HAL open science

DETERMINANTS OF LAPLACIANS ON RANDOM HYPERBOLIC SURFACES

Frédéric Naud

▶ To cite this version:

Frédéric Naud. DETERMINANTS OF LAPLACIANS ON RANDOM HYPERBOLIC SURFACES. 2023. hal-04012902

HAL Id: hal-04012902

https://hal.science/hal-04012902v1

Preprint submitted on 3 Mar 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

arXiv:2301.09965v3 [math.SP] 10 Feb 2023

DETERMINANTS OF LAPLACIANS ON RANDOM HYPERBOLIC SURFACES

BY FRÉDÉRIC NAUD

Abstract. For sequences (X_j) of random closed hyperbolic surfaces with volume $\mathrm{Vol}(X_j)$ tending to infinity, we prove that the regularized determinant of the Laplacian $\det(\Delta_{X_j})$ satisfies for all $\epsilon > 0$ and with high probability as $j \to +\infty$,

$$\frac{\log\det(\Delta_{X_j})}{\mathrm{Vol}(X_j)} \in [E-\epsilon, E+\epsilon],$$

where $E > 0$ is a universal constant. This result holds for various models of random surfaces, including the Weil-Petersson model.

1. Introduction and results

1.1. **On determinants.** Let $X = \Gamma\backslash\mathbb{H}^2$ be a compact connected hyperbolic surface obtained as a quotient of the hyperbolic plane $\mathbb{H}^2$ by a co-compact group of isometries. The hyperbolic Laplacian Δ_X has a pure point spectrum on $L^2(X)$ and it is denoted by

$$0 = \lambda_0 < \lambda_1 \leq \ldots \lambda_j \leq \ldots.$$

For all $\mathrm{Re}(s)$ large enough, we know by Weyl's law that the spectral zeta function

$$\zeta_X(s) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j^s}$$

is well defined and holomorphic. This zeta function $\zeta_X(s)$ actually has a meromorphic extension to $\mathbb{C}$ and is analytic at $s = 0$. The regularized determinant is then usually defined by

$$\log\det(\Delta_X) := -\zeta_X'(0),$$

which agrees with the formal computations. Practically, one performs the meromorphic continuation by noticing that for large $\mathrm{Re}(s)$ we have

$$\zeta_X(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1}(\mathrm{Tr}(e^{-t\Delta_X}) - 1)dt,$$

where $e^{-t\Delta_X}$ is the heat semi-group. The small time asymptotics of the heat kernel is the main tool which allows to "renormalize" the divergent behavior at $t = 0$ and obtain the meromorphic continuation. In the literature, Polyakov's string theory [32, 13] has emphasized the role of determinants on Riemann surfaces. In particular, the computation of "partition functions" in perturbative string theory involves formal sums over all genera of averages of determinants over the moduli space which have proved since then to be divergent, see Wolpert [38]. Several authors have provided [33, 6, 12] some "explicit" formulas for regularized determinants for various Laplace-like operators on Riemann surfaces. In small genus, it is possible to compute accurately such determinants by reducing to certain sums over closed geodesics which provide a fast convergence, see [31, 35].

In higher dimensions, determinants of Laplacians on differential forms are related to the so-called analytic torsion which in turn is related to important topological

invariants by results of Cheeger and Müller [11, 27]. In particular the work of Bergeron-Venkatesh [3] establishes exponential growth of the analytic torsion for certain families of covers of arithmetic manifolds.

In the case of Riemann surfaces, if Γ is a co-compact arithmetic Fuchsian group derived from a quaternion algebra, one can define "congruence covers"

$$X_{\mathscr{P}} := \Gamma(\mathscr{P})\backslash\mathbb{H}^2$$

of $X = \Gamma\backslash\mathbb{H}^2$ by looking at *prime ideals* $\mathscr{P}$ in the ring of integers of the corresponding number field. We denote by $\|\mathscr{P}\|$ the norm of ideals. Using the uniform spectral gap of these surfaces proved by Sarnak and Xue in [34], together with the fact that the injectivity radius goes to infinity as $\|\mathscr{P}\| \to \infty$, see in [18], one can readily show (for example by using the arguments from [2] or as a direct application of Theorem 3.1) that

$$\lim_{\|\mathscr{P}\|\to\infty} \frac{\log\det\Delta_{X_{\mathscr{P}}}}{\mathrm{Vol}(X_{\mathscr{P}})} = E,$$

where $E > 0$ is some universal constant. Arithmetic surfaces being highly non-generic, it is therefore natural to ask if this behavior is "typical" among larger families of surfaces whose volume (equivalently genus) goes to infinity.

1.2. **Models of random surfaces.** In this paper we will focus on behaviour of determinants of the Laplacian in the large volume (equivalently large genus) regime, using probabilistic tools. The first historical model of random compact Riemann surfaces in the mathematics literature is perhaps the model of **Brooks-Makover** [9] which is based on random 3-regular graphs as follows. Consider $\mathcal{G}_n$ a 3-regular graph on $2n$ vertices, endowed with an orientation $\mathcal{O}$. On the (finite) set of all possible pairs $(\mathcal{G}_n, \mathcal{O})$, one can put a probability measure (which is not the uniform measure) introduced first by Bollobás [4]. By glueing an ideal hyperbolic triangle (whose vertices are at infinity in $\mathbb{H}^2$) according to $(\mathcal{G}_n, \mathcal{O})$, see [9] for the precise recipe, one obtains a random finite area hyperbolic surface $S_n^O := S^O(\mathcal{G}_n, \mathcal{O})$ with $\mathrm{Vol}(S_n) = 2\pi n$. It is possible to show, see [8], that all surfaces in S_n^O are actually (non ramified) covers of the modular surface $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2$.

One can then conformally compactify S_n^O into a compact hyperbolic surface S_n^C by "cutting" the cusps and filling them with disks, see §3 in [9]. In §4, they also show that there exists a constant $C_0 > 0$ such that

$$\mathrm{Vol}(S_n^C) \geq C_0 n.$$

Most of the geometric properties of S_n^O (and then S_n^C, after a mild loss) can be read off from the combinatorics of $\mathcal{G}_n$.

Another more recent discrete model of random surface is the so-called **random cover model** which has been studied and used recently in [25, 20, 28]. In what follows, we fix a compact surface $X = \Gamma\backslash\mathbb{H}^2$, "the base surface". Let $\phi_n : \Gamma \to \mathcal{S}_n$ be a group homomorphism, where $\mathcal{S}_n$ is the symmetric group of permutations of $[n] := \{1, \ldots, n\}$. The discrete group Γ acts on $\mathbb{H}^2 \times [n]$ by

$$\gamma.(z, j) := (\gamma(z), \phi_n(\gamma)(j)).$$

The resulting quotient $X_n := \Gamma\backslash\mathbb{H}^2 \times [n]$ is then a finite cover of degree n of X, possibly not connected. By considering the (finite) space of all homomorphism $\phi_n : \Gamma \to \mathcal{S}_n$, endowed with the uniform probability measure, we obtain a notion of random covering surfaces of degree n, $X_n \to X$. Let us remark that we can also

view (up to isometry) the random cover X_n as

$$X_n = \bigsqcup_{k=1}^{p} \Gamma_k \backslash \mathbb{H}^2,$$

where each Γ_k is (a priori non-normal) subgroup of Γ given by

$$\Gamma_k = \mathrm{Stab}_\Gamma(i_k) = \{\gamma \in \Gamma \ : \ \phi_n(\gamma)(i_k) = i_k\},$$

where $i_1, \ldots, i_k \in [n]$ are representatives of the orbits of Γ (acting on $[n]$ via ϕ_n). In general, the cover X_n is not connected, but it follows directly from [19] that the probability that this cover is connected tends to 1 as n goes to infinity. In this model, we have $\mathrm{Vol}(X_n) = n\mathrm{Vol}(X)$.

A smooth model of random hyperbolic surfaces, is given by the **moduli space** $\mathscr{M}_g$ of closed hyperbolic surfaces with genus g. The moduli space $\mathscr{M}_g$ is the space of hyperbolic metrics on a fixed smooth closed surface S with genus g, up to isometry. It is often defined as the quotient

$$\mathscr{M}_g = \mathscr{T}_g / MCG,$$

where $\mathscr{T}_g$ is the Teichmüller space of hyperbolic metrics on a surface S of genus g and

$$MCG = \mathrm{Diff}(S)/\mathrm{Diff}_0(S)$$

is the group of isotopy classes of diffeomorphisms on S, aka the mapping class group. We refer the reader for example to [10], chapter 6 for more details. A symplectic form ω_{WP} lives naturally on $\mathscr{T}_g$ and descends to the moduli space, endowing it with a natural notion of volume, Weil-Petersson volume. The moduli space is a non-compact finite dimensional orbifold, but as a consequence of Bers' theorem it has a finite volume with respect to this Weil-Petersson volume. We can therefore normalize this measure and obtain a probability measure on $\mathscr{M}_g$. Notice that in this case if $X \in \mathscr{M}_g$, $\mathrm{Vol}(X) = 4\pi(g-1)$ by Gauss-Bonnet. The calculation of Weil-Petersson volumes of the moduli space by Mirzakhani [23] has made possible [24] the large genus asymptotic analysis of various geometric and spectral quantities, see for example [26, 40, 39] for recent works in that direction.

For all of the previous models, we will denote by $\mathbb{P}$ the associated probability measure, which depends either on n or g which are both proportional to the volume. We say that an event $\mathcal{A}$ is asymptoticaly almost sure (a.a.s.), or holds with high probability, if $\mathbb{P}(\mathcal{A})$ tends to 1 as the volume of surfaces tends to infinity. The expectation of any relevant random variable will also be denoted by $\mathbb{E}$.

1.3. **Main result.**

Theorem 1.1. *There exists a universal constant $E > 0$ such that for all the above models of random surfaces, for all $\epsilon > 0$, we have a.a.s as $\mathrm{Vol}(X) \to +\infty$,*

$$\frac{\log\det(\Delta_X)}{\mathrm{Vol}(X)} \in [E - \epsilon, E + \epsilon].$$

The constant E is actually explicit, see §2. This result shows that exponential growth of the determinant is typical when the genus goes to infinity. This low dimensional result is consistent, in a much simpler setting, with the conjectures on the exponential growth of the analytic torsion and the torsion homology for higher dimensional hyperbolic manifolds, see for example the paper of Bergeron-Venkatesh [3] and references herein.

Remark that the above statement says that the random variable $\frac{\log\det(\Delta_X)}{\mathrm{Vol}(X)}$ converges in probability to the constant E. What about other modes of convergence? It is possible to derive from Theorem 1.1 a convergence result for the expectation of $|\log\det(\Delta)|^\beta$, see section §5, Theorem 5.1: for both models of random covers and Weil-Petersson, we show the existence of exponents $\beta > 0$ such that

$$\lim_{\mathrm{Vol}(X)\to\infty} \mathbb{E}\left(\frac{|\log\det(\Delta_X)|^\beta}{\mathrm{Vol}(X)^\beta}\right) = E^\beta.$$

The paper is organized as follows. In §2 we recall how one establishes an identity for $\log\det(\Delta_X)$ which involves infinite sums over closed geodesics via the Heat trace formula. In §3 we prove an abstract Theorem which guarantees the exponential growth of determinants as long as a certain natural list of assumptions are satisfied. These hypotheses turn out to be valid a.a.s. for the probabilistic models listed above, and this is established in §4. In §5, we derive from Theorem 1.1 a convergence result for the expectation, based on some moments estimates for the systole and the smallest positive eigenvalue.

Acknowledgement. It is a pleasure to thank my colleague Bram Petri for several discussions around this work.

2. Heat kernels and determinants

On the hyperbolic plane $\mathbb{H}^2$, the heat kernel $p_t(x,y)$ (see for example [10] chapter 7) has an explicit formula given by

$$p_t(x,y) = \frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}} \int_{d(x,y)}^\infty \frac{re^{-r^2/4t}dr}{\sqrt{\cosh r - \cosh d(x,y)}},$$

where $d(x,y)$ denotes the hyperbolic distance in $\mathbb{H}^2$. On the quotient $X = \Gamma\backslash\mathbb{H}^2$, we can recover the heat kernel by summing over the group i.e.

$$h_t^X(x,y) = \sum_{\gamma\in\Gamma} p_t(x,\gamma y).$$

Convergence of the above series on any compact subset of $\mathbb{H}^2$ is garanteed by the lattice counting bound

$$N_\Gamma(x,y,T) := \#\{\gamma\in\Gamma \ : \ d(\gamma x, y) \le T\} = O(e^T), \tag{1}$$

which is standard and follows from a basic volume argument. The semi-group of operators $e^{-t\Delta_X}$ is then of trace class and one has the explicit "heat trace formula"

$$\mathrm{Tr}(e^{-t\Delta_X}) = \sum_j e^{-t\lambda_j} = \mathrm{Vol(X)}\frac{e^{-t/4}}{(4\pi t)^{3/2}}\int_0^\infty \frac{re^{-r^2/4t}}{\sinh(r/2)}dr \tag{2}$$

$$+\frac{e^{-t/4}}{(4\pi t)^{1/2}}\sum_{k\ge1}\sum_{\gamma\in\mathcal{P}}\frac{\ell(\gamma)}{2\sinh(k\ell(\gamma)/2)}e^{-(k\ell(\gamma))^2/4t},$$

where $\mathcal{P}$ stands for the set of primitive conjugacy classes in Γ (i.e. oriented primitive closed geodesics on X) and if $\gamma\in\mathcal{P}$, $\ell(\gamma)$ is the length. For more details on the calculation of this trace and more generally Selberg's formula, see [16, 10]. Setting

$$S_X(t) := \frac{e^{-t/4}}{(4\pi t)^{1/2}}\sum_{k\ge1}\sum_{\gamma\in\mathcal{P}}\frac{\ell(\gamma)}{2\sinh(k\ell(\gamma)/2)}e^{-(k\ell(\gamma))^2/4t},$$

it is easy to see that $|S_X(t)-1|$ is exponentially small as $t\to+\infty$. We also observe that $S_X(t)$ is exponentially small as $t\to 0$. Writing (for $\mathrm{Re}(s)$ large)

$$\zeta_X^{(2)}(s) := \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}(S_X(t)-1)dt$$

$$= \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}(S_X(t)-1)dt + \frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}(S_X(t)-1)dt$$

$$= \frac{-1}{\Gamma(s+1)} + \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}S_X(t)dt + \frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}(S_X(t)-1)dt,$$

we notice that the last two integrals make sense for all $s\in\mathbb{C}$. Therefore $\zeta_X^{(2)}(s)$ has an analytic extension to $\mathbb{C}$ and using elementary facts on the gamma function (in particular it has a simple pole at $s=0$ with residue 1), we have that

$$-\frac{d}{ds}\bigg|_{s=0}\zeta_X^{(2)}(s) = -\Gamma'(1) - \int_0^1 \frac{S_X(t)}{t}dt - \int_1^\infty \frac{(S_X(t)-1)}{t}dt.$$

On the other hand, setting for large $\mathrm{Re}(s)$

$$\zeta_X^{(1)}(s) := \mathrm{Vol(X)}\frac{1}{4\pi\Gamma(s)}\int_0^\infty t^{s-1}\frac{e^{-t/4}}{\sqrt{4\pi}t^{3/2}}\int_0^\infty \frac{re^{-r^2/4t}}{\sinh(r/2)}drdt,$$

and using that [1]

$$\frac{e^{-t/4}}{\sqrt{4\pi}t^{3/2}}\int_0^\infty \frac{re^{-r^2/4t}}{\sinh(r/2)}dr = e^{-t/4}\int_{-\infty}^{+\infty} x\tanh(\pi x)e^{-(x^2+1/4)t}dx,$$

one can compute the Mellin transform to obtain (again for $\mathrm{Re}(s)$ large)

$$\zeta_X^{(1)}(s) = \frac{2\mathrm{Vol}(X)}{4\pi}\int_0^\infty \frac{u\tanh(\pi u)}{(u^2+1/4)^s}du.$$

This function can be analytically continued to $s=0$, see for example [6], Appendix B, and the value can be actually computed as

$$-\frac{d}{ds}\bigg|_{s=0}\zeta_X^{(1)}(s) = \frac{\mathrm{Vol}(X)}{4\pi}\left(4\zeta'(-1) - 1/2 + \log(2\pi)\right) := \mathrm{Vol(X)}E,$$

with $\zeta'(-1) = 1/12 - \log(A)$ and A is the so-called *Glaisher-Kinkelin* constant, which is for example defined by

$$A = \lim_{n\to\infty}\frac{\prod_{k=1}^n k^k}{e^{-n^2/4}n^{n^2/2+n/2+1/12}}.$$

We have $E\approx 0,0538$. Using in addition that $\Gamma'(1) = -\gamma_0$, where

$$\gamma_0 = \lim_{n\to+\infty}\sum_{k=1}^n \frac{1}{k} - \log(n),$$

is the *Euler constant*, we have obtained the celebrated identity

$$\log\det\Delta_X = \mathrm{Vol}(X)E + \gamma_0 - \int_0^1 \frac{S_X(t)}{t}dt - \int_1^\infty \frac{(S_X(t)-1)}{t}dt. \tag{3}$$

This formula can be interpreted multiplicatively via Selberg zeta function at $s=1$, see [33, 6, 12].

[1] for example one can use the identity

$$\tanh(\pi x) = \frac{1}{\pi}\int_0^\infty \frac{\sin(ux)}{\sinh(u/2)}du.$$

3. An abstract deterministic statement

Theorem 1.1 actually follows from a more general deterministic result for sequences of compact surfaces satisfying certain hypotheses denoted as (H). More precisely, if (X_k) is a sequence of compact connected hyperbolic surfaces with $\mathrm{Vol}(X_k) \to +\infty$, let $\mathcal{P}_k$ denote the set of oriented primitive closed geodesics on X_k, and let Δ_k be the hyperbolic Laplacian on X_k. We also denote by $\ell_0(X_k)$ the length of the shortest closed geodesic on X_k. Let $C < 0$, $\eta > 0$, $L > 0$ and $0 < \alpha < 1/2$ be some constants.

We say that the sequence (X_k) satisfies hypothesis $\mathscr{H}_1(\eta)$ if for all $k \in \mathbb{N}$ we have the following.

$$\lambda_1(\Delta_k) \geq \eta.$$

We say that the sequence (X_k) satisfies hypothesis $\mathscr{H}_2(C, L, \alpha)$ if for all $k \in \mathbb{N}$ we have the following bound on the *number of closed geodesics*:

$$N_{X_k}(L) := \#\{(\gamma, m) \in \mathcal{P}_k \times \mathbb{N}^* \ : \ m\ell(\gamma) \leq L\} \leq C\mathrm{Vol}(X_k)^\alpha.$$

We point out that exponential growth of Laplace determinants is established in the litterature for families of covers for which a uniform spectral gap holds and the injectivity radius of the manifolds goes to infinity, see for example [3] and [2]. Typical examples are congruence covers of arithmetic hyperbolic manifolds and Laplacians twisted by a "strongly acyclic" representation which ensures a uniform spectral gap. In random models of surfaces, having the injectivity radius grow to infinity is *non typical* and we establish the result under the weaker assumption of small growth of the number of closed geodesics with bounded length.

Theorem 1.1 will follow from the following deterministic result.

Theorem 3.1. *Fix some $\eta > 0$ and $0 < \alpha < 1/2$. Assume that (X_k) satisfies $\mathscr{H}_1(\eta)$ and $\mathscr{H}_2(C_0, \alpha, L_0)$ with $L_0 = 2\mathrm{arcsinh}(1)$ for some $C_0 > 0$. Then for all $\epsilon > 0$, there exists $L_\epsilon > 0$ such that if in addition (X_k) satisfies $\mathscr{H}_2(C_\epsilon, L_\epsilon, \alpha)$ for some $C_\epsilon > 0$, then uniformly for all $\mathrm{Vol}(X_k)$ large,*

$$\frac{\log\det(\Delta_{X_k})}{\mathrm{Vol}(X_k)} \in [E - \epsilon, E + \epsilon],$$

where $E > 0$ is the universal constant from above.

We warn the reader that the positive constants denoted by $C_1, C_2, \ldots, C_j$ below *depend only on $C_0, L_0 > 0$* . We will make clear when constants have some additional dependence by adding suitable brackets. Before we give a proof of Theorem 3.1, we will need some preliminary Lemmas which are needed to control uniformly sums over closed geodesics.

Lemma 3.2. *Under hypothesis $\mathscr{H}_2(C_0, \alpha, L_0)$, there exists $c = c(C_0, L_0) > 0$ such that for all k,*

$$\ell_0(X_k) \geq c\mathrm{Vol}(X_k)^{-\alpha}.$$

Proof. Let m_0 be defined by

$$m_0 := \max\{m \geq 1 \ : \ m\ell_0(X_k) \leq L_0\}.$$

By definition of m_0 we have

$$(m_0 + 1)\ell_0(X_k) > L_0,$$

while

$$m_0 \leq N_k(L_0) \leq C_0\mathrm{Vol}(X_k)^\alpha.$$

therefore

$$\ell_0(X_k) \geq \frac{L_0}{C_0 \mathrm{Vol}(X_k)^\alpha + 1},$$

and the proof is done. □.

Lemma 3.3. *Under hypothesis $\mathscr{H}_2(C_0, \alpha, L_0)$ where*

$$L_0 = 2\mathrm{arcsinh}(1),$$

there exists $C_1 > 0$ such that for all k and all $T \geq 0$,

$$N_k(T) := \#\{(\gamma, m) \in \mathcal{P}_k \times \mathbb{N}^* \quad : \quad m\ell(\gamma) \leq T\} \leq C_1 \mathrm{Vol}(X_k) e^T.$$

Proof. A result of Buser ([10] Lemma 6.6.4) says that for any compact connected hyperbolic surface of genus g, the number of oriented closed geodesics with length $\leq T$ which are *not iterates of primitive closed geodesics of length* $\leq 2\mathrm{arcsinh}(1)$ is bounded from above by

$$(g-1)e^{T+6}.$$

Therefore we have

$$N_k(T) \leq \frac{\mathrm{Vol}(X_k)}{4\pi} e^{T+6}$$
$$+\#\{(\gamma, m) \in \mathcal{P}_k \times \mathbb{N}^* \quad : \quad m\ell(\gamma) \leq T \text{ and } \ell(\gamma) \leq 2\mathrm{arcsinh}(1)\}.$$

Since we have

$$\begin{aligned}
&\#\{(\gamma, m) \in \mathcal{P}_k \times \mathbb{N}^* \quad : \quad m\ell(\gamma) \leq T \text{ and } \ell(\gamma) \leq 2\mathrm{arcsinh}(1)\} \\
&\leq \sum_{\ell(\gamma) \leq \mathrm{arcsinh}(1)} \frac{T}{\ell(\gamma)} \\
&\leq \frac{T}{\ell_0(X_k)} \#\{\gamma \in \mathcal{P}_k \quad : \quad \ell(\gamma) \leq 2\mathrm{arcsinh}(1)\},
\end{aligned}$$

we can use Lemma 3.2 (with $L_0 = 2\mathrm{arcsinh}(1)$) to get

$$\begin{aligned}
&\#\{(\gamma, m) \in \mathcal{P}_k \times \mathbb{N}^* \quad : \quad m\ell(\gamma) \leq T \text{ and } \ell(\gamma) \leq 2\mathrm{arcsinh}(1)\} \\
&\leq Tc^{-1}C_0 \mathrm{Vol}(X_k)^{2\alpha} \leq Tc^{-1}C_0 \mathrm{Vol}(X_k),
\end{aligned}$$

and thus

$$N_k(T) \leq \mathrm{Vol}(X_k) e^T \left(\frac{e^6}{4\pi} + c^{-1}C_0 T e^{-T} \right) \leq \mathrm{Vol}(X_k) e^T \left(\frac{e^6}{4\pi} + c^{-1}C_0 e^{-1} \right),$$

the proof is done. □

Lemma 3.4. *Under hypothesis $\mathscr{H}_1(\eta)$ and $\mathscr{H}_2(C_0, \alpha, L_0)$ where L_0 is as above, there exists $C_2 > 0$ such that for all k and all $t \geq 1$,*

$$|S_{X_k}(t) - 1| \leq C_2 \mathrm{Vol}(X_k) e^{-\eta_0 t},$$

where $\eta_0 = \min(\eta, 1/4)$.

Proof. In this proof we will use Vinogradov's notation $A \ll B$ meaning that $A \leq CB$ where $C > 0$ is a *universal constant*. By formula (2) we have

$$\mathrm{tr}(e^{-t\Delta_{X_k}}) - 1 = \mathrm{Vol(X)} \frac{e^{-t/4}}{(4\pi t)^{3/2}} \int_0^\infty \frac{re^{-r^2/4t}}{\sinh(r/2)} dr + S_{X_k}(t) - 1,$$

and therefore we get for $t \geq 1$,

$$|S_{X_k}(t) - 1| \ll \mathrm{Vol}(X_k) e^{-t/4} + \sum_{j=1}^\infty e^{-t\lambda_j(X_k)}.$$

On the other hand using the uniform spectral gap we have for all $t \geq 1$,

$$\sum_{j=1}^{\infty} e^{-t\lambda_j(X_k)} \leq \sum_{j=1}^{\infty} e^{-(t-1)\lambda_j - \lambda_j} \leq e^{-(t-1)\eta} \mathrm{tr}(e^{-\Delta_{X_k}}).$$

Going back to formula (2) with $t = 1$, we have also

$$\mathrm{tr}(e^{-\Delta_{X_k}}) \ll \mathrm{Vol}(X_k) + S_{X_k}(1),$$

and

$$\begin{aligned} S_{X_k}(1) &\ll \sum_{m \geq 1} \sum_{\gamma \in \mathcal{P}_k} \frac{\ell(\gamma)}{2\sinh(m\ell(\gamma)/2)} e^{-(m\ell(\gamma))^2/4} \\ &\ll \int_0^{\infty} \frac{u}{2\sinh(u/2)} e^{-u^2/4} dN_k(u). \end{aligned}$$

We can use Lemma 3.3 to bound $N_k(u)$ as

$$N_k(u) \leq C_1 \mathrm{Vol}(X_k) e^u,$$

and a summation by parts shows that

$$\begin{aligned} \int_0^{\infty} \frac{u}{2\sinh(u/2)} e^{-u^2/4} dN_k(u) &= -\int_0^{\infty} N_k(u) \frac{d}{du} \left\{ \frac{u}{2\sinh(u/2)} e^{-u^2/4} \right\} du \\ &\ll C_1 \mathrm{Vol}(X_k), \end{aligned}$$

which ends the proof. □

We can now give a proof of Theorem 3.1. First notice that by formula (3), we have

$$\left| \frac{\log\det(\Delta_{X_k})}{\mathrm{Vol}(X_k)} - E \right| \leq O(\mathrm{Vol}(X_k)^{-1}) + \mathcal{D}_{X_k}^{(1)} + \mathcal{D}_{X_k}^{(2)},$$

where

$$\mathcal{D}_{X_k}^{(1)} = \frac{1}{\mathrm{Vol}(X_k)} \int_1^{\infty} \frac{|S_{X_k}(t) - 1|}{t} dt, \ \mathcal{D}_{X_k}^{(2)} = \frac{1}{\mathrm{Vol}(X_k)} \int_0^1 \frac{S_{X_k}(t)}{t} dt.$$

Let us fix $\epsilon > 0$. We first investigate $\mathcal{D}_{X_k}^{(1)}$. Using $\mathscr{H}_1(\eta)$ and $\mathscr{H}_2(C_0, \alpha, L_0)$, we can use Lemma 3.4 and write

$$\int_1^{\infty} \frac{|S_{X_k}(t) - 1|}{t} dt \leq \int_1^R \frac{S_{X_k}(t)}{t} dt + \log(R) + C_2 \mathrm{Vol}(X_k) \int_R^{\infty} \frac{e^{-\eta_0 t}}{t} dt.$$

Fixing $R = R(\epsilon)$ so large that

$$C_2 \int_R^{\infty} \frac{e^{-\eta_0 t}}{t} dt \leq \epsilon,$$

we have

$$\mathcal{D}_{X_k}^{(1)} \leq \frac{1}{\mathrm{Vol}(X_k)} \int_1^R \frac{S_{X_k}(t)}{t} dt + \frac{\log(R)}{\mathrm{Vol}(X_k)} + \epsilon.$$

We now pick $L_1 >> 1$ (to be adjusted later on) and write

$$\int_1^R \frac{S_{X_k}(t)}{t} dt = \int_1^R \frac{S_{X_k}^{L_1,-}(t)}{t} dt + \int_1^R \frac{S_{X_k}^{L_1,+}(t)}{t} dt,$$

where

$$S_{X_k}^{L_1,-}(t) = \frac{e^{-t/4}}{(4\pi t)^{1/2}} \sum_{m\ell(\gamma) \leq L_1} \frac{\ell(\gamma)}{2\sinh(m\ell(\gamma)/2)} e^{-(m\ell(\gamma))^2/4t},$$

$$S_{X_k}^{L_1,+}(t) = \frac{e^{-t/4}}{(4\pi t)^{1/2}} \sum_{m\ell(\gamma) > L_1} \frac{\ell(\gamma)}{2\sinh(m\ell(\gamma)/2)} e^{-(m\ell(\gamma))^2/4t}.$$

Clearly we have

$$\int_1^R \frac{S_{X_k}^{L_1,+}(t)}{t}dt \leq C_3 \sum_{m\ell(\gamma)>L} \frac{\ell(\gamma)}{2\sinh(m\ell(\gamma)/2)} e^{-(m\ell(\gamma))^2/4R},$$

for some universal constant $C_3 > 0$. Using Lemma 3.3 and a summation by parts, we deduce that

$$\int_1^R \frac{S_{X_k}^{L_1,+}(t)}{t}dt \leq C_4 \mathrm{Vol(X_k)} \int_{L_1}^\infty \left| \frac{d}{du}\left(\frac{u}{\sinh(u/2)} e^{-u^2/(4R)}\right)\right| e^u du.$$

we now take $L_1 = L_1(\epsilon)$ so large that

$$C_4 \int_{L_1}^\infty \left| \frac{d}{du}\left(\frac{u}{\sinh(u/2)} e^{-u^2/(4R)}\right)\right| e^u du < \epsilon.$$

We now observe that if $\mathscr{H}_2(C,\alpha,L_1)$ holds, we have

$$\int_1^R \frac{S_{X_k}^{L,-}(t)}{t}dt = \sum_{m\ell(\gamma)\leq L} \frac{\ell(\gamma)}{2\sinh(m\ell(\gamma)/2)} \int_1^R e^{-(m\ell(\gamma))^2/4t} \frac{e^{-t/4}}{\sqrt{4\pi}t^{3/2}}dt$$

$$\leq C_5 L_1 N_k(L_1) \leq C_5 C L_1 \mathrm{Vol}(X_k)^\alpha \leq C_6(L_1,C)\mathrm{Vol}(X_k)^\alpha,$$

for some possibly large constant $C_6(L_1,C) > 0$. In a nutshell, we have obtained, provided that $\mathscr{H}_2(C,\alpha,L_1)$ is satisfied with $L_1 = L_1(\epsilon)$ taken large enough,

$$\limsup_{\mathrm{Vol}(X_k)\to+\infty} \mathcal{D}_{X_k}^{(1)} \leq 2\epsilon.$$

We now turn our attention to $\mathcal{D}_{X_k}^{(2)}$, and this is where the control of the systole $\ell_0(X_k)$ will play a role. We first use the same idea as above by writing

$$\mathcal{D}_{X_k}^{(2)} = \frac{1}{\mathrm{Vol}(X_k)} \int_0^1 \frac{S_{X_k}(t)}{t}dt = \frac{1}{\mathrm{Vol}(X_k)} \int_0^1 \frac{S_{X_k}^{L_2,-}(t)}{t}dt + \frac{1}{\mathrm{Vol}(X_k)} \int_0^1 \frac{S_{X_k}^{L_2,+}(t)}{t}dt.$$

Writing for $t > 0$,

$$S_{X_k}^{L_2,+}(t) \leq C_7 t^{-1/2} \sum_{m\ell(\gamma)>L_2} \frac{\ell(\gamma)}{2\sinh(m\ell(\gamma)/2)} e^{-(m\ell(\gamma))^2/4t},$$

where $C_7 > 0$ is universal, we have by Fubini

$$\int_0^1 \frac{S_{X_k}^{L_2,+}(t)}{t}dt \leq C_7 \sum_{m\ell(\gamma)>L_2} \frac{\ell(\gamma)}{2\sinh(m\ell(\gamma)/2)} G(m\ell(\gamma)),$$

where for $u > 0$,

$$G(u) = \int_0^1 t^{-3/2} e^{-u^2/4t} dt.$$

Notice that $u \mapsto G(u)$ is a decreasing function and by a change of variable we have actually for all $u > 0$,

$$G(u) = \frac{4}{u} \int_{u/2}^\infty e^{-x^2} dx.$$

We have therefore the bound

$$G(u) = \frac{4}{u} \int_{u/2}^\infty e^{-x^2/2 - x^2/2} dx \leq \frac{4}{u} e^{-u^2/8} \int_0^\infty e^{-x^2/2} dx = \frac{2\sqrt{2\pi}}{u} e^{-u^2/8}.$$

As a consequence we get for $L_2 > 1$

$$\int_0^1 \frac{S_{X_k}^{L_2,+}(t)}{t}dt \leq C_8 \sum_{m\ell(\gamma)>L_2} \frac{\ell(\gamma)}{2\sinh(m\ell(\gamma)/2)} e^{-(m\ell(\gamma))^2/8},$$

and by using Lemma 3.3 and a summation by parts, we can definitely fix $L_2 = L_2(\epsilon)$ large enough so that

$$\int_0^1 \frac{S_{X_k}^{L_2,+}(t)}{t} dt \leq C_8 \sum_{m\ell(\gamma)>L_2} \frac{\ell(\gamma)}{2\sinh(m\ell(\gamma)/2)} e^{-(m\ell(\gamma))^2/8} \leq \mathrm{Vol}(X_k)\epsilon.$$

From the above bound on $G(u)$ we also deduce

$$\int_0^1 \frac{S_{X_k}^{L_2,-}(t)}{t} dt \leq \frac{C_9}{\ell_0(X_k)} \sum_{m\ell(\gamma)\leq L_2} \frac{\ell(\gamma)}{2\sinh(m\ell(\gamma)/2)}.$$

By using Lemma 3.2 and $\mathscr{H}_2(C, \alpha, L_2(\epsilon))$ we have again as above

$$\int_0^1 \frac{S_{X_k}^{L_2,-}(t)}{t} dt \leq C_{10}(L_2, C)\mathrm{Vol}(X_k)^{2\alpha},$$

where $C_{10}(L_2, C) > 0$ is some (possibly very large) constant depending on L_2, C. We have therefore shown, that whenever (X_k) satisfies $\mathscr{H}_2(C, \alpha, L(\epsilon))$ for some $C > 0$ and with $L(\epsilon) = \max\{L_1, L_2\}$ we have

$$\limsup_{\mathrm{Vol}(X_k)\to+\infty} \left| \frac{\log\det(\Delta_{X_k})}{\mathrm{Vol}(X_k)} - E \right| \leq 3\epsilon.$$

Theorem 3.1 is proved. □

4. Hypotheses $\mathscr{H}_1$ and $\mathscr{H}_2$ hold with high probability

Theorem 1.1 follows immediately from Theorem 3.1 if one can establish for all models that there exists $\eta > 0$ and $\alpha > 0$ with $\alpha < 1/2$ such that $\mathscr{H}_1(\eta)$ holds with high probability as $\mathrm{Vol}(X)$ goes to infinity and for all L large, one can find $C > 0$ such that $\mathscr{H}_2(C, \alpha, L)$ also holds with high probability as $\mathrm{Vol}(X)$ goes to infinity. Indeed we then have for all $\epsilon > 0$, and all $\mathrm{Vol}(X)$ large enough

$$\mathbb{P}\left(\frac{\log\det(\Delta_X)}{\mathrm{Vol}(X)} \in [E-\epsilon, E+\epsilon]\right) \geq$$
$$\mathbb{P}\left(X \in \mathscr{H}_1(\eta) \cap \mathscr{H}_2(C_0, \alpha, L_0) \cap \mathscr{H}_2(C, \alpha, L(\epsilon))\right),$$

with

$$\lim_{\mathrm{Vol}(X)\to\infty} \mathbb{P}\left(X \in \mathscr{H}_1(\eta) \cap \mathscr{H}_2(C_0, \alpha, L_0) \cap \mathscr{H}_2(C, \alpha, L(\epsilon))\right) = 1.$$

4.1. **Random covers.** First we recall that the cover $X_n \to X$ may not be connected but we know from [19] that

$$\lim_{n\to+\infty} \mathbb{P}(X_n \text{ connected}) = 1.$$

We can therefore either restrict ourselves to connected surfaces X_n or modify the definition of the regularized determinant by setting

$$\zeta_{X_n}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}(\mathrm{Tr}(e^{-t\Delta_X}) - d_n)dt,$$

where d_n is the number of connected component of X_n and use the fact that $d_n = 1$ with high probability.

It was recently shown in [20] that there exists a uniform spectral gap for random covers X_n a.a.s. as $n \to +\infty$. More precisely we have for all $0 < \eta < \min\{3/16, \lambda_1(X)\}$,

$$\lim_{n\to+\infty} \mathbb{P}(\lambda_1(X_n) \geq \eta) = 1.$$

Therefore $\mathscr{H}_1(\eta)$ holds a.a.s. provided η is taken small enough. On the other hand property $\mathscr{H}_2$ is less obvious from the existing litterature and will require some

explanations. We recall that given a random homomorphism $\phi_n : \Gamma \to \mathcal{S}_n$, one can define a unitary representation ρ_n of Γ by setting

$$\rho_n(\gamma)(f) := f \circ \phi_n(\gamma)^{-1},$$

where $f \in L^2([n])$, and the representation space is $L^2([n]) \simeq \mathbb{C}^n$. The main interest of this representation is the following fact, often called "Venkov-Zograf induction formula". For all $\mathrm{Re}(s) > 1$, one can define the Seberg zeta function of X_n by

$$Z_{X_n}(s) := \prod_{m \geq 0} \prod_{\gamma \in \mathcal{P}_{X_n}} \left(1 - e^{-(s+m)\ell(\gamma)}\right).$$

One can also look at the twisted Selberg zeta function of the base $X = \Gamma \backslash \mathbb{H}^2$ defined for $\mathrm{Re}(s) > 1$ by

$$Z_{X,\rho_n}(s) := \prod_{m \geq 0} \prod_{\gamma \in \mathcal{P}_X} \det\left(I - \rho_n(\gamma) e^{-(s+m)\ell(\gamma)}\right).$$

It turns out that we have for all s, $Z_{X_n}(s) = Z_{X,\rho_n}(s)$, see [37]. By computing logarithmic derivatives we have for all $\mathrm{Re}(s) > 1$,

$$\frac{Z'_{X_n}(s)}{Z_{X_n}(s)} = \sum_{\gamma \in \mathcal{P}_{X_n}} \sum_{q \geq 1} \frac{\ell(\gamma) e^{-sq\ell(\gamma)}}{1 - e^{-q\ell(\gamma)}} = \sum_{\gamma \in \mathcal{P}_X} \sum_{q \geq 1} \frac{\ell(\gamma) \mathrm{tr}(\rho_n(\gamma^q)) e^{-sq\ell(\gamma)}}{1 - e^{-q\ell(\gamma)}}.$$

Let $\phi \in C_0^\infty(\mathbb{R}^+)$ be a compactly supported smooth test function, and set

$$\psi(s) := \int_0^\infty e^{sx} \phi(x) dx.$$

One can check that $\psi(s)$ is actually analytic on $\mathbb{C}$ and by Fourier inversion formula we have for all $A > 1$,

$$\frac{1}{2i\pi} \int_{A-i\infty}^{A+i\infty} \frac{Z'_{X_n}(s)}{Z_{X_n}(s)} \psi(s) ds = \sum_{\gamma \in \mathcal{P}_{X_n}} \sum_{q \geq 1} \frac{\ell(\gamma)}{1 - e^{-q\ell(\gamma)}} \phi(q\ell(\gamma))$$

$$= \sum_{\gamma \in \mathcal{P}_X} \sum_{q \geq 1} \frac{\ell(\gamma) \mathrm{tr}(\rho_n(\gamma^q))}{1 - e^{-q\ell(\gamma)}} \phi(q\ell(\gamma)).$$

See for example [17] §3 for more details on the derivation of this formula. By carefully choosing the test function ϕ we deduce that for all $\mathcal{L} \in \mathbb{R}^+$, we recover the identity

$$\sum_{\gamma \in \mathcal{P}_{X_n}} \sum_{\substack{q \geq 1 \\ q\ell(\gamma) = \mathcal{L}}} \ell(\gamma) = \sum_{\gamma \in \mathcal{P}_X} \sum_{\substack{q \geq 1 \\ q\ell(\gamma) = \mathcal{L}}} \ell(\gamma) \mathrm{tr}(\rho_n(\gamma^q)).$$

Notice that this formula can be proved directly by group theoretic arguments, see for example in [15], in the proof of theorem 7.1. From this identity we deduce that for all $L > 0$, we have

$$\sum_{\gamma \in \mathcal{P}_{X_n}} \sum_{\substack{q \geq 1 \\ q\ell(\gamma) \leq L}} \ell(\gamma) = \sum_{\gamma \in \mathcal{P}_X} \sum_{\substack{q \geq 1 \\ q\ell(\gamma) \leq L}} \ell(\gamma) \mathrm{tr}(\rho_n(\gamma^q)).$$

In particular if we set

$$N_{X_n}(L) := \#\{(\gamma, m) \in \mathcal{P}_{X_n} \times \mathbb{N}^* \ : \ m\ell(\gamma) \leq L\},$$

we have

$$\ell_0(X) N_{X_n}(L) \leq L \sum_{\gamma \in \mathcal{P}_X} \sum_{\substack{q \geq 1 \\ q\ell(\gamma) \leq L}} \mathrm{tr}(\rho_n(\gamma^q)).$$

We point out that we have actually $\mathrm{tr}(\rho_n(\gamma^q)) = Fix(\phi_n(\gamma^q))$, where $Fix(\sigma)$ denotes the number of fixed points of the permutation σ acting on $[n]$. From the

combinatorial analysis of Magee-Puder [21, 20], we know that for all primitive $\gamma \in \Gamma$ and $q \geq 1$, we have

$$\lim_{n\to\infty} \mathbb{E}(Fix(\phi_n(\gamma^q))) = d(q),$$

where $d(q)$ stands for the number of divisors of q. This is enough to conclude that for all L, we have

$$\lim_{n\to\infty} \mathbb{E}(N_{X_n}(L)) \leq C(\Gamma, L),$$

where $C(\Gamma, L) > 0$ is some large constant. Applying Markov's inequality, we get that for all $\varepsilon > 0$ and L fixed,

$$\lim_{n\to\infty} \mathbb{P}\left(N_{X_n}(L) \leq \mathrm{Vol}(X_n)^{\varepsilon}\right) = 1.$$

As a conclusion, in the random cover model, $\mathscr{H}_2(C, \alpha, L)$ is satisfied a.a.s. for all L large and all $\alpha > 0$.

4.2. **Brooks-Makover model.** In the paper [9], they show (Theorem 2.2) that there exist a constant $C_1 > 0$ such that as $n \to \infty$,

$$\mathbb{P}(\lambda_1(S_n^C) \geq C_1) \to 1.$$

Notice that contrary to the model of random covers, the systole of S_n^C can be arbitrarily small, but we actually know that there exists $C_2 > 0$ such that as $n \to +\infty$,

$$\mathbb{P}(\ell_0(S_n^C) \geq C_2) \to 1.$$

In other words, property $\mathscr{H}_1(\eta)$ is satisfied a.a.s. for some $\eta > 0$. On the other hand, counting results for closed geodesics follow from the later work of Petri [29]. More precisely, one can derive from [29] the following fact.

Proposition 4.1. *For all $L > 0$ fixed, we can find an integer N_L and a finite set of words*

$$\mathcal{W}_L \subset \{l, r\}^{N_L},$$

such that with high probability as $n \to \infty$,

$$N_n(L) := \#\{(\gamma, m) \in \mathcal{P}_{S_n^C} \times \mathbb{N}^* \ : \ m\ell(\gamma) \leq L\} \leq \sum_{w \in \mathcal{W}_L} Z_{n,w},$$

where $Z_{n,w}$ are integer valued random variables. In addition each $Z_{n,w}$ converges in the sense of moments (and hence in distribution) as $n \to \infty$ to a Poisson variable with expectation $\lambda_w > 0$.

By applying Markov inequality, one deduces readily that for all $\varepsilon > 0$ we have a.a.s.

$$\sum_{w \in \mathcal{W}_L} Z_{n,w} \leq n^{\epsilon} \leq C_L \mathrm{Vol}(S_n^C)^{\varepsilon}.$$

This is enough to conclude that for all $\varepsilon > 0$, with high probability as $n \to \infty$, we have

$$N_n(L) \leq C\mathrm{Vol}(S_n^C)^{\varepsilon},$$

and therefore property $\mathscr{H}_2(C, \alpha, L)$ holds for any choice of $\alpha > 0$, just like in the previous model.

Let us now give some details on the proof of Proposition 4.1. The first step is to reduce the problem to a counting bound for S_n^O. In [9] §3, they introduce the notion of "large cusps" condition for S_n^O. This condition is satisfied a.a.s for S_n^O as $n \to \infty$ by Theorem 2.1 in [9]. The main interest of this condition is Theorem 3.2 in [9],

see also Lemma 2.5 in [29], which allows to show that provided this "large cusps" condition is satisfied, one can bound

$$N_n(L) \leq \#\{(\gamma, m) \in \mathcal{P}_{S_n^0} \times \mathbb{N}^* \ : \ m\ell(\gamma) \leq 2L\} =: N_n^O(2L).$$

As explained in [9], §4, closed geodesics and their length in S_n^O can be described via the combinatorial data of $(\mathcal{G}_n, \mathcal{O})$: any closed geodesic in S_n^O corresponds to a word $w \in \{l, r\}^N$, for some $N > 0$. To this word one can associate a matrix M_w in $SL_2(\mathbb{N})$ via the rule

$$M_w = W_1 \dots W_N,$$

where $W_j = \mathcal{L}$ if $w_j = l$ and $W_j = \mathcal{R}$ if $w_j = r$, where

$$\mathcal{L} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ \mathcal{R} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The length ℓ_w of the corresponding geodesic on S_n^O is then given by

$$\mathrm{tr}(M_w) = 2\cosh(\ell_w/2).$$

All we need to check is that fixing L implies finiteness of the corresponding set of words (by bounding their word length). This is done in Lemma 3.1 in [30]. One obtains therefore that L being fixed, there exists a finite subset $\mathcal{W}_L \subset \{l, r\}^{N_L}$ for some large $N_L > 0$, such that for all n, all closed geodesics with length $\leq 2L$ on S_n^O are given by words $w \in W_L$. Proposition 4.1 now follows directly from Theorem B in [29].

4.3. **Weil-Petersson model.** If $f(X)$ is a well-defined function on moduli space $\mathscr{M}_g$, we will denote by $\int_{\mathscr{M}_g} f(X)dX$ the corresponding integral with respect to Weil-Petersson volume. In the latter V_g will denote the Weil-Petersson volume of $\mathcal{M}_g$ so that the expectation of f is given by

$$\mathbb{E}_g(f) := \frac{1}{V_g} \int_{\mathscr{M}_g} f(X)dX.$$

In [24], the following fact was proved. There exists $\eta > 0$ such that as $g \to +\infty$, we have

$$\mathbb{P}(\lambda_1(X) \geq \eta) \to 1.$$

The constant η given by Mirzakhani follows from Cheeger's inequality and an estimate a.a.s. of Cheeger's isoperimetric constant. Recently it was shown independently in [40, 39] that one can actually take $\eta = 3/16 - \epsilon$. This shows that $\mathscr{H}_1(\eta)$ holds with high probability as $g \to +\infty$ for some universal $\eta > 0$. In [24], see Theorem 4.2 and the remark after, she proved that there exists a universal $\epsilon_0 > 0$ such that for all $0 < \epsilon \leq \epsilon_0$, one has for all g large

$$\mathbb{P}(\ell_0(X) \leq \epsilon) \leq C\epsilon^2,$$

where C is uniform in g. In particular for all $\varepsilon > 0$, we get for all g large

$$\mathbb{P}\left(\ell_0(X) \geq \mathrm{Vol}(X)^{-\varepsilon}\right) \geq 1 - O(\mathrm{Vol}(X)^{-2\varepsilon}).$$

On the other hand, in the paper [25], Mirzakhani and Petri showed that for all $L > 0$ fixed, the random variable

$$N_g^0(L) := \#\{\gamma \in \mathcal{P}_X \ : \ \ell(\gamma) \leq L\}$$

converges in distribution as $g \to +\infty$ to a Poisson variable Z_{λ_L} with parameter

$$\lambda_L := \int_0^L \frac{e^t + e^{-t} - 2}{2t} dt.$$

Moreover, we have also convergence of all moments with $p \in \mathbb{N}$

$$\lim_{g \to \infty} \mathbb{E}(\left(N_g^0(L)\right)^p) = \mathbb{E}(Z_{\lambda_L}^p).$$

An application of Markov's inequality then shows that for all $\varepsilon > 0$, for all g large enough

$$\mathbb{P}(N_g^0(L) \leq \mathrm{Vol}(X)^{\varepsilon}) \geq 1 - C_L \mathrm{Vol}(X)^{-\varepsilon}.$$

To control the counting function

$$N_g(L) := \#\{(\gamma, m) \in \mathcal{P}_X \times \mathbb{N}^* \ : \ m\ell(\gamma) \leq L\},$$

we write

$$N_g(L) \leq \sum_{m=1}^{[L/\ell_0(X)]+1} N_g^0(L/m) \leq N_g^0(L)\left(1 + \frac{L}{\ell_0(X)}\right).$$

For all $\varepsilon > 0$, we have with high probability as $g \to +\infty$

$$N_g(L) \leq \mathrm{Vol}(X)^{\varepsilon} + L\mathrm{Vol}(X)^{2\varepsilon} = O_L(\mathrm{Vol}(X)^{2\varepsilon}),$$

and thus for all L large, there exists $C_L > 0$ such that $\mathscr{H}_2(C_L, \alpha, L)$ holds with high probability for all $\alpha > 0$ on $\mathscr{M}_g$ as $g \to +\infty$.

5. Convergence results for moments of $\log\det(\Delta_X)$.

In this last section, we give the proof of the following fact.

Theorem 5.1. *In the Weil-Petersson model, for all $0 < \beta < 1$ we have*

$$\lim_{g\to\infty} \frac{1}{V_g(4\pi(g-1))^{\beta}} \int_{\mathscr{M}_g} |\log\det(\Delta_X)|^{\beta}\, dX = E^{\beta}.$$

In the random cover model, Let $\chi_0 = \mathbf{1}_{\{X_n \text{ connected}\}}$. Then as $n \to +\infty$, for all $\beta > 0$, we have

$$\lim_{n\to\infty} \mathbb{E}\left(\frac{|\log\det(\Delta_{X_n})|^{\beta}}{\mathrm{Vol}(X_n)^{\beta}}\chi_0\right) = E^{\beta}.$$

The proof for the Weil-Petersson model is a rather direct consequence of Theorem 1.1 and some estimates of Mirzakhani [23]. We first need an a priori estimate for $|\log\det(\Delta_X)|$ which follows from similar ideas as in Theorem 3.1, without the probabilistic input. We use Vinogradov's notation $\ll$ where the implied constant is universal.

By using Buser's counting bound [10], as in the proof of Lemma 3.3, we have the following universal bound for the number of closed geodesics of a surface with genus g:

$$\begin{aligned} N_X(L) &\leq (g-1)e^{L+6} + \frac{L}{\ell_0(X)}(3g-3) \\ &\ll \mathrm{Vol}(X)e^{L}\left(1 + \frac{1}{\ell_0(X)}\right), \end{aligned} \tag{4}$$

for some universal constant $A_1 > 0$. We now prove an a priori upper bound for $|\log\det(\Delta_X)|$. Following ideas of Wolpert [38], it is convenient to write for all $\mathrm{Re}(s)$ large,

$$\zeta_X(s) = \sum_{0<\lambda_j<1/4} \lambda_j^{-s} + \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}\left(\mathrm{tr}(e^{-t\Delta_X}) - \sum_{0\leq\lambda_j<1/4} e^{-t\lambda_j}\right) dt,$$

which can be rewritten as

$$\zeta_X(s) = \sum_{0<\lambda_j<1/4} \lambda_j^{-s} + \sum_{0\leq\lambda_j<1/4} H(s,\lambda_j) + \zeta_X^{(1)}(s) +$$

$$\frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}\left(S_X(t)-\sum_{0\le\lambda_j<1/4}e^{-t\lambda_j}\right)dt+\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}S_X(t)dt,$$

where we have set for $\mathrm{Re}(s)$ large,

$$H(s,\lambda):=\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}e^{-\lambda t}dt.$$

By integration by parts and elementary properties of the Euler gamma function, we then observe that for all $\lambda\in[0,1/4]$, $s\mapsto H(s,\lambda)$ has an analytic extension to $s=0$. Morevover, if we set

$$C(\lambda):=-\left.\frac{d}{ds}\right|_{s=0}H(\lambda,s),$$

the there exists a universal constant $A_1>0$ such that for all $\lambda\in[0,1/4]$,

$$|C(\lambda)|\le A_1.$$

This formula leads to the identity

$$\log\det(\Delta_X)=\sum_{0<\lambda_j<1/4}\log(\lambda_j)+\sum_{0<\lambda_j<1/4}C(\lambda_j)-\left.\frac{d}{ds}\right|_{s=0}\zeta_X^{(1)}(s)$$
$$-\int_0^1\frac{S_X(t)}{t}dt-\int_1^\infty\frac{(S_X(t)-\sum_{0\le\lambda_j<1/4}e^{-\lambda_j t})}{t}dt.$$

By mimicking the proof of Lemma 3.4 and using the above counting bound, we deduce that for all $t\ge 1$,

$$\left|S_X(t)-\sum_{0\le\lambda_j<1/4}e^{-t\lambda_j}\right|\ll\mathrm{Vol}(X)\left(1+\frac{1}{\ell_0(X)}\right)e^{-t/4}.$$

By Fubini and the estimate on $u\mapsto G(u)$ we have also

$$\int_0^1\frac{S_X(t)}{t}dt\ll\sum_{m,\gamma}\frac{e^{-(m\ell(\gamma))^2/8}}{m\ell(\gamma)}\ll\sum_{m\ell(\gamma)\le 1}\frac{1}{m\ell(\gamma)}+\sum_{m\ell(\gamma)>1}e^{-(m\ell(\gamma))^2/8}$$
$$\ll\sum_{m\ell(\gamma)\le 1}\frac{1}{m\ell(\gamma)}+\mathrm{Vol}(X)\left(1+\frac{1}{\ell_0(X)}\right).$$

By noticing that we have

$$\sum_{m\ell(\gamma)\le 1}\frac{1}{m\ell(\gamma)}\ll\frac{\log\ell_0^{-1}(X)}{\ell_0(X)}N_X^0(1),$$

where $N_X^0(L)$ is the counting function for primitive closed geodesics, we have obtained the estimate:

$$|\log\det(\Delta_X)|\ll\mathrm{Vol}(X)\left(1+|\log(\lambda_*(X))|+\frac{1}{\ell_0(X)}+\right)+\frac{\log\ell_0^{-1}(X)}{\ell_0(X)}N_X^0(1),$$

where we have set $\lambda_*(X)=\min\{\lambda_1(X),1/4\}$, and we have used the rough bound of Buser [10]:

$$\#\{\lambda_j<1/4\}\le 4g-3=1+\frac{\mathrm{Vol}(X)}{\pi}.$$

This estimate of $|\log\det(\Delta_X)|$ is consistent with the fact that $\det(\Delta_X)$ has exponential growth when X approaches certain boundary points of the (compactified) moduli space, a fact that was rigourously established by Wolpert in [38]. In particular the so-called "bosonic Polyakov integral" involving the determinant over the moduli space is indeed infinite, see [38].

Using the inequality for all $\beta > 0$ and all $a_j > 0$,

$$\left(\sum_{j=1}^{4} a_j\right)^{\beta} \leq 8^{\beta}\left(\sum_{j=1}^{4} a_j^{\beta}\right),$$

we end up with the estimate

$$\frac{|\log\det(\Delta_X)|^{\beta}}{\mathrm{Vol}(X)^{\beta}} \ll \left(1 + |\log(\lambda_*(X))|^{\beta} + \frac{1}{\ell_0(X)^{\beta}} + \mathrm{Vol}(X)^{-\beta}\left(\frac{\log \ell_0^{-1}(X)}{\ell_0(X)} N_X^0(1)\right)^{\beta}\right).$$

From Mirzakhani [23], Corollary 4.3, we know that

$$\int_{\mathscr{M}_g} \frac{1}{\ell_0(X)} dX \leq CV_g,$$

where $C > 0$ is independent of g, from which we can deduce easily that for all $0 < \alpha < 1$, we have

$$\int_{\mathscr{M}_g} \left(\frac{\log(\ell_0^{-1}(X))}{\ell_0(X)}\right)^{\alpha} dX \leq CV_g,$$

for some $C > 0$ *uniform in* g. We mention that the inverse of the systole is actually square integrable on moduli space [1], however it is unclear from [1] how this L^2 norm can depend on the genus. On the other hand, Cheeger's inequality says that

$$\lambda_1(X) \geq \frac{h_X^2}{4},$$

where h_X is the so-called Cheeger constant of X, which defined by an isoperimetric constant, see for example in [10], chapter 8. Again By Mirzakhani [23], Theorem 4.8, we know that for all $\alpha < 2$, we have

$$\int_{\mathscr{M}_g} \frac{1}{(h_X)^{\alpha}} dX \leq CV_g,$$

where $C > 0$ is again uniform with respect to g. In particular we deduce that for all $\beta > 0$,

$$\int_{\mathscr{M}_g} |\log(\lambda_*(X))|^{\beta} dX \leq CV_g.$$

Assuming that $\beta < 1$, we choose $p > 1$ such that $\beta < p\beta < 1$ and let q be such that $1/p + 1/q = 1$. By Hölder's inequality we get

$$\mathbb{E}_g\left(\left(\frac{\log \ell_0^{-1}(X)}{\ell_0(X)} N_X^0(1)\right)^{\beta}\right) \leq \left[\mathbb{E}_g\left(\left(\frac{\log \ell_0^{-1}(X)}{\ell_0(X)}\right)^{\beta p}\right)\right]^{1/p} \left[\mathbb{E}_g\left(N_X^0(1)^{\beta q}\right)\right]^{1/q},$$

which by the convergence of moments in Mirzakhani-Petri [25] and the above remarks is uniformly bounded as $g \to +\infty$. We therefore have shown that for all $\beta < 1$, there exists $C > 0$ independent of g such that

$$\int_{\mathscr{M}_g} \frac{|\log\det(\Delta_X)|^{\beta}}{\mathrm{Vol}(X)^{\beta}} dX \leq CV_g. \tag{5}$$

We now fix $\epsilon > 0$, and $0 < \beta < 1$. By Theorem 1.1, there exists a subset $\mathcal{A}_g(\epsilon) \subset \mathscr{M}_g$, with $\mathbb{P}(\mathcal{A}_g(\epsilon)) \to 1$ as $g \to +\infty$, such that for all $X \in \mathcal{A}_g(\epsilon)$,

$$(E - \epsilon)^{\beta} \leq \frac{|\log\det(\Delta_X)|^{\beta}}{\mathrm{Vol}(X)^{\beta}} \leq (E + \epsilon)^{\beta}.$$

Therefore we have

$$\frac{1}{V_g}\int_{\mathscr{M}_g} \frac{|\log\det(\Delta_X)|^{\beta}}{\mathrm{Vol}(X)^{\beta}} dX \leq (E + \epsilon)^{\beta} + \frac{1}{V_g}\int_{\mathcal{A}_g(\epsilon)^c} \frac{|\log\det(\Delta_X)|^{\beta}}{\mathrm{Vol}(X)^{\beta}} dX,$$

while

$$(E-\epsilon)^\beta \mathbb{P}(\mathcal{A}_g(\epsilon)) \leq \frac{1}{V_g}\int_{\mathscr{M}_g} \frac{|\log\det(\Delta_X)|^\beta}{\mathrm{Vol}(X)^\beta} dX.$$

We now apply Hölder's inequality and estimate (5). Since $0<\beta<1$, let $q>1$ be chosen such that $0<q\beta<1$ and let $0<p<\infty$ be such that $1/p+1/q=1$, we have

$$\frac{1}{V_g}\int_{\mathcal{A}_g(\epsilon)^c} \frac{|\log\det(\Delta_X)|^\beta}{\mathrm{Vol}(X)^\beta} dX \leq (\mathbb{P}(\mathcal{A}_g(\epsilon)^c))^{1/p}\left(\frac{1}{V_g}\int_{\mathscr{M}_g} \frac{|\log\det(\Delta_X)|^{q\beta}}{\mathrm{Vol}(X)^{q\beta}} dX\right)^{1/q}$$

$$\leq C\left(\mathbb{P}(\mathcal{A}_g(\epsilon)^c)\right)^{1/p}.$$

Since $\lim_{g\to+\infty}\mathbb{P}(\mathcal{A}_g(\epsilon)^c)=0$, we definitely have for all g large enough

$$(E-\epsilon)^\beta - \epsilon \leq \frac{1}{V_g}\int_{\mathscr{M}_g} \frac{|\log\det(\Delta_X)|^\beta}{\mathrm{Vol}(X)^\beta} dX \leq (E+\epsilon)^\beta + \epsilon,$$

and the proof is done for the Weil-Petterson case. We now deal with the random cover case. We recall that we have $\chi_0 = \mathbf{1}_{\{X_n \text{ connected}\}}$. In this model, the systole is bounded uniformly from below, so the a priori bound for $\log\det(\Delta_{X_n})$ whenever X_n is connected, is actually

$$\frac{|\log\det(\Delta_{X_n})|^\beta}{\mathrm{Vol}(X_n)^\beta} \leq A\left(1+|\log(\lambda_*(X))|^\beta\right),$$

for some constant $A>0$ independent of n. Using the same arguments as above based on Hölder's inequality, the result follows directly from the next fact.

Proposition 5.2. *Assume that X_n is connected, then we have for all n large,*

$$\lambda_1(X_n) \geq \frac{C_\Gamma}{n^{3/2}},$$

where C_Γ depends only on the base surface $X=\Gamma\backslash\mathbb{H}^2$. Consequently for all exponent β with $0<\beta$,

$$\mathbb{E}\left(|\log(\lambda_*(X))|^\beta \chi_0\right) \leq C,$$

where $C>0$ is uniform with respect to n.

We first need to prove a deterministic lower bound on $\lambda_1(X_n)$, provided that X_n is connected. By the induction formula, we know that the spectrum of Δ_{X_n} coincides, with multiplicity, with the spectrum of Δ_{ρ_n}, which is the Laplacian on the base surface twisted by the unitary representation ρ_n of Γ defined previously. If X_n is connected, then

$$\lambda_1(X_n) = \min\{\lambda_1(X), \lambda_0(\Delta_{\rho_n^0})\},$$

where ρ_n^0 is the representation given by

$$\rho_n^0(\gamma)U := U_{\phi_n^{-1}(\gamma)},$$

where $U\in V_n^0 := \{U\in\mathbb{C}^n \ : \ \sum_{j=1}^n U(j)=0\}$ and $U_{\phi_n^{-1}(\gamma)}(j) = U(\phi_n^{-1}(\gamma)(j))$. Notice that ρ_n^0 is just the orthogonal to the trivial representation in ρ_n. Let us fix a system of generators of Γ, denoted by S. A result of Sunada [36] then says that there exists C_S depending only on X and S such that

$$\lambda_0(\Delta_{\rho_n^0}) \geq C_S \inf_{\substack{U\in V_n^0\\ \|U\|=1}} \max_{\gamma\in S} \|\rho_n^0(\gamma)U - U\|.$$

Let us take $U\in V_n^0$ such that $\|U\|=1$. We therefore have

$$1 = \|U\|^2 \leq n\max_j(\mathrm{Re}(U(j))^2 + \mathrm{Im}(U(j))^2),$$

and we can assume without loss of generality that we have $\mathrm{Re}(U(j_0)) \geq \frac{1}{\sqrt{2n}}$ for some $j_0 \in [n]$. Because we have in addition

$$\sum_j \mathrm{Re}(U(j)) = 0,$$

there exists also $j_1 \in [n]$ such that $\mathrm{Re}(U(j_1)) \leq 0$. If X_n is connected, then Γ acts transitively on $[n]$ via ϕ_n and there exists $\gamma_0 \in \Gamma$ such that $\phi_n(\gamma_0)^{-1}(j_0) = j_1$. A simple induction argument shows that we can choose γ_0 with word length (with respect to S) less than $n-1$. We now have

$$\frac{1}{\sqrt{2n}} \leq |\mathrm{Re}(U(j_0)) - \mathrm{Re}(U(j_1))| \leq \|\rho_n^0(\gamma_0)U - U\|.$$

Writing

$$\gamma_0 = g_1 g_2 \dots g_m,$$

with $g_j \in S$ and $m \leq n-1$, we have therefore

$$\|\rho_n^0(\gamma_0)U - U\| \leq \sum_{j=1}^m \|\rho_n^0(g_j)U - U\| \leq (n-1) \max_{g \in S} \|\rho_n^0(g)U - U\|,$$

which yields

$$\max_{g \in S} \|\rho_n^0(g)U - U\| \geq \frac{1}{(n-1)\sqrt{2n}}.$$

The first claim of the proposition is proved. Alternatively, one can use directly a result of Brooks [7] which relates $\lambda_1(X_n)$ to the Cheeger constants of Schreier graphs of the covers (with a choice of generators of Γ) to obtain a similar lower bound $\lambda_1(X_n) \geq C_\Gamma n^{-2}$ which is slightly worse but good enough for our purpose.

From the proof of the uniform spectral gap in [20] one can directly derive that for all $r > 0$, for all $\epsilon > 0$, as $n \to \infty$,

$$\mathbb{P}(\lambda_1(X_n) \leq r) \leq \frac{C_\epsilon}{n^{4\sqrt{(1/4-r)}-1-\epsilon}}.$$

From that we deduce that for all $\alpha < 1$ we can find $r_\alpha > 0$ such that

$$\mathbb{P}(\lambda_1(X_n) \leq r_\alpha) \leq \frac{C_\alpha}{n^\alpha}.$$

We now fix any $\beta > 0$ and fix some $0 < \alpha < 1$, we can use the fact that we have (by the lower bound on $\lambda_*(X)$)

$$|\log(\lambda_*(X))|^\beta = O((\log(n))^\beta),$$

where the implied constant is uniform with respect to n. We therefore get as $n \to \infty$,

$$\mathbb{E}(|\log(\lambda_*(X))|^\beta \chi_0) \leq |\log(r_\alpha)|^\beta (1 + O(n^{-\alpha})) + O\left(n^{-\alpha} |\log(n)|^\beta\right) = O(1)$$

and the proof is done. We conclude with some comments.

- It would be interesting to know if a similar type of result can be proved for the Brooks-Makover model, in particular can one show that

$$\mathbb{E}\left(\frac{1}{\ell_0(S_n^C)}\right)$$

is finite and uniformly bounded with respect to n ? This will require an effective version of the compactification procedure, see the paper of Mangoubi [22].

- It is likely that our results can be extended to finite area surfaces, see Efrat [14] for the definition and properties of determinants in this context, or even geometrically finite surfaces where a notion of determinant also holds, see [5].

References

[1] Francisco Arana-Herrera and Jayadev S. Athreya. Square-integrability of the Mirzakhani function and statistics of simple closed geodesics on hyperbolic surfaces. *Forum Math. Sigma*, 8:Paper No. e9, 33, 2020.

[2] Nicolas Bergeron, Wolfgang Lück, and Roman Sauer. The asymptotic growth of twisted torsion. *Unpublished notes*.

[3] Nicolas Bergeron and Akshay Venkatesh. The asymptotic growth of torsion homology for arithmetic groups. *J. Inst. Math. Jussieu*, 12(2):391–447, 2013.

[4] Béla Bollobás. *Random graphs*, volume 73 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2001.

[5] David Borthwick, Chris Judge and Peter Perry. Determinants of Laplacians and isopolar metrics on surfaces of infinite area. *Duke Math. J.*, 118 (2003), no. 1, 61-102.

[6] J. Bolte and F. Steiner. Determinants of Laplace-like operators on Riemann surfaces. *Comm. Math. Phys.*, 130(3):581–597, 1990.

[7] Robert Brooks. The spectral geometry of a tower of coverings. *J. Differential Geom.* , 23 (1986) no. 1, 97-107.

[8] Robert Brooks and Eran Makover. Belyi surfaces. In *Entire functions in modern analysis (Tel-Aviv, 1997)*, volume 15 of *Israel Math. Conf. Proc.*, pages 37–46. Bar-Ilan Univ., Ramat Gan, 2001.

[9] Robert Brooks and Eran Makover. Random construction of Riemann surfaces. *J. Differential Geom.*, 68(1):121–157, 2004.

[10] Peter Buser. *Geometry and spectra of compact Riemann surfaces*. Modern Birkhäuser Classics. Birkhäuser Boston, Ltd., Boston, MA, 2010. Reprint of the 1992 edition.

[11] Jeff Cheeger. Analytic torsion and the heat equation. *Ann. of Math. (2)*, 109(2):259–322, 1979.

[12] Eric D'Hoker and D. H. Phong. On determinants of Laplacians on Riemann surfaces. *Comm. Math. Phys.*, 104(4):537–545, 1986.

[13] Eric D'Hoker and D. H. Phong. The geometry of string perturbation theory. *Rev. Modern Phys.*, 60(4):917–1065, 1988.

[14] Isaac Efrat. Determinants of Laplacians on surfaces of finite volume. *Comm. Math. Phys.* 119 (1988), no. 3,4434-451.

[15] Ksenia Fedosova and Anke Pohl. Meromorphic continuation of Selberg zeta functions with twists having non-expanding cusp monodromy. *Selecta Math. (N.S.)* 26 (2020), no. 1, Paper No. 9, 55 pp.

[16] Dennis A. Hejhal. *The Selberg trace formula for* $\mathrm{PSL}(2, R)$. *Vol. I.* Lecture Notes in Mathematics, Vol. 548. Springer-Verlag, Berlin-New York, 1976.

[17] Dmitry Jakobson and Frédéric Naud. On the critical line of convex co-compact hyperbolic surfaces. *Geom. Funct. Anal.*, 22(2):352–368, 2012.

[18] Mikhail G. Katz, Mary Schaps, and Uzi Vishne. Logarithmic growth of systole of arithmetic Riemann surfaces along congruence subgroups. *J. Differential Geom.*, 76(3):399–422, 2007.

[19] Martin W. Liebeck and Aner Shalev. Fuchsian groups, coverings of Riemann surfaces, subgroup growth, random quotients and random walks. *J. Algebra*, 276(2):552–601, 2004.

[20] Michael Magee, Frédéric Naud, and Doron Puder. A random cover of a compact hyperbolic surface has relative spectral gap $\frac{3}{16} - \epsilon$. *Geom. Funct. Anal.*, 32 (2022), no. 3, 595–661

[21] Michael Magee and Doron Puder. The asymptotic statistics of random covering surfaces. *Preprint*, 2020.

[22] Dan Mangoubi. Conformal extension of metrics of negative curvature. *J. Anal. Math.*, 91:193–209, 2003.

[23] Maryam Mirzakhani. Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces. *Invent. Math.*, 167(1):179–222, 2007.

[24] Maryam Mirzakhani. Growth of Weil-Petersson volumes and random hyperbolic surfaces of large genus. *J. Differential Geom.*, 94(2):267–300, 2013.

[25] Maryam Mirzakhani and Bram Petri. Lengths of closed geodesics on random surfaces of large genus. *Comment. Math. Helv.*, 94(4):869–889, 2019.

[26] Laura Monk. Benjamini-Schramm convergence and spectra of random hyperbolic surfaces of high genus. *Anal. PDE*, 15(3):727–752, 2022.

[27] Werner Müller. Analytic torsion and R-torsion of Riemannian manifolds. *Adv. in Math.*, 28(3):233–305, 1978.
[28] Frédéric Naud. Random covers of compact surfaces and smooth linear spectral statistics. *Preprint*, 2022.
[29] Bram Petri. Random regular graphs and the systole of a random surface. *J. Topol.*, 10(1):211–267, 2017.
[30] Bram Petri and Alexander Walker. Graphs of large girth and surfaces of large systole. *Math. Res. Lett.*, 25(6):1937–1956, 2018.
[31] Mark Pollicott and André C. Rocha. A remarkable formula for the determinant of the Laplacian. *Invent. Math.*, 130(2):399–414, 1997.
[32] A. M. Polyakov. Quantum geometry of bosonic strings. *Phys. Lett. B*, 103(3):207–210, 1981.
[33] Peter Sarnak. Determinants of Laplacians. *Comm. Math. Phys.*, 110(1):113–120, 1987.
[34] Peter Sarnak and Xiao Xi Xue. Bounds for multiplicities of automorphic representations. *Duke Math. J.*, 64(1):207–227, 1991.
[35] Alexander Strohmaier. Computation of eigenvalues, spectral zeta functions and zeta-determinants on hyperbolic surfaces. In *Geometric and computational spectral theory*, volume 700 of *Contemp. Math.*, pages 177–205. Amer. Math. Soc., Providence, RI, 2017.
[36] Toshikazu Sunada. Unitary representations of fundamental groups and the spectrum of twisted Laplacians. *Topology*, 28(2):125–132, 1989.
[37] Alexei B. Venkov. *Spectral theory of automorphic functions and its applications*, volume 51 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1990. Translated from the Russian by N. B. Lebedinskaya.
[38] Scott A. Wolpert. Asymptotics of the spectrum and the Selberg zeta function on the space of Riemann surfaces. *Comm. Math. Phys.*, 112(2):283–315, 1987.
[39] Alex Wright and Michael Lipnowski. Towards optimal spectral gap in large genus. *Preprint*, 2021.
[40] Yunhui Wu and Yuhao Xue. Random hyperbolic surfaces of large genus have first eigenvalues greater than $\frac{3}{16} - \epsilon$. *Geom. Funct. Anal.*, 32(2):340–410, 2022.

Frédéric Naud, Institut Mathématique de Jussieu, Université Pierre et Marie Curie, 4 place Jussieu, 75252 Paris Cedex 05, France.

Email address: `frederic.naud@imj-prg.fr`