{C}$ and, thus, Equation (26) is applied. \square

Remark 2.10. As a consequence of Proposition 2.9, for a differential operator $P \in \Psi_{\text{cl}}(M, \Sigma M)$ with Agmon angle θ , $\tilde{\omega}_{\log_\theta c^g(P)}^{\text{Res}}$ does define a global density since $\omega_{\log_\theta P}^{\text{sRes}}$ does.

Corollary 2.11. *For any differential operator $P \in \text{Diff}(M, \Sigma M)$ of Agmon angle θ and order m which is even for the \mathbb{Z}_2 -grading $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$, we have*

$$\tilde{\omega}_{\log_\theta U_\lambda^\# \mathfrak{f}_\lambda^\#(c^g(P))}^{\text{Res}} = (j_g \circ \mathfrak{f}_\lambda)(-2i)^{-n/2} \omega_{\log_\theta P}^{\text{sRes}} \circ \mathfrak{f}_\lambda.$$

Proof. We take $Q = \log_\theta c^g(P) \in \Psi_{\text{cl}}(M, \Lambda T^* M)$ in Proposition 2.7 (i.e. $\varphi_\theta = \log_\theta$) to get after, using $\mathfrak{f}_\lambda^\# \log_\theta = \log_\theta \mathfrak{f}_\lambda^\#$ (see Subsection 1.4) and $U_\lambda^\# \log_\theta = \log_\theta U_\lambda^\#$ by Definition (33), that

$$\tilde{\omega}_{\log_\theta c^g(P)}^{\text{Res}} \circ \mathfrak{f}_\lambda = \tilde{\omega}_{\log_\theta (U_\lambda^\# \mathfrak{f}_\lambda^\#(c^g(P)))}^{\text{Res}}.$$

Now we use Proposition 2.9 at the point $\mathfrak{f}_\lambda(\cdot)$ to deduce the result. \square

3. THE GEOMETRIC SET-UP

In this section, we review the geometric set up underlying Getzler rescaling. Specifically, in the language of [DS1, Paragraph 1.1], we deform the manifold M to a manifold \mathbb{M} via a deformation to the normal cone to a given point p , and pull back the Riemannian metric g on the manifold under the canonical projection $\hat{\pi} : \mathbb{M} \rightarrow M$ to a family $\{g_\lambda\}_{\lambda>0}$ of dilated metrics (see (42)). This family will play a crucial role when deforming operators.

3.1. Deformation to the normal cone to a point. For an embedding $M_0 \hookrightarrow M$ of two manifolds, the *deformation to the normal cone* is defined as

$$D(M_0, M) := (M \times \mathbb{R}_+) \cup (\mathcal{N}M_0 \times \{0\}),$$

where $\mathcal{N}M_0$ is the total space of the normal bundle to M_0 in M . The deformation to the normal cone extended to the embedding of the base of a groupoid into the groupoid gives rise to the tangent groupoid introduced by Connes [C] which proves useful in the context of manifolds with singularities. Here, choosing a reference

point $p \in M$ fixed throughout the paper, we take $M_0 = \{p\}$ so that $\mathcal{N}M_0 = T_pM$, in which case the deformation amounts to replacing M by the *deformed manifold around p* defined as

$$\mathbb{M} := (M \times \mathbb{R}_+) \cup (T_pM \times \{0\}).$$

The gluing of the two parts, namely $M \times \mathbb{R}_+$ and $T_pM \times \{0\}$ is carried out via the local diffeomorphism f_λ described in (6). Now, we build the map (denoted by Θ in [DS2, Paragraph 3.1], but here we adopt the notations of [vEY])

$$\mathbf{Exp}_p : T_{(p,\bullet)}\mathbb{M} \supset \mathbb{B}_{p,r} \longrightarrow \mathbb{M}$$

defined on

$$\mathbb{B}_{p,r} := (\{(\mathbf{x}, \lambda) \in T_pM \times \mathbb{R}_+, \mathbf{x} \in B_{r/\lambda}\}) \cup (T_pM \times \{0\}) \subset T_{(p,\bullet)}\mathbb{M}$$

by the identity map on $T_pM \times \{0\}$ and on the remaining part of $\mathbb{B}_{p,r}$ as follows

$$(39) \quad \begin{aligned} \mathbf{Exp}_p : \{(\mathbf{x}, \lambda) \in T_pM \times \mathbb{R}_+, \mathbf{x} \in B_{r/\lambda}\} &\longrightarrow U_p \times \mathbb{R}_+ \subset \mathbb{M} \\ (\mathbf{x}, \lambda) &\longmapsto (\exp_p(\lambda\mathbf{x}), \lambda) \quad \text{for } \lambda > 0. \end{aligned}$$

We consider the open set in \mathbb{M} ([DS1, Paragraph 1.1.])

$$W_p := (U_p \times \mathbb{R}_+) \cup (T_pM \times \{0\}) \subset \mathbb{M}.$$

The deformed manifold \mathbb{M} is endowed with the smooth (coarsest) structure for which the map $\mathbf{Exp}_p : \mathbb{B}_{p,r} \longrightarrow W_p$ is a diffeomorphism (we refer the reader to [H, Above Lemma 4.3] for further details). Via \mathbf{Exp}_p the point $(x = \exp_p(\mathbf{x}), \lambda)$ is identified with the point $f_\lambda(x) = \exp_p(\lambda\mathbf{x})$ and the point p is identified with \mathbf{x} . We refer to the coordinates given by (39) as the **λ -rescaled exponential coordinates**. To recover the manifold M from the deformed manifold \mathbb{M} , we consider the projection map

$$\hat{\pi} : \begin{array}{ccccc} \mathbb{M} & \xrightarrow{p_1} & M \times \mathbb{R}_{\geq 0} & \xrightarrow{\pi} & M \\ (x, \lambda) & \longmapsto & (x, \lambda) & \longmapsto & x \quad \text{if } \lambda > 0 \\ (\mathbf{x}, 0) & \longmapsto & (p, 0) & \longmapsto & p. \end{array}$$

With \mathbb{M} endowed with the smooth structure described above, the map $\hat{\pi}$ is smooth allowing to pull-back the geometry on M to \mathbb{M} . For any section s of a vector bundle E over M , its pull-back is a section of the pull-back bundle $\mathbb{E} := \hat{\pi}^*E \subset \mathbb{M} \times E$ over \mathbb{M} given by

$$(\hat{\pi}^*s)(x, \lambda) = s(x) \quad \text{for small enough } \lambda > 0, \forall x \in M$$

and

$$(\hat{\pi}^*s)(\mathbf{x}, 0) = s(p), \quad \forall \mathbf{x} \in T_pM.$$

In particular, the tangent bundle $TM \rightarrow M$ is pulled back to

$$\hat{\pi}^*TM = \{(x, \lambda, y, u) \in M \times \mathbb{R}_+ \times TM \mid x = y\} \cup \{(\mathbf{x}, 0, y, u) \in T_pM \times \{0\} \times TM \mid p = y\}$$

which is clearly isomorphic to $TM \times \mathbb{R}$. Also the local diffeomorphism (39) induces the isomorphism of vector bundles (see [DS2, Remark 3.4 (e)], [F, p.67-68])

$$\begin{aligned} TM \times \mathbb{R} &\longrightarrow T\mathbb{M} \\ (x, u, \lambda) &\longmapsto (x, \lambda u = h_\lambda(u), \lambda) \quad \text{if } \lambda > 0 \\ (p, \mathbf{x}, 0) &\longmapsto (p, \mathbf{x}) \in T_pM. \end{aligned}$$

Now, in the local exponential chart (39) of \mathbb{M} , the pull-back of a section s on E is the map $\hat{\pi}^*s \circ \mathbf{Exp}_p : \mathbb{B}_{p,r} \rightarrow \mathbb{E}$ that can be read as

$$(\hat{\pi}^*s \circ \mathbf{Exp}_p)(\mathbf{x}, \lambda) = (s \circ \hat{\pi})(\exp_p(\lambda\mathbf{x}), \lambda) = s(\exp_p(\lambda\mathbf{x})) = (s \circ \mathbf{f}_\lambda)(\exp_p(\mathbf{x}))$$

for any $\lambda > 0$ and $\mathbf{x} \in B_{r/\lambda}$. Also on the remaining part of $\mathbb{B}_{p,r}$, we have

$$(\hat{\pi}^*s \circ \mathbf{Exp}_p)(\mathbf{x}, 0) = (s \circ \hat{\pi})(\mathbf{x}, 0) = s(p) = (s \circ \mathbf{f}_0)(\exp_p(\mathbf{x})).$$

Therefore, by identifying the vector \mathbf{x} with the point $x := \exp_p(\mathbf{x})$, we write that

$$(\hat{\pi}^*s)(x, \lambda) \stackrel{\mathbf{Exp}_p}{=} (s \circ \mathbf{f}_\lambda)(x) \quad \text{for small enough } \lambda \geq 0.$$

3.2. Tensor bundles pulled back by $\hat{\pi}$. Coming back to the deformation to the normal cone, the tensor bundle $T_r^q M := TM^{\otimes q} \otimes T^*M^{\otimes r}$ is pulled back to $\hat{\pi}^*T_r^q M \rightarrow \mathbb{M}$ and a tensor t written in a normal geodesic coordinates chart with coordinates X at a point x as

$$(40) \quad t(x) = \sum t_{j_1 \dots j_r}^{i_1 \dots i_q} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_q}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_r} \Big|_x$$

is pulled back to

$$\begin{aligned} & (\hat{\pi}^*t)(x, \lambda) \stackrel{\mathbf{Exp}_p}{=} t \circ \mathbf{f}_\lambda(x) \\ &= \sum (t_{j_1 \dots j_r}^{i_1 \dots i_q} \circ \mathbf{f}_\lambda) \left(\frac{\partial}{\partial x^{i_1}} \circ \mathbf{f}_\lambda \right) \otimes \dots \otimes \left(\frac{\partial}{\partial x^{i_q}} \circ \mathbf{f}_\lambda \right) \otimes (dx^{j_1} \circ \mathbf{f}_\lambda) \otimes \dots \otimes (dx^{j_r} \circ \mathbf{f}_\lambda) \Big|_x, \end{aligned}$$

for small enough $\lambda \geq 0$. Combining (9) with (27), we deduce that

$$(\hat{\pi}^*t)(x, \lambda) \stackrel{\mathbf{Exp}_p}{=} \lambda^{q-r} (\mathbf{f}_\lambda^*t)(x) = U_\lambda(\mathbf{f}_\lambda^*t)(x),$$

for any small enough $\lambda > 0$ and $(\hat{\pi}^*t)(x, 0) \stackrel{\mathbf{Exp}_p}{=} t(p)$, for $\lambda = 0$. Specialising to $q = 0$ and $r = 2$, yields that the local description of the pull-back of the metric g on M , viewed as a covariant two tensor:

$$(41) \quad (\hat{\pi}^*g)(x, \lambda) \stackrel{\mathbf{Exp}_p}{=} \lambda^{-2} (\mathbf{f}_\lambda^*g)(x) = U_\lambda(\mathbf{f}_\lambda^*g)(x)$$

for small enough $\lambda > 0$ and it is $g(p)$ for $\lambda = 0$. It is therefore natural to introduce

$$(42) \quad g_\lambda := \lambda^{-2} \mathbf{f}_\lambda^*g = U_\lambda \mathbf{f}_\lambda^*g,$$

so that at any point in U_p

$$(g_\lambda)_{ij}(x) = (\mathbf{f}_\lambda^*g)_{ij}(y) = g_{ij}(\mathbf{f}_\lambda(x)).$$

As a consequence of the last identity and with the help of the Koszul formula, the Christoffel symbols $\Gamma_{ij}^k(\cdot, g) := g\left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right)$ satisfy for any small enough positive λ

$$\Gamma_{ij}^k(\cdot, g_\lambda) = \lambda \Gamma_{ij}^k(\mathbf{f}_\lambda(\cdot), g).$$

Similarly, the Christoffel symbols $\tilde{\Gamma}_{ls}^t(\cdot, g) := g(\nabla_{e_l} e_s, e_t)$ read in an orthonormal frame obtained by parallel transport along the geodesic curves, satisfy

$$(43) \quad \tilde{\Gamma}_{ls}^t(\cdot, g_\lambda) = \lambda \tilde{\Gamma}_{ls}^t(\mathbf{f}_\lambda(\cdot), g).$$

4. GEOMETRIC DIFFERENTIAL OPERATORS

In this section, we define the notion of geometric polynomials with respect to a given metric, as smooth sections (Definition 4.5) of a given vector bundle in terms of the corresponding vielbeins (see Appendix 6). To these polynomials, we assign an order called Gilkey order, inspired by Gilkey’s “order of jets” in the context of his invariance theory [G, Par. 2.4], see also [MP, Section 3], both of which use jets of metrics. Whereas geometric sections are defined in terms of the jets of the vielbein and hence metric tensor, the Gilkey order does not depend on the choice of metric. We then define geometric differential operators to be differential operators which take geometric sections to geometric sections with respect to some metric with an additive condition on the Gilkey order (Definition 4.10). In Proposition 4.14, we show that a geometric polynomial w.r.t. g transforms under a dilation f_λ to a geometric polynomial w.r.t. g_λ . In Proposition 4.15, we show a similar property for a geometric differential operator.

4.1. Polynomial expressions in the jets of the vielbeins. Let us recall some basic facts on the jets of a vector bundle. Given any vector bundle (E, π, M) where $\pi : E \rightarrow M$ is the orthogonal projection, we let $\Gamma(E)$ be the vector space of sections of E and $\Gamma_p(E)$ be the stalk ² of local sections at a point p . Two local sections s and s' in $\Gamma_p(E)$ have the same r -jet ($r \in \mathbb{Z}_+$) at p if

$$(D^\gamma s)|_p = (D^\gamma s')|_p$$

for any multiindex γ such that $0 \leq |\gamma| \leq r$. The relation

$$s \sim s' \iff s \text{ and } s' \text{ have the same } r\text{-jet at } p$$

defines an equivalence relation and we denote by $j_p^r s$ the equivalence class of s . The integer r is called the order of the jet. The set

$$J^r(E) := \{j_p^r s \mid p \in M, s \in \Gamma_p(E)\},$$

is a manifold, called the r -th jet manifold of π . The triple $(J^r(E), \pi_r, M)$ is a fiber bundle where $\pi_r : J^r(E) \rightarrow M; j_p^r s \rightarrow p$ and, in local coordinates,

$$j_p^r s = (s(p), D^\gamma s|_p; 1 \leq |\gamma| \leq r)$$

which can be locally represented by the polynomial $\sum_{|\gamma| \leq r} D^{|\gamma|} s|_p X^\gamma$ (here in the variable X). Inspired by Gilkey [G], we define the **Gilkey order**³ (at a point p) of $D^\gamma s$ of any local section s in $\Gamma_p(E)$

$$\text{ord}_p^{\text{Gi}}(D^\gamma s) := \text{ord}(j_p^{|\gamma|} s) = |\gamma|.$$

Note that whereas $D^\gamma s|_p$ depends on the choice of local coordinates, its Gilkey order does not as it takes into account the maximal order of derivation.

Remark 4.1. Clearly, the notion of Gilkey order is attached to a choice of vector bundle E but we do not emphasize this fact in the notation for the sake of simplicity.

²Let S_U denote the set of local sections of E defined on an open subset U of M containing p . The stalk of local sections at the point p is the set of \sim equivalence classes where for two elements s_U in S_U and s_V in S_V , $s_U \sim s_V$ if and only if s_U and s_V coincide in some neighborhood of p .

³relative to the vector bundle E .

In the sequel, we shall repeatedly use the additivity of the Gilkey order on products. For two local sections s_1, s_2 of the endomorphism bundle $\text{End}(E)$, we have

$$\text{ord}_p^{\text{Gi}}(D^{\gamma_1} s_1 D^{\gamma_2} s_2) = \text{ord}_p^{\text{Gi}}(D^{\gamma_1} s_1) + \text{ord}_p^{\text{Gi}}(D^{\gamma_2} s_2).$$

Remark 4.2. If $E = J^k F$ is the jet bundle of a vector bundle F , then a section s of E evaluated at the point p is of the form $s(p) = j_p^k t$ where t lies in $\Gamma_p(F)$. In this case, the Gilkey order of s at point p with respect to F is k added to its Gilkey order with respect to E .

The reference vector bundle E will often be implicit only when needed shall we mention it.

Definition 4.3. Given a normal geodesic coordinate system $X = (x^1, \dots, x^n)$ at a point p and for a non negative integer r , the r -valuation of a local section $s \in \Gamma_p(E)$ is defined by

$$\text{val}_{X,p}^r(s) = \min\{|\gamma| \leq r, D_X^\gamma s|_p \neq 0\}$$

with the notation of (19) provided such a minimum exists. Otherwise, following the usual convention we set $\text{val}_{X,p}^r(s) = +\infty$. Correspondingly, we define the valuation of s as being

$$(44) \quad \text{val}_{X,p}(s) = \min_{r \in \mathbb{Z}_{\geq 0}} \text{val}_{X,p}^r(s) = \min\{|\gamma| \mid D^\gamma s|_p \neq 0\} \in [0, +\infty].$$

Since we have fixed the point p , for simplicity, we henceforth omit the subscript p in the Gilkey order, abusively writing $\text{ord}^{\text{Gi}}(D^\gamma s) = |\gamma|$.

Example 4.4. We choose $E = T^*M \otimes T^*M$, and view g as a section of E trivialised above U_p by means of normal geodesic coordinates on U_p . In that trivialisation, the expansion of the metric around a point p is given by

$$(45) \quad g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{iklj}|_p x^k x^l - \frac{1}{6} R_{iklj;m}|_p x^k x^l x^m + O(|x|^4),$$

where $R_{iklj;m} = (\nabla R)_{miklj}$. Therefore, in this trivialisation, the valuation of $g - \text{Id}$ is at least 2. In contrast, in the trivialisation of E obtained by parallel transport, the valuation of $g - \text{Id}$ is $+\infty$.

Similarly the expansion of the inverse is given by

$$(46) \quad g^{ij}(x) = \delta_{ij} + \frac{1}{3} R_{iklj}|_p x^k x^l + \frac{1}{6} R_{iklj;t}|_p x^k x^l x^t + O(|x|^4),$$

so that, in these coordinates and with a slight abuse of notation, the valuation of $g^{-1} - \text{Id}$ is at least 2. Combining Equations (60) with (45) (resp. (59) with (46)) yields the following expansions [AGHM, Eq. (11)]

$$a_i^l(x, g) = \delta_{il} - \frac{1}{6} R_{ijkl}(p) x^j x^k - \frac{1}{12} \nabla_t R_{ijkl} x^j x^k x^t + O(|x|^4),$$

and

$$b_i^j(x, g) = \delta_{il} + \frac{1}{6} R_{ljk i}(p) x^j x^k + \frac{1}{12} \nabla_t R_{ljk i} x^j x^k x^t + O(|x|^4).$$

Hence, in the same way as before, the valuation of $A - \text{Id}$ (resp. $B - \text{Id}$ see Appendix 6) is at least 2 as well.

Finally, using the Koszul formula combining with (45) and the properties of the curvature operator, the Christoffel symbols $\Gamma_{ij}^k(\cdot, g) = g\left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right)$ have the following Taylor expansion at point p in the normal geodesic coordinates

$$\Gamma_{ij}^k(x, g) = \frac{1}{3} (R_{iklj}(p) + R_{ilkj}(p)) x^l + O(|x|^2).$$

Also the Christoffel symbols in an orthonormal frame $\tilde{\Gamma}_{ls}^t(\cdot, g) = g(\nabla_{e_l} e_s, e_t)$ have a similar Taylor expansion:

$$(47) \quad \tilde{\Gamma}_{ls}^t(x, g) = -\frac{1}{2} R_{list}(p) x^i + O(|x|^2),$$

which shows that both Christoffel symbols have valuation at least 1.

4.2. Geometric polynomials. We consider a rank k vector bundle $E \rightarrow M$ equipped with an affine connection. We trivialise the bundle E over an exponential neighborhood U_p of p using geodesic normal coordinates (x^1, \dots, x^n) at a point x in U_p by identifying the fibre E_x above $x = \exp_p(\mathbf{x}) \in U_p$ with the fibre E_p at point p via the parallel transport along geodesics $c(t) = \exp_p(t\mathbf{x})$, $\mathbf{x} \in V_p \subset T_p M$. We fix a basis $(s_1(p), \dots, s_k(p))$ of E_p , which is then transported to $(s_1(x, g), \dots, s_k(x, g))$. In this trivialisation, sections of E may be viewed as smooth functions on U_p with valued in the fixed fibre E_p .

Definition 4.5. We call a local section s of E over U_p a *geometric monomial* (resp. *polynomial*) or *simply geometric section* with respect to some metric g , if when $s = \sum_{j=1}^k \alpha_j s_j$ is written in the local trivialisation $s_j(\cdot, g)$, $j = 1, \dots, k$ of E above U_p , the coordinates $\alpha_j(\cdot, g)$ are monomials (resp. polynomials) in the jets of vielbein $A_p(\cdot, g)$ and $B_p(\cdot, g)$ for the metric g (resp. linear combinations of monomials) with the same Gilkey order which we refer to as the *Gilkey order* of s . Explicitly, the $\alpha_j(\cdot, g)$ are (resp. linear combinations of) expressions of the form

$$(48) \quad \prod_{s=1}^{S_j} D^{\beta_s^j}((a_j)_{i_s}^{t_s}(\cdot, g)) D^{\gamma_s^j}((b_j)_{n_s}^{l_s}(\cdot, g)),$$

all of which have the same Gilkey order $\sum_{s=1}^{S_j} |\beta_s^j| + |\gamma_s^j|$ independent of j . In this case, we shall write $s(\cdot, g)$.

One observes that the notion of geometric polynomial is invariant under transformations $g \mapsto f^*(e^\varphi g)$ of the metric g , where f is a diffeomorphism on M and φ is a smooth function on M .

Remark 4.6. When E is a subbundle of the tensor bundle, we can alternatively trivialise it over the exponential neighborhood U_p of p using geodesic normal coordinates (x^1, \dots, x^n) at a point in U_p . By (58), we have $\frac{\partial}{\partial x^i} = \sum_{\ell=1}^n a_i^\ell(\cdot, g) e_\ell(\cdot, g)$ and $dx^i = \sum_{m=1}^n b_i^m(\cdot, g) e^m(\cdot, g)$, where $(e_1(\cdot, g), \dots, e_n(\cdot, g))$ is the basis of TM obtained by parallel transport of some (fixed) orthonormal basis of $T_p M$. Inserting these relations in (40) yields an expression of

$$\mathbf{t}(\cdot) = \sum t_{j_1 \dots j_r}^{i_1 \dots i_q} P_{i_1 \dots i_q}^{\ell_1 \dots \ell_q} Q_{m_1 \dots m_r}^{j_1 \dots j_q} e_{\ell_1}(\cdot, g) \otimes \dots \otimes e_{\ell_q}(\cdot, g) \otimes e^{m_1}(\cdot, g) \otimes \dots \otimes e^{m_r}(\cdot, g),$$

where $P_{i_1 \dots i_q}^{\ell_1 \dots \ell_q}$ and $Q_{m_1 \dots m_r}^{j_1 \dots j_q}$ are linear combinations of expressions of the form (48). Thus, the coordinates $t_{j_1 \dots j_r}^{i_1 \dots i_q}$ of \mathbf{t} in (40) are linear combinations of expressions of the form (48) if and only if its coordinates $\tilde{t}_{m_1 \dots m_r}^{\ell_1 \dots \ell_q}$ in the basis $e_{\ell_1}(\cdot, g) \otimes \dots \otimes$

$e_{\ell_q}(\cdot, g) \otimes e^{m_1}(\cdot, g) \otimes \dots \otimes e^{m_r}(\cdot, g)$ are also linear combinations of expressions of the form (48). Consequently, we can use either trivialisation in this case.

Here are first examples of geometric polynomials.

- Example 4.7.** (1) Take $E = T^*M \otimes_s T^*M$. The metric g , which is a local section of E , is a geometric monomial with respect to the metric g of Gilkey order zero since its coordinates in the basis $dx^i \otimes dx^j$ induced by the normal geodesic coordinates x^1, \dots, x^n read $g_{ij}(\cdot) = \sum_{l=1}^n a_i^l(\cdot, g) a_j^l(\cdot, g)$ (see Equation (59)). So is its inverse g^{-1} a geometric monomial of Gilkey order zero since $g^{ij}(\cdot) = \sum_{l=1}^n b_l^i(\cdot, g) b_l^j(\cdot, g)$ (see Equation (60)).
- (2) Take $E = T^*M \otimes T^*M \otimes TM$. The Christoffel symbol which is a local section of E , is a geometric polynomial with respect to the metric g since its coordinates $\Gamma_{ij}^k(\cdot, g)$ in the basis $dx^i \otimes dx^j \otimes \frac{d}{dx^k}$ induced by the normal geodesic coordinates x^1, \dots, x^n read by Koszul's formula:

$$(49) \quad \Gamma_{ij}^k(\cdot, g) = \sum_{l=1}^n g^{kl}(\cdot) (\partial_{x^i}(g_{jl}(\cdot)) + \partial_{x^j}(g_{il}(\cdot)) - \partial_{x^l}(g_{ij}(\cdot))),$$

is a polynomial in the jets of vielbeins of Gilkey order one.

- (3) Similarly, the Christoffel symbols $\tilde{\Gamma}_{l_s}^t(\cdot, g) = g(\nabla_{e_l} e_s, e_t)$ written in the orthonormal frame $(e_1(\cdot, g), \dots, e_k(\cdot, g))$ obtained by parallel transport as in Remark 4.6 read as (use Einstein convention)

$$(50) \quad \tilde{\Gamma}_{l_s}^t(\cdot, g) = b_l^i(\cdot, g) b_t^j(\cdot, g) b_s^k(\cdot, g) \Gamma_{ik}^j(\cdot, g) + b_l^i(\cdot, g) b_t^j(\cdot, g) \partial_{x_i}(b_t^k(\cdot, g)) g_{kj}(\cdot).$$

are polynomials in the jets of the vielbeins of Gilkey order one.

Remark 4.8. Since jets are compatible with composition and differentiation, geometric monomials form an algebra stable under differentiation.

Remark 4.9. Due to Equations (59) and (60) which relate the metric to the vielbein, the class of polynomials we single out in Example 4.7, is consistent with the classes of polynomials in the jets of the metric considered in [ABP], [E, Theorem 1.2] and [G, Eq. 2.4.3]. There, the polynomials depend on the metric tensor, its inverse (or [ABP], see formula in item 1) on p. 282) its inverse determinant) and the derivatives of the metric tensor.

4.3. Geometric operators. In this subsection, we are going to define geometric differential operators on vector bundles based on the definition of geometric sections.

Definition 4.10. *Let E be a vector bundle over M of rank k equipped with an affine connection. We call a differential operator $P = \sum_{|\gamma| \leq m} P_\gamma D^\gamma$ in $\text{Diff}(M, E)$ of order m geometric with respect to a metric g if for any geometric polynomial local section $s \in \Gamma(E)$ with respect to g as in Definition 4.5, the section Ps is a geometric polynomial local section of E with respect to g such that*

$$\text{ord}^{\text{Gi}}(Ps) = \text{ord}(P) + \text{ord}^{\text{Gi}}(s).$$

In other words, and by using the local expression of Ps in Equation (21), a geometric operator P with respect to the metric g is equivalent to saying that $P_\gamma = (P_\gamma)_{ij}(x) \in \text{End}(E_x)$, written in the basis obtained by parallel transport of some fixed basis of E_p , are polynomials in the jets at x of vielbeins in a way that

$$(51) \quad \text{ord}^{\text{Gi}}((P_\gamma)_{ij}(x)) = \text{ord}(P) - |\gamma|,$$

for all γ .

Remark 4.11. As for geometric sections, when E is a subbundle of the tensor bundle, we can alternatively trivialise it over the exponential neighborhood U_p of p using geodesic normal coordinates (x^1, \dots, x^n) at a point in U_p .

Example 4.12. For $E = \Lambda T^*M$, resp. $E = \Sigma M$, for any $X \in TM$, the covariant differentiation ∇_X defines a geometric operator with respect to g of order 1.

- (1) When $E = \Lambda T^*M$, we express the covariant derivative on a differential form $\alpha = \sum_I \alpha_I dx^I$ of degree k in normal geodesic coordinates (x^1, \dots, x^n) around $p \in M$ as follows (here $I = \{i_1 < \dots < i_k\}$)

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x^i}} \alpha &= \sum_I \left(\frac{\partial}{\partial x^i} \alpha_I \right) dx^I \\ &+ \sum_{t,I} \alpha_I \left(\sum_{s,l=1}^n g \left(\nabla_{\frac{\partial}{\partial x^i}} dx^{i_s}, dx^l \right) g_{tl}(\cdot) \right) dx^{i_1} \wedge \dots \wedge \underbrace{dx^t}_{s^{th} \text{-slot}} \wedge \dots \wedge dx^{i_k}. \end{aligned}$$

This shows that $\nabla_{\frac{\partial}{\partial x^i}}$ is a geometric differential operator with respect to the metric g of order 1 whose zero-th order part $\sum_{m,l=1}^n g \left(\nabla_{\frac{\partial}{\partial x^i}} dx^{i_s}, dx^l \right) g_{tl}(\cdot)$ has coefficients given by linear combinations of monomials (48) with Gilkey order 1.

- (2) When $E = \Sigma M$ is equipped with the spin connection induced by the Levi-Civita connection, the corresponding $\text{End}(\Sigma_p M)$ -valued functions

$$(e^i \cdot_g : \sigma \mapsto e^i \cdot_g \sigma, i = 1, \dots, n)$$

on U_p are constant along the geodesics and hence in the trivialisation induced by parallel transport [BGV, Lemma 4.14]. Using the normal geodesic coordinates x^1, \dots, x^n on U_p , the spinorial connection $\nabla_{\frac{\partial}{\partial x^i}}$ acting on smooth functions from U_p to E_p reads

$$(52) \quad \nabla_{\frac{\partial}{\partial x^i}} = \frac{\partial}{\partial x^i} + \frac{1}{4} \sum_{s,t} \underbrace{g(\nabla_{\frac{\partial}{\partial x^i}} e_s, e_t)}_{\tilde{\Gamma}_{is}^t(\cdot, g)} e^s \cdot_g e^t \cdot_g$$

and therefore defines a geometric differential operator with respect to g of order 1. Indeed, the coefficients $\tilde{\Gamma}_{is}^t(\cdot, g)$ are smooth real functions from U_p and, by (50), are polynomials in the jets of the vielbeins of Gilkey order 1.

In the following, we consider again a vector bundle E over M of rank k equipped with a connection ∇ . We consider the pull-back bundle $f_\lambda^* E$ over U_p^λ which we equip with the following connection: For any local section $s \in \Gamma(E)$ and $X \in TU_p^\lambda$

$$(53) \quad \nabla_X^{f_\lambda^* E} (f_\lambda^*(s)) := f_\lambda^* (\nabla_{(f_\lambda)_* X}^E s).$$

Lemma 4.13. *Let E be a vector bundle over M of rank k and let $s(p) \in E_p$, for some fixed $p \in M$. We denote by $s(\cdot, g)$ the section in $\Gamma(E)$ obtained by parallel transport of $s(p)$ along the exponential curve $c(t) = \exp_p(t\mathbf{x})$ corresponding to the metric g with $\mathbf{x} \in B_r \subset T_p M$. We also denote by $\bar{s}(\cdot, g_\lambda)$ the section in $\Gamma(f_\lambda^* E)$ obtained by parallel transport of $s(p)$ along the exponential curve $\gamma(t)$ corresponding to the metric g_λ . Then, we have*

$$\bar{s}(\cdot, g_\lambda) = s(f_\lambda(\cdot), g).$$

Proof. First, we notice that if $c(t) = \exp_p(t\mathbf{x})$ is the exponential curve corresponding to the metric g with $\mathbf{x} \in B_r \subset T_pM$, then the curve $\gamma : I \rightarrow U_p^\lambda$ given by

$$\gamma(t) := \mathfrak{f}_\lambda^{-1} \circ c(t) = \exp_p(t\lambda^{-1}\mathbf{x})$$

is the exponential curve associated with the metric \mathfrak{f}_λ^*g as well for the metric g_λ . The section $s(\cdot, g) \circ \mathfrak{f}_\lambda$ in $\Gamma(\mathfrak{f}_\lambda^*E)$ is parallel along the curve $\gamma(t)$ as a direct consequence from Equation (53) and the fact that $s(\cdot, g)$ is parallel on E along the curve $c(t)$. Now the initial condition and the uniqueness of the parallel transport allow to deduce the result. \square

Proposition 4.14. *Let E be a vector bundle over M of rank k . Let s be a local section of E which is a geometric monomial (resp. polynomial) with respect to the metric g of Gilkey order $\text{ord}^{\text{Gi}}(s)$. The local section \mathfrak{f}_λ^*s in the bundle \mathfrak{f}_λ^*E is a geometric monomial (resp. polynomial) with respect to the metric g_λ of the same Gilkey order $\text{ord}^{\text{Gi}}(s)$.*

Proof. Since the local section s is a geometric monomial with respect to the metric g , it can be written as $s(\cdot, g) = \sum_{j=1}^k \alpha_j(\cdot, g) s_j(\cdot, g)$ where $\alpha_j(\cdot, g)$ is a monomial in the jets of the vielbeins. The section \mathfrak{f}_λ^*s is equal to $\mathfrak{f}_\lambda^*s = \sum_{j=1}^k \alpha_j(\mathfrak{f}_\lambda(\cdot), g) s_j(\mathfrak{f}_\lambda(\cdot), g)$. In order to express \mathfrak{f}_λ^*s in terms of the metric g_λ , we first differentiate the relations from Lemma 6.2 in the Appendix,

$$a_i^l(\cdot, g_\lambda) = a_i^l(\mathfrak{f}_\lambda(\cdot), g) \quad \text{and} \quad b_i^i(\cdot, g_\lambda) = b_i^i(\mathfrak{f}_\lambda(\cdot), g),$$

to get that $\alpha_j(\cdot, g_\lambda) = \lambda^{\text{ord}^{\text{Gi}}(s)} \alpha_j(\mathfrak{f}_\lambda(\cdot), g)$. On the other hand, by Lemma 4.13, we have that $\bar{s}_j(\cdot, g_\lambda) = s_j(\mathfrak{f}_\lambda(\cdot), g)$ for all j . Therefore, we deduce that

$$\mathfrak{f}_\lambda^*s = \lambda^{-\text{ord}^{\text{Gi}}(s)} \sum_{j=1}^k \alpha_j(\cdot, g_\lambda) \bar{s}_j(\cdot, g_\lambda).$$

That mainly means \mathfrak{f}_λ^*s is a geometric monomial with respect to the metric g_λ and that s and \mathfrak{f}_λ^*s have the same Gilkey order. \square

As a direct consequence, we get the following result on geometric differential operators.

Proposition 4.15. *Let E be a vector bundle over M of rank k equipped with an affine connection. Let P be a differential operator in $\text{Diff}(M, E)$ of order m geometric with respect to the metric g . The differential operator $\mathfrak{f}_\lambda^\sharp P$ in $\text{Diff}(M, \mathfrak{f}_\lambda^*E)$ of order m is geometric with respect to the metric g_λ .*

Proof. Let s be a local section of E which is a geometric polynomial with respect to the metric g . As P is a geometric differential operator with respect to g , the local section Ps is geometric with respect to the same metric. By Proposition 4.14, the section $\mathfrak{f}_\lambda^*(Ps) = Ps \circ \mathfrak{f}_\lambda$ is geometric with respect to the metric g_λ . Now the relation $(\mathfrak{f}_\lambda^\sharp P)(s \circ \mathfrak{f}_\lambda) = Ps \circ \mathfrak{f}_\lambda$ which can be proven straightforwardly allows to deduce that $(\mathfrak{f}_\lambda^\sharp P)(s \circ \mathfrak{f}_\lambda)$ is geometric with respect to the metric g_λ . We just need

now to show the equality with the Gilkey orders. Indeed, we compute

$$\begin{aligned} \text{ord}^{\text{Gi}}\left(\left(\mathfrak{f}_\lambda^\sharp P\right)(s \circ \mathfrak{f}_\lambda)\right) &= \text{ord}^{\text{Gi}}(\mathfrak{f}_\lambda^*(Ps)) \\ &\stackrel{\text{Prop.4.14}}{=} \text{ord}^{\text{Gi}}(Ps) \\ &\stackrel{P \text{ geom.}}{=} m + \text{ord}^{\text{Gi}}(s) \\ &\stackrel{\widetilde{\text{Prop.4.14}}}{=} m + \text{ord}^{\text{Gi}}(\mathfrak{f}_\lambda^* s). \end{aligned}$$

Recall here that the operator $\mathfrak{f}_\lambda^\sharp P$ has the same order as P . This ends the proof. \square

5. GETZLER RESCALED GEOMETRIC DIFFERENTIAL OPERATORS

In this section, we focus on geometric differential operators P in $\text{Diff}(M, E)$ for $E = \Sigma M$, resp. $E = \Lambda T^*M$ given in Definition 4.10. We study their behaviour under the combined action of a dilation \mathfrak{f}_λ and the map U_λ defined in (28) as well as the limit as λ tends to zero of the resulting operator. For that purpose we introduce the operators $\widetilde{\mathbb{P}}^{\text{Ge}}|_{(\cdot, \lambda)}$ in (54), resp. (55), where the superscript Ge stands for Getzler. We call P rescalable if $\widetilde{\mathbb{P}}^{\text{Ge}}|_{(\cdot, \lambda)}$ admits a limit $\widetilde{\mathbb{P}}^{\text{lim}}$ when λ tends to zero. In Proposition 5.6 in the case $E = \Lambda T^*M$, resp. in Proposition 5.9 in the case $E = \Sigma M$, we give a necessary and sufficient condition for P to be rescalable and show that the coefficients of the limit operator $\widetilde{\mathbb{P}}^{\text{lim}}$ are polynomials in the jets of the curvature tensor on M . For P in $\text{Diff}(M, \Lambda T^*M)$, resp. in $\text{Diff}(M, \Sigma M)$, we give in Theorem 5.7, resp. Corollary 5.10 (for an even order operator) a localisation formula similar to the one in Proposition 1.5. This time instead of the local residue form $\omega_{\log_\theta}^{\text{Res}}|_{\widetilde{\mathbb{P}}^{\text{lim}}}$, our localisation formula involves the local n -form $\widetilde{\omega}_{\log_\theta}^{\text{Res}}|_{\widetilde{\mathbb{P}}^{\text{lim}}}$.

In the same spirit as (22) and (23), we set for $P \in \text{Diff}(M, \Lambda T^*M)$ of order m

$$(54) \quad \mathbb{P}^{\text{Ge}}|_{(\cdot, \lambda)} := \lambda^m U_\lambda^\sharp P|_{(\cdot)}, \quad \text{and} \quad \widetilde{\mathbb{P}}_\lambda^{\text{Ge}} := \mathfrak{f}_\lambda^\sharp \mathbb{P}^{\text{Ge}},$$

resp. for $P \in \text{Diff}(M, \Sigma M)$

$$(55) \quad \mathbb{P}^{\text{Ge}}|_{(\cdot, \lambda)} := \lambda^m \left(U_\lambda^\sharp \circ c^g(P) \right)|_{(\cdot)}, \quad \text{and} \quad \widetilde{\mathbb{P}}_\lambda^{\text{Ge}} := \mathfrak{f}_\lambda^\sharp \mathbb{P}^{\text{Ge}}.$$

where $U_\lambda^\sharp P$ and $c^g(P)$ are given in Equations (32) and (37) respectively.

Definition 5.1. *We call a differential operator P in $\text{Diff}(M, E)$ with $E = \Lambda T^*M$, resp. $E = \Sigma M$ rescalable at a point p if and only if $\widetilde{\mathbb{P}}_\lambda^{\text{Ge}}|_{U_p}$ introduced in (54) and (55) admits a limit $\widetilde{\mathbb{P}}^{\text{lim}}$ when λ goes to zero.*

Remark 5.2. Note that rescalability is a local notion valid at a point, and is defined via a local normal geodesic coordinates.

Proposition 5.3. *Rescalable operators in $\text{Diff}(M, E)$ at point p for $E = \Lambda T^*M$, resp. $E = \Sigma M$, form a subalgebra.*

Proof. Let P_1, P_2 be two operators in $\text{Diff}(M, \Lambda T^*M)$ of order m_1 and m_2 respectively. Since the order is additive on products of operators and the degree is also additive on wedge products of forms, we have

$$\lambda^{m_1+m_2} \mathfrak{f}_\lambda^\sharp U_\lambda^\sharp (P_1 P_2) = \left(\lambda^{m_1} \mathfrak{f}_\lambda^\sharp U_\lambda^\sharp (P_1) \right) \left(\lambda^{m_2} \mathfrak{f}_\lambda^\sharp U_\lambda^\sharp (P_2) \right).$$

If P_1 and P_2 are rescalable, the limits as $\lambda \rightarrow 0$ exist on the r.h.s., and hence so do they on the l.h.s., which shows that the product $P_1 P_2$ is rescalable. Replacing U_λ^\sharp

by $U_\lambda^\sharp \circ c^g$ and using the fact that c^g is an algebra morphism yields the result for $E = \Sigma M$. \square

The following technical lemma will be useful:

Lemma 5.4. *In local normal geodesic coordinates X , and with the notations of (44), let $q := \text{val}_{X,p}(h)$ be the valuation of a local section $h \in \Gamma_p(E)$ around p , where E is the trivial bundle $E := M \times \mathbb{R} \rightarrow M$ and $p \in M$.*

- (1) *For any real number θ , as λ tends to zero, the expression $\lambda^{-\theta} D^\gamma(h \circ \mathfrak{f}_\lambda)$
 - converges if and only if $\theta \leq \max(|\gamma|, q)$;
 - if $\theta < \max(|\gamma|, q)$, it converges to zero.*
- (2) *If $\theta = \max(|\gamma|, q)$, the expression $\lambda^{-\theta} D^\gamma(h \circ \mathfrak{f}_\lambda)$ converges to the coefficient of order $\theta - |\gamma|$ in the Taylor expansion of h at point p .*

Proof. The proof is based on the fact that $\lambda^{-\theta} D^\gamma(h \circ \mathfrak{f}_\lambda) = \lambda^{|\gamma|-\theta} (D^\gamma h) \circ \mathfrak{f}_\lambda$. By definition of q , we have $h = O(|x|^q)$ so that for $q \geq |\gamma|$, we have $(D^\gamma h) \circ \mathfrak{f}_\lambda = \lambda^{q-|\gamma|} O(|x|^{q-|\gamma|})$. If, $q < |\gamma|$, we have $(D^\gamma h) \circ \mathfrak{f}_\lambda = O(1)$, which ends the proof of (1) and (2) observing that in the convergent case, the limit corresponds to the $\theta - |\gamma|$ coefficient in the Taylor expansion. \square

We now specialise to a monomial in the jets of the vielbeins as in (48) written in normal geodesic coordinates. We set

$$\Theta_{(x,g)}^{X,p} := \sum_s \max(|\alpha_s|, \text{val}_{X,p}(a_{i_s}^{t_s}(x, g))) + \max(|\beta_s|, \text{val}_{X,p}(b_{n_s}^{l_s}(x, g))).$$

In the following, we shall often drop the explicit mention of X, p, x, g .

- Example 5.5.** (1) Recall that $g_{ij}(x) = \sum_l a_i^l(x, g) a_j^l(x, g)$. We have $\Theta_{g_{ij}} = 0$ if $i = j = l$ and it is larger than 2 otherwise.
- (2) Using the Koszul formula (49), the Christoffel symbols $\Gamma_{ij}^k(x, g)$ can be written $\Gamma_{ij}^k(x, g) = \sum_l P_{i,j,k}^l(x, g)$ with $\Theta_{P_{i,j,k}^l} = 1$ for $l = i = j = k$ and larger than 2 otherwise.
- (3) Similarly, by relation (50) the Christoffel symbols $\tilde{\Gamma}_{ls}^t(x, g)$ can be written

$$\tilde{\Gamma}_{ls}^t(x, g) = \sum_k P_{l,s,t}^k(x, g)$$

with $\Theta_{P_{l,s,t}^k} \geq 2$. Notice here that $\Theta_{P_{l,s,t}^k}$ cannot be equal to 1, since this corresponds to $l = s = t = k$ which would imply $\tilde{\Gamma}_{ls}^t(x, g) = \tilde{\Gamma}_{ls}^s(x, g) = 0$ and hence would yield a contradiction.

Proposition 5.6. *Let $P \in \text{Diff}(M, \Lambda T^* M)$ be a geometric differential operator with respect to a metric g of order m . In a local trivialisation around a point p of $\Lambda T^* M$ induced by normal geodesic coordinates (x^1, \dots, x^n) on U_p , the operator P applied to a section $s = \sum_I \alpha_I dx^I$ ($dx^I := dx^{i_1} \wedge \dots \wedge dx^{i_{|I|}}$ for $I = \{i_1, \dots, i_{|I|}\}$) reads*

$$P \left(\sum_{I \subset \mathbb{N}} \alpha_I dx^I \right) = \sum_{|\gamma| \leq m} \sum_{I, J \subset \mathbb{N}} (P_\gamma)_{IJ}(\cdot, g) D^\gamma(\alpha_I) dx^J.$$

where $(P_\gamma)_{IJ}(\cdot, g)$ are polynomials as in (48). The operator P is rescalable at a point p if and only if $|J| - |I| \leq \Theta_{(P_\gamma)_{IJ}}$. In this case, the limit rescaled operator

reads

$$\tilde{\mathbb{P}}^{\text{lim}} = \sum_{|\gamma| \leq m} \sum_{|J|-|I|=\Theta_{(P_\gamma)_{IJ}}} (P_\gamma)_{IJ}^{\text{lim}}(p, g) ((dx^I)^* \otimes dx^J) D^\gamma$$

where

$$(P_\gamma)_{IJ}^{\text{lim}}(p, g) = \lim_{\lambda \rightarrow 0} \left(\lambda^{|I|-|J|} (P_\gamma)_{IJ}(\cdot, g_\lambda) \right)$$

is a polynomial expression in the jets of the Riemannian curvature tensor.

Proof. The local expression of P in the theorem results from Remark 4.6, Equations (21) and (40). Hence, from the definition of $U_\lambda^\sharp P = U_\lambda P U_\lambda^{-1}$, we get

$$(U_\lambda^\sharp P) \left(\sum_{I \subset \mathbb{N}} \alpha_I dx^I \right) = \sum_{|\gamma| \leq m} \sum_{I, J \subset \mathbb{N}} \lambda^{|I|-|J|} (P_\gamma)_{IJ}(\cdot, g) D^\gamma(\alpha_I) dx^J.$$

Since P is a geometric differential operator, the coefficients $(P_\gamma)_{IJ}(\cdot, g)$ are in the jets of the vielbeins as in (48) with $\text{ord}^{\text{Gi}}((P_\gamma)_{IJ}) + |\gamma| = m$. Now, we write

$$\begin{aligned} \tilde{\mathbb{P}}_\lambda^{\text{Ge}}(s \circ \mathfrak{f}_\lambda) &= \lambda^m (\mathfrak{f}_\lambda^\sharp U_\lambda^\sharp P)(s \circ \mathfrak{f}_\lambda) \\ &= \lambda^m (U_\lambda^\sharp P)(s) \circ \mathfrak{f}_\lambda \\ &= \sum_{|\gamma| \leq m} \sum_{I, J \subset \mathbb{N}} \lambda^{m+|I|-|J|} (P_\gamma)_{IJ}(\mathfrak{f}_\lambda(\cdot), g) D^\gamma(\alpha_I) dx^J|_{\mathfrak{f}_\lambda(\cdot)} \\ &= \sum_{|\gamma| \leq m} \sum_{I, J \subset \mathbb{N}} \lambda^{|\gamma|+|I|-|J|} (P_\gamma)_{IJ}(\cdot, g_\lambda) \lambda^{-|\gamma|} D^\gamma(\alpha_I \circ \mathfrak{f}_\lambda) dx^J \circ \mathfrak{f}_\lambda. \end{aligned}$$

In the last equality, we use the fact that $(P_\gamma)_{IJ}(\cdot, g_\lambda) = \lambda^{\text{ord}^{\text{Gi}}((P_\gamma)_{IJ})} (P_\gamma)_{IJ}(\mathfrak{f}_\lambda(\cdot), g)$. Hence, we deduce that

$$\tilde{\mathbb{P}}_\lambda^{\text{Ge}} = \sum_{|\gamma| \leq m} \sum_{I, J \subset \mathbb{N}} \lambda^{|I|-|J|} (P_\gamma)_{IJ}(\cdot, g_\lambda) ((dx^I)^* \otimes dx^J)|_{\mathfrak{f}_\lambda} D^\gamma.$$

Now by Lemma 6.2 in the Appendix, we write for $(P_\gamma)_{IJ}(\cdot, g_\lambda)$

$$\prod_{s=1}^S D^{\alpha_s}(a_{i_s}^{t_s}(\cdot, g_\lambda)) D^{\beta_s}(b_{n_s}^{l_s}(\cdot, g_\lambda)) = \prod_{s=1}^S D^{\alpha_s}(a_{i_s}^{t_s}(\cdot, g) \circ \mathfrak{f}_\lambda) D^{\beta_s}(b_{n_s}^{l_s}(\cdot, g) \circ \mathfrak{f}_\lambda).$$

For convenience, we have dropped the explicit mention of the indices I and J . Applying Lemma 5.4 to $h_a := a_{i_s}^{t_s}(\cdot, g)$ and $h_b := b_{n_s}^{l_s}(\cdot, g)$ with both θ_a and θ_b non negative integers such that $\theta_a + \theta_b = |J| - |I|$, it tells us that the expression $\lambda^{-\theta_a} D^\gamma(\mathfrak{f}_\lambda^* h_a)$ converges if and only if $\theta_a \leq \max(|\alpha_s|, q_a)$, with $q_a := \text{val}_{X,p}(a_{i_s}^{t_s}(x, g))$ and that the limit vanishes if we have a strict inequality. If $\theta_a = \max(|\alpha_s|, q_a)$, the limit is a polynomial in the jets of the curvature tensor. Similarly for h_b and θ_b . Hence, the only non zero terms which survive in the limit of (56) as $\lambda \rightarrow 0$, correspond to $\theta_a = \max(|\alpha_s|, q_a)$ and $\theta_b = \max(|\beta_s|, q_b)$ and hence $\Theta_{(P_\gamma)_{IJ}} = |J| - |I|$. This yields the statement of the theorem. \square

We prove a localisation formula for the local form $\tilde{\omega}_{\log_\theta}^{\text{Res}} P$ when $E = \Lambda T^* M$.

Theorem 5.7. *Let P in $\text{Diff}(M, \Lambda T^* M)$ be a geometric differential operator with respect to the metric g of Agmon angle θ . If P is rescalable, then*

$$\tilde{\omega}_{\log_\theta}^{\text{Res}} P(p) = \tilde{\omega}_{\log_\theta}^{\text{Res}}(\tilde{\mathbb{P}}^{\text{lim}})(x), \quad \forall x \in U_p,$$

where $\widetilde{\mathbb{P}}^{\text{lim}} := \lim_{\lambda \rightarrow 0} \widetilde{\mathbb{P}}_\lambda^{\text{Ge}}$.

Proof. Formula (36) applied to $Q = \log_\theta P$ ($\varphi_\theta = \log_\theta$) combined with the fact that the logarithm commutes with $\mathfrak{f}_\lambda^\sharp$ and U_λ^\sharp yields

$$\widetilde{\omega}_{\log_\theta P}^{\text{Res}}(\mathfrak{f}_\lambda(x)) = \widetilde{\omega}_{\log_\theta U_\lambda^\sharp(\mathfrak{f}_\lambda^\sharp P)}^{\text{Res}}(x) = \widetilde{\omega}_{\log_\theta(\widetilde{\mathbb{P}}_\lambda^{\text{Ge}})}^{\text{Res}}(x).$$

By Proposition 5.6, the limit $\widetilde{\mathbb{P}}^{\text{lim}} := \lim_{\lambda \rightarrow 0} \widetilde{\mathbb{P}}_\lambda^{\text{Ge}}$ exists, from which we deduce the statement of the theorem by letting λ tend to zero in the above identities. \square

Example 5.8. The exterior differential $d : \Lambda T^*M \rightarrow \Lambda T^*M$ (as well as its L^2 -adjoint δ) is not a rescalable operator. Indeed, by writing $d = \sum_{j=1}^n dx^j \wedge \nabla_{\frac{\partial}{\partial x^j}}$ and using the local expression of $\nabla_{\frac{\partial}{\partial x^j}}$ in Example 4.12, we have for any $\alpha = \sum_I \alpha_I dx^I$ that

$$\begin{aligned} d \left(\sum_I \alpha_I dx^I \right) &= \sum_{j,I} \left(\frac{\partial}{\partial x^j} \alpha_I \right) dx^j \wedge dx^I \\ &+ \sum_{t,I} \alpha_I \left(\sum_{s,l=1}^n g \left(\nabla_{\frac{\partial}{\partial x^j}} dx^{i_s}, dx^l \right) g_{tl}(\cdot) \right) dx^j \wedge dx^{i_1} \wedge \dots \wedge \underbrace{dx^t}_{s^{th}\text{-slot}} \wedge \dots \wedge dx^{i_k}. \end{aligned}$$

Hence for the j 's that do not belong to I in the first sum of the r.h.s., we have $|J| - |I| = 1$ and the corresponding $\Theta = 0$. Therefore the condition in Proposition 5.6 is not fulfilled. However, the Hodge operator $\Delta = d\delta + \delta d$ is a geometric rescalable operator. This results from the Bochner-Weitzenböck formula on k -forms: $\Delta = \nabla^* \nabla + W^{[k]}$, where $\nabla^* \nabla$ is given by

$$\nabla^* \nabla = -g^{ij}(\cdot) \left(\nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} - \Gamma_{ij}^k(\cdot, g) \nabla_{\frac{\partial}{\partial x^k}} \right)$$

and $W^{[k]} = \sum_{i,j=1}^n e_j^* \wedge (e_i \lrcorner R(e_i, e_j))$ is the Bochner operator. Here R is the curvature operator of the manifold M . Indeed, by replacing $\nabla_{\frac{\partial}{\partial x^i}}$ by its expression and performing some computations, one can easily see that Δ (we use Einstein convention) has the form

$$\begin{aligned} \Delta(\alpha_I dx^I) &= -g^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} (\alpha_I) dx^I - g^{ij} \frac{\partial}{\partial x^j} (\alpha_I) \Gamma_{i_s}^k g_{tk} dx^{i_1} \wedge \dots \wedge \underbrace{dx^t}_{s^{th}\text{-slot}} \wedge \dots \wedge dx^{i_k} \\ &+ \dots \end{aligned}$$

Since the Laplacian preserves the degree, then $|J| - |I| = 0 \leq \Theta$ is always satisfied. The limit of the rescaled operator corresponds to polynomials with $\Theta = 0$. Hence by the computation of Θ in Example 5.5, we get that

$$\widetilde{\mathbb{P}}^{\text{lim}} \left(\sum_I \alpha_I dx^I \right) = - \sum_{i,I} \frac{\partial^2 \alpha_I}{(\partial x^i)^2} dx^I.$$

Therefore the localisation formula in Theorem 5.7 can be applied for the Hodge Laplacian and we get

$$\widetilde{\omega}_{\log_\theta \Delta}^{\text{Res}}(p) = \widetilde{\omega}_{\log_\theta(\widetilde{\mathbb{P}}^{\text{lim}})}^{\text{Res}}(x), \quad \forall x \in U_p.$$

Proposition 5.9. *Let $P \in \text{Diff}(M, \Sigma M)$ be a geometric differential operator with respect to the metric g of order m . In the trivialisation $\{e^1, \dots, e^n\}$ induced by parallel transport, the operator reads*

$$(56) \quad P = \sum_{|\gamma| \leq m} \sum_I (P_\gamma)_I(\cdot, g) e^I \cdot_g D^\gamma,$$

where $(P_\gamma)_I(\cdot, g)$ are polynomials as in (48) and $e^I = e^{i_1} \cdot_g \dots \cdot_g e^{i_k}$ with $i_1 < \dots < i_k$ and $|I| = i_1 + \dots + i_k$. The operator P is rescalable if and only if $|I| \leq \Theta_{(P_\gamma)_I}$. In this case, the limit rescaled operator in (55) reads

$$\tilde{\mathbb{P}}^{\text{lim}} = \sum_{|\gamma| \leq m} \sum_{|I| = \Theta_{(P_\gamma)_I}} (P_\gamma)_I^{\text{lim}}(p, g) e^I \wedge D^\gamma$$

where

$$(P_\gamma)_I^{\text{lim}}(p, g) := \lim_{\lambda \rightarrow 0} \left(\lambda^{-|I|} (P_\gamma)_I(\cdot, g_\lambda) \right)$$

is a polynomial expression in the jets of the Riemannian curvature tensor.

Proof. In any local trivialization of ΣM , the operator P can be written as $P = \sum_{|\gamma| \leq m} P_\gamma(\cdot, g) D^\gamma$, where $P_\gamma(\cdot, g) \in \text{End}(\Sigma M) \simeq C\ell(TM)$. In the local trivialisation above a normal geodesic chart induced by parallel transport, we write

$$P_\gamma(\cdot, g) = \sum_I (P_\gamma)_I(\cdot, g) e^I,$$

Recall here that the $\text{End}(\Sigma_p M)$ -valued functions $e^i \cdot_g : \sigma \mapsto e^i \cdot_g \sigma$ for each $i = 1, \dots, n$ on U_p are constant along the geodesics and hence in this trivialisation [BGV, Lemma 4.14]. Now, as P is geometric, we get that $(P_\gamma)_I(\cdot, g)$ are in the jets of the vielbeins as in (48). Hence, we get (56). In particular, we deduce that

$$c^g(P) = \sum_{|\gamma| \leq m} \sum_I (P_\gamma)_I(\cdot, g) c^g(e^I) D^\gamma.$$

Now, we apply Equation (55) to a section s to get

$$\begin{aligned} \tilde{\mathbb{P}}_\lambda^{\text{Ge}}(s \circ \mathfrak{f}_\lambda) &= \lambda^m (\mathfrak{f}_\lambda^\sharp U_\lambda^\sharp c^g(P))(s \circ \mathfrak{f}_\lambda) \\ &= \lambda^m (U_\lambda^\sharp c^g(P) s) \circ \mathfrak{f}_\lambda \\ &= \sum_{|\gamma| \leq m} \sum_{I \subset \mathbb{N}} \lambda^m (P_\gamma)_I(\mathfrak{f}_\lambda(\cdot), g) U_\lambda^\sharp(c^g(e^I))(D^\gamma s)|_{\mathfrak{f}_\lambda(\cdot)}. \\ &= \sum_{|\gamma| \leq m} \sum_{I \subset \mathbb{N}} \lambda^{m-|I|} (P_\gamma)_I(\mathfrak{f}_\lambda(\cdot), g) \lambda^{|I|} U_\lambda^\sharp(c^g(e^I)) \lambda^{-|\gamma|} D^\gamma (s \circ \mathfrak{f}_\lambda) \\ &= \sum_{|\gamma| \leq m} \sum_{I \subset \mathbb{N}} \lambda^{-|I|} (P_\gamma)_I(\cdot, g_\lambda) \lambda^{|I|} U_\lambda^\sharp(c^g(e^I)) D^\gamma (s \circ \mathfrak{f}_\lambda). \end{aligned}$$

Here, we use the fact that $(P_\gamma)_I(\cdot, g_\lambda) = \lambda^{\text{ord}^{\text{Gi}}((P_\gamma)_I)} (P_\gamma)_I(\mathfrak{f}_\lambda(\cdot), g)$ and that $\text{ord}^{\text{Gi}}((P_\gamma)_I) + |\gamma| = m$, since $(P_\gamma)_I(\cdot, g)$ are in the jets of the vielbeins. Therefore, we deduce that

$$\tilde{\mathbb{P}}_\lambda^{\text{Ge}} = \sum_{|\gamma| \leq m} \sum_{I \subset \mathbb{N}} \lambda^{-|I|} (P_\gamma)_I(\cdot, g_\lambda) \lambda^{|I|} U_\lambda^\sharp(c^g(e^I)) D^\gamma.$$

Now, using (30), we have that $\lambda^{|I|}U_\lambda^\sharp(c^g(e^I))$ converges to $e^I \wedge$ as $\lambda \rightarrow 0$. Also by Lemma 5.4, the term $\lambda^{-|I|}(P_\gamma)_I(\cdot, g_\lambda)$ converges if and only if $|I| \leq \Theta_{(P_\gamma)_I}$. Thus, the operator P is rescalable if and only if $|I| \leq \Theta_{(P_\gamma)_I}$. The limit rescaled operator follows then easily. \square

We prove a localisation formula for the local form $\tilde{\omega}_{\log_\theta P}^{\text{Res}}$ when $E = \Lambda \Sigma M$.

Corollary 5.10. *Let P in $\text{Diff}(M, \Sigma M)$ be a geometric differential operator with respect to the metric g , of Agmon angle θ which is even for the \mathbb{Z}_2 -grading $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$. If P is rescalable, then we have*

$$\tilde{\omega}_{\log_\theta(\tilde{\mathbb{P}}^{\text{lim}})}^{\text{Res}}(x) = (-2i)^{-n/2} \omega_{\log_\theta P}^{\text{sRes}}(p), \quad \forall x \in U_p,$$

where $\tilde{\mathbb{P}}^{\text{lim}} := \lim_{\lambda \rightarrow 0} \tilde{\mathbb{P}}_\lambda^{\text{Ge}}$.

Proof. Using the relation in Corollary 2.11, the fact that P is rescalable and that $j_g \circ \mathfrak{f}_\lambda \rightarrow 1$ as $\lambda \rightarrow 0$ yield the result. \square

6. THE RESCALED SQUARE OF THE DIRAC OPERATOR

In this paragraph, we show that whereas the Dirac operator (which is a geometric operator and hence so is its square) is not rescalable, its square is. We then apply the results of the previous section to $P = \not{D}$ and compute $\tilde{\mathbb{P}}^{\text{lim}}$ with the help of Proposition 5.9. This allows to find the expression of $\tilde{\mathbb{P}}^{\text{lim}}$ in terms of the curvautre operator of M as in [Ge]. We then derive from Corollary 5.10 a localisation formula (57) for the graded residue of the logarithm of \not{D}^2 .

We recall that the **Dirac operator** on a spin manifold (M^n, g) is the differential operator of order one given by (using Einstein's summation convention)

$$\not{D} := \sum_{i=1}^n e^i \cdot_g \nabla_{e_i},$$

where ∇_{e_i} is the spinorial Levi-Civita covariant derivative in the direction of e_i . Using the expression (52), the Dirac operator reads

$$\begin{aligned} \not{D} &= \sum_{i=1}^n e^i \cdot_g \nabla_{e_i} \\ &= \sum_{i=l} a_i^l(\cdot, g) e^i \cdot_g \partial_{x^l} + \sum_{i \neq l} a_i^l(\cdot, g) e^i \cdot_g \partial_{x^l} \\ &\quad + \frac{1}{4} \sum_{i,l,s,t=1}^n \tilde{\Gamma}_{ls}^t(\cdot, g) a_i^l(\cdot, g) e^i \cdot_g e^s \cdot_g e^t \cdot_g \end{aligned}$$

The operator \not{D} is geometric. Indeed, the above expression involves a sum of three terms, each of which is expressed in terms of jets of vielbeins (see Example 4.7 for $\tilde{\Gamma}$) and satisfies Condition (51). Yet it is not rescalable. Indeed, with the notations of Proposition 5.9 with $P = \not{D}$, the condition $|I| \leq \Theta_{(P_\gamma)_I}$ is not satisfied in the first of the three sums since $|I| = 1$ and $\Theta_{P_\gamma} = 0$. Recall that the valuation of $a_i^l(\cdot, g)$ is zero if $l = i$ and at least 2 otherwise.

Proposition 6.1. *The square of the Dirac operator \mathcal{D}^2 is a rescalable geometric differential operator. Setting $P := \mathcal{D}^2$, the operator $\tilde{\mathbb{P}}_\lambda^{\text{Ge}}$ in (55), read in a local trivialisation of ΣM at point x in U_p obtained by parallel transport along geodesics, converges to*

$$-(\partial_{x^i} - \frac{1}{8}R_{ijst}(p)x^j e^s \wedge e^t \wedge)^2,$$

where $\{e_1(p), \dots, e_n(p)\}$ is an orthonormal frame $T_p M$.

Proof. Since \mathcal{D} is geometric so is its square \mathcal{D}^2 . We now show it is rescalable. Since the action by Clifford multiplication $e^i \cdot_g$ of the vectors of an orthonormal frame of $T_x M$ obtained from $\{e_1(p), \dots, e_n(p)\}$ by parallel transport is constant in x , in the following we will simply write $\{e_1, \dots, e_n\}$. We use the Schrödinger-Lichnerowicz formula [LM] to write (we use the Einstein convention in the sequel)

$$\begin{aligned} \mathcal{D}^2|_x &= -g^{ij}(x) (\nabla_{x^i} \nabla_{x^j} - \Gamma_{ij}^k(x, g) \nabla_{x^k}) + \frac{1}{4} \text{Scal}(x) \\ &= -g^{ij}(x) \left(\partial_{x^i} + \frac{1}{4} \tilde{\Gamma}_{ik}^l(x, g) e^k \cdot_g e^l \cdot_g \right) \left(\partial_{x^j} + \frac{1}{4} \tilde{\Gamma}_{js}^t(x, g) e^s \cdot_g e^t \cdot_g \right) \\ &\quad + g^{ij}(x) \Gamma_{ij}^k(x, g) \left(\partial_{x^k} + \frac{1}{4} \tilde{\Gamma}_{ks}^t(x, g) e^s \cdot_g e^t \cdot_g \right) + \frac{1}{4} \text{Scal}(x) \\ &= \underbrace{-g^{ij}(x) \partial_{x^i x^j}^2}_{(I)} - \underbrace{\frac{1}{4} g^{ij}(x) \partial_{x^i} (\tilde{\Gamma}_{js}^t(x, g)) e^s \cdot_g e^t \cdot_g}_{(II)} - \underbrace{\frac{1}{2} g^{ij}(x) \tilde{\Gamma}_{js}^t(x, g) e^s \cdot_g e^t \cdot_g \partial_{x^i}}_{(III)} \\ &\quad - \underbrace{\frac{1}{16} \sum_{k \neq l, s \neq t} g^{ij}(x) \tilde{\Gamma}_{ik}^l(x, g) \tilde{\Gamma}_{js}^t(x, g) e^k \cdot_g e^l \cdot_g e^s \cdot_g e^t \cdot_g}_{(IV)} \\ &\quad + \underbrace{g^{ij}(x) \Gamma_{ij}^k(x, g) \left(\partial_{x^k} + \frac{1}{4} \tilde{\Gamma}_{ks}^t(x, g) e^s \cdot_g e^t \cdot_g \right)}_{(V)} + \frac{1}{4} \text{Scal}(x), \end{aligned}$$

where Scal is the scalar curvature of the metric g . Combining Equations (59) and (50) with the Koszul formula, we can express \mathcal{D}^2 in terms of the vielbeins. To avoid lengthy computations, we only sketch the computation for the third term in the above equation to show that the relation $|I| \leq (\Theta_{P_\gamma})_I$ holds. Thus, the operator is rescalable. To show the above inequality, we first observe that $|I|$ equals 2. According to Examples 5.5, the coefficient in (III) can be written as a sum of polynomials of the vielbeins such that the corresponding Θ is at least 2 (since the one corresponding to $\tilde{\Gamma}_{im}^l(x, g)$ is at least 2). With the help of (43) and (47), the limit of the rescaled operator of (III) is equal to

$$\lim_{\lambda \rightarrow 0} \lambda^{-2} \sum g_\lambda^{ij}(x) \tilde{\Gamma}_{js}^t(x, g_\lambda) (e^s \wedge e^t \wedge) \partial_{x^i} = -\frac{1}{2} \sum \delta^{ij} R_{jkst}(p) x^k (e^s \wedge e^t \wedge) \partial_{x^i}.$$

The same thing can be done for the first term (I) which converges to $-\sum_i \partial_{x^i}^2$. The second term (II) converges to $\frac{1}{8} \sum_{i,j,s,t} \delta^{ij} R_{ijst}(p) e^s \wedge e^t \wedge$. The fourth term (IV) tends to

$$-\frac{1}{64} \sum \delta^{ij} R_{iqkl}(p) R_{jlst}(p) x^q x^l e^k \wedge e^l \wedge e^s \wedge e^t \wedge.$$

The other terms converge to 0. Therefore, we deduce that

$$\begin{aligned}
 \lim_{\lambda \rightarrow 0} \widetilde{\mathbb{P}}_\lambda^{\text{Ge}} &= - \sum_i \partial_{x^i}^2 + \frac{1}{8} \sum_{i,j,s,t} \delta^{ij} R_{ijst}(p) e^s \wedge e^t \\
 &\quad + \frac{1}{4} \sum \delta^{ij} R_{jkst}(p) x^k (e^s \wedge e^t \wedge) \partial_{x^i} \\
 &\quad - \frac{1}{64} \sum \delta^{ij} R_{iqkl}(p) R_{jlst}(p) x^q x^l e^k \wedge e^l \wedge e^s \wedge e^t \wedge \\
 &= - \sum_i (\partial_{x^i} - \frac{1}{8} R_{ijst}(p) x^j e^s \wedge e^t \wedge)^2,
 \end{aligned}$$

which confirms the fact, which we checked by hand in the previous tedious computations, that the operator is rescalable. \square

We can apply Corollary 5.10 to the square of the Dirac operator $P = \mathbb{D}^2$. Since the operator \mathbb{D}^2 is non negative self-adjoint, it has a well defined logarithm $\log_\theta \mathbb{D}^2$ (here $\theta = \pi$). Hence, we have that

$$(57) \quad \widetilde{\omega}_{\log_\theta(\widetilde{\mathbb{P}}^{\text{lim}})}^{\text{Res}}(x) = (-2i)^{-n/2} \omega_{\log_\theta \mathbb{D}^2}^{\text{sRes}}(p),$$

for any $x \in U_p$. The computation of the Wodzicki residue on the r.h.s is tedious. In [Sc], it is derived from the heat-kernel asymptotics of $\widetilde{\mathbb{P}}^{\text{lim}}$.

APPENDIX: VIELBEIN

Let (M^n, g) be a Riemannian manifold. Let $F_p(\cdot, g) := \{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ be the cartesian frame on U_p built from the geodesic coordinates around p defined in (8). From a given orthonormal basis $e_1(p), \dots, e_n(p)$ of $T_p M$ at $p \in M$, we build a local orthonormal frame $O_p(\cdot, g) := \{e_1(\cdot, g), \dots, e_n(\cdot, g)\}$, of TM obtained by the parallel transport along small geodesics as in (7).

A linear map $A_p(\cdot, g) : TM \rightarrow TM$ (resp. its inverse $B_p(\cdot, g)$) which takes the basis $F_p(\cdot, g)$ to $O_p(\cdot, g)$ (resp. $O_p(\cdot, g)$ to $F_p(\cdot, g)$) can be represented by a $n \times n$ matrix $A = (a_i^l(\cdot, g))$ (resp. $B = (b_l^i(\cdot, g))$) with

$$(58) \quad \frac{\partial}{\partial x^i} = \sum_{l=1}^n a_i^l(\cdot, g) e_l(\cdot, g), \forall i \in [[1, n]] \quad \text{and} \quad e_l(\cdot, g) = \sum_{j=1}^n b_l^j(\cdot, g) \frac{\partial}{\partial x^j}, \forall l \in [[1, n]].$$

Also, we have that

$$dx^i = \sum_{m=1}^n b_m^i(\cdot, g) e^m(\cdot, g), \forall i \in [[1, n]] \quad \text{and} \quad e^l(\cdot, g) = \sum_{j=1}^n a_j^l(\cdot, g) dx^j, \forall l \in [[1, n]].$$

With these conventions, and dropping the explicit mention of p whenever this does not lead to confusion, we write $A(\cdot, g) = (a_i^l(\cdot, g))_{i,l}$ and we have that

$$(59) \quad \sum_{l=1}^n a_i^l(\cdot, g) a_j^l(\cdot, g) = g_{ij}(\cdot) \quad \text{or equivalently} \quad A A^t = G$$

where G has entries $g_{ij}(\cdot)$. Similarly, we have

$$(60) \quad \sum_{l=1}^n b_l^i(\cdot, g) b_l^j(\cdot, g) = g^{ij}(\cdot) \quad \text{or equivalently} \quad B^t B = G^{-1}.$$

Taking the scalar product of the first equation in (58) with e_m yields after inserting the second equation

$$(61) \quad a_i^l(\cdot, g) = \sum_{j=1}^n g_{ij}(\cdot) b_l^j(\cdot, g) \quad \text{and} \quad b_l^j(\cdot, g) = \sum_{i=1}^n g^{ij}(\cdot) a_i^l(\cdot, g).$$

The second equation in (61) is also derived by multiplying by the inverse of the metric.

Lemma 6.2. *At any point in U_p and $\lambda > 0$, we have*

$$a_i^l(\cdot, g_\lambda) = a_i^l(f_\lambda(\cdot), g) \quad \text{and} \quad b_l^i(\cdot, g_\lambda) = b_l^i(f_\lambda(\cdot), g).$$

Proof. To prove the equality $a_i^l(\cdot, g_\lambda) = a_i^l(f_\lambda(\cdot), g)$, we let $\{\bar{e}_1(\cdot, g_\lambda), \dots, \bar{e}_n(\cdot, g_\lambda)\}$ the orthonormal frame obtained by parallel transport from $\{e_1(p), \dots, e_n(p)\}$ with respect to the metric g_λ . We know from Lemma 4.13 that

$$\bar{e}_i(\cdot, g_\lambda) = e_i(f_\lambda(\cdot), g).$$

Using (58), we write with respect to the metric g_λ that

$$\frac{\partial}{\partial x^i} \circ f_\lambda = \sum_{l=1}^n a_i^l(\cdot, g_\lambda) \bar{e}_l(\cdot, g_\lambda).$$

Also, (58) applied to the point $f_\lambda(\cdot)$ gives that

$$\frac{\partial}{\partial x^i} \circ f_\lambda = \sum_{l=1}^n a_i^l(f_\lambda(\cdot), g) e_l(f_\lambda(\cdot), g).$$

Comparing both equations yields the results. The second equality can be proven in the same way. \square

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