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An extended Darboux transformation to get families of solutions to the KPI equation

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Abstract

By means of a Darboux transform with particular generating function solutions to the Kadomtsev-Petviashvili equation (KPI) are constructed. We give a method that provides different types of solutions in terms of particular determinants of order N . For any order, these solutions depend of the degree of summation and the degree of derivation of the generating functions.

We study the patterns of their modulus in the plane (x, y) and their evolution according time and parameters.

Keywords: KPI equation; Darboux transformation; wronskians; rational solutions.

PACS numbers :

33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd

1 Introduction

Kadomtsev-Petviashvili equation (KPI) is considered in the following normalization

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0, \quad (1)$$

where subscripts x , y and t denote partial derivatives.

In 1970 Kadomtsev and Petviashvili [1] proposed first this equation as a universal model describing weakly nonlinear waves in media with dispersion of velocity. It is used in hydrodynamics for surface and internal water waves [2, 3],

and in nonlinear optics [4]. The inverse scattering transform (IST) was used by Zakharov to solve this KPI equation.

The first rational solutions were constructed by Manakov, Zakharov, Bordag and Matveev [5] in 1977. The works of Krichever [6, 7] showed that the dynamics of the lumps of the KPI equation is governed by the Calogero-Moser system. Among many researches, we can mention the following works of Satsuma and Ablowitz in 1979 [8], Matveev in 1979 [9], Freeman and Nimmo in 1983 [10, 11].

Solutions of the KPI equation are constructed in terms of wronskians of order N depending on several real parameters. We construct here some explicit solutions of order 1, depending on several real parameters, and the representations of their modulus in the plane of the coordinates (x, y) according to parameters and time t .

2 Multi-parametric solutions of order N to the KPI equation

Let the Kadomtsev-Petviashvili (KPI) equation be written in the following normalization

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0.$$

We denote $W(f_1, \dots, f_N)$ the wronskian of order N of the functions f_1, \dots, f_N , defined by $\det(\partial_x^{i-1} f_j)_{1 \leq i \leq N, 1 \leq j \leq N}$, ∂_x^i being the partial derivative of order i with respect to x and $\partial_x^0 f_j$ being the function f_j .

Let $a_j, b_{j,s,d}, c_{j,s}, d_{j,s}, e_{j,s}$ be arbitrary real numbers.

Let $f_{j,s}$ be the following elementary functions defined by

$$f_{j,s}(x, y, t) = e^{ic_{j,s}x - ic_{j,s}^2 y + ic_{j,s}^3 t}. \quad (2)$$

Then, we have the following result :

Theorem 2.1 *Let ψ_j , be the functions defined by*

$$\psi_j(x, y, t) = a_j + \sum_{s=1}^S \sum_{d=0}^D b_{j,s,d} \partial_{c_{j,s}}^d e^{ic_{j,s}x - ic_{j,s}^2 y + ic_{j,s}^3 t} = a_j + \sum_{s=1}^S \sum_{d=0}^D b_{j,s,d} \partial_{c_{j,s}}^d f_{j,s}(x, y, t). \quad (3)$$

then the function u defined by

$$u(x, y, t) = -2\partial_x^2 \ln(W(\psi_1, \dots, \psi_N)) \quad (4)$$

is a solution to the (KPI) equation (1) depending on real $N(S(D+4)+1)$ parameters $a_j, b_{j,s,d}, c_{j,s}, d_{j,s}, e_{j,s}, 1 \leq j \leq N, 1 \leq s \leq S, 0 \leq d \leq D$.

The order N of the wronskian will be called the order of the solution. The degree of the summation of the solution will be the number S of terms of the

summation and the degree of derivation of the solution with the number D . For each order N of the solution there is a lot of choices concerning the degrees of the summation and derivation.

In the following, we give some examples.

3 Solutions of order 1 depending on $4S + 1$ parameters without derivative

In this section, we take equal N to 1. We make variations on the degrees of summation.

3.1 Case $S = 1, (D = 0)$

In this case solutions to the KPI equation can be written as

$$v(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

with

$$n(x, y, t) = -b_{1,1,0}c_{1,1}^2 e^{ic_{1,1}x - ic_{1,1}^2 y + ic_{1,1}^3 t + d_{1,1} + ie_{1,1} a_1}$$

and

$$d(x, y, t) = b_{1,1,0} e^{ic_{1,1}x + itc_{1,1}^3 - ic_{1,1}^2 y + d_{1,1} + ie_{1,1} a_1} + a_1$$

We give some figures in the (x, y) plane of coordinates

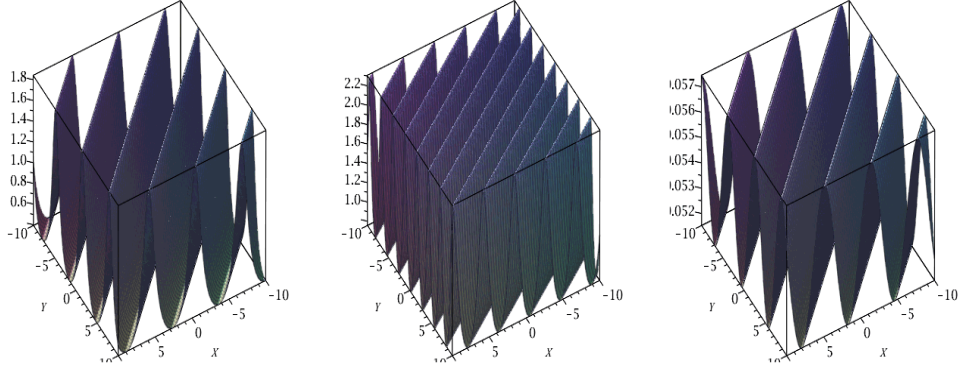


Figure 1. Solution of order 1 to KPI, on the left for $t = 0, a_1 = 1, b_{1,1,0} = 1, c_{1,1} = 1, d_{1,1} = 1, e_{1,1} = 1$; in the center for $t = 1, a_1 = 10^2, b_{1,1,0} = 10, c_{1,1} = 1.5, d_{1,1} = 1, e_{1,1} = 1$; on the right for $t = 10^3, a_1 = 10^2, b_{1,1,0} = 1, c_{1,1} = 1, d_{1,1} = 1, e_{1,1} = 1$.

3.2 Case $S = 2, (D = 0)$

In this case solutions to the KPI equation can be written as

$$v(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

with

$$\begin{aligned}
n(x, y, t) = & -b_{1,1,0}c_{1,1}^2 e^{ic_{1,1}x - ic_{1,1}^2 y + ic_{1,1}^3 t + d_{1,1} + ie_{1,1}} a_1 \\
& -b_{1,1,0}c_{1,1}^2 e^{ic_{1,1}x + ic_{1,2}x + ic_{1,1}^3 t + d_{1,1} + ie_{1,1} - ic_{1,1}^2 y - ic_{1,2}^2 y + ic_{1,2}^3 t + d_{1,2} + ie_{1,2}} b_{1,2,0} \\
& -b_{1,2,0}c_{1,2}^2 e^{ic_{1,2}x - ic_{1,2}^2 y + ic_{1,2}^3 t + d_{1,2} + ie_{1,2}} a_1 \\
& -b_{1,2,0}c_{1,2}^2 e^{ic_{1,1}x + ic_{1,2}x + ic_{1,1}^3 t + d_{1,1} + ie_{1,1} - ic_{1,1}^2 y - ic_{1,2}^2 y + ic_{1,2}^3 t + d_{1,2} + ie_{1,2}} b_{1,1,0} \\
& + 2b_{1,1,0}c_{1,1} e^{ic_{1,1}x + ic_{1,2}x + ic_{1,1}^3 t + d_{1,1} + ie_{1,1} - ic_{1,1}^2 y - ic_{1,2}^2 y + ic_{1,2}^3 t + d_{1,2} + ie_{1,2}} b_{1,2,0} c_{1,2}
\end{aligned}$$

and

$$\begin{aligned}
d(x, y, t) = & b_{1,1,0} e^{ic_{1,1}x + itc_{1,1}^3 - ic_{1,1}^2 y + d_{1,1} + ie_{1,1}} \\
& + b_{1,2,0} e^{ic_{1,2}x + itc_{1,2}^3 - ic_{1,2}^2 y + d_{1,2} + ie_{1,2}} + a_1
\end{aligned}$$

We give some figures in the (x, y) plane of coordinates

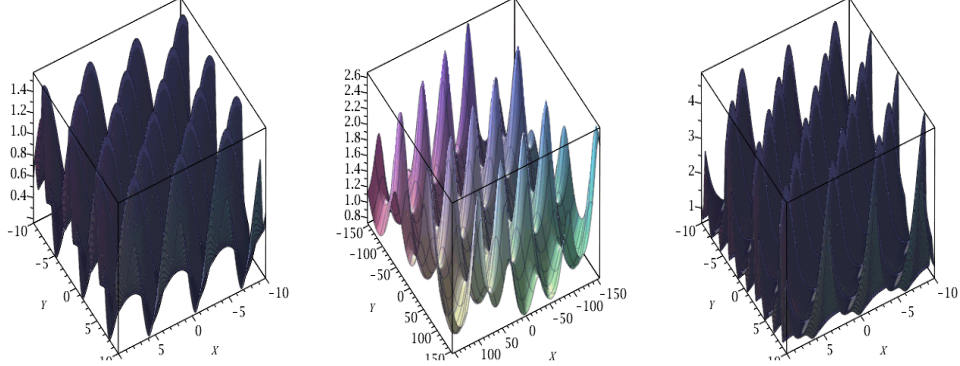


Figure 2. Solution of order 1 to KPI, on the left for $t = 0$, $a_1 = 1$, $b_{1,1,0} = 1$, $b_{1,2,0} = 2$, $c_{1,1} = 1$, $c_{1,2} = 2$, $d_{1,1} = 1$, $d_{1,2} = 2$, $e_{1,1} = 1$, $e_{1,2} = 2$; in the center for $t = 1$, $a_1 = 10^2$, $b_{1,1,0} = 10$, $b_{1,2,0} = 2$, $c_{1,1} = 0, 1$, $c_{1,2} = 2$, $d_{1,1} = 1$, $d_{1,2} = 2$, $e_{1,1} = 1$, $e_{1,2} = 2$; on the right for $t = 10^2$, $a_1 = 10^3$, $b_{1,1,0} = 10$, $b_{1,2,0} = 2$, $c_{1,1} = 1$, $c_{1,2} = 2$, $d_{1,1} = 1$, $d_{1,2} = 2$, $e_{1,1} = 1$, $e_{1,2} = 2$.

3.3 Case $S = 3$, ($D = 0$)

In this case solutions to the KPI equation can be written as

$$v(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

with

$$\begin{aligned}
n(x, y, t) = & -b_{1,1,0}c_{1,1}^2 e^{ic_{1,1}x - ic_{1,1}^2 y + ic_{1,1}^3 t + d_{1,1} + ie_{1,1}} a_1 \\
& -b_{1,1,0}c_{1,1}^2 e^{ic_{1,1}x + ic_{1,2}x + ic_{1,1}^3 t + d_{1,1} + ie_{1,1} - ic_{1,1}^2 y - ic_{1,2}^2 y + ic_{1,2}^3 t + d_{1,2} + ie_{1,2}} b_{1,2,0} \\
& -b_{1,1,0}c_{1,1}^2 e^{ic_{1,1}x + ic_{1,3}x + ic_{1,1}^3 t + d_{1,1} + ie_{1,1} - ic_{1,1}^2 y - ic_{1,3}^2 y + ic_{1,3}^3 t + d_{1,3} + ie_{1,3}} b_{1,3,0} \\
& -b_{1,2,0}c_{1,2}^2 e^{ic_{1,2}x - ic_{1,2}^2 y + ic_{1,2}^3 t + d_{1,2} + ie_{1,2}} a_1 \\
& -b_{1,2,0}c_{1,2}^2 e^{ic_{1,1}x + ic_{1,2}x + ic_{1,1}^3 t + d_{1,1} + ie_{1,1} - ic_{1,1}^2 y - ic_{1,2}^2 y + ic_{1,2}^3 t + d_{1,2} + ie_{1,2}} b_{1,1,0} \\
& -b_{1,2,0}c_{1,2}^2 e^{ic_{1,2}x + ic_{1,3}x + ic_{1,2}^3 t + d_{1,2} + ie_{1,2} - ic_{1,2}^2 y - ic_{1,3}^2 y + ic_{1,3}^3 t + d_{1,3} + ie_{1,3}} b_{1,3,0} \\
& -b_{1,3,0}c_{1,3}^2 e^{ic_{1,3}x - ic_{1,3}^2 y + ic_{1,3}^3 t + d_{1,3} + ie_{1,3}} a_1 \\
& -b_{1,3,0}c_{1,3}^2 e^{ic_{1,1}x + ic_{1,3}x + ic_{1,1}^3 t + d_{1,1} + ie_{1,1} - ic_{1,1}^2 y - ic_{1,3}^2 y + ic_{1,3}^3 t + d_{1,3} + ie_{1,3}} b_{1,1,0}
\end{aligned}$$

$$\begin{aligned}
& -b_{1,3,0}c_{1,3}^2 e^{ic_{1,2}x+ic_{1,3}x+ic_{1,2}^3t+d_{1,2}+ie_{1,2}-ic_{1,2}^2y-ic_{1,3}^2y+ic_{1,3}^3t+d_{1,3}+ie_{1,3}} b_{1,2,0} \\
& +2b_{1,1,0}c_{1,1} e^{ic_{1,1}x+ic_{1,2}x+ic_{1,1}^3t+d_{1,1}+ie_{1,1}-ic_{1,1}^2y-ic_{1,2}^2y+ic_{1,2}^3t+d_{1,2}+ie_{1,2}} b_{1,2,0}c_{1,2} + \\
& 2b_{1,1,0}c_{1,1} e^{ic_{1,1}x+ic_{1,3}x+ic_{1,1}^3t+d_{1,1}+ie_{1,1}-ic_{1,1}^2y-ic_{1,3}^2y+ic_{1,3}^3t+d_{1,3}+ie_{1,3}} b_{1,3,0}c_{1,3} + \\
& 2b_{1,2,0}c_{1,2} e^{ic_{1,2}x+ic_{1,3}x+ic_{1,2}^3t+d_{1,2}+ie_{1,2}-ic_{1,2}^2y-ic_{1,3}^2y+ic_{1,3}^3t+d_{1,3}+ie_{1,3}} b_{1,3,0}c_{1,3}
\end{aligned}$$

and

$$\begin{aligned}
d(x, y, t) &= b_{1,1,0} e^{ic_{1,1}x+itc_{1,1}^3-ic_{1,1}^2y+d_{1,1}+ie_{1,1}} \\
&+ b_{1,2,0} e^{ic_{1,2}x+itc_{1,2}^3-ic_{1,2}^2y+d_{1,2}+ie_{1,2}} \\
&+ b_{1,3,0} e^{ic_{1,3}x+itc_{1,3}^3-ic_{1,3}^2y+d_{1,3}+ie_{1,3}} + a_1
\end{aligned}$$

We give some figures in the (x, y) plane of coordinates

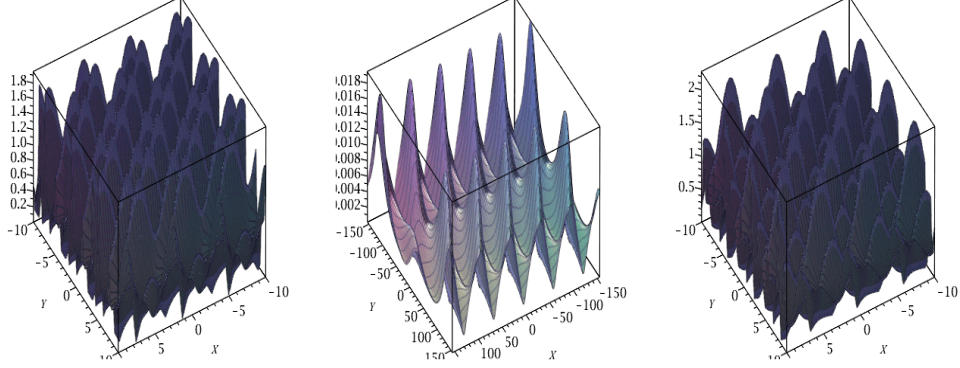


Figure 3. Solution of order 1 to KPI, on the left for $t = 0$, $a_1 = 1$, $b_{1,1,0} = 1$, $b_{1,2,0} = 2$, $b_{1,3,0} = 3$, $c_{1,1} = 1$, $c_{1,2} = 2$, $c_{1,3} = 3$, $d_{1,1} = 1$, $d_{1,2} = 2$, $d_{1,3} = 3$, $e_{1,1} = 1$, $e_{1,2} = 2$, $e_{1,3} = 3$; in the center for $t = 1$, $a_1 = 1$, $b_{1,1,0} = 1$, $b_{1,2,0} = 2$, $b_{1,3,0} = 3$, $c_{1,1} = 0, 1$, $c_{1,2} = 0, 2$, $c_{1,3} = 0, 3$, $d_{1,1} = 1$, $d_{1,2} = 2$, $e_{1,3} = 3$, $e_{1,1} = 1$, $e_{1,2} = 2$, $d_{1,3} = 3$; on the right for $t = 10$, $a_1 = 1$, $b_{1,1,0} = 1$, $b_{1,2,0} = 2$, $b_{1,3,0} = 3$, $c_{1,1} = 1$, $c_{1,2} = 2$, $c_{1,3} = 3$, $d_{1,1} = 1$, $d_{1,2} = 2$, $d_{1,3} = 3$, $e_{1,1} = 0, 1$, $e_{1,2} = 0, 2$, $e_{1,3} = 0, 3$.

4 Solutions of order 1 depending on $S(D + 4) + 1$ parameters with derivatives

In this section, we take the order N equal to 1. We make variations on the degrees of summation and derivation.

4.1 Case $S = 1$, $D = 1$

In this case solutions to the KPI equation can be written as

$$v(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

with

$$n(x, y, t) = -ie^{ic_{1,1}x-iy c_{1,1}^2+ic_{1,1}^3t+d_{1,1}+ie_{1,1}} b_{1,1,1} c_{1,1}^2 a_1 x - e^{ic_{1,1}x-iy c_{1,1}^2+ic_{1,1}^3t+d_{1,1}+ie_{1,1}}$$

$(3ib_{1,1,1}c_{1,1}^4ta_1 - 2ib_{1,1,1}c_{1,1}^3ya_1 + b_{1,1,0}c_{1,1}^2a_1 + 2c_{1,1}b_{1,1,1}a_1 - e^{ic_{1,1}x - iyc_{1,1}^2 + ic_{1,1}^3t + d_{1,1} + ie_{1,1}}b_{1,1,1}^2)$
 and
 $d(x, y, t) = ib_{1,1,1}e^{ic_{1,1}x + itc_{1,1}^3 - iyc_{1,1}^2 + d_{1,1} + ie_{1,1}}x + a_1 + b_{1,1,0}e^{ic_{1,1}x + itc_{1,1}^3 - iyc_{1,1}^2 + d_{1,1} + ie_{1,1}} -$
 $2ib_{1,1,1}e^{ic_{1,1}x + itc_{1,1}^3 - iyc_{1,1}^2 + d_{1,1} + ie_{1,1}}yc_{1,1} + 3itb_{1,1,1}c_{1,1}^2e^{ic_{1,1}x + itc_{1,1}^3 - iyc_{1,1}^2 + d_{1,1} + ie_{1,1}}$
 We give some figures in the (x, y) plane of coordinates

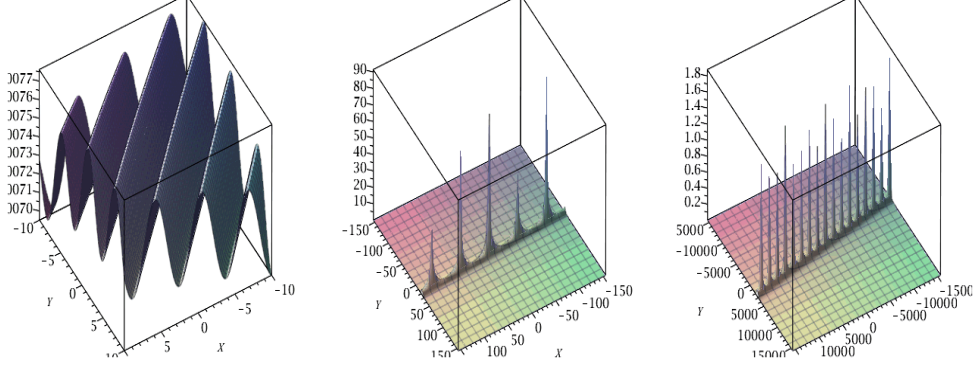


Figure 4. Solution of order 1 to KPI, on the left for $t = 0$, $a_1 = 1$, $b_{1,1,0} = 1$, $b_{1,1,1} = 1$, $c_{1,1} = 1$, $d_{1,1} = 1$, $e_{1,1} = 1$; in the center for $t = 1$, $a_1 = 1$, $b_{1,1,0} = 10^2$, $b_{1,1,1} = 2$, $c_{1,1} = 10$, $d_{1,1} = 1$, $e_{1,1} = 1$; on the right for $t = 10$, $a_1 = 10$, $b_{1,1,0} = 10$, $b_{1,1,1} = 2$, $c_{1,1} = 3$, $d_{1,1} = 1$, $e_{1,1} = 1$.

4.2 Case $S = 1$, $D = 2$

In this case solutions to the KPI equation can be written as

$$v(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

with

$$\begin{aligned}
 n(x, y, t) = & e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}} (-2b_{1,1,2}^2 e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}} + \\
 & b_{1,1,2}c_{1,1}^2a_1)x^2 + e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}} (6b_{1,1,2}c_{1,1}^4ta_1 - ib_{1,1,1}c_{1,1}^2a_1 \\
 & - 4ib_{1,1,2}c_{1,1}a_1 + 8b_{1,1,2}^2c_{1,1}e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}}y - 4b_{1,1,2}c_{1,1}^3ya_1 - \\
 & 12b_{1,1,2}^2c_{1,1}^2e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}}t + 2ib_{1,1,2}b_{1,1,1}e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}})x + \\
 & e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}} (6ib_{1,1,2}b_{1,1,1}e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}}tc_{1,1}^2 - \\
 & 18ib_{1,1,2}c_{1,1}^3ta_1 - 2b_{1,1,2}b_{1,1,0}e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}} + e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}}b_{1,1,1}^2 - \\
 & 2b_{1,1,1}c_{1,1}a_1 - b_{1,1,0}c_{1,1}^2a_1 - 2b_{1,1,2}a_1 - 12ib_{1,1,2}^2e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}}tc_{1,1} - \\
 & 12b_{1,1,2}c_{1,1}^5tya_1 - 3ib_{1,1,1}c_{1,1}^4ta_1 - 4ib_{1,1,2}b_{1,1,1}e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}}yc_{1,1} + \\
 & 2ib_{1,1,1}c_{1,1}^3ya_1 - 8b_{1,1,2}^2c_{1,1}^2y^2e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}} + 9b_{1,1,2}c_{1,1}^6t^2a_1 + \\
 & 24b_{1,1,2}^2c_{1,1}^3te^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}}y + 4b_{1,1,2}c_{1,1}^4y^2a_1 \\
 & + 4ib_{1,1,2}^2e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}}y + 10ib_{1,1,2}c_{1,1}^2ya_1 \\
 & - 18b_{1,1,2}^2c_{1,1}^4t^2e^{ic_{1,1}x - ic_{1,1}^2y + ic_{1,1}^3t + d_{1,1} + ie_{1,1}})
 \end{aligned}$$

and

$$d(x, y, t) = -b_{1,1,2}e^{ic_{1,1}x + itc_{1,1}^3 - ic_{1,1}^2y + d_{1,1} + ie_{1,1}}x^2 + (-6tb_{1,1,2}c_{1,1}^2e^{ic_{1,1}x + itc_{1,1}^3 - ic_{1,1}^2y + d_{1,1} + ie_{1,1}} +$$

$4yb_{1,1,2}c_{1,1}e^{ic_{1,1}x+itc_{1,1}^3-ic_{1,1}^2y+d_{1,1}+ie_{1,1}+ib_{1,1,1}e^{ic_{1,1}x+itc_{1,1}^3-ic_{1,1}^2y+d_{1,1}+ie_{1,1}})x+$
 $a_1+b_{1,1,0}e^{ic_{1,1}x+itc_{1,1}^3-ic_{1,1}^2y+d_{1,1}+ie_{1,1}}-2ib_{1,1,2}e^{ic_{1,1}x+itc_{1,1}^3-ic_{1,1}^2y+d_{1,1}+ie_{1,1}}y+$
 $3itb_{1,1,1}c_{1,1}^2e^{ic_{1,1}x+itc_{1,1}^3-ic_{1,1}^2y+d_{1,1}+ie_{1,1}}-2ib_{1,1,1}e^{ic_{1,1}x+itc_{1,1}^3-ic_{1,1}^2y+d_{1,1}+ie_{1,1}}yc_{1,1}+$
 $6itb_{1,1,2}c_{1,1}e^{ic_{1,1}x+itc_{1,1}^3-ic_{1,1}^2y+d_{1,1}+ie_{1,1}}-9t^2b_{1,1,2}c_{1,1}^4e^{ic_{1,1}x+itc_{1,1}^3-ic_{1,1}^2y+d_{1,1}+ie_{1,1}}+$
 $12tyb_{1,1,2}c_{1,1}^3e^{ic_{1,1}x+itc_{1,1}^3-ic_{1,1}^2y+d_{1,1}+ie_{1,1}}-4y^2b_{1,1,2}c_{1,1}^2e^{ic_{1,1}x+itc_{1,1}^3-ic_{1,1}^2y+d_{1,1}+ie_{1,1}}$
 We give some figures in the (x, y) plane of coordinates

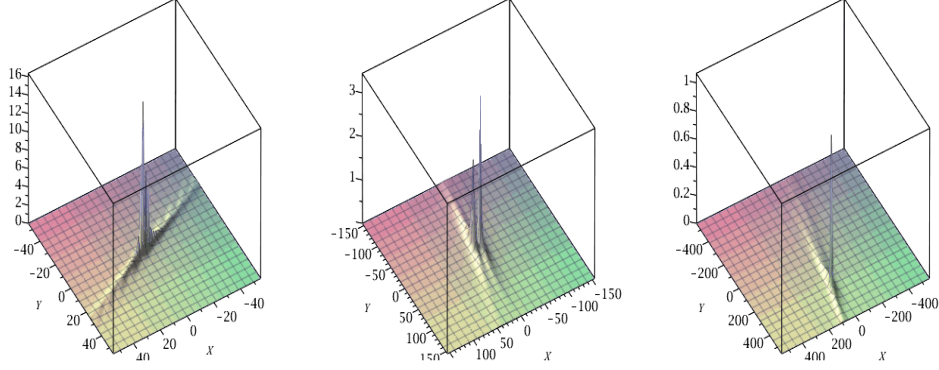


Figure 5. Solution of order 1 to KPI, on the left for $t = 0$, $a_1 = 1$, $b_{1,1,0} = 1$, $b_{1,1,1} = 1$, $b_{1,1,2} = 1$, $b_{1,1,3} = 1$, $c_{1,1} = 1$, $d_{1,1} = 1$, $e_{1,1} = 1$; in the center for $t = 1$, $a_1 = 10^2$, $b_{1,1,0} = 10$, $b_{1,1,1} = 1$, $b_{1,1,2} = 1$, $c_{1,1} = 0.1$, $d_{1,1} = 1$, $e_{1,1} = 1$; on the right for $t = 10^3$, $a_1 = 10^2$, $b_{1,1,0} = 10$, $b_{1,1,1} = 1$, $b_{1,1,2} = 1$, $c_{1,1} = 1$, $d_{1,1} = 1$, $e_{1,1} = 1$.

4.3 Case $S = 1$, $D = 3$

In this case solutions to the KPI equation can be written as

$$v(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

with

$$\begin{aligned}
 n(x, y, t) = & 3e^{ic_{1,1}x-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}b_{1,1,3}^2e^{ic_{1,1}x+23_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}x^4 + \\
 & e^{ic_{1,1}x-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}(ib_{1,1,3}c_{1,1}^2e^{3_{1,1}}a_1+36b_{1,1,3}^2e^{ic_{1,1}x+23_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}tc_{1,1}^2- \\
 & 4ib_{1,1,3}e^{ic_{1,1}x+3_{1,1}+2_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}b_{1,1,2}-24b_{1,1,3}^2e^{ic_{1,1}x+23_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}yc_{1,1})x^3+ \\
 & e^{ic_{1,1}x-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}(-2b_{1,1,2}^2e^{ic_{1,1}x+22_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}-6ib_{1,1,3}c_{1,1}^3e^{3_{1,1}}ya_1+ \\
 & 6b_{1,1,3}e^{3_{1,1}}c_{1,1}a_1+72b_{1,1,3}^2c_{1,1}^2e^{ic_{1,1}x+23_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}y^2 \\
 & +162b_{1,1,3}^2c_{1,1}^4e^{ic_{1,1}x+23_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}t^2-216b_{1,1,3}^2c_{1,1}^3e^{ic_{1,1}x+23_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}yt+ \\
 & 24ib_{1,1,3}c_{1,1}e^{ic_{1,1}x+3_{1,1}+2_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}b_{1,1,2}y+9ib_{1,1,3}c_{1,1}^4e^{3_{1,1}}ta_1+b_{1,1,2}c_{1,1}^2e^{2_{1,1}}a_1- \\
 & 36ib_{1,1,2}c_{1,1}^2e^{ic_{1,1}x+3_{1,1}+2_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}b_{1,1,3}t)x^2 \\
 & +e^{ic_{1,1}x-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}(12ib_{1,1,3}c_{1,1}^4e^{3_{1,1}}y^2a_1+2ib_{1,1,2}e^{ic_{1,1}x+2_{1,1}+1_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}b_{1,1,1}+ \\
 & 432b_{1,1,3}^2c_{1,1}^4e^{ic_{1,1}x+23_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}ty^2+8b_{1,1,2}^2c_{1,1}e^{ic_{1,1}x+22_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}y- \\
 & 108ib_{1,1,2}c_{1,1}^4e^{ic_{1,1}x+3_{1,1}+2_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}t^2b_{1,1,3}-36ib_{1,1,3}c_{1,1}^5e^{3_{1,1}}tya_1- \\
 & 48ib_{1,1,2}c_{1,1}^2e^{ic_{1,1}x+3_{1,1}+2_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}y^2b_{1,1,3}-96b_{1,1,3}^2e^{ic_{1,1}x+23_{1,1}-iy_{c_{1,1}}^2+ic_{1,1}^3t+ie_{1,1}}y^3c_{1,1}^3+
 \end{aligned}$$

$$\begin{aligned}
& 324 b_{1,1,3}^2 c_{1,1}^6 e^{ic_{1,1}x+23_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t^3} - 4 ib_{1,1,2} c_{1,1} e^{2i_{1,1}a_1} + 54 b_{1,1,3} c_{1,1}^3 e^{3i_{1,1}ta_1} - \\
& 30 b_{1,1,3} c_{1,1}^2 e^{3i_{1,1}ya_1} - 648 b_{1,1,3}^2 c_{1,1}^5 e^{ic_{1,1}x+23_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t^2} y - 6 ib_{1,1,3} e^{3i_{1,1}a_1} - \\
& 4 b_{1,1,2} c_{1,1}^3 e^{2i_{1,1}ya_1} + 36 b_{1,1,3}^2 e^{ic_{1,1}x+23_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} \\
& - 6 ib_{1,1,3} e^{ic_{1,1}x+3_{1,1}+d_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} b_{1,1,0} - ib_{1,1,1} c_{1,1}^2 e^{1i_{1,1}a_1} \\
& - 12 b_{1,1,2}^2 c_{1,1}^2 e^{ic_{1,1}x+22_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} + 27 ib_{1,1,3} c_{1,1}^6 e^{3i_{1,1}t^2} a_1 \\
& + 144 ib_{1,1,2} c_{1,1}^3 e^{ic_{1,1}x+3_{1,1}+2_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} y b_{1,1,3} t + 6 b_{1,1,2} c_{1,1}^4 e^{2i_{1,1}ta_1} x + \\
& e^{ic_{1,1}x-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} (32 ib_{1,1,2} c_{1,1}^3 e^{ic_{1,1}x+3_{1,1}+2_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} y^3 b_{1,1,3} + \\
& 6 ib_{1,1,1} c_{1,1}^2 e^{ic_{1,1}x+2_{1,1}+1_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} b_{1,1,2} t \\
& - 4 ib_{1,1,1} c_{1,1} e^{ic_{1,1}x+2_{1,1}+1_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} b_{1,1,2} y \\
& + 12 ib_{1,1,3} e^{ic_{1,1}x+3_{1,1}+d_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} y c_{1,1} b_{1,1,0} \\
& - 18 ib_{1,1,0} c_{1,1}^2 e^{ic_{1,1}x+3_{1,1}+d_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} b_{1,1,3} t \\
& + 36 ib_{1,1,1} c_{1,1} e^{ic_{1,1}x+3_{1,1}+1_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} b_{1,1,3} t \\
& - 108 ib_{1,1,2} c_{1,1}^6 e^{ic_{1,1}x+3_{1,1}+2_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} t^3 b_{1,1,3} - 60 ib_{1,1,3} e^{3i_{1,1}tc_{1,1}^2} a_1 + \\
& 24 ib_{1,1,3} e^{3i_{1,1}yc_{1,1}a_1} - 3 ib_{1,1,1} c_{1,1}^4 e^{1i_{1,1}ta_1} + 2 ib_{1,1,1} c_{1,1}^3 e^{1i_{1,1}ya_1} - 18 ib_{1,1,2} c_{1,1}^3 e^{2i_{1,1}ta_1} + \\
& 10 ib_{1,1,2} c_{1,1}^2 e^{2i_{1,1}ya_1} + 27 ib_{1,1,3} c_{1,1}^8 e^{3i_{1,1}t^3} a_1 - 8 ib_{1,1,3} c_{1,1}^5 e^{3i_{1,1}y^3} a_1 - 12 b_{1,1,2} c_{1,1}^5 e^{2i_{1,1}tya_1} - \\
& 126 b_{1,1,3} c_{1,1}^4 e^{3i_{1,1}tya_1} + b_{1,1,1}^2 e^{ic_{1,1}x+2_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} + 4 ib_{1,1,2}^2 e^{ic_{1,1}x+22_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} y - \\
& 216 b_{1,1,3}^2 e^{ic_{1,1}x+23_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} t^2 c_{1,1}^2 + 48 b_{1,1,3}^2 c_{1,1}^4 e^{ic_{1,1}x+23_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} y^4 + \\
& 243 b_{1,1,3}^2 c_{1,1}^8 e^{ic_{1,1}x+23_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} t^4 - 18 b_{1,1,2}^2 e^{ic_{1,1}x+22_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} t^2 c_{1,1}^4 - \\
& 8 b_{1,1,2}^2 e^{ic_{1,1}x+22_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} y^2 c_{1,1}^2 - 12 ib_{1,1,3} e^{ic_{1,1}x+3_{1,1}+1_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} y b_{1,1,1} - \\
& 12 ib_{1,1,2} e^{ic_{1,1}x+3_{1,1}+2_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} b_{1,1,3} t - 12 ib_{1,1,2}^2 e^{ic_{1,1}x+22_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} t c_{1,1} - \\
& b_{1,1,0} c_{1,1}^2 e^{d_{1,1}a_1} - 2 b_{1,1,1} c_{1,1} e^{1i_{1,1}a_1} - 36 b_{1,1,3}^2 e^{ic_{1,1}x+23_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} y^2 \\
& - 2 b_{1,1,2} e^{2i_{1,1}a_1} - 2 b_{1,1,2} e^{ic_{1,1}x+2_{1,1}+d_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} b_{1,1,0} \\
& + 216 ib_{1,1,2} c_{1,1}^5 e^{ic_{1,1}x+3_{1,1}+2_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} y b_{1,1,3} t^2 \\
& - 144 ib_{1,1,2} c_{1,1}^4 e^{ic_{1,1}x+3_{1,1}+2_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} y^2 b_{1,1,3} t - 54 ib_{1,1,3} c_{1,1}^7 e^{3i_{1,1}t^2} y a_1 + \\
& 36 ib_{1,1,3} c_{1,1}^6 e^{3i_{1,1}ty^2} a_1 + 144 b_{1,1,3}^2 e^{ic_{1,1}x+23_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} y c_{1,1} t \\
& - 288 b_{1,1,3}^2 c_{1,1}^5 e^{ic_{1,1}x+23_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} y^3 t - 648 b_{1,1,3}^2 c_{1,1}^7 e^{ic_{1,1}x+23_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} t^3 y + \\
& 648 b_{1,1,3}^2 c_{1,1}^6 e^{ic_{1,1}x+23_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} t^2 y^2 + 24 b_{1,1,2}^2 c_{1,1}^3 e^{ic_{1,1}x+22_{1,1}-iyc_{1,1}^2+ic_{1,1}^3t+ie_{1,1}t} y t + \\
& 9 b_{1,1,2} c_{1,1}^6 e^{2i_{1,1}t^2} a_1 + 36 b_{1,1,3} c_{1,1}^3 e^{3i_{1,1}y^2} a_1 + 108 b_{1,1,3} c_{1,1}^5 e^{3i_{1,1}t^2} a_1 + 4 b_{1,1,2} c_{1,1}^4 e^{2i_{1,1}y^2} a_1) \\
& \text{and} \\
& d(x, y, t) = -ib_{1,1,3} e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+3_{1,1}+ie_{1,1}x^3} + (-9 itb_{1,1,3} c_{1,1}^2 e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+3_{1,1}+ie_{1,1}x} + \\
& 6 i y b_{1,1,3} c_{1,1} e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+3_{1,1}+ie_{1,1}x} - b_{1,1,2} e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+2_{1,1}+ie_{1,1}x} x^2 + \\
& (-27 it^2 b_{1,1,3} c_{1,1}^4 e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+3_{1,1}+ie_{1,1}x} + 36 it y b_{1,1,3} c_{1,1}^3 e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+3_{1,1}+ie_{1,1}x} - \\
& 12 ib_{1,1,3} c_{1,1}^2 e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+3_{1,1}+ie_{1,1}x} y^2 - 6 t b_{1,1,2} c_{1,1}^2 e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+2_{1,1}+ie_{1,1}x} - \\
& 18 t b_{1,1,3} c_{1,1} e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+3_{1,1}+ie_{1,1}x} + 4 y b_{1,1,2} c_{1,1} e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+2_{1,1}+ie_{1,1}x} + \\
& 6 y b_{1,1,3} e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+3_{1,1}+ie_{1,1}x} + ib_{1,1,1} e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+1_{1,1}+ie_{1,1}x} x - \\
& 2 ib_{1,1,2} e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+2_{1,1}+ie_{1,1}x} y - 54 t^2 b_{1,1,3} c_{1,1}^3 e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+3_{1,1}+ie_{1,1}x} + \\
& 54 it^2 y b_{1,1,3} c_{1,1}^5 e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+3_{1,1}+ie_{1,1}x} + 3 it b_{1,1,1} c_{1,1}^2 e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+1_{1,1}+ie_{1,1}x} + \\
& a_1 + 6 it b_{1,1,2} c_{1,1} e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+2_{1,1}+ie_{1,1}x} + 54 t y b_{1,1,3} c_{1,1}^2 e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+3_{1,1}+ie_{1,1}x} - \\
& 4 y^2 b_{1,1,2} c_{1,1}^2 e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+2_{1,1}+ie_{1,1}x} - 2 ib_{1,1,1} e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+1_{1,1}+ie_{1,1}x} y c_{1,1} - \\
& 36 ib_{1,1,3} e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+3_{1,1}+ie_{1,1}x} t c_{1,1}^4 y^2 - 9 t^2 b_{1,1,2} c_{1,1}^4 e^{ic_{1,1}x+itc_{1,1}^3-iyc_{1,1}^2+2_{1,1}+ie_{1,1}x} +
\end{aligned}$$

$12 tyb_{1,1,2}c_{1,1}^3 e^{ic_{1,1}x+itc_{1,1}^3-iy c_{1,1}^2+2i_{1,1}+ie_{1,1}} - 12 y^2 b_{1,1,3}c_{1,1} e^{ic_{1,1}x+itc_{1,1}^3-iy c_{1,1}^2+3i_{1,1}+ie_{1,1}} -$
 $27 ib_{1,1,3} e^{ic_{1,1}x+itc_{1,1}^3-iy c_{1,1}^2+3i_{1,1}+ie_{1,1}} t^3 c_{1,1}^6 + 6 itb_{1,1,3} e^{ic_{1,1}x+itc_{1,1}^3-iy c_{1,1}^2+3i_{1,1}+ie_{1,1}} +$
 $b_{1,1,0} e^{ic_{1,1}x+itc_{1,1}^3-iy c_{1,1}^2+d_{1,1}+ie_{1,1}} + 8 iy^3 b_{1,1,3}c_{1,1}^3 e^{ic_{1,1}x+itc_{1,1}^3-iy c_{1,1}^2+3i_{1,1}+ie_{1,1}}$
 We give some figures in the (x, y) plane of coordinates

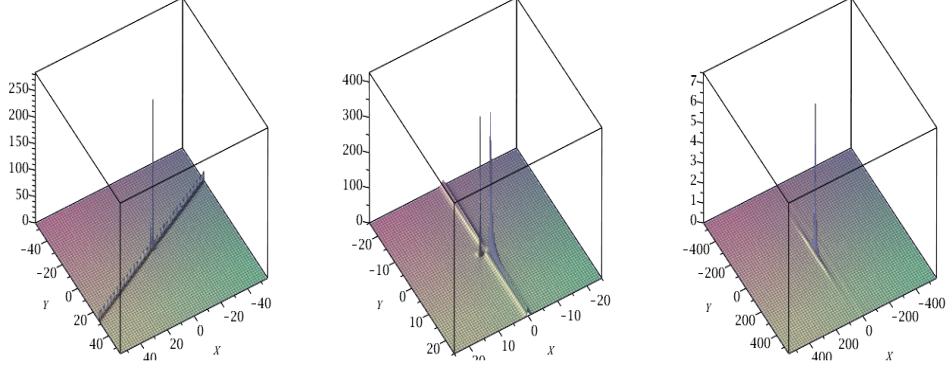


Figure 6. Solution of order 1 to KPI, on the left for $t = 0$, $a_1 = 1$, $b_{1,1,0} = 1$, $b_{1,1,1} = 1$, $b_{1,1,2} = 1$, $b_{1,1,3} = 1$, $c_{1,1} = 1$, $d_{1,1} = 1$, $e_{1,1} = 1$; in the center for $t = 1$, $a_1 = 1$, $b_{1,1,0} = 1$, $b_{1,1,1} = 1$, $b_{1,1,2} = 1$, $c_{1,1} = 0$, $d_{1,1} = 1$, $e_{1,1} = 1$; on the right for $t = 10^4$, $a_1 = 1$, $b_{1,1,0} = 1$, $b_{1,1,1} = 1$, $b_{1,1,2} = 1$, $c_{1,1} = 0$, $d_{1,1} = 1$, $e_{1,1} = 1$.

In the following, in the case where S is greater or equal to 2, to shortened the expressions of the solution we fix parameters a_i , $b_{j_s d}$, c_{j_s} equal to 1, and d_{j_s} , e_{j_s} equal to 0 for all the values of the indices.

4.4 Case $S = 2$, $D = 1$

In this case solutions to the KPI equation can be written as

$$v(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

with

$$n(x, y, t) = -2ie^{ix-iy+it}x + 4ie^{ix-iy+it}y - 6ie^{ix-iy+it}t + 4e^{2ix-2iy+2it} - 6e^{ix-iy+it}$$

and

$$d(x, y, t) = 2ie^{i(x+t-y)}x + 1 + 2e^{i(x+t-y)} - 4iye^{i(x+t-y)} + 6ite^{i(x+t-y)}$$

We give some figures in the (x, y) plane of coordinates

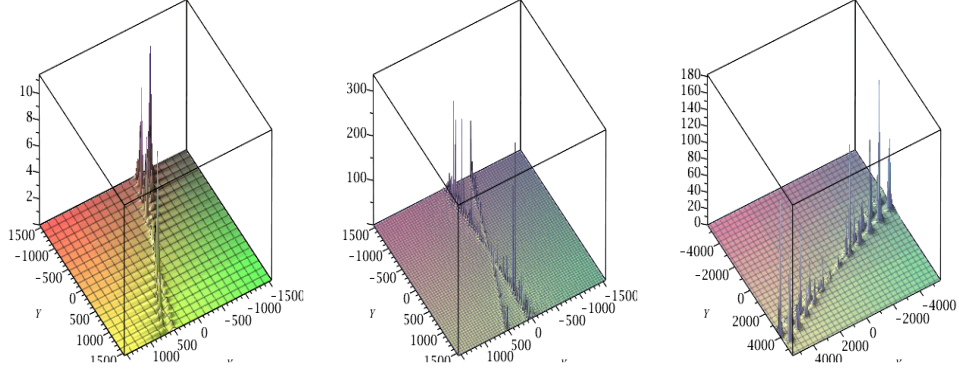


Figure 7. Solution of order 1 to KPI, on the left for $t = 0$, $a_1 = 1$, $b_{1,1,0} = 1$, $b_{1,1,1} = 1$, $b_{1,2,0} = 2$, $b_{1,1,1} = 2$, $c_{1,1} = 0.5$, $c_{1,2} = 0.2$, $d_{1,1} = 1$, $d_{1,2} = 2$, $e_{1,1} = 1$, $e_{1,2} = 2$; in the center for $t = 1$, $a_1 = 1$, $b_{1,1,0} = 1$, $b_{1,1,1} = 1$, $b_{1,2,0} = 2$, $b_{1,1,1} = 2$, $c_{1,1} = 0.5$, $c_{1,2} = 0.1$, $d_{1,1} = 1$, $d_{1,2} = 2$, $e_{1,1} = 1$, $e_{1,2} = 2$; on the right for $t = 10^3$, $a_1 = 1$, $b_{1,1,0} = 1$, $b_{1,1,1} = 1$, $b_{1,2,0} = 2$, $b_{1,1,1} = 2$, $c_{1,1} = 0.1$, $c_{1,2} = 1$, $d_{1,1} = 1$, $d_{1,2} = 2$, $e_{1,1} = 1$, $e_{1,2} = 2$.

4.5 Case $S = 2$, $D = 2$

In this case solutions to the KPI equation can be written as

$$v(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

with

$$n(x, y, t) = (2e^{ix-iy+it} - 8e^{2ix-2iy+2it})x^2 + (-10ie^{ix-iy+it} + 12e^{ix-iy+it}t + 32e^{2ix-2iy+2it}y - 48e^{2ix-2iy+2it}t + 8ie^{2ix-2iy+2it} - 8e^{ix-iy+it}y)x - 24e^{ix-iy+it}yt + 8e^{ix-iy+it}y^2 + 18e^{ix-iy+it}t^2 - 32e^{2ix-2iy+2it}y^2 - 72e^{2ix-2iy+2it}t^2 - 24ie^{2ix-2iy+2it}t - 10e^{ix-iy+it} - 4e^{2ix-2iy+2it} + 24ie^{ix-iy+it}y + 96e^{2ix-2iy+2it}yt - 42ie^{ix-iy+it}t$$

and

$$d(x, y, t) = -2e^{i(x+t-y)}x^2 + (2ie^{i(x+t-y)} + 8ye^{i(x+t-y)} - 12te^{i(x+t-y)})x + 1 + 2e^{i(x+t-y)} - 8iye^{i(x+t-y)} - 8y^2e^{i(x+t-y)} - 18t^2e^{i(x+t-y)} + 18ite^{i(x+t-y)} + 24tye^{i(x+t-y)}$$

We give some figures in the (x, y) plane of coordinates

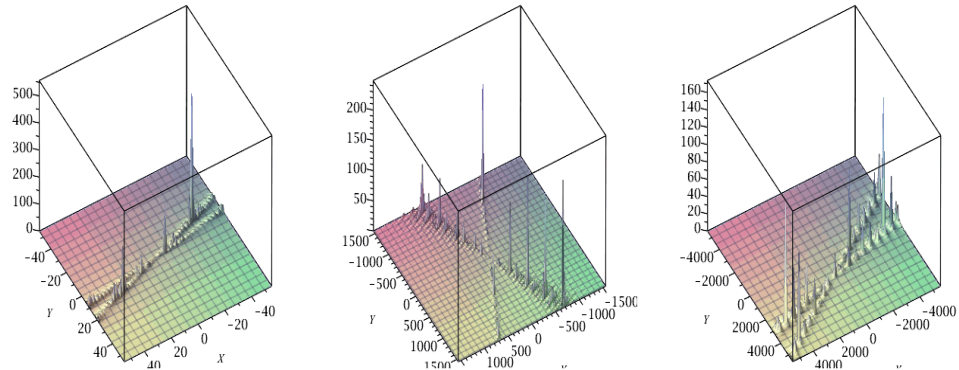


Figure 8. Solution of order 1 to KPI, on the left for $t = 0$, $a_1 = 1$, $b_{1,1,0} = 1$,
 $b_{1,1,1} = 1$, $b_{1,1,2} = 1$, $b_{1,2,0} = 2$, $b_{1,2,1} = 2$, $b_{1,2,2} = 2$, $c_{1,1} = 0.1$, $c_{1,2} = 2$,
 $d_{1,1} = 1$, $d_{1,2} = 2$, $e_{1,1} = 1$, $e_{1,2} = 2$; in the center for $t = 1$, $a_1 = 1$, $b_{1,1,0} = 1$,
 $b_{1,1,1} = 1$, $b_{1,1,2} = 1$, $b_{1,2,0} = 2$, $b_{1,2,1} = 2$, $b_{1,2,2} = 2$, $c_{1,1} = 0.5$, $c_{1,2} = 0.2$,
 $d_{1,1} = 1$, $d_{1,2} = 2$, $e_{1,1} = 1$, $e_{1,2} = 2$; on the right for $t = 10^3$, $a_1 = 1$, $b_{1,1,0} = 1$,
 $b_{1,1,1} = 1$, $b_{1,1,2} = 1$, $b_{1,2,0} = 2$, $b_{1,2,1} = 2$, $b_{1,2,2} = 2$, $c_{1,1} = 0.1$, $c_{1,2} = 1$,
 $d_{1,1} = 1$, $d_{1,2} = 2$, $e_{1,1} = 1$, $e_{1,2} = 2$.

4.6 Case $S = 2$, $D = 3$

In this case solutions to the KPI equation can be written as

$$v(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

with

$$\begin{aligned} n(x, y, t) = & 12 e^{2ix-2iy+2it} x^4 + (-96 e^{2ix-2iy+2it} y - 16 i e^{2ix-2iy+2it} + 2 i e^{ix-iy+it} + \\ & 144 e^{2ix-2iy+2it} t) x^3 + (96 i e^{2ix-2iy+2it} y - 8 e^{2ix-2iy+2it} - 864 e^{2ix-2iy+2it} t y - \\ & 12 i e^{ix-iy+it} y + 18 i e^{ix-iy+it} t + 14 e^{ix-iy+it} - 144 i e^{2ix-2iy+2it} t^2 + 648 e^{2ix-2iy+2it} t^2 + \\ & 288 e^{2ix-2iy+2it} y^2) x^2 + (-432 i e^{2ix-2iy+2it} t^2 - 22 i e^{ix-iy+it} + 1296 e^{2ix-2iy+2it} t^3 - \\ & 2592 e^{2ix-2iy+2it} t^2 y + 576 i e^{2ix-2iy+2it} t y + 54 i e^{ix-iy+it} t^2 + 120 e^{ix-iy+it} t - \\ & 68 e^{ix-iy+it} y + 96 e^{2ix-2iy+2it} t + 32 e^{2ix-2iy+2it} y - 384 e^{2ix-2iy+2it} y^3 + 24 i e^{ix-iy+it} y^2 + \\ & 1728 e^{2ix-2iy+2it} t y^2 - 16 i e^{2ix-2iy+2it} - 192 i e^{2ix-2iy+2it} y^2 - 72 i e^{ix-iy+it} t y) x - \\ & 10 e^{ix-iy+it} - 4 e^{2ix-2iy+2it} - 16 i e^{ix-iy+it} y^3 + 54 i e^{ix-iy+it} t^3 + 2592 e^{2ix-2iy+2it} t^2 y^2 - \\ & 1152 e^{2ix-2iy+2it} t y^3 - 276 e^{ix-iy+it} y t + 672 e^{2ix-2iy+2it} t y - 162 i e^{ix-iy+it} t + \\ & 72 i e^{ix-iy+it} y - 432 i e^{2ix-2iy+2it} t^3 + 128 i e^{2ix-2iy+2it} y^3 + 192 e^{2ix-2iy+2it} y^4 - \\ & 936 e^{2ix-2iy+2it} t^2 + 80 e^{ix-iy+it} y^2 + 234 e^{ix-iy+it} t^2 - 176 e^{2ix-2iy+2it} y^2 - 2592 e^{2ix-2iy+2it} t^3 y + \\ & 972 e^{2ix-2iy+2it} t^4 + 72 i e^{ix-iy+it} y^2 t - 108 i e^{ix-iy+it} t^2 y - 576 i e^{2ix-2iy+2it} t y^2 + \\ & 864 i e^{2ix-2iy+2it} t^2 y \end{aligned}$$

and

$$\begin{aligned} d(x, y, t) = & 2 i e^{i(x+t-y)} x^3 + (2 e^{i(x+t-y)} + 18 i t e^{i(x+t-y)} - 12 i y e^{i(x+t-y)}) x^2 + \\ & (54 i e^{i(x+t-y)} t^2 + 24 i y^2 e^{i(x+t-y)} + 48 t e^{i(x+t-y)} - 20 y e^{i(x+t-y)} - 2 i e^{i(x+t-y)} - \\ & 72 i e^{i(x+t-y)} y t) x - 1 - 2 e^{i(x+t-y)} - 108 i t^2 y e^{i(x+t-y)} + 54 i t^3 e^{i(x+t-y)} + 72 i t y^2 e^{i(x+t-y)} + \\ & 8 i y e^{i(x+t-y)} - 16 i y^3 e^{i(x+t-y)} - 132 t y e^{i(x+t-y)} - 30 i t e^{i(x+t-y)} + 32 y^2 e^{i(x+t-y)} + \\ & 126 t^2 e^{i(x+t-y)} \end{aligned}$$

We give some figures in the (x, y) plane of coordinates

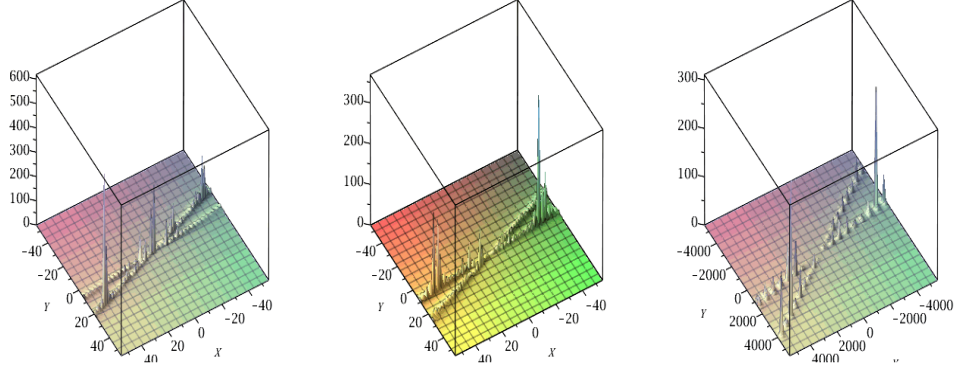


Figure 9. Solution of order 1 to KPI, on the left for $t = 0$, $a_1 = 1$, $b_{1,1,0} = 1$, $b_{1,1,1} = 1$, $b_{1,1,2} = 1$, $b_{1,1,3} = 1$, $b_{1,2,0} = 2$, $b_{1,2,1} = 2$, $b_{1,2,2} = 2$, $b_{1,2,3} = 1$, $c_{1,1} = 1$, $c_{1,2} = 2$, $d_{1,1} = 1$, $d_{1,2} = 2$, $e_{1,1} = 1$, $d_{1,2} = 2$; in the center for $t = 1$, $a_1 = 1$, $b_{1,1,0} = 1$, $b_{1,1,1} = 1$, $b_{1,1,2} = 1$, $b_{1,1,3} = 1$, $b_{1,2,0} = 2$, $b_{1,2,1} = 2$, $b_{1,2,2} = 2$, $b_{1,2,3} = 1$, $c_{1,1} = 0$, $c_{1,2} = 2$, $d_{1,1} = 1$, $d_{1,2} = 2$, $e_{1,1} = 1$, $d_{1,2} = 2$; on the right for $t = 10$, $a_1 = 1$, $b_{1,1,0} = 1$, $b_{1,1,1} = 1$, $b_{1,1,2} = 1$, $b_{1,1,3} = 1$, $b_{1,2,0} = 2$, $b_{1,2,1} = 2$, $b_{1,2,2} = 2$, $b_{1,2,3} = 1$, $c_{1,1} = 0.1$, $c_{1,2} = 2$, $d_{1,1} = 1$, $d_{1,2} = 2$, $e_{1,1} = 1$, $d_{1,2} = 2$.

5 Conclusion

With the Darboux transform, multi-parametric solutions to the KPI equation have been constructed. These solutions depend on multi-parameters; first, on the order of determinant N . They depend also on the degree of summation S and the degree of derivation D . In the case general case of an expression with a second derivative in x of a determinant of order N , we obtain solutions depending on $N(S(D + 4) + 1)$ real parameters.

All these solutions are different from these already published in particular by the present author [42, 44, 50, 58].

We restrict the study to the case where c_{js} is a real parameter. We postpone to another publication the study in the case where $c_{js}g_{js} + ih_{js}$ is a complex parameter.

We only give the expressions of the solution in the simple case of the first order $N = 1$. It would important to study in more details this first order and try to classify these solutions.

It would be also relevant to study the cases of order greater or equal to 2 and to realize an exhaustive classification of the solutions to the KPI equation.

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