



**HAL**  
open science

# An extended Darboux transformation to get families of solutions to the KPI equation

Pierre Gaillard

► **To cite this version:**

Pierre Gaillard. An extended Darboux transformation to get families of solutions to the KPI equation. 2023. hal-04014108

**HAL Id: hal-04014108**

**<https://hal.science/hal-04014108>**

Preprint submitted on 3 Mar 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# An extended Darboux transformation to get families of solutions to the KPI equation

Pierre Gaillard,

Université de Bourgogne Franche Comté,  
Institut de mathématiques de Bourgogne,  
9 avenue Alain Savary BP 47870  
21078 Dijon Cedex, France :  
E-mail : Pierre.Gaillard@u-bourgogne.fr

March 3, 2023

## Abstract

By means of a Darboux transform with particular generating function solutions to the Kadomtsev-Petviashvili equation (KPI) are constructed. We give a method that provides different types of solutions in terms of particular determinants of order  $N$ . For any order, these solutions depend of the degree of summation and the degree of derivation of the generating functions.

We study the patterns of their modulus in the plane  $(x, y)$  and their evolution according time and parameters.

Keywords: KPI equation; Darboux transformation; wronskians; rational solutions.

PACS numbers :

33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd

## 1 Introduction

Kadomtsev-Petviashvili equation (KPI) is considered in the following normalization

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0, \quad (1)$$

where subscripts  $x$ ,  $y$  and  $t$  denote partial derivatives.

In 1970 Kadomtsev and Petviashvili [1] proposed first this equation as a universal model describing weakly nonlinear waves in media with dispersion of velocity. It is used in hydrodynamics for surface and internal water waves [2, 3],

and in nonlinear optics [4]. The inverse scattering transform (IST) was used by Zakharov to solve this KPI equation.

The first rational solutions were constructed by Manakov, Zakharov, Bordag and Matveev [5] in 1977. The works of Krichever [6, 7] showed that the dynamics of the lumps of the KPI equation is governed by the Calogero-Moser system. Among many researches, we can mention the following works of Satsuma and Ablowitz in 1979 [8], Matveev in 1979 [9], Freeman and Nimmo in 1983 [10, 11].

Solutions of the KPI equation are constructed in terms of wronskians of order  $N$  depending on several real parameters. We construct here some explicit solutions of order 1, depending on several real parameters, and the representations of their modulus in the plane of the coordinates  $(x, y)$  according to parameters and time  $t$ .

## 2 Multi-parametric solutions of order $N$ to the KPI equation

Let the Kadomtsev-Petviashvili (KPI) equation be written in the following normalization

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0.$$

We denote  $W(f_1, \dots, f_N)$  the wronskian of order  $N$  of the functions  $f_1, \dots, f_N$ , defined by  $\det(\partial_x^{i-1} f_j)_{1 \leq i \leq N, 1 \leq j \leq N}$ ,  $\partial_x^i$  being the partial derivative of order  $i$  with respect to  $x$  and  $\partial_x^0 f_j$  being the function  $f_j$ .

Let  $a_j, b_{j,s,d}, c_{j,s}, d_{j,s}, e_{j,s}$  be arbitrary real numbers.

Let  $f_{j,s}$  be the following elementary functions defined by

$$f_{j,s}(x, y, t) = e^{ic_{j,s}x - ic_{j,s}^2 y + ic_{j,s}^3 t}. \quad (2)$$

Then, we have the following result :

**Theorem 2.1** *Let  $\psi_j$ , be the functions defined by*

$$\psi_j(x, y, t) = a_j + \sum_{s=1}^S \sum_{d=0}^D b_{j,s,d} \partial_{c_{j,s}}^d e^{ic_{j,s}x - ic_{j,s}^2 y + ic_{j,s}^3 t} = a_j + \sum_{s=1}^S \sum_{d=0}^D b_{j,s,d} \partial_{c_{j,s}}^d f_{j,s}(x, y, t). \quad (3)$$

*then the function  $u$  defined by*

$$u(x, y, t) = -2\partial_x^2 \ln(W(\psi_1, \dots, \psi_N)) \quad (4)$$

*is a solution to the (KPI) equation (1) depending on real  $N(S(D+4)+1)$  parameters  $a_j, b_{j,s,d}, c_{j,s}, d_{j,s}, e_{j,s}, 1 \leq j \leq N, 1 \leq s \leq S, 0 \leq d \leq D$ .*

The order  $N$  of the wronskian will be called the order of the solution. The degree of the summation of the solution will be the number  $S$  of terms of the

summation and the degree of derivation of the solution with the number  $D$ . For each order  $N$  of the solution there is a lot of choices concerning the degrees of the summation and derivation.

In the following, we give some examples.

### 3 Solutions of order 1 depending on $4S + 1$ parameters without derivative

In this section, we take equal  $N$  to 1. We make variations on the degrees of summation.

#### 3.1 Case $S = 1$ , ( $D = 0$ )

In this case solutions to the KPI equation can be written as

$$v(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

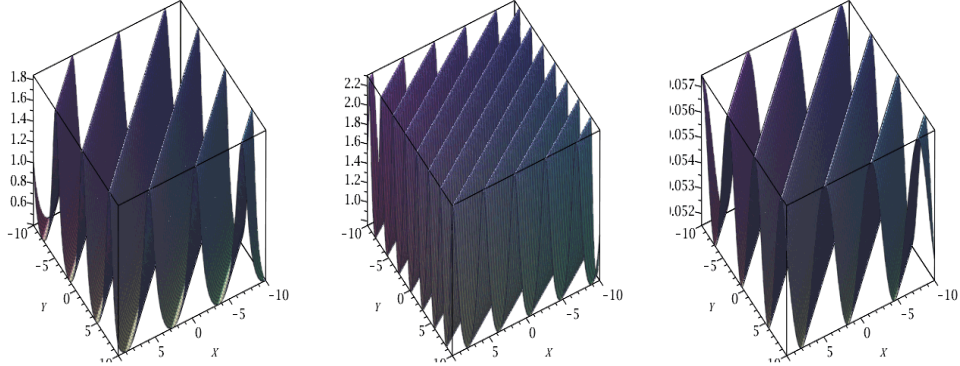
with

$$n(x, y, t) = -b_{1,1,0}c_{1,1}^2 e^{ic_{1,1}x - ic_{1,1}^2 y + ic_{1,1}^3 t + d_{1,1} + ie_{1,1}} a_1$$

and

$$d(x, y, t) = b_{1,1,0} e^{ic_{1,1}x + itc_{1,1}^3 - ic_{1,1}^2 y + d_{1,1} + ie_{1,1}} + a_1$$

We give some figures in the  $(x, y)$  plane of coordinates



**Figure 1.** Solution of order 1 to KPI, on the left for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $c_{1,1} = 1$ ,  $d_{1,1} = 1$ ,  $e_{1,1} = 1$ ; in the center for  $t = 1$ ,  $a_1 = 10^2$ ,  $b_{1,1,0} = 10$ ,  $c_{1,1} = 1.5$ ,  $d_{1,1} = 1$ ,  $e_{1,1} = 1$ ; on the right for  $t = 10^3$ ,  $a_1 = 10^2$ ,  $b_{1,1,0} = 1$ ,  $c_{1,1} = 1$ ,  $d_{1,1} = 1$ ,  $e_{1,1} = 1$ .

#### 3.2 Case $S = 2$ , ( $D = 0$ )

In this case solutions to the KPI equation can be written as

$$v(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

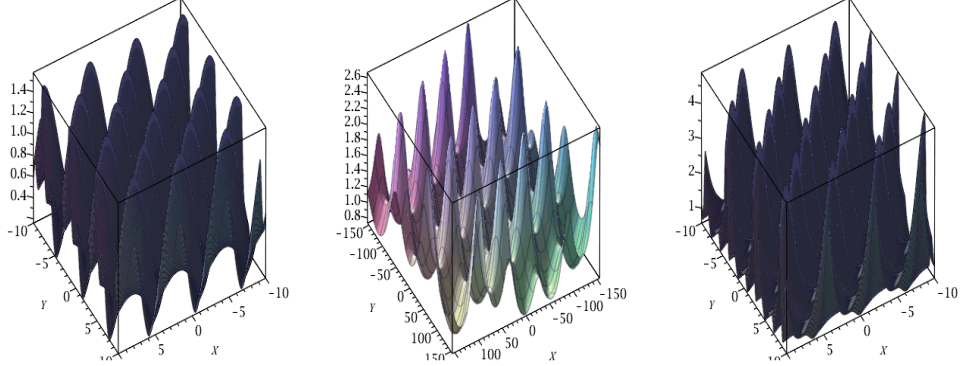
with

$$\begin{aligned}
n(x, y, t) = & -b_{1,1,0}c_{1,1}^2 e^{ic_{1,1}x - ic_{1,1}^2 y + ic_{1,1}^3 t + d_{1,1} + ie_{1,1}} a_1 \\
& -b_{1,1,0}c_{1,1}^2 e^{ic_{1,1}x + ic_{1,2}x + ic_{1,1}^3 t + d_{1,1} + ie_{1,1} - ic_{1,1}^2 y - ic_{1,2}^2 y + ic_{1,2}^3 t + d_{1,2} + ie_{1,2}} b_{1,2,0} \\
& -b_{1,2,0}c_{1,2}^2 e^{ic_{1,2}x - ic_{1,2}^2 y + ic_{1,2}^3 t + d_{1,2} + ie_{1,2}} a_1 \\
& -b_{1,2,0}c_{1,2}^2 e^{ic_{1,1}x + ic_{1,2}x + ic_{1,1}^3 t + d_{1,1} + ie_{1,1} - ic_{1,1}^2 y - ic_{1,2}^2 y + ic_{1,2}^3 t + d_{1,2} + ie_{1,2}} b_{1,1,0} \\
& + 2b_{1,1,0}c_{1,1} e^{ic_{1,1}x + ic_{1,2}x + ic_{1,1}^3 t + d_{1,1} + ie_{1,1} - ic_{1,1}^2 y - ic_{1,2}^2 y + ic_{1,2}^3 t + d_{1,2} + ie_{1,2}} b_{1,2,0} c_{1,2}
\end{aligned}$$

and

$$\begin{aligned}
d(x, y, t) = & b_{1,1,0} e^{ic_{1,1}x + itc_{1,1}^3 - ic_{1,1}^2 y + d_{1,1} + ie_{1,1}} \\
& + b_{1,2,0} e^{ic_{1,2}x + itc_{1,2}^3 - ic_{1,2}^2 y + d_{1,2} + ie_{1,2}} + a_1
\end{aligned}$$

We give some figures in the  $(x, y)$  plane of coordinates



**Figure 2.** Solution of order 1 to KPI, on the left for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $c_{1,1} = 1$ ,  $c_{1,2} = 2$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ; in the center for  $t = 1$ ,  $a_1 = 10^2$ ,  $b_{1,1,0} = 10$ ,  $b_{1,2,0} = 2$ ,  $c_{1,1} = 0, 1$ ,  $c_{1,2} = 2$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ; on the right for  $t = 10^2$ ,  $a_1 = 10^3$ ,  $b_{1,1,0} = 10$ ,  $b_{1,2,0} = 2$ ,  $c_{1,1} = 1$ ,  $c_{1,2} = 2$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ .

### 3.3 Case $S = 3$ , ( $D = 0$ )

In this case solutions to the KPI equation can be written as

$$v(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

with

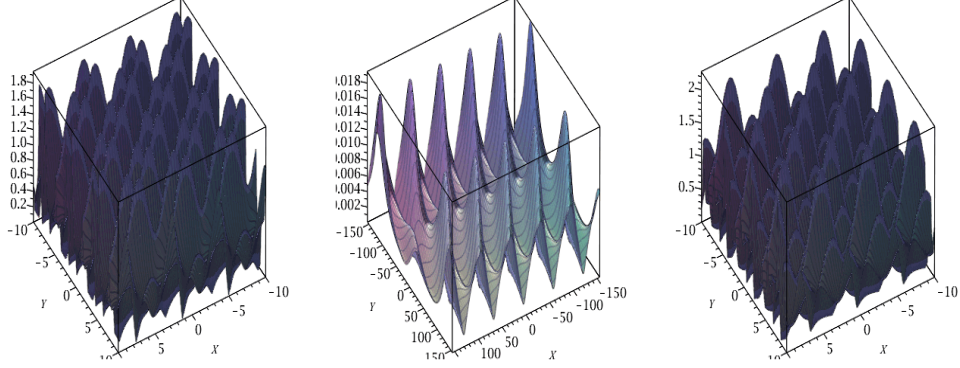
$$\begin{aligned}
n(x, y, t) = & -b_{1,1,0}c_{1,1}^2 e^{ic_{1,1}x - ic_{1,1}^2 y + ic_{1,1}^3 t + d_{1,1} + ie_{1,1}} a_1 \\
& -b_{1,1,0}c_{1,1}^2 e^{ic_{1,1}x + ic_{1,2}x + ic_{1,1}^3 t + d_{1,1} + ie_{1,1} - ic_{1,1}^2 y - ic_{1,2}^2 y + ic_{1,2}^3 t + d_{1,2} + ie_{1,2}} b_{1,2,0} \\
& -b_{1,1,0}c_{1,1}^2 e^{ic_{1,1}x + ic_{1,3}x + ic_{1,1}^3 t + d_{1,1} + ie_{1,1} - ic_{1,1}^2 y - ic_{1,3}^2 y + ic_{1,3}^3 t + d_{1,3} + ie_{1,3}} b_{1,3,0} \\
& -b_{1,2,0}c_{1,2}^2 e^{ic_{1,2}x - ic_{1,2}^2 y + ic_{1,2}^3 t + d_{1,2} + ie_{1,2}} a_1 \\
& -b_{1,2,0}c_{1,2}^2 e^{ic_{1,1}x + ic_{1,2}x + ic_{1,1}^3 t + d_{1,1} + ie_{1,1} - ic_{1,1}^2 y - ic_{1,2}^2 y + ic_{1,2}^3 t + d_{1,2} + ie_{1,2}} b_{1,1,0} \\
& -b_{1,2,0}c_{1,2}^2 e^{ic_{1,2}x + ic_{1,3}x + ic_{1,2}^3 t + d_{1,2} + ie_{1,2} - ic_{1,2}^2 y - ic_{1,3}^2 y + ic_{1,3}^3 t + d_{1,3} + ie_{1,3}} b_{1,3,0} \\
& -b_{1,3,0}c_{1,3}^2 e^{ic_{1,3}x - ic_{1,3}^2 y + ic_{1,3}^3 t + d_{1,3} + ie_{1,3}} a_1 \\
& -b_{1,3,0}c_{1,3}^2 e^{ic_{1,1}x + ic_{1,3}x + ic_{1,1}^3 t + d_{1,1} + ie_{1,1} - ic_{1,1}^2 y - ic_{1,3}^2 y + ic_{1,3}^3 t + d_{1,3} + ie_{1,3}} b_{1,1,0}
\end{aligned}$$

$$\begin{aligned}
& -b_{1,3,0}c_{1,3}^2 e^{ic_{1,2}x+ic_{1,3}x+ic_{1,2}^3t+d_{1,2}+ie_{1,2}-ic_{1,2}^2y-ic_{1,3}^2y+ic_{1,3}^3t+d_{1,3}+ie_{1,3}} b_{1,2,0} \\
& +2b_{1,1,0}c_{1,1} e^{ic_{1,1}x+ic_{1,2}x+ic_{1,1}^3t+d_{1,1}+ie_{1,1}-ic_{1,1}^2y-ic_{1,2}^2y+ic_{1,2}^3t+d_{1,2}+ie_{1,2}} b_{1,2,0}c_{1,2} + \\
& 2b_{1,1,0}c_{1,1} e^{ic_{1,1}x+ic_{1,3}x+ic_{1,1}^3t+d_{1,1}+ie_{1,1}-ic_{1,1}^2y-ic_{1,3}^2y+ic_{1,3}^3t+d_{1,3}+ie_{1,3}} b_{1,3,0}c_{1,3} + \\
& 2b_{1,2,0}c_{1,2} e^{ic_{1,2}x+ic_{1,3}x+ic_{1,2}^3t+d_{1,2}+ie_{1,2}-ic_{1,2}^2y-ic_{1,3}^2y+ic_{1,3}^3t+d_{1,3}+ie_{1,3}} b_{1,3,0}c_{1,3}
\end{aligned}$$

and

$$\begin{aligned}
d(x, y, t) &= b_{1,1,0} e^{ic_{1,1}x+itc_{1,1}^3-ic_{1,1}^2y+d_{1,1}+ie_{1,1}} \\
&+ b_{1,2,0} e^{ic_{1,2}x+itc_{1,2}^3-ic_{1,2}^2y+d_{1,2}+ie_{1,2}} \\
&+ b_{1,3,0} e^{ic_{1,3}x+itc_{1,3}^3-ic_{1,3}^2y+d_{1,3}+ie_{1,3}} + a_1
\end{aligned}$$

We give some figures in the  $(x, y)$  plane of coordinates



**Figure 3.** Solution of order 1 to KPI, on the left for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,3,0} = 3$ ,  $c_{1,1} = 1$ ,  $c_{1,2} = 2$ ,  $c_{1,3} = 3$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $d_{1,3} = 3$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ,  $e_{1,3} = 3$ ; in the center for  $t = 1$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,3,0} = 3$ ,  $c_{1,1} = 0, 1$ ,  $c_{1,2} = 0, 2$ ,  $c_{1,3} = 0, 3$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $e_{1,3} = 3$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ,  $d_{1,3} = 3$ ; on the right for  $t = 10$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,3,0} = 3$ ,  $c_{1,1} = 1$ ,  $c_{1,2} = 2$ ,  $c_{1,3} = 3$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $d_{1,3} = 3$ ,  $e_{1,1} = 0, 1$ ,  $e_{1,2} = 0, 2$ ,  $e_{1,3} = 0, 3$ .

## 4 Solutions of order 1 depending on $S(D + 4) + 1$ parameters with derivatives

In this section, we take the order  $N$  equal to 1. We make variations on the degrees of summation and derivation.

### 4.1 Case $S = 1$ , $D = 1$

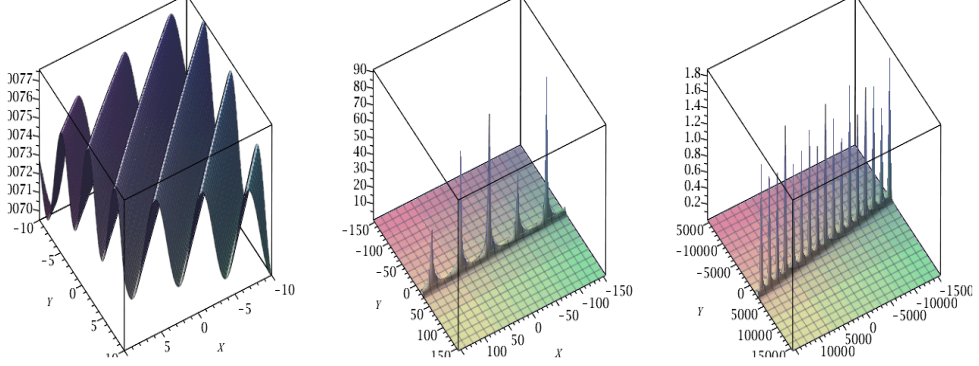
In this case solutions to the KPI equation can be written as

$$v(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

with

$$n(x, y, t) = -ie^{ic_{1,1}x-iy c_{1,1}^2+ic_{1,1}^3t+d_{1,1}+ie_{1,1}} b_{1,1,1} c_{1,1}^2 a_1 x - e^{ic_{1,1}x-iy c_{1,1}^2+ic_{1,1}^3t+d_{1,1}+ie_{1,1}}$$

$(3ib_{1,1,1}c_{1,1}^4ta_1 - 2ib_{1,1,1}c_{1,1}^3ya_1 + b_{1,1,0}c_{1,1}^2a_1 + 2c_{1,1}b_{1,1,1}a_1 - e^{ic_{1,1}x - icy_{1,1}^2 + ic_{1,1}^3t + d_{1,1} + ie_{1,1}}b_{1,1,1}^2)$   
 and  
 $d(x, y, t) = ib_{1,1,1}e^{ic_{1,1}x + itc_{1,1}^3 - icy_{1,1}^2 + d_{1,1} + ie_{1,1}}x + a_1 + b_{1,1,0}e^{ic_{1,1}x + itc_{1,1}^3 - icy_{1,1}^2 + d_{1,1} + ie_{1,1}} -$   
 $2ib_{1,1,1}e^{ic_{1,1}x + itc_{1,1}^3 - icy_{1,1}^2 + d_{1,1} + ie_{1,1}}yc_{1,1} + 3itb_{1,1,1}c_{1,1}^2e^{ic_{1,1}x + itc_{1,1}^3 - icy_{1,1}^2 + d_{1,1} + ie_{1,1}}$   
 We give some figures in the  $(x, y)$  plane of coordinates



**Figure 4.** Solution of order 1 to KPI, on the left for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  
 $b_{1,1,1} = 1$ ,  $c_{1,1} = 1$ ,  $d_{1,1} = 1$ ,  $e_{1,1} = 1$ ; in the center for  $t = 1$ ,  $a_1 = 1$ ,  
 $b_{1,1,0} = 10^2$ ,  $b_{1,1,1} = 2$ ,  $c_{1,1} = 10$ ,  $d_{1,1} = 1$ ,  $e_{1,1} = 1$ ; on the right for  $t = 10$ ,  
 $a_1 = 10$ ,  $b_{1,1,0} = 10$ ,  $b_{1,1,1} = 2$ ,  $c_{1,1} = 3$ ,  $d_{1,1} = 1$ ,  $e_{1,1} = 1$ .

## 4.2 Case $S = 1$ , $D = 2$

In this case solutions to the KPI equation can be written as

$$v(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

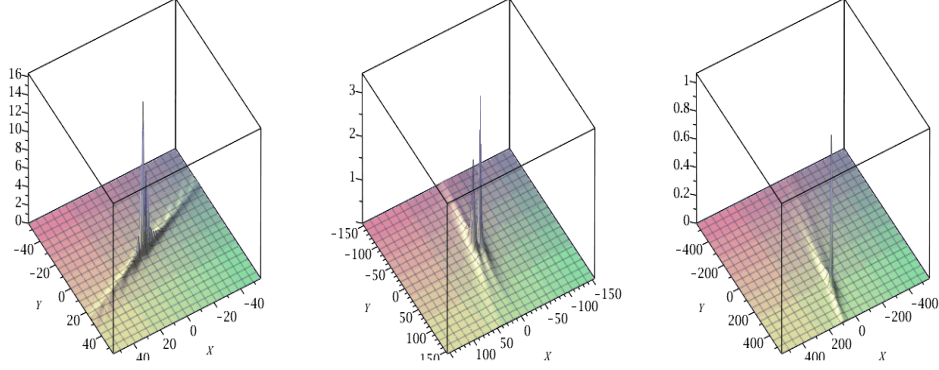
with

$$\begin{aligned}
 n(x, y, t) = & e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}} (-2b_{1,1,2}^2 e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}} + \\
 & b_{1,1,2}c_{1,1}^2a_1)x^2 + e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}} (6b_{1,1,2}c_{1,1}^4ta_1 - ib_{1,1,1}c_{1,1}^2a_1 \\
 & - 4ib_{1,1,2}c_{1,1}a_1 + 8b_{1,1,2}^2c_{1,1}e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}}y - 4b_{1,1,2}c_{1,1}^3ya_1 - \\
 & 12b_{1,1,2}^2c_{1,1}^2e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}}t + 2ib_{1,1,2}b_{1,1,1}e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}})x + \\
 & e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}} (6ib_{1,1,2}b_{1,1,1}e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}}tc_{1,1}^2 - \\
 & 18ib_{1,1,2}c_{1,1}^3ta_1 - 2b_{1,1,2}b_{1,1,0}e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}} + e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}}b_{1,1,1}^2 - \\
 & 2b_{1,1,1}c_{1,1}a_1 - b_{1,1,0}c_{1,1}^2a_1 - 2b_{1,1,2}a_1 - 12ib_{1,1,2}^2e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}}tc_{1,1} - \\
 & 12b_{1,1,2}c_{1,1}^5tya_1 - 3ib_{1,1,1}c_{1,1}^4ta_1 - 4ib_{1,1,2}b_{1,1,1}e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}}yc_{1,1} + \\
 & 2ib_{1,1,1}c_{1,1}^3ya_1 - 8b_{1,1,2}^2c_{1,1}^2y^2e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}} + 9b_{1,1,2}c_{1,1}^6t^2a_1 + \\
 & 24b_{1,1,2}^2c_{1,1}^3te^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}}y + 4b_{1,1,2}c_{1,1}^4y^2a_1 \\
 & + 4ib_{1,1,2}^2e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}}y + 10ib_{1,1,2}c_{1,1}^2ya_1 \\
 & - 18b_{1,1,2}^2c_{1,1}^4t^2e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}})
 \end{aligned}$$

and

$$d(x, y, t) = -b_{1,1,2}e^{ic_{1,1}x + itc_{1,1}^3 - ic_{1,1}^2y + d_{1,1} + ie_{1,1}}x^2 + (-6tb_{1,1,2}c_{1,1}^2e^{ic_{1,1}x + itc_{1,1}^3 - ic_{1,1}^2y + d_{1,1} + ie_{1,1}} +$$

$4yb_{1,1,2}c_{1,1}e^{ic_{1,1}x+itc_{1,1}^3-ic_{1,1}^2y+d_{1,1}+ie_{1,1}+ib_{1,1,1}e^{ic_{1,1}x+itc_{1,1}^3-ic_{1,1}^2y+d_{1,1}+ie_{1,1}})x+$   
 $a_1+b_{1,1,0}e^{ic_{1,1}x+itc_{1,1}^3-ic_{1,1}^2y+d_{1,1}+ie_{1,1}}-2ib_{1,1,2}e^{ic_{1,1}x+itc_{1,1}^3-ic_{1,1}^2y+d_{1,1}+ie_{1,1}}y+$   
 $3itb_{1,1,1}c_{1,1}^2e^{ic_{1,1}x+itc_{1,1}^3-ic_{1,1}^2y+d_{1,1}+ie_{1,1}}-2ib_{1,1,1}e^{ic_{1,1}x+itc_{1,1}^3-ic_{1,1}^2y+d_{1,1}+ie_{1,1}}yc_{1,1}+$   
 $6itb_{1,1,2}c_{1,1}e^{ic_{1,1}x+itc_{1,1}^3-ic_{1,1}^2y+d_{1,1}+ie_{1,1}}-9t^2b_{1,1,2}c_{1,1}^4e^{ic_{1,1}x+itc_{1,1}^3-ic_{1,1}^2y+d_{1,1}+ie_{1,1}}+$   
 $12tyb_{1,1,2}c_{1,1}^3e^{ic_{1,1}x+itc_{1,1}^3-ic_{1,1}^2y+d_{1,1}+ie_{1,1}}-4y^2b_{1,1,2}c_{1,1}^2e^{ic_{1,1}x+itc_{1,1}^3-ic_{1,1}^2y+d_{1,1}+ie_{1,1}}$   
 We give some figures in the  $(x, y)$  plane of coordinates



**Figure 5.** Solution of order 1 to KPI, on the left for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,1,1} = 1$ ,  $b_{1,1,2} = 1$ ,  $b_{1,1,3} = 1$ ,  $c_{1,1} = 1$ ,  $d_{1,1} = 1$ ,  $e_{1,1} = 1$ ; in the center for  $t = 1$ ,  $a_1 = 10^2$ ,  $b_{1,1,0} = 10$ ,  $b_{1,1,1} = 1$ ,  $b_{1,1,2} = 1$ ,  $c_{1,1} = 0.1$ ,  $d_{1,1} = 1$ ,  $e_{1,1} = 1$ ; on the right for  $t = 10^3$ ,  $a_1 = 10^2$ ,  $b_{1,1,0} = 10$ ,  $b_{1,1,1} = 1$ ,  $b_{1,1,2} = 1$ ,  $c_{1,1} = 1$ ,  $d_{1,1} = 1$ ,  $e_{1,1} = 1$ .

### 4.3 Case $S = 1$ , $D = 3$

In this case solutions to the KPI equation can be written as

$$v(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

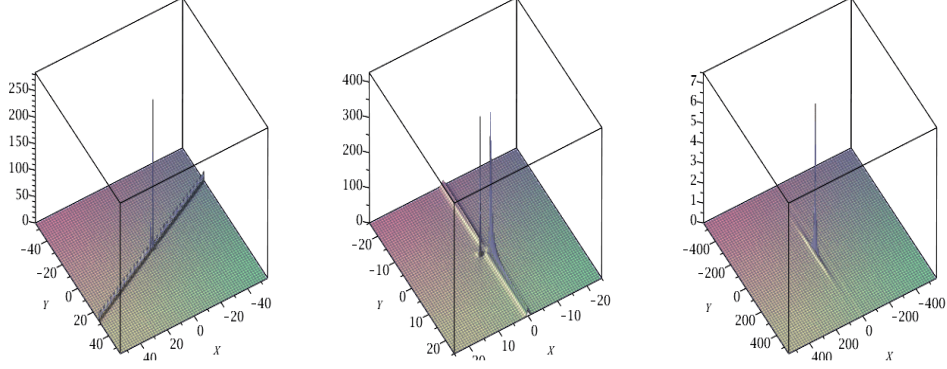
with

$$\begin{aligned}
 n(x, y, t) = & 3e^{ic_{1,1}x-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}b_{1,1,3}^2e^{ic_{1,1}x+23_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}x^4 + \\
 & e^{ic_{1,1}x-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}(ib_{1,1,3}c_{1,1}^2e^{3_{1,1}}a_1+36b_{1,1,3}^2e^{ic_{1,1}x+23_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}tc_{1,1}^2- \\
 & 4ib_{1,1,3}e^{ic_{1,1}x+3_{1,1}+2_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}b_{1,1,2}-24b_{1,1,3}^2e^{ic_{1,1}x+23_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}yc_{1,1})x^3+ \\
 & e^{ic_{1,1}x-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}(-2b_{1,1,2}^2e^{ic_{1,1}x+22_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}-6ib_{1,1,3}c_{1,1}^3e^{3_{1,1}}ya_1+ \\
 & 6b_{1,1,3}e^{3_{1,1}}c_{1,1}a_1+72b_{1,1,3}^2c_{1,1}^2e^{ic_{1,1}x+23_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}y^2 \\
 & +162b_{1,1,3}^2c_{1,1}^4e^{ic_{1,1}x+23_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}t^2-216b_{1,1,3}^2c_{1,1}^3e^{ic_{1,1}x+23_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}yt+ \\
 & 24ib_{1,1,3}c_{1,1}e^{ic_{1,1}x+3_{1,1}+2_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}b_{1,1,2}y+9ib_{1,1,3}c_{1,1}^4e^{3_{1,1}}ta_1+b_{1,1,2}c_{1,1}^2e^{2_{1,1}}a_1- \\
 & 36ib_{1,1,2}c_{1,1}^2e^{ic_{1,1}x+3_{1,1}+2_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}b_{1,1,3}t)x^2 \\
 & +e^{ic_{1,1}x-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}(12ib_{1,1,3}c_{1,1}^4e^{3_{1,1}}y^2a_1+2ib_{1,1,2}e^{ic_{1,1}x+2_{1,1}+1_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}b_{1,1,1}+ \\
 & 432b_{1,1,3}^2c_{1,1}^4e^{ic_{1,1}x+23_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}ty^2+8b_{1,1,2}^2c_{1,1}e^{ic_{1,1}x+22_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}y- \\
 & 108ib_{1,1,2}c_{1,1}^4e^{ic_{1,1}x+3_{1,1}+2_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}t^2b_{1,1,3}-36ib_{1,1,3}c_{1,1}^5e^{3_{1,1}}tya_1- \\
 & 48ib_{1,1,2}c_{1,1}^2e^{ic_{1,1}x+3_{1,1}+2_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}y^2b_{1,1,3}-96b_{1,1,3}^2e^{ic_{1,1}x+23_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}y^3c_{1,1}^3+
 \end{aligned}$$



$$\begin{aligned}
& 324 b_{1,1,3}^2 c_{1,1}^6 e^{ic_{1,1}x+23_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t^3} - 4 ib_{1,1,2} c_{1,1} e^{2i_{1,1}a_1} + 54 b_{1,1,3} c_{1,1}^3 e^{3i_{1,1}ta_1} - \\
& 30 b_{1,1,3} c_{1,1}^2 e^{3i_{1,1}ya_1} - 648 b_{1,1,3}^2 c_{1,1}^5 e^{ic_{1,1}x+23_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t^2} y - 6 ib_{1,1,3} e^{3i_{1,1}a_1} - \\
& 4 b_{1,1,2} c_{1,1}^3 e^{2i_{1,1}ya_1} + 36 b_{1,1,3}^2 e^{ic_{1,1}x+23_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} \\
& - 6 ib_{1,1,3} e^{ic_{1,1}x+3_{1,1}+d_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} b_{1,1,0} - ib_{1,1,1} c_{1,1}^2 e^{1i_{1,1}a_1} \\
& - 12 b_{1,1,2}^2 c_{1,1}^2 e^{ic_{1,1}x+22_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} + 27 ib_{1,1,3} c_{1,1}^6 e^{3i_{1,1}t^2} a_1 \\
& + 144 ib_{1,1,2} c_{1,1}^3 e^{ic_{1,1}x+3_{1,1}+2_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} y b_{1,1,3} t + 6 b_{1,1,2} c_{1,1}^4 e^{2i_{1,1}ta_1} x + \\
& e^{ic_{1,1}x-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} (32 ib_{1,1,2} c_{1,1}^3 e^{ic_{1,1}x+3_{1,1}+2_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} y^3 b_{1,1,3} + \\
& 6 ib_{1,1,1} c_{1,1}^2 e^{ic_{1,1}x+2_{1,1}+1_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} b_{1,1,2} t \\
& - 4 ib_{1,1,1} c_{1,1} e^{ic_{1,1}x+2_{1,1}+1_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} b_{1,1,2} y \\
& + 12 ib_{1,1,3} e^{ic_{1,1}x+3_{1,1}+d_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} y c_{1,1} b_{1,1,0} \\
& - 18 ib_{1,1,0} c_{1,1}^2 e^{ic_{1,1}x+3_{1,1}+d_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} b_{1,1,3} t \\
& + 36 ib_{1,1,1} c_{1,1} e^{ic_{1,1}x+3_{1,1}+1_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} b_{1,1,3} t \\
& - 108 ib_{1,1,2} c_{1,1}^6 e^{ic_{1,1}x+3_{1,1}+2_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} t^3 b_{1,1,3} - 60 ib_{1,1,3} e^{3i_{1,1}ta_1} t c_{1,1}^2 a_1 + \\
& 24 ib_{1,1,3} e^{3i_{1,1}ya_1} y c_{1,1} a_1 - 3 ib_{1,1,1} c_{1,1}^4 e^{1i_{1,1}ta_1} + 2 ib_{1,1,1} c_{1,1}^3 e^{1i_{1,1}ya_1} - 18 ib_{1,1,2} c_{1,1}^3 e^{2i_{1,1}ta_1} + \\
& 10 ib_{1,1,2} c_{1,1}^2 e^{2i_{1,1}ya_1} + 27 ib_{1,1,3} c_{1,1}^8 e^{3i_{1,1}t^3} a_1 - 8 ib_{1,1,3} c_{1,1}^5 e^{3i_{1,1}y^3} a_1 - 12 b_{1,1,2} c_{1,1}^5 e^{2i_{1,1}tya_1} - \\
& 126 b_{1,1,3} c_{1,1}^4 e^{3i_{1,1}tya_1} + b_{1,1,1}^2 e^{ic_{1,1}x+2_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} + 4 ib_{1,1,2}^2 e^{ic_{1,1}x+22_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} y - \\
& 216 b_{1,1,3}^2 e^{ic_{1,1}x+23_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} t^2 c_{1,1}^2 + 48 b_{1,1,3}^2 c_{1,1}^4 e^{ic_{1,1}x+23_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} y^4 + \\
& 243 b_{1,1,3}^2 c_{1,1}^8 e^{ic_{1,1}x+23_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} t^4 - 18 b_{1,1,2}^2 e^{ic_{1,1}x+22_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} t^2 c_{1,1}^4 - \\
& 8 b_{1,1,2}^2 e^{ic_{1,1}x+22_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} y^2 c_{1,1}^2 - 12 ib_{1,1,3} e^{ic_{1,1}x+3_{1,1}+1_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} y b_{1,1,1} - \\
& 12 ib_{1,1,2} e^{ic_{1,1}x+3_{1,1}+2_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} b_{1,1,3} t - 12 ib_{1,1,2}^2 e^{ic_{1,1}x+22_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} t c_{1,1} - \\
& b_{1,1,0} c_{1,1}^2 e^{d_{1,1}a_1} - 2 b_{1,1,1} c_{1,1} e^{1i_{1,1}a_1} - 36 b_{1,1,3}^2 e^{ic_{1,1}x+23_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} y^2 \\
& - 2 b_{1,1,2} e^{2i_{1,1}a_1} - 2 b_{1,1,2} e^{ic_{1,1}x+2_{1,1}+d_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} b_{1,1,0} \\
& + 216 ib_{1,1,2} c_{1,1}^5 e^{ic_{1,1}x+3_{1,1}+2_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} y b_{1,1,3} t^2 \\
& - 144 ib_{1,1,2} c_{1,1}^4 e^{ic_{1,1}x+3_{1,1}+2_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} y^2 b_{1,1,3} t - 54 ib_{1,1,3} c_{1,1}^7 e^{3i_{1,1}t^2} y a_1 + \\
& 36 ib_{1,1,3} c_{1,1}^6 e^{3i_{1,1}ty^2} a_1 + 144 b_{1,1,3}^2 e^{ic_{1,1}x+23_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} y c_{1,1} t \\
& - 288 b_{1,1,3}^2 c_{1,1}^5 e^{ic_{1,1}x+23_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} y^3 t - 648 b_{1,1,3}^2 c_{1,1}^7 e^{ic_{1,1}x+23_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} t^3 y + \\
& 648 b_{1,1,3}^2 c_{1,1}^6 e^{ic_{1,1}x+23_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} t^2 y^2 + 24 b_{1,1,2}^2 c_{1,1}^3 e^{ic_{1,1}x+22_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} y t + \\
& 9 b_{1,1,2} c_{1,1}^6 e^{2i_{1,1}t^2} a_1 + 36 b_{1,1,3} c_{1,1}^3 e^{3i_{1,1}y^2} a_1 + 108 b_{1,1,3} c_{1,1}^5 e^{3i_{1,1}t^2} a_1 + 4 b_{1,1,2} c_{1,1}^4 e^{2i_{1,1}y^2} a_1) \\
& \text{and} \\
& d(x, y, t) = -ib_{1,1,3} e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+3_{1,1}+ie_{1,1}t} x^3 + (-9 itb_{1,1,3} c_{1,1}^2 e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+3_{1,1}+ie_{1,1}t} + \\
& 6 i y b_{1,1,3} c_{1,1} e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+3_{1,1}+ie_{1,1}t} - b_{1,1,2} e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+2_{1,1}+ie_{1,1}t} x^2 + \\
& (-27 it^2 b_{1,1,3} c_{1,1}^4 e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+3_{1,1}+ie_{1,1}t} + 36 it y b_{1,1,3} c_{1,1}^3 e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+3_{1,1}+ie_{1,1}t} - \\
& 12 ib_{1,1,3} c_{1,1}^2 e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+3_{1,1}+ie_{1,1}t} y^2 - 6 t b_{1,1,2} c_{1,1}^2 e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+2_{1,1}+ie_{1,1}t} - \\
& 18 t b_{1,1,3} c_{1,1} e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+3_{1,1}+ie_{1,1}t} + 4 y b_{1,1,2} c_{1,1} e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+2_{1,1}+ie_{1,1}t} + \\
& 6 y b_{1,1,3} e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+3_{1,1}+ie_{1,1}t} + ib_{1,1,1} e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+1_{1,1}+ie_{1,1}t} x - \\
& 2 ib_{1,1,2} e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+2_{1,1}+ie_{1,1}t} y - 54 t^2 b_{1,1,3} c_{1,1}^3 e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+3_{1,1}+ie_{1,1}t} + \\
& 54 it^2 y b_{1,1,3} c_{1,1}^5 e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+3_{1,1}+ie_{1,1}t} + 3 it b_{1,1,1} c_{1,1}^2 e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+1_{1,1}+ie_{1,1}t} + \\
& a_1 + 6 it b_{1,1,2} c_{1,1} e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+2_{1,1}+ie_{1,1}t} + 54 t y b_{1,1,3} c_{1,1}^2 e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+3_{1,1}+ie_{1,1}t} - \\
& 4 y^2 b_{1,1,2} c_{1,1}^2 e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+2_{1,1}+ie_{1,1}t} - 2 ib_{1,1,1} e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+1_{1,1}+ie_{1,1}t} y c_{1,1} - \\
& 36 ib_{1,1,3} e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+3_{1,1}+ie_{1,1}t} t c_{1,1}^4 y^2 - 9 t^2 b_{1,1,2} c_{1,1}^4 e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+2_{1,1}+ie_{1,1}t} +
\end{aligned}$$

$12 tyb_{1,1,2}c_{1,1}^3 e^{ic_{1,1}x+itc_{1,1}^3-iy c_{1,1}^2+2i_{1,1}+ie_{1,1}} - 12 y^2 b_{1,1,3}c_{1,1} e^{ic_{1,1}x+itc_{1,1}^3-iy c_{1,1}^2+3i_{1,1}+ie_{1,1}} -$   
 $27 ib_{1,1,3} e^{ic_{1,1}x+itc_{1,1}^3-iy c_{1,1}^2+3i_{1,1}+ie_{1,1}} t^3 c_{1,1}^6 + 6 itb_{1,1,3} e^{ic_{1,1}x+itc_{1,1}^3-iy c_{1,1}^2+3i_{1,1}+ie_{1,1}} +$   
 $b_{1,1,0} e^{ic_{1,1}x+itc_{1,1}^3-iy c_{1,1}^2+d_{1,1}+ie_{1,1}} + 8 iy^3 b_{1,1,3}c_{1,1}^3 e^{ic_{1,1}x+itc_{1,1}^3-iy c_{1,1}^2+3i_{1,1}+ie_{1,1}}$   
 We give some figures in the  $(x, y)$  plane of coordinates



**Figure 6.** Solution of order 1 to KPI, on the left for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,1,1} = 1$ ,  $b_{1,1,2} = 1$ ,  $b_{1,1,3} = 1$ ,  $c_{1,1} = 1$ ,  $d_{1,1} = 1$ ,  $e_{1,1} = 1$ ; in the center for  $t = 1$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,1,1} = 1$ ,  $b_{1,1,2} = 1$ ,  $c_{1,1} = 0$ ,  $d_{1,1} = 1$ ,  $e_{1,1} = 1$ ; on the right for  $t = 10^4$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,1,1} = 1$ ,  $b_{1,1,2} = 1$ ,  $c_{1,1} = 0$ ,  $d_{1,1} = 1$ ,  $e_{1,1} = 1$ .

In the following, in the case where  $S$  is greater or equal to 2, to shortened the expressions of the solution we fix parameters  $a_i$ ,  $b_{j_s d}$ ,  $c_{j_s}$  equal to 1, and  $d_{j_s}$ ,  $e_{j_s}$  equal to 0 for all the values of the indices.

#### 4.4 Case $S = 2$ , $D = 1$

In this case solutions to the KPI equation can be written as

$$v(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

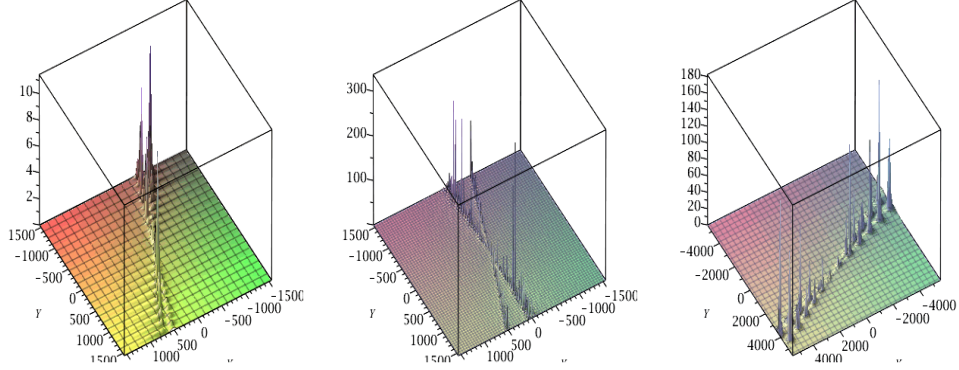
with

$$n(x, y, t) = -2ie^{ix-iy+it}x + 4ie^{ix-iy+it}y - 6ie^{ix-iy+it}t + 4e^{2ix-2iy+2it} - 6e^{ix-iy+it}$$

and

$$d(x, y, t) = 2ie^{i(x+t-y)}x + 1 + 2e^{i(x+t-y)} - 4iye^{i(x+t-y)} + 6ite^{i(x+t-y)}$$

We give some figures in the  $(x, y)$  plane of coordinates



**Figure 7.** Solution of order 1 to KPI, on the left for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,1,1} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,1,1} = 2$ ,  $c_{1,1} = 0.5$ ,  $c_{1,2} = 0.2$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ; in the center for  $t = 1$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,1,1} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,1,1} = 2$ ,  $c_{1,1} = 0.5$ ,  $c_{1,2} = 0.1$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ; on the right for  $t = 10^3$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,1,1} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,1,1} = 2$ ,  $c_{1,1} = 0.1$ ,  $c_{1,2} = 1$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ .

#### 4.5 Case $S = 2$ , $D = 2$

In this case solutions to the KPI equation can be written as

$$v(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

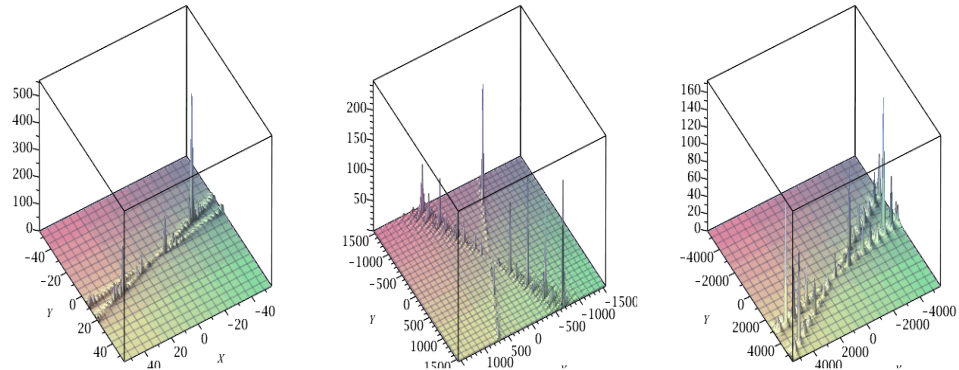
with

$$n(x, y, t) = (2e^{ix-iy+it} - 8e^{2ix-2iy+2it})x^2 + (-10ie^{ix-iy+it} + 12e^{ix-iy+it}t + 32e^{2ix-2iy+2it}y - 48e^{2ix-2iy+2it}t + 8ie^{2ix-2iy+2it} - 8e^{ix-iy+it}y)x - 24e^{ix-iy+it}yt + 8e^{ix-iy+it}y^2 + 18e^{ix-iy+it}t^2 - 32e^{2ix-2iy+2it}y^2 - 72e^{2ix-2iy+2it}t^2 - 24ie^{2ix-2iy+2it}t - 10e^{ix-iy+it} - 4e^{2ix-2iy+2it} + 24ie^{ix-iy+it}y + 96e^{2ix-2iy+2it}yt - 42ie^{ix-iy+it}t$$

and

$$d(x, y, t) = -2e^{i(x+t-y)}x^2 + (2ie^{i(x+t-y)} + 8ye^{i(x+t-y)} - 12te^{i(x+t-y)})x + 1 + 2e^{i(x+t-y)} - 8iye^{i(x+t-y)} - 8y^2e^{i(x+t-y)} - 18t^2e^{i(x+t-y)} + 18ite^{i(x+t-y)} + 24tye^{i(x+t-y)}$$

We give some figures in the  $(x, y)$  plane of coordinates



**Figure 8.** Solution of order 1 to KPI, on the left for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  
 $b_{1,1,1} = 1$ ,  $b_{1,1,2} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,2,1} = 2$ ,  $b_{1,2,2} = 2$ ,  $c_{1,1} = 0.1$ ,  $c_{1,2} = 2$ ,  
 $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ; in the center for  $t = 1$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  
 $b_{1,1,1} = 1$ ,  $b_{1,1,2} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,2,1} = 2$ ,  $b_{1,2,2} = 2$ ,  $c_{1,1} = 0.5$ ,  $c_{1,2} = 0.2$ ,  
 $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ; on the right for  $t = 10^3$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  
 $b_{1,1,1} = 1$ ,  $b_{1,1,2} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,2,1} = 2$ ,  $b_{1,2,2} = 2$ ,  $c_{1,1} = 0.1$ ,  $c_{1,2} = 1$ ,  
 $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ .

#### 4.6 Case $S = 2$ , $D = 3$

In this case solutions to the KPI equation can be written as

$$v(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

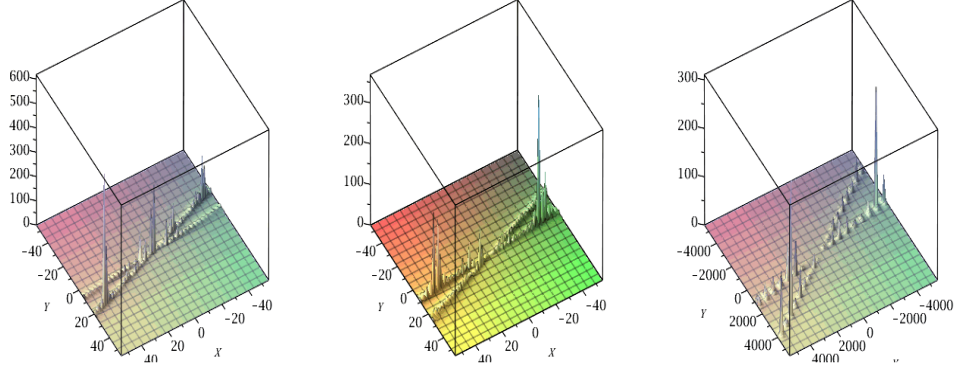
with

$$\begin{aligned} n(x, y, t) = & 12 e^{2ix-2iy+2it} x^4 + (-96 e^{2ix-2iy+2it} y - 16 i e^{2ix-2iy+2it} + 2 i e^{ix-iy+it} + \\ & 144 e^{2ix-2iy+2it} t) x^3 + (96 i e^{2ix-2iy+2it} y - 8 e^{2ix-2iy+2it} - 864 e^{2ix-2iy+2it} t y - \\ & 12 i e^{ix-iy+it} y + 18 i e^{ix-iy+it} t + 14 e^{ix-iy+it} - 144 i e^{2ix-2iy+2it} t^2 + 648 e^{2ix-2iy+2it} t^2 + \\ & 288 e^{2ix-2iy+2it} y^2) x^2 + (-432 i e^{2ix-2iy+2it} t^2 - 22 i e^{ix-iy+it} + 1296 e^{2ix-2iy+2it} t^3 - \\ & 2592 e^{2ix-2iy+2it} t^2 y + 576 i e^{2ix-2iy+2it} t y + 54 i e^{ix-iy+it} t^2 + 120 e^{ix-iy+it} t - \\ & 68 e^{ix-iy+it} y + 96 e^{2ix-2iy+2it} t + 32 e^{2ix-2iy+2it} y - 384 e^{2ix-2iy+2it} y^3 + 24 i e^{ix-iy+it} y^2 + \\ & 1728 e^{2ix-2iy+2it} t y^2 - 16 i e^{2ix-2iy+2it} - 192 i e^{2ix-2iy+2it} y^2 - 72 i e^{ix-iy+it} t y) x - \\ & 10 e^{ix-iy+it} - 4 e^{2ix-2iy+2it} - 16 i e^{ix-iy+it} y^3 + 54 i e^{ix-iy+it} t^3 + 2592 e^{2ix-2iy+2it} t^2 y^2 - \\ & 1152 e^{2ix-2iy+2it} t y^3 - 276 e^{ix-iy+it} y t + 672 e^{2ix-2iy+2it} t y - 162 i e^{ix-iy+it} t + \\ & 72 i e^{ix-iy+it} y - 432 i e^{2ix-2iy+2it} t^3 + 128 i e^{2ix-2iy+2it} y^3 + 192 e^{2ix-2iy+2it} y^4 - \\ & 936 e^{2ix-2iy+2it} t^2 + 80 e^{ix-iy+it} y^2 + 234 e^{ix-iy+it} t^2 - 176 e^{2ix-2iy+2it} y^2 - 2592 e^{2ix-2iy+2it} t^3 y + \\ & 972 e^{2ix-2iy+2it} t^4 + 72 i e^{ix-iy+it} y^2 t - 108 i e^{ix-iy+it} t^2 y - 576 i e^{2ix-2iy+2it} t y^2 + \\ & 864 i e^{2ix-2iy+2it} t^2 y \end{aligned}$$

and

$$\begin{aligned} d(x, y, t) = & 2 i e^{i(x+t-y)} x^3 + (2 e^{i(x+t-y)} + 18 i t e^{i(x+t-y)} - 12 i y e^{i(x+t-y)}) x^2 + \\ & (54 i e^{i(x+t-y)} t^2 + 24 i y^2 e^{i(x+t-y)} + 48 t e^{i(x+t-y)} - 20 y e^{i(x+t-y)} - 2 i e^{i(x+t-y)} - \\ & 72 i e^{i(x+t-y)} y t) x - 1 - 2 e^{i(x+t-y)} - 108 i t^2 y e^{i(x+t-y)} + 54 i t^3 e^{i(x+t-y)} + 72 i t y^2 e^{i(x+t-y)} + \\ & 8 i y e^{i(x+t-y)} - 16 i y^3 e^{i(x+t-y)} - 132 t y e^{i(x+t-y)} - 30 i t e^{i(x+t-y)} + 32 y^2 e^{i(x+t-y)} + \\ & 126 t^2 e^{i(x+t-y)} \end{aligned}$$

We give some figures in the  $(x, y)$  plane of coordinates



**Figure 9.** Solution of order 1 to KPI, on the left for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,1,1} = 1$ ,  $b_{1,1,2} = 1$ ,  $b_{1,1,3} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,2,1} = 2$ ,  $b_{1,2,2} = 2$ ,  $b_{1,2,3} = 1$ ,  $c_{1,1} = 1$ ,  $c_{1,2} = 2$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $e_{1,1} = 1$ ,  $d_{1,2} = 2$ ; in the center for  $t = 1$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,1,1} = 1$ ,  $b_{1,1,2} = 1$ ,  $b_{1,1,3} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,2,1} = 2$ ,  $b_{1,2,2} = 2$ ,  $b_{1,2,3} = 1$ ,  $c_{1,1} = 0$ ,  $c_{1,2} = 2$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $e_{1,1} = 1$ ,  $d_{1,2} = 2$ ; on the right for  $t = 10$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,1,1} = 1$ ,  $b_{1,1,2} = 1$ ,  $b_{1,1,3} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,2,1} = 2$ ,  $b_{1,2,2} = 2$ ,  $b_{1,2,3} = 1$ ,  $c_{1,1} = 0.1$ ,  $c_{1,2} = 2$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $e_{1,1} = 1$ ,  $d_{1,2} = 2$ .

## 5 Conclusion

With the Darboux transform, multi-parametric solutions to the KPI equation have been constructed. These solutions depend on multi-parameters; first, on the order of determinant  $N$ . They depend also on the degree of summation  $S$  and the degree of derivation  $D$ . In the case general case of an expression with a second derivative in  $x$  of a determinant of order  $N$ , we obtain solutions depending on  $N(S(D + 4) + 1)$  real parameters.

All these solutions are different from these already published in particular by the present author [42, 44, 50, 58].

We restrict the study to the case where  $c_{js}$  is a real parameter. We postpone to another publication the study in the case where  $c_{js}g_{js} + ih_{js}$  is a complex parameter.

We only give the expressions of the solution in the simple case of the first order  $N = 1$ . It would important to study in more details this first order and try to classify these solutions.

It would be also relevant to study the cases of order greater or equal to 2 and to realize an exhaustive classification of the solutions to the KPI equation.

## References

- [1] B.B. Kadomtsev, V.I. Petviashvili, On the stability of solitary waves in weakly dispersing media, *Sov. Phys. Dokl.*, V. 15, N. 6, 539-541, 1970
- [2] M.J. Ablowitz, H. Segur On the evolution of packets of water waves, *J. Fluid Mech.*, V. 92, 691-715, 1979
- [3] M.J. Ablowitz, P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, 1991
- [4] D.E. Pelinovsky, Y.A. Stepanyants, Y.A. Kivshar, Self-focusing of plane dark solitons in nonlinear defocusing media, *Phys. Rev. E*, V. 51, 5016-5026, 1995
- [5] S.V. Manakov, V.E. Zakharov, L.A. Bordag, V.B. Matveev, Two-dimensional solitons of the Kadomtsev-Petviashvili equation and their interaction, *Phys. Letters*, V. 63A, N. 3, 205-206, 1977
- [6] I. Krichever, Rational solutions of the Kadomtsev-Petviashvili equation and integrable systems of  $n$  particles on a line, *Funct. Anal. and Appl.*, V. 12, N. 1 : 76-78, 1978
- [7] I. Krichever, S.P. Novikov Holomorphic bundles over Riemann surfaces and the KPI equation, *Funkt. Ana. E Pril.*, V. 12, 41-52, 1979
- [8] J. Satsuma, M.J. Ablowitz, Two-dimensional lumps in nonlinear dispersive systems, *J. Math. Phys.*, V. 20, 1496-1503, 1979
- [9] V.B. Matveev, Darboux transformation and explicit solutions of the Kadomtsev-Petviashvili equation depending on functional parameters, *Letters in Mathematical Physics*, V. 3, 213-216, 1979
- [10] N. C Freeman, J.J.C. Nimmo Rational solutions of the KdV equation in wronskian form, *Phys. Letters*, V. 96 A, N. 9, 443-446, 1983
- [11] N. C Freeman, J.J.C. Nimmo The use of Bäcklund transformations in obtaining  $N$ -soliton solutions in wronskian form, *J. Phys. A : Math. Gen.*, V. 17 A, 1415-1424, 1984
- [12] P. Gaillard, V.B. Matveev, Wronskian addition formula and its applications, *Max-Planck-Institut für Mathematik, MPI 02-31*, V. **161**, 2002
- [13] P. Gaillard, A new family of deformations of Darboux-Pöschl-Teller potentials, *Lett. Math. Phys.*, V. **68**, 77-90, 2004
- [14] P. Gaillard, V.B. Matveev, New formulas for the eigenfunctions of the two-particle Calogero-Moser system, *Lett. Math. Phys.*, V. **89**, 1-12, 2009
- [15] P. Gaillard, V.B. Matveev, Wronskian and Casorai determinant representations for Darboux-Pöschl-Teller potentials and their difference extensions, *J. Phys A : Math. Theor.*, V. **42**, 404409-1-16, 2009

- [16] P. Dubard, P. Gaillard, C. Klein, V. B. Matveev, On multi-rogue wave solutions of the NLS equation and positon solutions of the KdV equation, *Eur. Phys. J. Spe. Top.*, V. **185**, 247-258, 2010
- [17] P. Gaillard, Families of quasi-rational solutions of the NLS equation and multi-rogue waves, *J. Phys. A : Meth. Theor.*, V. **44**, 435204-1-15, 2011
- [18] P. Gaillard, Wronskian representation of solutions of the NLS equation and higher Peregrine breathers, *J. Math. Sciences : Adv. Appl.*, V. **13**, N. 2, 71-153, 2012
- [19] P. Gaillard, Degenerate determinant representation of solution of the NLS equation, higher Peregrine breathers and multi-rogue waves, *J. Math. Phys.*, V. **54**, 013504-1-32, 2013
- [20] P. Gaillard, Wronskian representation of solutions of NLS equation and seventh order rogue waves, *J. Mod. Phys.*, V. **4**, N. 4, 246-266, 2013
- [21] P. Gaillard, V.B. Matveev, Wronskian addition formula and Darboux-Pöschl-Teller potentials, *J. Math.*, V. **2013**, ID 645752, 1-10, 2013
- [22] P. Gaillard, Two-parameters determinant representation of seventh order rogue waves solutions of the NLS equation, *J. Theor. Appl. Phys.*, V. **7**, N. **45**, 1-6, 2013
- [23] P. Gaillard, Deformations of third order Peregrine breather solutions of the NLS equation with four parameters, *Phys. Rev. E*, V. **88**, 042903-1-9, 2013
- [24] P. Gaillard, Two parameters deformations of ninth Peregrine breather solution of the NLS equation and multi rogue waves, *J. Math.*, V. **2013**, 1-111, 2013
- [25] P. Gaillard, Six-parameters deformations of fourth order Peregrine breather solutions of the NLS equation, *J. Math. Phys.*, V. **54**, 073519-1-22, 2013
- [26] P. Gaillard, The fifth order Peregrine breather and its eight-parameters deformations solutions of the NLS equation, *Commun. Theor. Phys.*, V. **61**, 365-369, 2014
- [27] P. Gaillard, Ten parameters deformations of the sixth order Peregrine breather solutions of the NLS equation, *Phys. Scripta*, V. **89**, 015004-1-7, 2014
- [28] P. Gaillard, Higher order Peregrine breathers, their deformations and multi-rogue waves, *J. Of Phys. : Conf. Ser.*, V. **482**, 012016-1-7, 2014
- [29] P. Gaillard, M. Gastineau, Eighteen parameter deformations of the Peregrine breather of order ten solutions of the NLS equation, *Int. J. Mod. Phys. C*, V. **26**, N. 2, 1550016-1-14, 2014

- [30] P. Gaillard, Two parameters wronskian representation of solutions of nonlinear Schrödinger equation, eight Peregrine breather and multi-rogue waves, *J. Math. Phys.*, V. **5**, 093506-1-12, 2014
- [31] P. Gaillard, Hierarchy of solutions to the NLS equation and multi-rogue waves, *J. Phys. : Conf. Ser.*, V. **574**, 012031-1-5, 2015
- [32] P. Gaillard, Tenth Peregrine breather solution of the NLS, *Ann. Phys.*, V. **355**, 293-298, 2015
- [33] P. Gaillard, M. Gastineau, The Peregrine breather of order nine and its deformations with sixteen parameters solutions of the NLS equation *Phys. Lett. A*, V. **379**, 1309-1313, 2015
- [34] P. Gaillard, Other  $2N-2$  parameters solutions to the NLS equation and  $2N+1$  highest amplitude of the modulus of the  $N$ -th order AP breather, *J. Phys. A: Math. Theor.*, V. **48**, 145203-1-23, 2015
- [35] P. Gaillard, Multi-parametric deformations of the Peregrine breather of order  $N$  solutions to the NLS equation and multi-rogue waves, *Adv. Res.*, V. **4**, N. **5**, 346-364, 2015
- [36] P. Gaillard, Higher order Peregrine breathers solutions to the NLS equation, *Jour. Phys. : Conf. Ser.*, V. **633**, 012106-1-6, 2016
- [37] P. Gaillard, M. Gastineau Patterns of deformations of Peregrine breather of order 3 and 4, solutions to the NLS equation with multi-parameters, *Journal of Theoretical and Applied Physics*, V. **10**,1-7, 2016
- [38] P. Gaillard, M. Gastineau Twenty parameters families of solutions to the NLS equation and the eleventh Peregrine breather, *Commun. Theor. Phys*, V. **65**, N. **2**, 136-144, 2016
- [39] P. Gaillard, Rational solutions to the KPI equation and multi rogue waves, *Annals Of Physics*, V. **367**, 1-5, 2016
- [40] P. Gaillard, M. Gastineau Twenty two parameters deformations of the twelfth Peregrine breather solutions to the NLS equation, *Adv. Res.*, V. **10**, 83-89, 2016
- [41] P. Gaillard, Towards a classification of the quasi rational solutions to the NLS equation, *Theor. And Math. Phys.*, V. **189**, N. **1**, 1440-1449, 2016
- [42] P. Gaillard, Fredholm and Wronskian representations of solutions to the KPI equation and multi-rogue waves, *Jour. of Math. Phys.*, V. **57**, 063505-1-13, doi: 10.1063/1.4953383, 2016
- [43] P. Gaillard, M. Gastineau Families of deformations of the thirteenth Peregrine breather solutions to the NLS equation depending on twenty four parameters, *Jour. Of Bas. And Appl. Res. Int.*, V. **21**, N. 3, 130-139, 2017



- [44] P. Gaillard, From Fredholm and Wronskian representations to rational solutions to the KPI equation depending on  $2N^2$  parameters, *Int. Jour. of Appl. Sci. And Math.*, V. **4**, N. 3, 60-70, 2017
- [45] P. Gaillard, Families of rational solutions of order 5 to the KPI equation depending on 8 parameters, *New Hor. in Math. Phys.*, V. 1, N. 1, 26-31, 2017
- [46] P. Gaillard, 6-th order rational solutions to the KPI Equation depending on 10 parameters, *Jour. Of Bas. And Appl. Res. Int.*, V. **21**, N. 2, 92-98, 2017
- [47] P. Gaillard,  $N$ -Order rational solutions to the Johnson equation depending on  $2N - 2$  parameters, *Int. Jour. of Adv. Res. in Phys. Sci.*, V. **4**, N. 9, 19-37, 2017
- [48] P. Gaillard, Families of rational solutions to the KPI equation of order 7 depending on 12 parameters, *Int. Jour. of Adv. Res. in Phys. Sci.*, V. **4**, N. 11, 24-30, 2017
- [49] P. Gaillard, Rational solutions to the Johnson equation and rogue waves, *Int. Jour. of Inn. In Sci. and Math.*, V. **6**, N. 1, 14-19, 2018
- [50] P. Gaillard, Multiparametric families of solutions of the KPI equation, the structure of their rational representations and multi-rogue waves, *Theo. And Mat. Phys.*, V. **196**, N. 2, 1174-1199, 2018
- [51] P. Gaillard, The Johnson equation, Fredholm and wronskian representations of solutions and the case of order three, *Adv. In Math. Phys.*, V. **2018**, 1-18, 2018
- [52] P. Gaillard, Families of Solutions of Order 5 to the Johnson Equation Depending on 8 Parameters, *NHIMP*, V. **2**, N. 4, 53-61, 2018
- [53] P. Gaillard, Multiparametric families of solutions to the Johnson equation, *J. Phys. : Conf. Series*, V. **1141**, 012102-1-10, 2018
- [54] P. Gaillard, Rational solutions to the Boussinesq equation, *Fund. Jour. Of Math. And Appl.*, V. , 109-112, 2019
- [55] P. Gaillard, Differential relations for the solutions to the NLS equation and their different representations, *Comm. In Adv. Math. Sci.*, V. **2**, N. 4, 1-4, 2019
- [56] P. Gaillard, Multi-parametric families of solutions of order  $N$  to the Boussinesq and KP equations and the degenerate rational case, *UJMA*, V. **3**, N. 2, 44-52, 2020
- [57] P. Gaillard, The mKdV equation and multi-parameters rational solutions, *Wave Motion*, V. **100**, 102667-1-9, 2021

- [58] P. Gaillard, Degenerate Riemann theta functions, Fredholm and wronskian representations of the solutions to the KdV equation and the degenerate rational case, *Jour. Of Geom. And Phys.*, V. **161**, 104059-1-12, 2021
- [59] P. Gaillard, 2N parameter solutions to the Burgers equation, *Jour. Of Appl. Nonlin. Dyn.*, V. **1**, N. **1**, 1-6, 2021
- [60] P. Gaillard, Multi-Parametric rational Solutions to the KdV equation, *Asi. Jour. Of Res. and Rev. In Phys.*, V. **3**, N. **3**, 14-21, 2021
- [61] P. Gaillard, Multiparametric solutions to the Gardner equation and the degenerate rational case, *Jour. Of Appl. Ana. And Appl.*, V. **11**, N. **4**, 2102-2113, 2021
- [62] P. Gaillard, Other families of rational solutions to the KPI equation, *Asi. Res. Jour Of Math.*, V. **17**, N. **6**, 27-34, 2021
- [63] P. Gaillard, Rational solutions to the mKdV equation associated to particular polynomials, *Wave Motion*, V. **107**, 102824-1-11, 2021
- [64] P. Gaillard, Multiparametric Rational Solutions of Order  $N$  to the KPI Equation and the Explicit Case of Order 3, *Arch. Of cur. Res. Int.*, V. **21**, N. **6**, 58-71, 2021
- [65] P. Gaillard, Rational solutions to the KPI equation from particular polynomials, *Wave Motion*, V. **108**, 102828-1-9, 2022
- [66] P. Gaillard, Rogue Waves of the Lakshmanan Porsezian Daniel Equation Depending on Multi-parameters, *As. Jour. Of Adv. Res. And Rep.*, V **16**, N. **3**, 32-40, 2022
- [67] P. Gaillard, Quasi-rational and rational solutions to the defocusing NLS equation, *Jour. Of Math. Sci. And Mod. Res. And Rep.*, V **5**, N. **1**, 24-34, 2022
- [68] P. Gaillard, Rational solutions to the KdV equation in terms of particular polynomials, *Jour. Of Nonlin. Mod. and Ana.*, V. **4**, N. **4**, 615-626, 2022
- [69] P. Gaillard, Rational solutions to the Painlevé II equation from particular polynomials, *Int. J. Of Appl. Nonlin. Sci.*, V. **3**, N. **3**, 189-196, 2022
- [70] P. Gaillard, Multi-lump solutions to the KPI equation with a zero degree of derivation, *Jour. Of Theor. And Appl. Phys.*, V. **16**, N. **4**, 1-8, 2022