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Discrete-Time Mean-Field Stochastic Control with Partial Observations

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Abstract

We study the optimal control of discrete time mean field dynamical systems under partial observations. We express the global law of the filtered process as a controlled system with its own dynamics. Following a dynamic programming approach, we prove a verification result providing a solution to the optimal control of the filtered system. As an application, we study a general linear quadratic example for which an explicit solution is given. We also describe an algorithm for the numerical approximation of the optimal value and provide numerical experiments on a financial example.

MSC Classification: 60K35, 60G35, 49L20.

Keywords: Optimal control, mean-field interaction, partial observation, dynamic programming.

1 Introduction

We consider an optimal control problem for a system with mean field discrete time dynamics under partial observations. The question of how to optimally control a system with mean field dependence in the dynamics is related to the modelization of the optimal behavior for populations with large number of interacting individuals.

The case where each individual chooses an own action in order to optimise a non-cooperative reward, has led to the theory of mean-field games (MFGs), introduced in [12] and [13]. The Nash equilibrium is then described by two equations, the first one corresponding to the optimal behavior of a representative agent and the second one corresponding to the evolution of the whole population under the optimal choice of each individual.

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In the case where all the individuals follow the same goal, the cooperative behavior leads to the optimal control of mean field dynamical systems. A growing literature has emerged on the subject for the continuous time case with two main approaches. The first one follows a maximum principle method to provide necessary and sufficient conditions for optimality, see e.g. [2, 14, 4, 5, 11]. The second approach consists in proving a dynamic programming principle to solve the problem, see e.g. [1, 3, 18].

Such models with law dependency of the dynamics appear in mathematical finance. The calibration of local volatility models to market smiles leads indeed to mean field stochastic differential equations. We refer to [10, Chapter 11] for more details.

Concerning the discrete-time case, [6] studied the case of a linear-quadratic problem and turn it into a quadratic optimization problem in an Hilbert space, allowing to get necessary and sufficient conditions for solving the problem. A dynamic programming approach is used in [17] to the case where controls are restricted to feedback ones. By considering the law of the controlled process as a state variable, it allows to get a verification theorem and to solve explicitly the linear quadratic case.

In this paper, we investigate the optimal control problem of mean-field discrete time systems under partial information. This question has already been studied for systems without mean-field interaction. We refer to the book [7] for a detailed presentation of the estimation and the control of systems under partial information. The common approach to deal with optimal control of partially observable systems consists in computing the conditional law of the unobserved component given the observation and to derive its dynamic to retrieve a completely observable controlled system called filtered controlled system.

In the case where the unobservable component admits a mean-field dependence, this approach leads to an intractable problem since the filtered system involves two different marginal laws, the original law of the unobservable component and the filtered law given the observations.

To overcome this issue, we consider the law of the global system composed by both observed and unobserved components as a state variable. By keeping in mind the past, it also allows to consider general controls that might not be in the feedback form. We then rewrite the initial problem as a new control problem with respect to the global law of the system starting from the initial time and with controls as functions depending only on the observed variables. We derive a dynamic programming principle and a verification theorem for this new formulation. A second verification theorem is also provided for feedback controls. As an application of the verification theorem, we provide the explicit solution of a linear-quadratic optimal control problem with partial observations. As it is not possible to get explicit solutions in many cases, we describe an algorithm for the numerical approximation of the optimal value of our problem. We then provide numerical experiments for this algorithm. We first test our algorithm on a benchmark given by the explicit solution of the linear-quadratic model. We then apply our algorithm to a model inspired by the optimal investment problems for private equities in finance and compare the results with standard strategies.

The remainder of the paper is organized as follows. In Section 2, we present the control problem and the computation of the filter. In Section 3, we extend the original filtered problem to get a tractable problem. We then provide a dynamic programming principle for

the extended problem and a verification theorem. We also present a feedback version of this verification theorem. Finally, we present in Section 4 some applications. We first study a linear quadratic model for which we give an explicit solution via the verification Theorem. An algorithm is then provided to approximate the optimal value in the general case. We test this algorithm on the linear-quadratic problem and run an experiment on a financial model inspired by private equity investments.

2 The control problem under partial observation

2.1 The model

We fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and three Banach spaces E , F and C . We denote by $|\cdot|$ the norm on those spaces and by $\mathcal{B}(E)$, $\mathcal{B}(F)$ and $\mathcal{B}(C)$ their respective Borel σ -algebras. We also denote by $\mathcal{P}_2(E)$ the set of probability measure μ on $(E, \mathcal{B}(E))$ such that $\int_E |x|^2 d\mu(x) < +\infty$. We similarly define $\mathcal{P}_2(E \times E)$ and $\mathcal{P}_2(E \times F)$. We endow the set $\mathcal{P}_2(E)$ with the 2-Wasserstein distance \mathcal{W}_2 defined by

$$\mathcal{W}_2(\mu, \mu') = \left(\inf \left\{ \int_{E^2} |x - y|^2 \pi(dx, dy), \pi \in \mathcal{P}_2(E^2) : \pi(\cdot \times E) = \mu, \pi(E \times \cdot) = \mu' \right\} \right)^{\frac{1}{2}}$$

and denote by $\mathcal{B}(\mathcal{P}_2(E))$ its related Borel σ -algebra. We also similarly define \mathcal{W}_2 on $E \times F$ and $\mathcal{B}(\mathcal{P}_2(E \times F))$. We fix a terminal time T and we define the partially observed control system.

Controls. A control is a sequence $\alpha := (\alpha_n)_{0 \leq n \leq T-1}$ of random variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$ and valued in C such that

$$\mathbb{E}[|\alpha_n|^2] < +\infty \tag{2.1}$$

for all $n = 0, \dots, T-1$. We denote by \mathfrak{C} the set of such controls.

Hidden system. For a given control $\alpha \in \mathfrak{C}$, we consider a controlled process $(X_k)_{0 \leq k \leq T}$ defined by its initial condition $X_0^\alpha = \xi$ where $\xi \in L^2(\Omega, \mathcal{A}, \mathbb{P}; E)$ and the dynamics

$$X_{n+1}^\alpha = G_{n+1}(X_n^\alpha, \mathbb{P}_{X_n^\alpha}, \alpha_n, \varepsilon_{n+1}), \quad 0 \leq n \leq T-1$$

for some measurable functions G_1, \dots, G_T from $E \times \mathcal{P}_2(E) \times C \times \mathbb{R}^d$ to E , where $(\varepsilon_n)_{1 \leq n \leq T}$ is a sequence of square integrable i.i.d. random variables valued in \mathbb{R}^d and independent of ξ . We make the following assumption on the functions G_1, \dots, G_T .

(H1) There exist a constant C such that

$$G_n(x, \mu, a, e) \leq C(1 + |x|^2 + \mathcal{W}_2(\mu, \delta_0)^2 + |a|^2 + |e|^2),$$

for all $(x, a, \mu, e) \in E \times C \times \mathcal{P}_2(E) \times \mathbb{R}^d$ and $n = 1, \dots, T$.

Observed System. For a given control $\alpha \in \mathfrak{C}$, the observation is given by a process $(Y_n^\alpha)_{0 \leq n \leq T}$ defined by its initial condition $Y_0^\alpha = \zeta$ where $\zeta \in L^2(\Omega, \mathcal{A}, \mathbb{P}; F)$ is independent of $(\varepsilon_n)_{1 \leq n \leq T}$ and the dynamics

$$Y_{n+1}^\alpha = H_{n+1}(X_{n+1}^\alpha, Y_n^\alpha, \alpha_n, \eta_{n+1}), \quad 0 \leq n \leq T-1$$

for some measurable functions H_1, \dots, H_T from $E \times F \times C \times \mathbb{R}^d$ into F , where $(\eta_n)_{1 \leq n \leq T}$ is a sequence of i.i.d. random variables valued in \mathbb{R}^d , independent of ξ, ζ and $(\varepsilon_n)_{1 \leq n \leq T}$. We make the following assumption on the functions H_1, \dots, H_T .

(H2) There exist a constant C such that

$$H_n(x, y, e) \leq C(1 + |x|^2 + |y|^2 + |e|^2),$$

for all $(x, y, e) \in E \times F \times \mathbb{R}^d$ and $n = 1, \dots, T$.

Under Assumptions **(H1)** and **(H2)**, we get that the controlled process $(X_n^\alpha, Y_n^\alpha)_{0 \leq n \leq T}$ is square integrable

$$\mathbb{E}[|X_n^\alpha|^2] + \mathbb{E}[|Y_n^\alpha|^2] < +\infty$$

for any control $\alpha \in \tilde{\mathfrak{C}}$ and any $n = 0, \dots, T$.

Optimization problem. We fix T measurable functions c_0, \dots, c_{T-1} from $E \times \mathcal{P}_2(E) \times C$ to \mathbb{R} and a function γ from $E \times \mathcal{P}_2(E)$ to \mathbb{R} . on which we make the following assumption.

(H3) There exist a constant C such that

$$\begin{aligned} c_n(x, \mu, a) &\leq C(1 + |x|^2 + \mathcal{W}_2^2(\mu, \delta_0) + |a|^2), \quad n = 0, \dots, T-1, \\ \gamma(x, \mu) &\leq C(1 + |x|^2 + \mathcal{W}_2^2(\mu, \delta_0)) \end{aligned}$$

for all $(x, \mu, a) \in E \times \mathcal{P}_2(E) \times C$.

We next introduce the cost function J by

$$J(\alpha) := \mathbb{E} \left[\sum_{k=0}^{T-1} c_k(X_k^\alpha, \mathbb{P}_{X_k^\alpha}, \alpha_k) + \gamma(X_T^\alpha, \mathbb{P}_{X_T^\alpha}) \right]$$

for a strategy $\alpha \in \mathfrak{C}$. We notice that under **(H1)**, **(H2)** and **(H3)** $J(\alpha)$ is well defined. We now define the subset $\bar{\mathfrak{C}}$ of \mathfrak{C} by

$$\bar{\mathfrak{C}} = \left\{ \alpha = (\alpha_k)_{0 \leq k \leq T-1} \in \mathfrak{C} \text{ adapted to the filtration generated by } Y^\alpha \right\}$$

The problem is to compute

$$V_0 = \inf_{\alpha \in \bar{\mathfrak{C}}} J(\alpha). \quad (2.2)$$

In the sequel we use the following notation : for $N \geq 0$ and (x_0, \dots, x_N) a given vector, we write $x_{n:m}$ for (x_n, \dots, x_m) where $0 \leq n \leq m \leq N$.

We notice that for $\alpha \in \bar{\mathfrak{C}}$, there exists measurable functions $a_k : F^k \rightarrow A$, $k = 0, \dots, T-1$ such that

$$\alpha_k = a(Y_{0:k}^\alpha), \quad k = 0, \dots, T-1.$$

We therefore identify in the sequel, the set $\bar{\mathfrak{C}}$ to the set $\tilde{\mathfrak{C}}$ defined by

$$\tilde{\mathfrak{C}} := \left\{ a = (a_k)_{0 \leq k \leq T-1} \text{ such that } a_k : F^k \rightarrow A \text{ measurable} \right. \\ \left. \text{and } \mathbb{E}[|a(Y_{0:k}^a)|^2] < +\infty \text{ for } k = 0, \dots, T-1 \right\}.$$

with $X^a = X^\alpha$ and $Y^a = Y^\alpha$ for $\alpha_k = a_k(Y_{0:k}^a)$, $k = 0, \dots, T-1$. Let us stress the well posedness of the controlled processes X^a and Y^a as the components X_k^a and Y_k^a depend only on $\alpha_0, \dots, \alpha_{k-1}$ for $k = 1, \dots, T$.

More precisely, we have the following identity

$$V_0 = \sup_{a \in \tilde{\mathfrak{C}}} \tilde{J}(a)$$

where

$$\tilde{J}(a) = \mathbb{E} \left[\sum_{k=0}^{T-1} c_k(X_k^a, \mathbb{P}_{X_k^a}, a_k(Y_{0:k}^a)) + \gamma(X_T^a, \mathbb{P}_{X_T^a}) \right] = J(\alpha).$$

We next define the sets $\tilde{\mathfrak{C}}_0, \dots, \tilde{\mathfrak{C}}_{T-1}$ by

$$\tilde{\mathfrak{C}}_k(\mu) = \left\{ a_k : F^{k+1} \rightarrow C \text{ measurable such that } \int_{F^{k+1}} |a_k(y_{0:k})|^2 d\mu(x_{0:k}, y_{0:k}) < +\infty \right\},$$

for $\mu \in \mathcal{P}_2((E \times F)^{k+1})$ and $k = 0, \dots, T-1$. In particular, we have

$$\tilde{\mathfrak{C}} = \left\{ a = (a_k)_{0 \leq k \leq T-1} \text{ such that } a_k \in \tilde{\mathfrak{C}}_k(\mathbb{P}_{X_{0:k}^a, Y_{0:k}^a}) \text{ for } k = 0, \dots, T-1 \right\}.$$

We take in the rest of the paper $E = \mathbb{R}^p$ and $F = \mathbb{R}^{p'}$ and $C = \mathbb{R}^{p''}$ for some integers $p, p', p'' \geq 1$.

2.2 Filtered system

To compute the value V_0 defined by (2.2), we would like to compute the conditional law of X_k given $Y_{0:k}$ for $k = 1, \dots, T$. For that, we make the following assumption.

(H4) For $(x, y, a) \in E \times F \times C$ and $k = 1, \dots, T$, the random variable $H_k(x, y, a, \eta_k)$ admits a density

$$e \mapsto h_k(x, y, a, e)$$

where the function h_k is a $\mathcal{B}(E) \otimes \mathcal{B}(F) \otimes \mathcal{B}(C) \otimes \mathcal{B}(\mathbb{R}^{d'})$ -measurable.

Denote by P the controlled transition probability of the process X . It is given by

$$P_k^u(x_{k-1}, \mu, dx_k) = \mathbb{P}(G_k(x_{k-1}, \mu, u, \varepsilon_k) \in dx_k)$$

for $k \geq 1$, $x_{k-1} \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

For a given control $a \in \tilde{\mathcal{C}}$, we get from assumption **(H4)** that the pair $(X_k^a, Y_k^a)_{0 \leq k \leq T-1}$ admits the following transition

$$\begin{aligned} \mathbb{P}((X_k^a, Y_k^a) \in dx_k dy_k | X_{0:k-1}^a, Y_{0:k-1}^a) &= \\ h_k(x_k, Y_{k-1}^a, a_{k-1}(Y_{0:k-1}^a), y_k) P_k^{a_{k-1}(Y_{0:k-1}^a)}(X_{k-1}^a, \mathbb{P}_{X_{k-1}^a}, dx_k) dy_k & \end{aligned} \quad (2.3)$$

for $k = 1, \dots, T$. Therefore, the joint law $\mathbb{P}_{(X_{0:k}^a, Y_{0:k}^a)}$ of $(X_{0:k}^a, Y_{0:k}^a)$ is given by

$$\mathbb{P}_{(X_{0:k}^a, Y_{0:k}^a)}(dx_{0:k}, dy_{0:k}) = \mathbb{P}_{X_0, Y_0}(dx_0, dy_0) \prod_{\ell=1}^k h_\ell(x_\ell, y_{\ell-1}, a_{\ell-1}, y_\ell) P_\ell^{a_{\ell-1}(y_{0:\ell-1})}(x_{\ell-1}, \mu_{\ell-1}^a, dx_\ell) dy_\ell .$$

where $\mu_\ell^a = \mathbb{P}_{X_\ell^a}$ for $\ell = 1, \dots, k$.

We next introduce the measures Π_k^a , $k = 0, \dots, T$ defined by

$$\Pi_{k, y_{0:k}}^a(dx_k) = \mathbb{P}(X_k^a \in dx_k | Y_{0:k}^a = y_{0:k}) .$$

From (2.3) we get the Kallianpur Streibel formula

$$\Pi_{k, y_{0:k}}^a(dx_k) = \frac{\int_{x_{0:k-1}} \prod_{\ell=1}^k h_\ell(x_\ell, y_{\ell-1}, a_{\ell-1}, y_\ell) P_\ell^{a_{\ell-1}(y_{0:\ell-1})}(x_{\ell-1}, \mu_{\ell-1}^a, dx_\ell) \mathbb{P}_{X_0}(dx_0 | Y_0 = y_0)}{\int_{x'_{0:k}} \prod_{\ell=1}^k h_\ell(x'_\ell, y_{\ell-1}, a_{\ell-1}, y_\ell) P_\ell^{a_{\ell-1}(y_{0:\ell-1})}(x'_{\ell-1}, \mu_{\ell-1}^a, dx_\ell) \mathbb{P}_{X_0}(dx'_0 | Y_0 = y_0)} .$$

We also define the unnormalized measures $\pi_{k, y_{0:k}}^a$ by

$$\pi_{k, y_{0:k}}^a(dx_k) = \int_{x_{0:k-1}} \prod_{\ell=1}^k h_\ell(x_\ell, y_{\ell-1}, a_{\ell-1}, y_\ell) P_\ell^{a_{\ell-1}(y_{0:\ell-1})}(x_{\ell-1}, \mu_{\ell-1}^a, dx_\ell) \mathbb{P}_{X_0}(dx_0 | Y_0 = y_0) .$$

Still using (2.3) we get

$$\Pi_{k, y_{0:k}}^a(dx_k) = \frac{\pi_{k, y_{0:k}}^a(dx_k)}{\pi_{k, y_{0:k}}^a(\mathbb{R}^d)} .$$

We now compute the probability measures μ_k^a . From (2.3) we have

$$\begin{aligned} \mu_k^a(dx_k) &= \int_{x_{0:k-1}} \int_{y_{0:k-1}} P_k^{a_{k-1}(y_{0:k-1})}(x_{k-1}, \mu_{k-1}^a, dx_k) h_k(x_k, y_{k-1}, a_{k-1}, y_k) \\ &\quad \prod_{\ell=1}^{k-1} h_\ell(x_\ell, y_{\ell-1}, a_{\ell-1}, y_\ell) P_\ell^{a_{\ell-1}(y_{0:\ell-1})}(x_{\ell-1}, \mu_{\ell-1}^a, dx_\ell) dy_{1:k-1} \mathbb{P}_{X_0, Y_0}(dx_0, dy_0) . \end{aligned}$$

2.3 The filtered problem

We now turn to the computation of the value V_0 . By definition we have

$$\begin{aligned} V_0 &= \inf_{a \in \tilde{\mathcal{C}}} J(a) \\ &= \inf_{a \in \tilde{\mathcal{C}}} \mathbb{E} \left[\sum_{k=0}^{T-1} \mathbb{E} \left[c_k(X_k^a, \mathbb{P}_{X_k^a}, a_k(Y_{0:k}^a)) | Y_{0:k}^a \right] + \mathbb{E} \left[\gamma(X_T^a, \mathbb{P}_{X_T^a}) | Y_{0:T}^a \right] \right] . \end{aligned}$$

Using the previous notations, we get

$$V_0 = \inf_{a \in \mathfrak{C}} \mathbb{E} \left[\sum_{k=0}^{T-1} \int c_k(x_k, \mu_k^a, a_k(Y_{0:k}^a)) \Pi_{k, Y_{0:k}^a}^a(dx_k) + \int \gamma(x_T, \mu_T^a) \Pi_{T, Y_{0:T}^a}^a(dx_T) \right].$$

Unfortunately, this form is not time consistent. Indeed, the costs involve the marginal laws of the unobservable controlled process X^a but also the controlled conditional laws $\Pi_{k, Y_{0:k}^a}$.

3 Extended filtered control problem

3.1 Extended problem

Under the previous form the problem is not tractable as it involves the laws μ_n^a and $\Pi_{n, Y_{0:n}^a}$. Indeed, we cannot get μ_n^a from $\Pi_{n, Y_{0:n}^a}$ and conversely. This prevents from an application of a dynamic programming approach. To overcome this issue, we introduce the controlled measures $M_n^a \in \mathcal{P}_2((E \times F)^{n+1})$ defined by

$$M_n^a(dx_{0:n}, dy_{0:n}) = \prod_{\ell=1}^n h_\ell(x_\ell, y_{\ell-1}, \alpha_{\ell-1}, y_\ell) P_\ell^{a_{\ell-1}(y_{0:\ell-1})}(x_{\ell-1}, \mu_{\ell-1}^a, dx_\ell) dy_{1:n} \mathbb{P}_{X_0, Y_0}(dx_0, dy_0)$$

for $n = 0, \dots, T$. We observe that μ_n^a and $\Pi_{n, Y_{0:n}^a}$ can be computed from the measure M_n^a . Indeed, we first have

$$\mu_n^a(dx_{0:n}) = \int_{y_{0:n}} M_n^a(dx_{0:n}, dy_{0:n}).$$

Secondly, we have

$$\pi_{n, y_{0:n}}^a(dx_n) = \int_{x_{0:n-1}} \frac{dM_n^a}{dy_{0:n}}(y_{0:n}, dx_{0:n}).$$

Therefore, we get

$$\Pi_{n, y_{0:n}}^a(dx_n) = \frac{\int_{x_{0:n-1}} \frac{dM_n^a}{dy_{0:n}}(y_{0:n}, dx_{0:n})}{\int_{x_{0:n}} \frac{dM_n^a}{dy_{0:n}}(y_{0:n}, dx_{0:n})}.$$

We now introduce some notations. For $n \in \{0, \dots, T\}$ and $M \in \mathcal{P}_2((E \times F)^{n+1})$ we denote by ${}_\ell M$ the ℓ -th marginal of M :

$${}_\ell M(dx_\ell, dy_\ell) = \int_{x_0, y_0} \cdots \int_{x_{\ell-1}, y_{\ell-1}} \int_{x_{\ell+1}, y_{\ell+1}} \cdots \int_{x_n, y_n} M(dx_{0:n}, dy_{0:n})$$

and ${}^1_\ell M$ and ${}^2_\ell M$ the first and second marginals of ${}_\ell M$ respectively:

$$\begin{aligned} {}^1_\ell M(dx_\ell) &= \int_{y_\ell} {}_\ell M(dx_\ell, dy_\ell) \\ {}^2_\ell M(dy_\ell) &= \int_{x_\ell} {}_\ell M(dx_\ell, dy_\ell) \end{aligned}$$

for $\ell \in \{0, \dots, n\}$. We define the controlled transition probability $\bar{P}_n : (E \times F)^n \times \tilde{\mathfrak{C}} \times \mathcal{P}_2((E \times F)^n) \rightarrow \mathcal{P}_2(E \times F)$ by

$$\bar{P}_n^{a_{n-1}}(z, M, dw) = P_n^{a_{n-1}(y_{0:n-1})}(x_{n-1}, \frac{1}{n-1}M, dw_1)h_n(w_1, y_{n-1}, a_{n-1}(y_{0:n-1}), w_2)dw_2$$

for $z = (x_\ell, y_\ell)_{0 \leq \ell \leq n-1} \in (E \times F)^n$, $M \in \mathcal{P}_2((E \times F)^n)$, $a_{n-1} \in \tilde{\mathfrak{C}}_{n-1}(M)$ and $w = (w_1, w_2) \in E \times F$ and $n = 1, \dots, T$. The controlled measures $\mathbb{P}_{X_n^a, Y_n^a}$ have the following dynamics

$$\begin{aligned} & \mathbb{P}_{(X_{0:n+1}^a, Y_{0:n+1}^a)}(dx_{0:n+1}, dy_{0:n+1}) = \\ & \bar{P}_{n+1}^{a_n}(x_{0:n}, y_{0:n}, a_n(Y_{0:n}), \mathbb{P}_{(X_{0:n}^a, Y_{0:n}^a)}, dx_{n+1}, dy_{n+1})\mathbb{P}_{(X_{0:n}^a, Y_{0:n}^a)}(dx_{0:n}, dy_{0:n}) \end{aligned}$$

for $n = 0, \dots, T-1$. In the sequel, we use the following notation : for $\mu \in \mathcal{P}_2((E \times F)^{n+1})$ and $a_n \in \tilde{\mathfrak{C}}_n(\mu)$, we define $\bar{P}_{n+1}^{a_n}\mu \in \mathcal{P}_2((E \times F)^{n+2})$ by

$$\bar{P}_{n+1}^{a_n}\mu(dx_{0:n+1}, dy_{0:n+1}) = \bar{P}_{n+1}^{a_n}(x_{0:n}, y_{0:n}, \mu, dx_{n+1}, dy_{n+1})\mu(dx_{0:n}, dy_{0:n})$$

for $n = 0, \dots, T-1$. The dynamics of the controlled measures $\mathbb{P}_{(X_{0:n}^a, Y_{0:n}^a)}$ can be rewritten under the following simplified form

$$\mathbb{P}_{(X_{0:n+1}^a, Y_{0:n+1}^a)}(dx_{0:n+1}, dy_{0:n+1}) = \bar{P}_{n+1}^{a_n}\mathbb{P}_{(X_{0:n}^a, Y_{0:n}^a)}(dx_{0:n}, dy_{0:n})$$

for $n = 0, \dots, T-1$.

We now turn to the filtered problem. We define the cost functions C_n , $n = 0, \dots, T-1$ and Γ by

$$C_n(M, a_n) = \int c_n\left(x_n, \frac{1}{n}M, a_n(y_{0:n})\right)dM(x_{0:n}, y_{0:n})$$

for $M \in \mathcal{P}_2((E \times F)^{n+1})$, $a_n \in \tilde{\mathfrak{C}}_n(M)$ and

$$\Gamma(M) = \int \gamma\left(x_T, \frac{1}{T}M\right)dM(x_{0:T}, y_{0:T}),$$

for $M \in \mathcal{P}_2((E \times F)^{T+1})$. A straightforward computation gives

$$V_0 = \inf_{a \in \tilde{\mathfrak{C}}} \bar{J}(a).$$

where the criteria \bar{J} is defined by

$$\bar{J}(a) := \sum_{n=0}^{T-1} C_k(\mathbb{P}_{(X_{0:n}^a, Y_{0:n}^a)}, a_n) + \Gamma(\mathbb{P}_{(X_{0:T}^a, Y_{0:T}^a)}), \quad a \in \tilde{\mathfrak{C}}. \quad (3.4)$$

3.2 Dynamic programming

We dynamically extend the value V_0 . For that, we define for $n = 0, \dots, T$ the functions $V_n : \mathcal{P}_2((E \times F)^{n+1}) \rightarrow \mathbb{R}$ by

$$V_n(\mu) = \inf_{a \in \tilde{\mathfrak{C}}(\mu)} \bar{J}_n(\mu, a)$$

where

$$\bar{J}_n(\mu, a) := \sum_{k=n}^{T-1} C_k(M_k^{n,\mu,a}, a_k) + \Gamma(M_T^{n,\mu,a})$$

with $(M_k^{n,\mu,a})_{k=n,\dots,T}$ defined by $M_n^{n,\mu,a} = \mu$ and

$$M_{k+1}^{n,\mu,a} = \bar{P}_{k+1}^a M_k^{n,\mu,a} \quad (3.5)$$

for $k = n, \dots, T-1$, and

$${}_n\tilde{\mathcal{C}}(\mu) = \left\{ a = (a_k)_{0 \leq k \leq T-1} \text{ such that } a_k \in \tilde{\mathcal{C}}_k(M_k^{n,\mu,a}) \text{ for } k = n, \dots, T-1 \right\}$$

for any $\mu \in \mathcal{P}_2((E \times F)^{n+1})$

Lemma 3.1 (Dynamic programming). *The value functions V_0, \dots, V_T satisfy the following dynamic programming principle*

$$V_n(\mu) = \inf_{a_n \in \tilde{\mathcal{C}}_n(\mu)} \left\{ C_n(\mu, a_n) + V_{n+1}(\bar{P}_{n+1}^{a_n} \mu) \right\}, \quad \mu \in \mathcal{P}_2((E \times F)^{n+1}), \quad (3.6)$$

for $n = 0, \dots, T-1$

Proof. Denote by $Z_n(\mu)$ the right-hand-side of (3.6). Fix a strategy $a \in {}_n\tilde{\mathcal{C}}(\mu)$. We then have $a \in {}_{n+1}\tilde{\mathcal{C}}(\bar{P}_n^{a_n} \mu)$. We then notice that

$$M_k^{n+1, \bar{P}_{n+1}^{a_n} \mu, a} = M_k^{n, \mu, a}$$

for $k = n+1, \dots, T$. Therefore, we have

$$V_{n+1}(\bar{P}_{n+1}^{a_n} \mu) \leq \sum_{k=n+1}^{T-1} C_k(M_k^{n,\mu,a}, a_k) + \Gamma(M_T^{n,\mu,a})$$

and

$$C_n(\mu, a_n) + V_{n+1}(\bar{P}_{n+1}^{a_n} \mu) \leq \bar{J}_n(\mu, a).$$

Taking the infimum over $a \in {}_n\tilde{\mathcal{C}}(\mu)$, we get

$$Z_n(\mu) \leq V_n(\mu).$$

We turn to the reverse inequality. Fix $\varepsilon > 0$ and a strategy $a_n^\varepsilon \in \tilde{\mathcal{C}}_n(\mu)$ such that

$$C_n(\mu, a_n^\varepsilon) + V_{n+1}(\bar{P}_{n+1}^{a_n^\varepsilon} \mu) - \frac{\varepsilon}{2} \leq Z_n(\mu). \quad (3.7)$$

We now fix a strategy $\tilde{a}^\varepsilon \in {}_{n+1}\tilde{\mathcal{C}}(\bar{P}_{n+1}^{a_n^\varepsilon} \mu)$ such that

$$\bar{J}_{n+1}(\bar{P}_{n+1}^{a_n^\varepsilon} \mu, \tilde{a}^\varepsilon) - \frac{\varepsilon}{2} \leq V_{n+1}(\bar{P}_{n+1}^{a_n^\varepsilon} \mu). \quad (3.8)$$

We define \hat{a}^ε as the concatenation of a_n^ε and \tilde{a}^ε :

$$\hat{a}_k^\varepsilon(\cdot) = a_n^\varepsilon(\cdot)\mathbb{1}_{k=n} + \tilde{a}_k^\varepsilon(\cdot)\mathbb{1}_{k \neq n}.$$

Then we have $\hat{a} \in {}_n\tilde{\mathfrak{C}}(\mu)$, $M_{n+1}^{n,\mu,\hat{a}^\varepsilon} = \bar{P}_{n+1}^{a_n^\varepsilon}\mu$ and

$$M_k^{n,\mu,\hat{a}^\varepsilon} = M_k^{n+1,\bar{P}_{n+1}^{a_n^\varepsilon}\mu,\tilde{a}^\varepsilon}$$

for $k = n + 1, \dots, T$. Therefore we get from (3.7) and (3.8)

$$\bar{J}_n(\mu, \hat{a}^\varepsilon) - \varepsilon \leq Z_n(\mu)$$

and

$$V_n(\mu) - \varepsilon \leq Z_n(\mu).$$

Since ε is arbitrarily chosen, we get $V_n(\mu) \leq Z_n(\mu)$. □

We now provide a verification result for the optimal value and an optimal strategy.

Theorem 3.1 (Verification). *Consider the functions W_0, \dots, W_T defined by*

$$W_T(\mu) = \Gamma(\mu), \quad \mu \in \mathcal{P}_2((E \times F)^{T+1}),$$

and

$$W_n(\mu) = \inf_{a_n \in \tilde{\mathfrak{C}}_n(\mu)} \{C_n(\mu, a_n) + W_{n+1}(\bar{P}_{n+1}^{a_n}\mu)\}, \quad \mu \in \mathcal{P}_2((E \times F)^{n+1}) \quad (3.9)$$

for $n = 0, \dots, T - 1$. Then $W_n = V_n$ for $n = 0, \dots, T$.

Suppose that for any $n = 0, \dots, T - 1$ and any $\mu \in \mathcal{P}_2((E \times F)^n)$, there exist a function $a_n^{\mu,*} \in \tilde{\mathfrak{C}}_n(\mu)$ such that

$$W_n(\mu) = C_n(\mu, a_n^{\mu,*}) + W_{n+1}(\bar{P}_{n+1}^{a_n^{\mu,*}}\mu). \quad (3.10)$$

Then for a given starting measure $\mu \in \mathcal{P}_2(E \times F)$, the strategy a^* defined by $a_0^* = a_0^{*,\mu}$ and

$$a_n^* = a_n^{M_n^{0,\mu,a^*},*}, \quad n = 1, \dots, T - 1, \quad (3.11)$$

is optimal:

$$V_0(\mu) = \bar{J}(\mu, a^*).$$

Proof. Fix $n = 0, \dots, T - 1$, $\mu \in \mathcal{P}_2((E \times F)^{n+1})$ and a control $a \in {}_n\tilde{\mathfrak{C}}(\mu)$. Since $W_T = \Gamma$, we get by (3.9) and a straightforward backward induction

$$W_n(\mu) \leq \sum_{k=n}^{T-1} C_k(M_k^{n,\mu,a}, a_k) + \Gamma(M_T^{n,\mu,a}).$$

Since $a \in {}_n\tilde{\mathfrak{C}}(\mu)$ is arbitrarily chosen, we get

$$W_n(\mu) \leq V_n(\mu)$$

for $\mu \in \mathcal{P}_2((E \times F)^{n+1})$ and $n = 0, \dots, T$.

We now prove the reverse inequality by a backward induction. First we have $V_T = \Gamma = W_T$, so $W_T \geq V_T$.

Suppose that we have $W_{n+1} \geq V_{n+1}$. Using Lemma 3.1 and (3.9) we get $W_n \geq V_n$. Therefore, the result holds true for any $n = 0, \dots, T$.

Fix now $\mu \in \mathcal{P}_2(E \times F)$ and consider the strategy a^* defined by (3.11). Writing the equality (3.9) at each step we get

$$W_0(\mu) = \hat{J}(\mu, a^*)$$

and a^* is optimal since $W_0(\mu) = V_0(\mu)$. \square

In the previous result, the verification condition (3.9) is written over the set $\tilde{\mathfrak{C}}_n$ which might be too large. Indeed, as the dynamics (3.5) of $M^{n,\mu,a}$ involves the past only through the control a , one may wonder if condition (3.9) can be reduced to closed loop controls, *i.e.* controls depending on the the present value of $M^{n,\mu,a}$, and if an optimal strategy of this form can be derived. This is possible under the condition that the starting position measure μ already has this structure.

More precisely, we introduce, for $n = 0, \dots, T-1$ and $\mu \in \mathcal{P}_2((E \times F)^n)$ the subset $\tilde{\mathfrak{C}}_{n,\text{cl}}(\mu)$ of $\tilde{\mathfrak{C}}_n(\mu)$ composed by control functions a_n depending only on the last component

$$a_n(y_{0:n}) = a_n(y_n), \quad y_{0:n} \in F^{n+1}.$$

We then define for $\mu \in \mathcal{P}_2(E \times F)$ the set $\tilde{\mathfrak{C}}_{\text{cl}}(\mu)$ as the set of controls $a = (a_0, \dots, a_{T-1}) \in {}_0\tilde{\mathfrak{C}}(\mu)$ such that $a_n \in \tilde{\mathfrak{C}}_{n,\text{cl}}(M_n^{0,\mu,a})$ for $n = 0, \dots, T-1$. We finally denote by $\mathcal{P}_{\text{cl}}((E \times F)^n)$ the subset of $\mathcal{P}_2((E \times F)^n)$ composed by laws of controlled processes $(X_{0:n}^a, Y_{0:n}^a)$ for $a \in \tilde{\mathfrak{C}}_{\text{cl}}(\mu)$ with $\mathbb{P}_{(X_0^a, Y_0^a)} = \mu$ and $\mu \in \mathcal{P}_2(E \times F)$. We notice that $\mathcal{P}_{\text{cl}}(E \times F) = \mathcal{P}_2(E \times F)$ as the initial position does not depend on the control. We can now state the verification result for closed loop controls.

Theorem 3.2 (Verification with closed loop controls). *Consider the functions W_0, \dots, W_T defined by*

$$W_T(\mu) = \Gamma(\mu), \quad \mu \in \mathcal{P}_{\text{cl}}((E \times F)^{T+1}),$$

and

$$W_n(\mu) = \inf_{a_n \in \tilde{\mathfrak{C}}_n(\mu)} \{C_n(\mu, a_n) + W_{n+1}(\bar{P}_{n+1}^{a_n} \mu)\}, \quad \mu \in \mathcal{P}_{\text{cl}}((E \times F)^{n+1}), \quad (3.12)$$

for $0 \leq n \leq T-1$. Then $W_n = V_n$ for $n = 0, \dots, T$ on $\mathcal{P}_{\text{cl}}((E \times F)^{n+1})$. Suppose that for any $n = 0, \dots, T-1$ there exists a measurable function $\mathfrak{a}_n : \mathcal{P}_2((E \times F)^n) \times F \rightarrow C$ such that

$$W_n(\mu) = C_n(\mu, \mathfrak{a}_n(\mu, \cdot)) + W_{n+1}(\bar{P}_{n+1}^{\mathfrak{a}_n(\mu, \cdot)} \mu). \quad (3.13)$$

and $\mathbf{a}_n(\mu, \cdot) \in \tilde{\mathcal{C}}_n$ for all $\mu \in \mathcal{P}_{cl}((E \times F)^n)$. Then for a given starting measure $\mu \in \mathcal{P}_2(E \times F)$, the strategy a^* defined by $a_0^* = \mathbf{a}_0(\mu, \cdot)$ and

$$a_n^* = \mathbf{a}_n(M_n^{0,\mu,a^*}, \cdot), \quad n = 1, \dots, T-1,$$

is optimal:

$$V_0(\mu) = \bar{J}(\mu, a^*).$$

Proof. The proof follows exactly the same lines as that of Theorem 3.1 and is therefore omitted. \square

4 Applications

4.1 Linear-quadratic case

We take $p = p' = p'' = d' = d$ in this section. We suppose that the processes X and Y have the following dynamics

$$X_{k+1}^a = B_k X_k^a + \bar{B}_k \mathbb{E}[X_k^a] + D_k a_k(Y_{0:k}^a) + \varepsilon_{k+1} \quad (4.14)$$

$$Y_{k+1}^a = J_{k+1} X_{k+1}^a + \eta_{k+1}, \quad (4.15)$$

where B_k, \bar{B}_k, D_k and J_k are deterministic $d \times d$ matrices and ε_k and η_k follow $\mathcal{N}(0, I_d)$. We also suppose that X_0 and Y_0 are independent and follow $\mathcal{N}(0, I_d)$. We then define the cost function \tilde{J} by

$$\begin{aligned} \tilde{J}(a) := & \mathbb{E} \left[\sum_{k=0}^{T-1} X_k^{a\top} Q_k X_k^a + \mathbb{E}[X_k^a]^\top \bar{Q}_k \mathbb{E}[X_k^a] + a(Y_{0:k}^a)^\top R_k a(Y_{0:k}^a) \right. \\ & \left. + X_T^{a\top} Q_T X_T^a + \mathbb{E}[X_T^a]^\top \bar{Q}_T \mathbb{E}[X_T^a] \right], \end{aligned}$$

for any strategy $a \in \tilde{\mathcal{C}}$. In this case **(H1)**-**(H2)**-**(H3)** are satisfied and we have

$$h_k(x, y, a, e) = h_k(x, e) = \frac{1}{(\sqrt{2\pi})^d} \exp\left(-\frac{1}{2}|e - J_k x|^2\right), \quad x, y, a, e \in \mathbb{R}^d,$$

which is the density of the law $\mathcal{N}(J_k x, I_d)$. We now introduce some notations. For $\mu \in \mathcal{P}_2(E \times F)$ and $\Lambda \in \mathbb{R}^{2d \times 2d}$, we set

$$\bar{\mu} = \int_{\mathbb{R}^{2d}} \begin{pmatrix} x \\ y \end{pmatrix} d\mu(x, y)$$

and

$$\langle \mu \rangle(\Lambda) = \int_{\mathbb{R}^{2d}} \begin{pmatrix} x \\ y \end{pmatrix}^\top \Lambda \begin{pmatrix} x \\ y \end{pmatrix} d\mu(x, y).$$

We also define the matrices $\mathbf{Q}_k, \bar{\mathbf{Q}}_k \in \mathbb{R}^{2d \times 2d}$ by

$$\mathbf{Q}_k = \begin{pmatrix} Q_k & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \bar{\mathbf{Q}}_k = \begin{pmatrix} \bar{Q}_k & 0 \\ 0 & 0 \end{pmatrix}$$

for $k = 0, \dots, T$. The functions C_k and Γ appearing in the definition (3.4) of \bar{J} are then given by

$$C_k(\mu, a) = \langle {}_k\mu \rangle(\mathbf{Q}_k) + {}_k\bar{\mu}^\top \mathbf{Q}_{kk} \bar{\mu} + \int a(y_{0:k})^\top R_k a(y_{0:k}) d\mu(x_{0:k}, y_{0:k})$$

for $\mu \in \mathcal{P}_2((E \times F)^{k+1})$, $a \in \tilde{\mathcal{C}}_k(\mu)$, $k = 0, \dots, T-1$, and

$$\Gamma(\mu) = \langle {}_T\mu \rangle(\mathbf{Q}_T) + {}_T\bar{\mu}^\top (\mathbf{Q}_T) {}_T\bar{\mu}, \quad \mu \in \mathcal{P}_2((E \times F)^{T+1}),$$

where we recall that ${}_k\mu$ stands for the k -th marginal of μ for $k = 0, \dots, T$.

We look for candidates W_k , $k = 0, \dots, T$, satisfying the verification Theorem. For that we chose an ansatz in the following quadratic form:

$$W_k(\mu) = \langle {}_k\mu \rangle(\Lambda_k) + {}_k\bar{\mu}^\top \Theta_k \bar{\mu} + \chi_k \quad (4.16)$$

for $\mu \in \mathcal{P}_2((E \times F)^{k+1})$. We next suppose that J_1, \dots, J_T and D_0, \dots, D_{T-1} are all invertible and that $Q_1, Q_1 + \bar{Q}_1, \dots, Q_T, Q_T + \bar{Q}_T$ are symmetric nonnegative and R_1, \dots, R_T are all symmetric positive. We then have the following result.

Proposition 4.1. *There exists $\Lambda_k, \Theta_k \in \mathbb{R}^{2d \times 2d}$ symmetric, with Λ_k and $\Theta_k + \Lambda_k$ nonnegative, and $\chi_k \in \mathbb{R}$, $k = 0, \dots, T$, symmetric nonnegative such that the functions W_k , $k = 0, \dots, T$, given by (4.16) satisfy the verification Theorem 3.2 with a feedback optimal strategy \mathbf{a} of the form*

$$\mathbf{a}_n(\mu, y) = G_n \left(\Xi_n \Phi_n(\mu, y) + \hat{\mathbf{S}}_{nn}^\top \bar{\mu} \right), \quad y \in \mathbb{R}^d, \quad \mu \in \mathcal{P}_{cl}((\mathbb{R}^d \times \mathbb{R}^d)^{n+1}),$$

with Φ_n given by

$$\Phi_n(\mu, y) = \int_{\mathbb{R}^d} x_n \frac{e^{-\frac{1}{2}|J_n x_n - y_n|^2}}{\int_{\mathbb{R}^d} e^{-\frac{1}{2}|J_n x'_n - y_n|^2} d {}_n^1 \mu(x'_n)} d {}_n^1 \mu(x_n), \quad (4.17)$$

and $G_n \in \mathbb{R}^{d \times d}$, $\Xi_n \in \mathbb{R}^{d \times d}$ and $\hat{\mathbf{S}}_n \in \mathbb{R}^{2d \times d}$ for $n = 0, \dots, T-1$

Proof. We use a backward induction on k to prove the following statement:

For $n = 0, \dots, N$, there exists $\Lambda_k, \Theta_k \in \mathbb{R}^{2d \times 2d}$ symmetric, with Λ_k and $\Theta_k + \Lambda_k$ nonnegative, and $\chi_k \in \mathbb{R}$, $k = 0, \dots, T$, such that the functions W_k , $k = n, \dots, T$, given by (4.16) satisfy the verification Theorem 3.2 with a feedback optimal strategy \mathbf{a} of the form

$$\mathbf{a}_k(\mu, y) = G_k \left(\Xi_k \Phi_k(\mu, y) + \hat{\mathbf{S}}_{kk}^\top \bar{\mu} \right), \quad y \in \mathbb{R}^d, \quad \mu \in \mathcal{P}_{cl}((\mathbb{R}^d \times \mathbb{R}^d)^{k+1}),$$

with $G_k \in \mathbb{R}^{d \times d}$, $\Xi_k \in \mathbb{R}^{d \times d}$ and $\hat{\mathbf{S}}_k \in \mathbb{R}^{2d \times d}$ some matrices depending on the coefficients for $k = n, \dots, T-1$

For $n = T$ a straightforward computation gives

$$\Lambda_T = \mathbf{Q}_T, \Theta_T = \bar{\mathbf{Q}}_T \quad \text{and} \quad \chi_T = 0.$$

Therefore, the property holds for $n = T$.

Suppose that the property holds for $n + 1$. Fix $\mu \in \mathcal{P}_{\text{cl}}((E \times F)^{n+1})$. From the induction assumption, we have

$$\begin{aligned} & \inf_{a_n \in \tilde{\mathcal{C}}_n(\mu)} \{C_n(\mu, a_n) + W_{n+1}(M_{n+1}^{n,\mu,a_n})\} = \\ & \inf_{a_n \in \tilde{\mathcal{C}}_n(\mu)} \{C_n(\mu, a_n) + \langle_{n+1} M_{n+1}^{n,\mu,a_n} \rangle (\Lambda_{n+1}) \\ & + ({}_{n+1} \bar{M}_{n+1}^{n,\mu,a})^\top \Theta_{n+1} ({}_{n+1} \bar{M}_{n+1}^{n,\mu,a_n}) + \chi_{n+1}\} = \\ & \inf_{a_n \in \tilde{\mathcal{C}}_n(\mu)} \{ \langle_n \mu \rangle (\mathbf{Q}_n) + {}_n \bar{\mu}^\top \bar{\mathbf{Q}}_{nn} \bar{\mu} \\ & + \int a_n(y_{0:n})^\top R_n a_n(y_{0:n}) d\mu(x_{0:n}, y_{0:n}) + \langle_{n+1} M_{n+1}^{n,\mu,a_n} \rangle (\Lambda_{n+1}) \\ & + ({}_{n+1} \bar{M}_{n+1}^{n,\mu,a})^\top \Theta_{n+1} ({}_{n+1} \bar{M}_{n+1}^{n,\mu,a_n}) + \chi_{n+1} \}. \end{aligned}$$

From (3.5) we have

$$\begin{aligned} {}_{n+1} \bar{M}_{n+1}^{n,\mu,a_n} &= \int_{\mathbb{R}^{2d}} \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} d {}_{n+1} M_{n+1}^{n,\mu,a_n}(x_{n+1}, y_{n+1}) \\ &= \int_{\mathbb{R}^{2d}} \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} \int_{(\mathbb{R}^{2d})^n} P_n^{a_n(y_{0:n})}(x_n, {}^1_n \mu, dx_{n+1}) h_n(x_{n+1}, y_{n+1}) dy_{n+1} d\mu(x_{0:n}, y_{0:n}) \end{aligned}$$

From (4.14)-(4.15) we get

$$\begin{aligned} {}_{n+1} \bar{M}_{n+1}^{n,\mu,a_n} &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} \int_{(\mathbb{R}^{2d})^n} e^{-\frac{1}{2}|x_{n+1} - (B_n x_n + \bar{B}_n {}^1_n \bar{\mu} + D_n a_n(y_{0:n}))|^2} \\ & \quad e^{-\frac{1}{2}|y_{n+1} - J_{n+1} x_{n+1}|^2} d\mu(x_{0:n}, y_{0:n}) dx_{n+1} dy_{n+1} \\ &= \int_{(\mathbb{R}^{2d})^n} \begin{pmatrix} (B_n x_n + \bar{B}_n {}^1_n \bar{\mu} + D_n a_n(y_{0:n})) \\ J_{n+1} (B_n x_n + \bar{B}_n {}^1_n \bar{\mu} + D_n a_n(y_{0:n})) \end{pmatrix} d\mu(x_{0:n}, y_{0:n}) \\ &= \mathbf{J}_{n+1} (B_n + \bar{B}_n) {}^1_n \bar{\mu} + \mathbf{J}_{n+1} D_n \int_{(\mathbb{R}^{2d})^n} a_n(y_{0:n}) d\mu(x_{0:n}, y_{0:n}) \end{aligned}$$

where

$$\mathbf{J}_{n+1} = \begin{pmatrix} I_d \\ J_{n+1} \end{pmatrix} \in \mathbb{R}^{2d \times d}.$$

Therefore we get

$$\begin{aligned} & ({}_{n+1} \bar{M}_{n+1}^{n,\mu,a_n})^\top \Theta_{n+1} ({}_{n+1} \bar{M}_{n+1}^{n,\mu,a_n}) = \\ & {}_n \bar{\mu}^\top (\mathbf{B}_n + \bar{\mathbf{B}}_n)^\top \mathbf{J}_{n+1}^\top \Theta_{n+1} \mathbf{J}_{n+1} (\mathbf{B}_n + \bar{\mathbf{B}}_n) {}^1_n \bar{\mu} \\ & + 2 {}_n \bar{\mu}^\top (\mathbf{B}_n + \bar{\mathbf{B}}_n)^\top \mathbf{J}_{n+1}^\top \Theta_{n+1} \mathbf{J}_{n+1} D_n \int_{(\mathbb{R}^{2d})^n} a_n(y_{0:n}) d\mu(x_{0:n}, y_{0:n}) \\ & + \left(\int_{(\mathbb{R}^{2d})^n} a_n(y_{0:n}) d\mu(x_{0:n}, y_{0:n}) \right)^\top D_n^\top \mathbf{J}_{n+1}^\top \Theta_{n+1} \mathbf{J}_{n+1} D_n \int_{(\mathbb{R}^{2d})^n} a_n(y_{0:n}) d\mu(x_{0:n}, y_{0:n}) \end{aligned}$$

with

$$\mathbf{B}_n = \begin{pmatrix} B_n & 0 \end{pmatrix} \in \mathbb{R}^{d \times 2d} \quad \text{and} \quad \bar{\mathbf{B}}_n = \begin{pmatrix} \bar{B}_n & 0 \end{pmatrix} \in \mathbb{R}^{d \times 2d}.$$

We turn to the computation of the second order moment. Still using (4.14)-(4.15), a computation gives

$$\begin{aligned} & \int_{\mathbb{R}^{2d}} \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}^\top \Lambda_{n+1} \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} d_{(n+1)} M_{n+1}^{n, \mu, a_n}(x_{n+1}, y_{n+1}) = \\ & \int_{(\mathbb{R}^{2d})^n} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}^\top \Lambda_{n+1} \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} e^{-\frac{1}{2}|x_{n+1} - (B_n x_n + \bar{B}_n^1 \bar{\mu} + D_n a_n(y_{0:n}))|^2} \\ & \qquad \qquad \qquad e^{-\frac{1}{2}|y_{n+1} - J_{n+1} x_{n+1}|^2} dx_{n+1} dy_{n+1} d\mu(x_{0:n}, y_{0:n}) = \\ & \int_{(\mathbb{R}^{2d})^n} (B_n x_n + \bar{B}_n^1 \bar{\mu} + D_n a_n(y_{0:n}))^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} (B_n x_n + \bar{B}_n^1 \bar{\mu} + D_n a_n(y_{0:n})) d\mu(x_{0:n}, y_{0:n}) \\ & \qquad \qquad \qquad + \text{Tr}(\mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1}) + \text{Tr}(\mathbf{I}^\top \Lambda_{n+1} \mathbf{I}) \end{aligned}$$

where

$$\mathbf{I} = \begin{pmatrix} 0 \\ I_d \end{pmatrix} \in \mathbb{R}^{2d \times d}.$$

We therefore finally get

$$\begin{aligned} & \langle_{n+1} M_{n+1}^{n, \mu, a_n} \rangle (\Lambda_{n+1}) = \\ & \langle_n \mu \rangle (\mathbf{B}_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} \mathbf{B}_n) \\ & +_n \bar{\mu}^\top (\bar{\mathbf{B}}_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} \bar{\mathbf{B}}_n + \mathbf{B}_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} \bar{\mathbf{B}}_n + \bar{\mathbf{B}}_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} \mathbf{B}_n)_n \bar{\mu} \\ & \qquad \qquad \qquad + \text{Tr}(\mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1}) + \text{Tr}(\mathbf{I}^\top \Lambda_{n+1} \mathbf{I}) \\ & + \int_{(\mathbb{R}^{2d})^n} a_n(y_{0:n})^\top D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n a_n(y_{0:n}) d\mu(x_{0:n}, y_{0:n}) \\ & \qquad \qquad \qquad + 2 \int_{(\mathbb{R}^{2d})^n} x_n^\top B_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n a_n(y_{0:n}) d\mu(x_{0:n}, y_{0:n}) \\ & \qquad \qquad \qquad + 2_n \bar{\mu}^\top \int_{(\mathbb{R}^{2d})^n} \bar{\mathbf{B}}_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n a_n(y_{0:n}) d\mu(x_{0:n}, y_{0:n}). \end{aligned}$$

We now go back to the definition of W_n :

$$\begin{aligned}
W_n(\mu) &= \inf_{a_n \in \tilde{\mathcal{E}}_n(\mu)} \left\{ \langle {}_n\mu \rangle (\mathbf{Q}_n) + {}_n\bar{\mu}^\top \bar{\mathbf{Q}}_{nn} \bar{\mu} + \int a_n(y_{0:n})^\top R_n a_n(y_{0:n}) d\mu(x_{0:n}, y_{0:n}) \right. \\
&\quad \left. + \langle {}_{n+1}M_{n+1}^{n,\mu,a_n} \rangle (\Lambda_{n+1}) + ({}_{n+1}\bar{M}_{n+1}^{n,\mu,a})^\top \Theta_{n+1} ({}_{n+1}\bar{M}_{n+1}^{n,\mu,a_n}) + \chi_{n+1} \right\} \\
&= \inf_{a_n \in \tilde{\mathcal{E}}_n(\mu)} \left\{ \langle {}_n\mu \rangle (\mathbf{Q}_n + \mathbf{B}_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} \mathbf{B}_n) + ({}_n\bar{\mu})^\top \mathbf{K}_n ({}_n\bar{\mu}) \right. \\
&\quad + \text{Tr}(\mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1}) + \text{Tr}(\mathbf{I}^\top \Lambda_{n+1} \mathbf{I}) + \chi_{n+1} \\
&\quad + \left(\int_{(\mathbb{R}^{2d})^n} a_n(y_{0:n}) d\mu(x_{0:n}, y_{0:n}) \right)^\top \mathbf{N}_n \int_{(\mathbb{R}^{2d})^n} a_n(y_{0:n}) d\mu(x_{0:n}, y_{0:n}) \\
&\quad + 2 {}_n\bar{\mu}^\top \mathbf{S}_n \int_{(\mathbb{R}^{2d})^n} a_n(y_{0:n}) d\mu(x_{0:n}, y_{0:n}) \\
&\quad + 2 \int_{(\mathbb{R}^{2d})^n} x_n^\top B_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n a_n(y_{0:n}) d\mu(x_{0:n}, y_{0:n}) \\
&\quad \left. + \int_{(\mathbb{R}^{2d})^n} a_n(y_{0:n})^\top (D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n + R_n) a_n(y_{0:n}) d\mu(x_{0:n}, y_{0:n}) \right\}
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{K}_n &= \bar{\mathbf{Q}}_n + (\mathbf{B}_n + \bar{\mathbf{B}}_n)^\top \mathbf{J}_{n+1}^\top \Theta_{n+1} \mathbf{J}_{n+1} (\mathbf{B}_n + \bar{\mathbf{B}}_n) + \bar{\mathbf{B}}_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} \bar{\mathbf{B}}_n, \\
&\quad + \mathbf{B}_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} \bar{\mathbf{B}}_n + \bar{\mathbf{B}}_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} \mathbf{B}_n, \\
\mathbf{N}_n &= D_n^\top \mathbf{J}_{n+1}^\top \Theta_{n+1} \mathbf{J}_{n+1} D_n, \\
\mathbf{S}_n &= (\mathbf{B}_n + \bar{\mathbf{B}}_n)^\top \mathbf{J}_{n+1}^\top \Theta_{n+1} \mathbf{J}_{n+1} D_n + \bar{\mathbf{B}}_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n.
\end{aligned}$$

Since $\mu \in \mathcal{P}_{\text{cl}}((E \times F)^n)$, there exists $\bar{a} \in \tilde{\mathcal{E}}_{\text{cl}}(\mathbb{P}_{(X_0^a, Y_0^a)})$ such that μ is the law of $(X_{0:n}^a, Y_{0:n}^a)$. Therefore, we get

$$\begin{aligned}
&\left(\int_{(\mathbb{R}^{2d})^n} a_n(y_{0:n}) d\mu(x_{0:n}, y_{0:n}) \right)^\top \mathbf{N}_n \int_{(\mathbb{R}^{2d})^n} a_n(y_{0:n}) d\mu(x_{0:n}, y_{0:n}) \\
&\quad + 2 {}_n\bar{\mu}^\top \mathbf{S}_n \int_{(\mathbb{R}^{2d})^n} a_n(y_{0:n}) d\mu(x_{0:n}, y_{0:n}) \\
&\quad + 2 \int_{(\mathbb{R}^{2d})^n} x_n^\top B_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n a_n(y_{0:n}) d\mu(x_{0:n}, y_{0:n}) \\
&+ \int_{(\mathbb{R}^{2d})^n} a_n(y_{0:n})^\top (D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n + R_n) a_n(y_{0:n}) d\mu(x_{0:n}, y_{0:n}) = \\
&\quad \mathbb{E} \left[a_n(Y_{0:n}^{\bar{a}}) \right]^\top \mathbf{N}_n \mathbb{E} \left[a_n(Y_{0:n}^{\bar{a}}) \right] \\
&\quad + 2 {}_n\bar{\mu}^\top \mathbf{S}_n \mathbb{E} \left[a_n(Y_{0:n}^{\bar{a}}) \right] \\
&\quad + 2 \mathbb{E} \left[(X_n^{\bar{a}})^\top B_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n a_n(Y_{0:n}^{\bar{a}}) \right] \\
&\quad + \mathbb{E} \left[a_n(Y_{0:n}^{\bar{a}})^\top (D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n + R_n) a_n(Y_{0:n}^{\bar{a}}) \right]
\end{aligned}$$

for any $a_n \in \tilde{\mathfrak{C}}_n(\mu)$. Since Λ_{n+1} and R_n are positive and from the definition of \mathbf{S}_n , we can apply Jensen conditional inequality given Y_n and we get

$$\begin{aligned}
& \mathbb{E} \left[a_n(Y_{0:n}^{\bar{a}}) \right]^\top \mathbf{N}_n \mathbb{E} \left[a_n(Y_{0:n}^{\bar{a}}) \right] + 2_n \bar{\mu}^\top \mathbf{S}_n \mathbb{E} \left[a_n(Y_{0:n}^{\bar{a}}) \right] \\
& \quad + 2 \mathbb{E} \left[(X_n^{\bar{a}})^\top B_n^\top \tilde{\Lambda}_{n+1} D_n a_n(Y_{0:n}^{\bar{a}}) \right] \\
& + \mathbb{E} \left[a_n(Y_{0:n}^{\bar{a}})^\top (D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n + R_n) a_n(Y_{0:n}^{\bar{a}}) \right] \geq \\
& \quad \mathbb{E} \left[\hat{a}_n(Y_n^{\bar{a}}) \right]^\top \mathbf{N}_n \mathbb{E} \left[\hat{a}_n(Y_n^{\bar{a}}) \right] + 2_n \bar{\mu}^\top \mathbf{S}_n \mathbb{E} \left[\hat{a}_n(Y_n^{\bar{a}}) \right] \\
& \quad + 2 \mathbb{E} \left[(X_n^{\bar{a}})^\top B_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n \hat{a}_n(Y_n^{\bar{a}}) \right] \\
& + \mathbb{E} \left[\hat{a}_n(Y_n^{\bar{a}})^\top (D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n + R_n) \hat{a}_n(Y_n^{\bar{a}}) \right]
\end{aligned}$$

where \hat{a}_n is defined by

$$\hat{a}_n(y) = \mathbb{E} \left[a_n(Y_{0:n}^{\bar{a}}) \middle| Y_n = y \right], \quad y \in \mathbb{R}^d.$$

Therefore, the infimum in the definition of W_n can be restricted to $\tilde{\mathfrak{C}}_{n,\text{cl}}(\mu)$:

$$\begin{aligned}
W_n(\mu) = \inf_{a_n \in \tilde{\mathfrak{C}}_{n,\text{cl}}(\mu)} & \left\{ \langle {}_n \mu \rangle (\mathbf{Q}_n + \mathbf{B}_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} \mathbf{B}_n) + ({}_n \bar{\mu})^\top \mathbf{K}_n ({}_n \bar{\mu}) \right. \\
& + \text{Tr}(\mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1}) + \text{Tr}(\mathbf{I}^\top \Lambda_{n+1} \mathbf{I}) + \chi_{n+1} \\
& + \left(\int_{\mathbb{R}^{2d}} a_n(y_n) d_n \mu(x_n, y_n) \right)^\top \mathbf{N}_n \int_{\mathbb{R}^{2d}} a_n(y_n) d_n \mu(x_n, y_n) \\
& + 2_n \bar{\mu}^\top \mathbf{S}_n \int_{\mathbb{R}^{2d}} a_n(y_n) d_n \mu(x_n, y_n) \\
& + 2 \int_{\mathbb{R}^{2d}} x_n^\top B_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n a_n(y_n) d_n \mu(x_n, y_n) \\
& \left. + \int_{\mathbb{R}^{2d}} a_n(y_n)^\top (D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n + R_n) a_n(y_n) d_n \mu(x_n, y_n) \right\}.
\end{aligned}$$

From this last identity we deduce that W_n depends only on ${}_n \mu$. Using (4.15), we have

$$\begin{aligned}
W_n(\mu) = \inf_{a_n \in \tilde{\mathfrak{C}}_{n,\text{cl}}(\mu)} & \left\{ \langle {}_n \mu \rangle (\mathbf{Q}_n + \mathbf{B}_n^\top \Lambda_{n+1} \mathbf{B}_n) + \text{Tr}(\mathbf{I}^\top \Lambda_{n+1} \mathbf{I}) + \chi_{n+1} + ({}_n \bar{\mu})^\top \mathbf{K}_n ({}_n \bar{\mu}) \right. \\
& + \left(\int_{\mathbb{R}^d} a_n(y_n) d_n^2 \mu(y_n) \right)^\top \mathbf{N}_n \int_{\mathbb{R}^d} a_n(y_n) d_n^2 \mu(y_n) \\
& + 2_n \bar{\mu}^\top \mathbf{S}_n \int_{\mathbb{R}^{2d}} a_n(y_n) d_n^2 \mu(y_n) \\
& + 2 \int_{\mathbb{R}^d} \Phi_n(\mu, y_n)^\top B_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n a_n(y_n) d_n^2 \mu(y_n) \\
& \left. + \int_{\mathbb{R}^d} a_n(y_n)^\top (D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n + R_n) a_n(y_n) d_n^2 \mu(y_n) \right\}.
\end{aligned}$$

where Φ_n is given by (4.17). We then notice that the function of a_n inside the infimum is continuous and goes to $+\infty$ as $\int |a_n|^2 d_n^2 \mu$ goes to infinity since $\Lambda_{n+1} + \Theta_{n+1}$ is nonnegative and R_n is positive. Hence, this function admits a global minimum. Since this function is continuously differentiable we can compute the first order condition and we get

$$\begin{aligned} \mathbf{N}_n \left(\int_{\mathbb{R}^{2d}} a_n(y'_n) d_n^2 \mu(y'_n) \right) + \mathbf{S}_{nn}^\top \bar{\mu} + D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} B_n \Phi(\mu, y_n) \\ + (D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n + R_n) a_n(y_n) = 0, \quad y_n \in \mathbb{R}^d. \end{aligned} \quad (4.18)$$

Therefore we get

$$\begin{aligned} a_n^*(y_n) = & -(D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n + R_n)^{-1} (D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} B_n \Phi(\mu, y_n) \\ & + \mathbf{S}_{nn}^\top \bar{\mu} + \mathbf{N}_n \int_{\mathbb{R}^d} a_n(y_n) d_n^2 \mu(y_n)). \end{aligned}$$

Taking the integral with respect to $({}_n^2 \mu)$ on both sides of (4.18), we get

$$\int_{\mathbb{R}^d} a_n(y_n) d_n^2 \mu(y_n) = -(D_n^\top \mathbf{J}_{n+1}^\top (\Lambda_{n+1} + \Theta_{n+1}) \mathbf{J}_{n+1} D_n + R_n)^{-1} \tilde{\mathbf{S}}_{nn}^\top \bar{\mu}$$

with

$$\tilde{\mathbf{S}}_n = \mathbf{S}_n + \mathbf{B}_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n$$

and

$$a_n^*(y_n) = G_n \left(\Xi_n \Phi(\mu, y_n) + \hat{\mathbf{S}}_{nn}^\top \bar{\mu} \right).$$

with

$$\begin{aligned} G_n &= -(D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n + R_n)^{-1} \\ \Xi_n &= D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} B_n J_{n+1}^{-1} \\ \hat{\mathbf{S}}_n &= \mathbf{S}_n - \tilde{\mathbf{S}}_n (D_n^\top \mathbf{J}_{n+1}^\top (\Lambda_{n+1} + \Theta_{n+1}) \mathbf{J}_{n+1} D_n + R_n)^{-1} \mathbf{N}_n. \end{aligned}$$

We then get

$$\begin{aligned} W_n(\mu) &= \langle {}_n \mu \rangle (\mathbf{Q}_n + \mathbf{B}_n^\top \Lambda_{n+1} \mathbf{B}_n) + \text{Tr}(\mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1}) + \text{Tr}(\mathbf{I}^\top \Lambda_{n+1} \mathbf{I}) \\ &+ \chi_{n+1} + ({}_n \bar{\mu})^\top \mathbf{K}_n ({}_n \bar{\mu}) \\ &+ \left(\int_{(\mathbb{R}^{2d})} a_n^*(y_n) d_n \mu(x_n, y_n) \right)^\top \mathbf{N}_n \int_{(\mathbb{R}^{2d})} a_n^*(y_n) d_n \mu(x_n, y_n) \\ &+ 2 {}_n \bar{\mu}^\top \mathbf{S}_n \int_{\mathbb{R}^{2d}} a_n^*(y_n) d_n \mu(x_n, y_n) \\ &+ 2 \int_{(\mathbb{R}^{2d})} x_n^\top B_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n a_n^*(y_n) d_n \mu(x_n, y_n) \\ &+ \int_{\mathbb{R}^{2d}} a_n^*(y_n)^\top (D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n + R_n) a_n^*(y_n) d_n \mu(x_n, y_n) \\ &= \langle {}_n \mu \rangle (\Lambda_n) + {}_n \bar{\mu}^\top \Theta_n {}_n \bar{\mu} + \chi_n \end{aligned}$$

with

$$\begin{aligned}
\Lambda_n &= \mathbf{Q}_n + \mathbf{B}_n^\top \Lambda_{n+1} \mathbf{B}_n + \mathbf{I} \Xi_n^\top (D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n + R_n) \Xi_n \mathbf{I}^\top, \\
\Theta_n &= \mathbf{K}_n \\
&\quad + \hat{\mathbf{S}}_n (D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n + R_n)^{-1} \mathbf{N}_n (D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n + R_n)^{-1} \hat{\mathbf{S}}_n^\top \\
&\quad + \mathbf{I} \Xi_n^\top (D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n + R_n)^{-1} \mathbf{N}_n (D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n + R_n)^{-1} \Xi_n \mathbf{I}^\top \\
&\quad + \mathbf{I} \Xi_n^\top (D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n + R_n)^{-1} \mathbf{N}_n (D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n + R_n)^{-1} \hat{\mathbf{S}}_n^\top \\
&\quad + \hat{\mathbf{S}}_n (D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n + R_n)^{-1} \mathbf{N}_n (D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n + R_n)^{-1} \Xi_n \mathbf{I}^\top \\
&\quad + 2\mathbf{S}_n (D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n + R_n)^{-1} (\Xi_n \mathbf{I}^\top + \hat{\mathbf{S}}_n^\top) \\
&\quad + 2\mathbf{I} (B_n J_{n+1}^{-1})^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n (D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n + R_n)^{-1} \hat{\mathbf{S}}_n^\top \\
&\quad + \hat{\mathbf{S}}_n (D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n + R_n)^{-1} \hat{\mathbf{S}}_n^\top \\
&\quad + \mathbf{I} \Xi_n^\top (D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n + R_n)^{-1} \hat{\mathbf{S}}_n^\top \\
&\quad + \hat{\mathbf{S}}_n (D_n^\top \mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1} D_n + R_n)^{-1} \Xi_n \mathbf{I}^\top, \\
\chi_n &= \text{Tr}(\mathbf{J}_{n+1}^\top \Lambda_{n+1} \mathbf{J}_{n+1}) + \text{Tr}(\mathbf{I}^\top \Lambda_{n+1} \mathbf{I}) + \chi_{n+1}.
\end{aligned}$$

We then easily have Λ_n nonnegative. A straightforward computation shows that $\Theta_n + \Lambda_n$ is also nonnegative. Therefore, the induction property holds true at rank n , and it holds for any $n = 0, \dots, T$. \square

4.2 Numerical approximation of the optimal value

4.2.1 The algorithm

We present a numerical algorithm for the approximation of the optimal value base on the verification Theorem 3.1. For that, we define two finite subsets Λ_E and Λ_F of E and F respectively by

$$\begin{aligned}
\Lambda_E &= \{x^1, \dots, x^N\}, \\
\Lambda_F &= \{y^1, \dots, y^N\}.
\end{aligned}$$

We next introduce two Voronı tessellations $(C(\Lambda_E)^i)_{1 \leq i \leq N}$ and $(C(\Lambda_F)^i)_{1 \leq i \leq N}$ of subsets of E and F respectively. This means that $(C(\Lambda_E)^i)_{1 \leq i \leq N}$ and $(C(\Lambda_F)^i)_{1 \leq i \leq N}$ satisfy

$$\begin{aligned}
\bigcup_{1 \leq i \leq N} C(\Lambda_E)^i &= E, \quad C(\Lambda_E)^i \cap C(\Lambda_E)^j = \emptyset \text{ for } i \neq j, \\
\bigcup_{1 \leq i \leq N} C(\Lambda_F)^i &= F, \quad C(\Lambda_F)^i \cap C(\Lambda_F)^j = \emptyset \text{ for } i \neq j,
\end{aligned}$$

and

$$\begin{aligned}
C(\Lambda_E)^i &\subset \{x \in E : |x - x^i| = \min_{1 \leq j \leq N} |x - x^j|\}, \\
C(\Lambda_F)^i &\subset \{y \in F : |y - y^i| = \min_{1 \leq j \leq N} |y - y^j|\},
\end{aligned}$$

for $i = 1, \dots, N$. We then define the projection operators Proj_{Λ_E} and Proj_{Λ_F} by

$$\text{Proj}_{\Lambda_E}(x) = x^i$$

for $x \in C^i(\Lambda_E)$ and

$$\text{Proj}_{\Lambda_F}(y) = y^i$$

for $y \in C^i(\Lambda_F)$ and $1 \leq i \leq N$. Our goal is to provide a discrete version of the dynamic programming equation defining the functions W_0, \dots, W_T in Theorem 3.1. We first discretize the initial conditions ξ and ζ by defining

$$\begin{aligned}\hat{\xi} &= \text{Proj}_{\Lambda_E}(\xi), \\ \hat{\zeta} &= \text{Proj}_{\Lambda_F}(\zeta).\end{aligned}$$

We then define the processes (\hat{X}^a, \hat{Y}^a) as the quantizer (X^a, Y^a) according to Λ_E and Λ_F . This means that (\hat{X}^a, \hat{Y}^a) is the approximation of (X^a, Y^a) valued in $\Lambda_E \times \Lambda_F$ by $(\hat{X}_0^a, \hat{Y}_0^a) = (\hat{\xi}, \hat{\zeta})$ and

$$\begin{aligned}\hat{X}_{n+1}^a &= \text{Proj}_{\Lambda_E}\left(G_{n+1}(\hat{X}_n^a, \mathbb{P}_{\hat{X}_n^a}, a(\hat{Y}_{0:n}^a), \varepsilon_{n+1})\right), \\ \hat{Y}_{n+1}^a &= \text{Proj}_{\Lambda_F}\left(H_{n+1}(\hat{X}_n^a, \hat{Y}_n^a, a(\hat{Y}_{0:n}^a), \eta_{n+1})\right)\end{aligned}$$

for $n = 0, \dots, T-1$. A straightforward computation give the dynamics of the process (\hat{X}^a, \hat{Y}^a) as

$$\mathbb{P}((\hat{X}_{n+1}^a, \hat{Y}_{n+1}^a) = (x^i, y^j) | \hat{X}_{0:n}^a, \hat{Y}_{0:n}^a) = \hat{h}_{n+1}(x^i, \hat{Y}_n^a, a_n(\hat{Y}_{0:n}^a), y^j) \hat{P}_{n+1}^{a_n(\hat{Y}_{0:n}^a)}(\hat{X}_n^a, \mathbb{P}_{\hat{X}_n^a}, \hat{x}^j)$$

for $n = 0, \dots, T-1$ and $(x^i, y^j) \in \Lambda_E \times \Lambda_F$, where

$$\hat{h}_{n+1}(x^i, \hat{Y}_n^a, a_n(\hat{Y}_{0:n}^a), y^j) = \int_{C_j(\Lambda_F)} h_{n+1}(x^i, \hat{Y}_n^a, a_n(\hat{Y}_{0:n}^a), y) dy$$

and

$$\hat{P}_{n+1}^{a_n(\hat{Y}_{0:n}^a)}(\hat{X}_n^a, \mathbb{P}_{\hat{X}_n^a}, x^j) = \int_{C_i(\Lambda_E)} P_{n+1}^{a_n(\hat{Y}_{0:n}^a)}(\hat{X}_n^a, \mathbb{P}_{\hat{X}_n^a}, dx).$$

Then, the global controlled transition \bar{P} is replaced by \hat{P} defined by

$$\hat{P}_{n+1}^{a_n}(z, M, w) = \hat{P}_{n+1}^{a_n(y_{0:n})}(x_n, \frac{1}{n}M, w_1) \hat{h}_{n+1}(w_1, y_n, a_n(y_{0:n}), w_2)$$

for $n = 0, \dots, T-1$, $z = (x_\ell, y_\ell)_{0 \leq \ell \leq n} \in (\Lambda_E \times \Lambda_F)^{n+1}$, $M \in \mathcal{P}_2((\Lambda_E \times \Lambda_F)^{n+1})$, $a_n \in \tilde{\mathfrak{C}}_n(M)$ and $w = (w_1, w_2) \in \Lambda_E \times \Lambda_F$. For $n = 0, \dots, T-1$ and $\mu \in \mathcal{P}_2((\Lambda_E \times \Lambda_F)^{n+1})$ $a_n \in \tilde{\mathfrak{C}}_n(\mu)$, we define the measure $\hat{P}_{n+1}^{a_n} \mu \in \mathcal{P}_2((\Lambda_E \times \Lambda_F)^{n+2})$ by

$$\hat{P}_{n+1}^{a_n} \mu(x_{0:n+1}, y_{0:n+1}) = \hat{P}_{n+1}^{a_n}(x_{0:n}, y_{0:n}, \mu, x_{n+1}, y_{n+1}) \mu(x_{0:n}, y_{0:n})$$

for $(x_{0:n+1}, y_{0:n+1}) \in (\Lambda_E \times \Lambda_F)^{n+2}$. The dynamics of the controlled measures $\mathbb{P}_{\hat{X}_n^a, \hat{Y}_n^a}$ can be written under the following simplified form

$$\mathbb{P}_{\hat{X}_{0:n+1}^a, \hat{Y}_{0:n+1}^a}(x_{0:n+1}, y_{0:n+1}) = \hat{P}_{n+1}^{a_n} \mathbb{P}_{\hat{X}_{0:n}^a, \hat{Y}_{0:n}^a}(x_{0:n}, y_{0:n}).$$

We then define the related discretized cost coefficients \hat{C}_n for $n = 0, \dots, T-1$ and γ by

$$\hat{C}_n(M, a_n) = \sum_{(x_{0:n}, y_{0:n}) \in \Lambda_E \times \Lambda_F} c_n(x_n, \frac{1}{n}M, a_n(y_{0:n})) M(x_{0:n}, y_{0:n})$$

for $M \in \mathcal{P}_2((\Lambda_E \times \Lambda_F)^{n+1})$, $a_n \in \tilde{\mathfrak{C}}_n(M)$ and

$$\hat{\Gamma}(M) = \sum_{(x_{0:T}, y_{0:T}) \in \Lambda_E \times \Lambda_F} \gamma(x_T, \frac{1}{T}M) dM_T(x_{0:T}, y_{0:T}),$$

for $M \in \mathcal{P}_2((\Lambda_E \times \Lambda_F)^{T+1})$. Then, the related approximated value functions \hat{W}_n , $n = 0, \dots, T$ are given by

$$\hat{W}_T(\mu) = \hat{\Gamma}(\mu), \quad \mu \in \mathcal{P}_2((\Lambda_E \times \Lambda_F)^{T+1}),$$

and

$$\hat{W}_n(\mu) = \inf_{a_n \in \tilde{\mathfrak{C}}_n(\mu)} \{ \hat{C}_n(\mu, a_n) + \hat{W}_{n+1}(\hat{P}_{n+1}^{a_n} \mu) \}, \quad \mu \in \mathcal{P}_2((\Lambda_E \times \Lambda_F)^{n+1}),$$

for $n = 0, \dots, T-1$. To get tractable versions \tilde{W}_n of values functions \hat{W}_n , $n = 0, \dots, T$, we define finite subsets Π_{n+1} of $\mathcal{P}_2((\Lambda_E \times \Lambda_F)^{n+1})$ as follows. We first introduce L sequences $(p_{i,j}^\ell)_{1 \leq i,j \leq N}$ for $\ell = 1, \dots, L$ such that $p_{i,j}^\ell \in \mathbb{R}_+$ for $i, j = 1, \dots, N$ and

$$\sum_{1 \leq i,j \leq N} p_{i,j}^\ell = 1$$

for $\ell = 1, \dots, L$. We next define the set Π_n by

$$\begin{aligned} \Pi_n &= \left\{ \sum_{1 \leq i_1, j_1, \dots, i_n, j_n \leq N} \delta_{(x^{i_0}, \dots, x^{i_n}, y^{i_0}, \dots, y^{i_n})} \prod_{r=0}^n p_{i_r, j_r}^{\ell_r}, \ell_1, \dots, \ell_n \in \{1, \dots, L\} \right\} \\ &\subset \mathcal{P}_2((\Lambda_E \times \Lambda_F)^n) \end{aligned}$$

for $n = 1, \dots, T$. We next define the approximation \tilde{P} of \hat{P} defined by

$$\tilde{P}_{n+1}^{a_n}(z, M, w) = \text{Proj}_{\{p_{i,j}^1, \dots, p_{i,j}^L\}} \left(\hat{P}_{n+1}^{a_n}(z, M, w) \right)$$

for $n = 0, \dots, T-1$, $z = (x_\ell, y_\ell)_{0 \leq \ell \leq n} \in (\Lambda_E \times \Lambda_F)^{n+1}$ with $(x_n, y_n) = (x^i, y^j)$, $M \in \mathcal{P}_2((\Lambda_E \times \Lambda_F)^{n+1})$, $a_n \in \tilde{\mathfrak{C}}_n(M)$ and $w = (w_1, w_2) \in \Lambda_E \times \Lambda_F$. We observe that

$$\tilde{P}_{n+1}^{a_n} \mu \in \Pi_{n+2}$$

for $n = 0, \dots, T-1$, $\mu \in \Pi^{n+1}$ and $a_n \in \tilde{\mathfrak{C}}_n(\mu)$. In particular, the functions \tilde{W}_n , $n = 0, \dots, T$ are given by

$$\tilde{W}_T(\mu) = \hat{\Gamma}(\mu), \quad \mu \in \Pi_{T+1},$$

and

$$\tilde{W}_n(\mu) = \inf_{a_n \in \tilde{\mathfrak{C}}_n(\mu)} \left\{ \hat{C}_n(\mu, a_n) + \tilde{W}_{n+1}(\tilde{P}_{n+1}^{a_n} \mu) \right\}, \quad \mu \in \Pi_{n+1},$$

for $n = 0, \dots, T-1$, provide a computable approximation of the functions W_n , $n = 0, \dots, T$.

4.2.2 Test of the algorithm

It is possible to compare this approximated algorithm with the linear quadratic case developed earlier. We choose to fix Λ_E and Λ_F to be the centers of the Voronoï cells $C(\Lambda_E)^i$ and $C(\Lambda_F)^i$ for $1 \leq i \leq N$ respectively. Precisely, we apply Lloyd Algorithm on $\mathcal{N}(0, I_d)$ to obtain $C(\Lambda_E)^i$ and $C(\Lambda_F)^i$. Thus, $\Lambda_E = \Lambda_F$ and $C(\Lambda_E)^i = C(\Lambda_F)^i$. We refer to the book [15] for a description of quantization methods and their related algorithms.

We fix $d = 2$. In order to be complete, we describe all the matrices we used for this experiment.

For the main dynamics we fix:

$$\begin{aligned} B_k &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ \bar{B}_k &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ D_k &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \\ J_{k+1} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

for $k = 1, \dots, T$. For the cost function \bar{J} , we fix:

$$\begin{aligned} Q_k &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \\ \bar{Q}_k &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \\ R_k &= I_d \end{aligned}$$

for $k = 1, \dots, T$. We approximate the integral needed to compute \hat{h}_{n+1} and $\hat{P}_{n+1}^{a_n(\hat{Y}_{0:n}^a)}$ through Monte Carlo simulations. Controls α are restricted to the following set

$$\tilde{\mathfrak{C}}_n(M) := \left\{ \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}.$$

Finally, we fix $T = 3$ and we compute W_t at time $t = 0$ using the formula derived in the proof of Proposition 4.1 and \tilde{W}_0 using the algorithm described in Section 4.2.1. The relative error $\frac{|\tilde{W}_0 - W_0|}{W_0}$ is computed for $N = 2, 4, 10$ and 20. The results are presented in the Figure 1.

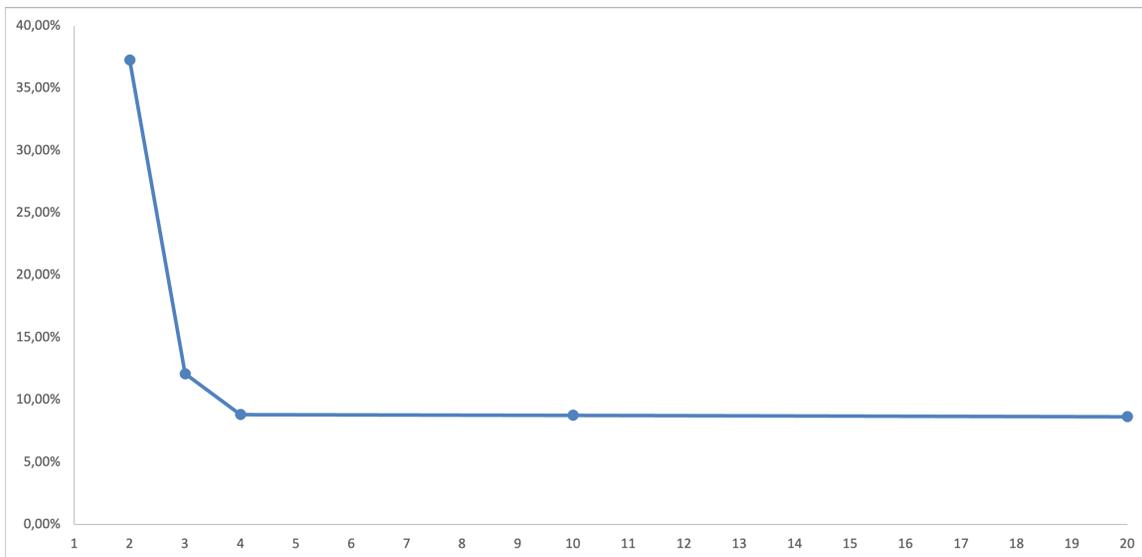


Figure 1: Relative Error of Value function as N grows

Errors are expressed as percentages. Increasing N decreases the relative error. For small N , we achieve good results. For example, with $N \geq 4$, the absolute error is less than 10% (round 8%). However, the error does not decrease anymore for $N \geq 4$. This might be explained by the restriction done on the control set.

4.3 A Mean-Variance optimal investment problem

We refer to [16] for a review on portfolio optimization in partial observation framework. We consider a financial market over the horizon $\{0, \dots, T\}$. We suppose that this market is composed by one asset with return process (R_0, \dots, R_T) satisfying

$$R_k = b_0 \Delta + \sigma \sqrt{\Delta} \epsilon_{k+1}, \quad k = 0, \dots, T-1,$$

where the drift and the variance b_0 and σ are known. We assume that $(\epsilon_1, \dots, \epsilon_T)$ is a sequence of independent $\mathcal{N}(0, 1)$ -distributed random variables independent from X_0 . We consider an investor who can invest at each time on this asset. The wealth process $(X_0^\alpha, \dots, X_T^\alpha)$ is given by

$$X_{k+1}^\alpha = X_k^\alpha + \alpha_k \left(b_0 \Delta + \sigma \sqrt{\Delta} \epsilon_{k+1} \right), \quad k = 0, \dots, T-1,$$

Due to a lack of liquidity on this asset, we suppose that the investor does not observe its portfolio directly, but rather an approximate representation $(X_0^\alpha, \dots, X_T^\alpha)$ is given by

$$Y_k^\alpha = X_k^\alpha + \eta_k, \quad k = 1, \dots, T.$$

Such a situation can be faced by private equity investors since they only have some intuition about the value of their portfolios. We refer to [8], [9], [19] for more details. The studied

system is then:

$$\begin{cases} X_{k+1}^\alpha = X_k^\alpha + \alpha_k \left(b_0 \Delta + \sigma \sqrt{\Delta} \epsilon_{k+1} \right), \\ Y_{k+1}^\alpha = X_{k+1}^\alpha + \eta_{k+1}, \quad k = 0, \dots, T-1. \end{cases} \quad (4.19)$$

We suppose that η_{k+1} is a sequence of independent $\mathcal{N}(0, 1)$ -distributed random variables, also independent from (η_1, \dots, η_T) , X_0 and Y_0 .

The investor goal is to find a portfolio allocation that minimize a mean-variance criterum:

$$\begin{aligned} V_0 &= \inf_{\alpha \in \bar{\mathfrak{C}}} J(\alpha) \\ &= \inf_{\alpha \in \bar{\mathfrak{C}}} \left[\frac{\gamma}{2} \text{Var}[X_T^\alpha] - \mathbb{E}[X_T^\alpha] \right] \end{aligned}$$

for a $\gamma > 0$.

We propose to apply the numerical approximation algorithm of Section 4.2.1. As a starting point, we set the $d = 1$, $T = 5$. Additionally, we fix $b_0 = 0.02$, $\sigma = 0.05$ for the Return process, so that the investor can expect a return of 0.02 with a volatility of 0.05 for this asset. Furthermore, we assume the control space to be $\{0.5, 0.75, 1, 2\}$. Investing only a quarter, half, three quarters, or the entire wealth of an investor is permitted. Using the proposed algorithm with $N = 2$, we compute the optimal control at every time $t = 0, \dots, T$. This control leads to a portfolio. Our proposed allocation is benchmarked against two strategies. The first one is known as 'buy and hold': the investor remains invested in the asset at all times. For the second strategy, it can be viewed as a classical trending strategy: if the asset return is positive at a given time t , the investor invests, else the investor shorts the asset. Based on 250 trajectory simulations, we compute the empirical final wealth mean, denoted by $\bar{\mathbb{E}}[X_T^\alpha]$ and the empirical final wealth variance denoted by $\bar{\text{Var}}[X_T^\alpha]$. We present in Tables 1, 2, 3 and 4 the results for several values of γ .

	Proposed Strategy	Buy and Hold	Trending Strategy
$\bar{\mathbb{E}}[X_T^\alpha]$	1,02027868	1,04535767	1,01155139
$\bar{\text{Var}}[X_T^\alpha]$	0,00481573	0,00680738	0,00688046
V_0	-1,01546295	-1,03855029	-1,00467093

Table 1: Empirical Results for $\gamma = 2$

	Proposed Strategy	Buy and Hold	Trending Strategy
$\bar{\mathbb{E}}[X_T^\alpha]$	1,02681514	1,03881421	1,01034027
$\bar{\text{Var}}[X_T^\alpha]$	0,00467356	0,00653589	0,00636125
V_0	-1,01746802	-1,02574243	-0,99761777

Table 2: Empirical Results for $\gamma = 4$

	Proposed Strategy	Buy and Hold	Trending Strategy
$\mathbb{E}[X_T^\alpha]$	1,02314645	1,03975832	1,00942357
$\text{Var}[X_T^\alpha]$	0,00452504	0,00651461	0,00684134
V_0	-1,00504629	-1,01369988	-0,98205821

Table 3: Empirical Results for $\gamma = 8$

	Proposed Strategy	Buy and Hold	Trending Strategy
$\mathbb{E}[X_T^\alpha]$	1,01748989	1,03562335	1,01643678
$\text{Var}[X_T^\alpha]$	0,00433524	0,00693694	0,006672365
V_0	-0,98280797	-0,98012783	-0,96305786

Table 4: Empirical Results for $\gamma = 16$

With the proposed approach, the variance of the final wealth is systematically smaller for every γ . The buy-and-hold strategy provides the best returns for investors: however, the proposed strategy reduces variance compared to a buy-and-hold strategy. For example, when $\gamma = 2$, the proposed strategy reduces volatility by 41% compared to buy and hold for a return's cost of 2.45%. We note that the proposed strategy is better in terms of V_0 value when γ is equal to 16. As a result, we have proposed an interesting allocation that can be used to reduce portfolio risk using only two quantization points.

5 Conclusion

A partially observed optimal control problem for a system with mean field discrete time dynamics is presented and solved. We extend the linear-quadratic case (also known as Kalman Bucy) to deal with the mean field dependence. We also propose a general algorithmic approach based on optimal quantization to approximate the optimal value. We check the robustness of the algorithm empirically with a financial example. Some extensions of the work can be proposed. A first natural question is the estimation of the error of the proposed algorithm. The extension of the results to the continuous time case can also be addressed. This leads to the question of the approximation of the continuous time case by a discrete-time model using an Euler discretization of the continuous problem.

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