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Abstract

We show the stabilization by a finite number of controllers of a fluid-structure interaction system where the fluid is modeled by the Navier-Stokes system into a periodical canal and where the structure is an elastic wall localized on top of the fluid domain. The elastic deformation of the structure follows a damped beam equation. We also assume that the fluid can slip on its boundaries and we model this by using the Navier slip boundary conditions. Our result states the local exponential stabilization around a stationary state of strong solutions by using dynamical controllers in order to handle the compatibility conditions at initial time. The proof is based on a change of variables to write the fluid-structure interaction system in a fixed domain and on the stabilization of the linearization of the corresponding system around the stationary state. One of the main difficulties consists in handling the nonlinear terms coming from the change of variables in the boundary conditions.

Keywords: Navier-Stokes system, damped beam equation, strong solutions, stabilization, Navier slip boundary conditions
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1 Introduction

We consider the stabilization of a fluid-structure interaction system coupling the Navier-Stokes equations with a damped beam equation. Such a system has been introduced in [46] as a first model for the blood flow into arteries. The corresponding system has been studied by many authors, see at the end of this section for some references on the subject. In most of these works, the fluid is assumed to satisfy a no-slip boundary condition that can be modeled by the Dirichlet boundary condition. Nevertheless several works in fluid-structure interaction systems suggest to modify this hypothesis (see, for instance, [21, 22, 27, 28, 29, 30]). Here we assume that the fluid can slip tangentially on its boundaries and that the fluid satisfies the Navier slip boundary conditions. These conditions were introduced in [42] by Navier in 1823. Our aim here is to stabilize the corresponding fluid-structure interaction system by using boundary controls localized on the boundary of fluid, on a non-moving part.

To simplify the presentation, we assume that the fluid motion is bi-dimensional so that the elastic motion at its boundary is one-dimensional. We also focus in a particular geometry where all the quantities are periodic in the horizontal direction. We have considered that all the physical parameters are equal to 1 (viscosity and density of the fluid, elasticity parameters of the plate, friction coefficients in the Navier slip boundary conditions, etc.) More precisely, let us present the geometry of our problem: we assume $L > 0$ and we denote by $\mathcal{I} := \mathbb{R}/L\mathbb{Z}$ the corresponding torus. Then, for any function $\eta : \mathcal{I} \to (-1, \infty)$ (that corresponds to an elastic deformation), we define the corresponding fluid domain

$$\Omega_\eta := \{(x_1, x_2) \in \mathcal{I} \times \mathbb{R} \ : \ 0 < x_2 < 1 + \eta(x_1)\},$$

whose boundary can be split into a fixed bottom and an upper deformable part, that is $\partial\Omega_\eta = \Gamma_\eta \cup \Gamma_{\rm b}$ where

$$\Gamma_\eta := \{(x_1, x_2) \in \mathcal{I} \times \mathbb{R} \ : \ x_2 = 1 + \eta(x_1)\}, \quad \Gamma_{\rm b} := \mathcal{I} \times \{0\}.$$

The equations of motion of the fluid-structure interaction system we consider write as follows:

$$\left\{\begin{array}{rl} \partial_t V + (V \cdot \nabla)V - \operatorname{div} \mathbb{T}(V, P) = f^{\rm S} & t > 0, \ x \in \Omega_{\eta(t)}, \\ \operatorname{div} V = 0 & t > 0, \ x \in \Omega_{\eta(t)}, \\ \partial_t^2 \eta + \partial_{x_1}^4 \eta - \partial_t \partial_{x_1}^2 \eta = \widetilde{\mathbb{H}}_\eta(V, P) + h^{\rm S} & t > 0, \ x_1 \in \mathcal{I}, \end{array}\right. \tag{1.1}$$

$$\left\{\begin{array}{rl} [V - U_{\rm cont}]_\nu = 0 & t > 0, \ x \in \Gamma_{\rm b}, \\ [2\mathbb{D}(V)\nu + (V - U_{\rm cont})]_\tau = 0 & t > 0, \ x \in \Gamma_{\rm b}, \\ [V - \partial_t \eta e_2]_\nu = 0 & t > 0, \ x \in \Gamma_{\eta(t)}, \\ [2\mathbb{D}(V)\nu + (V - \partial_t \eta e_2)]_\tau = 0 & t > 0, \ x \in \Gamma_{\eta(t)}, \end{array}\right. \tag{1.2}$$

$$\eta(0, \cdot) = \eta_1^0 \text{ in } \mathcal{I}, \quad \partial_t \eta(0, \cdot) = \eta_2^0 \text{ in } \mathcal{I}, \quad V(0, \cdot) = V^0 \text{ in } \Omega_{\eta^0}. \tag{1.3}$$

In the above system, the Cauchy stress tensor $\mathbb{T}$ is given by

$$\mathbb{T}(V, P) = 2\mathbb{D}(V) - PI_2, \quad \mathbb{D}(V)_{i,j} = \frac{1}{2}\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right),$$

so that

$$\operatorname{div} \mathbb{T}(V, P) = \Delta V - \nabla P.$$

The force exerted by the fluid on the structure is

$$\widetilde{\mathbb{H}}_\eta(V, P) = -\sqrt{1 + |\partial_{x_1}\eta|^2}\,(\mathbb{T}(V, P)\nu \cdot e_2),$$

where $\tau = \tau_\eta$ and $\nu = \nu_\eta$ are unit vectors for the tangent and exterior normal vectors on $\partial\Omega$:

$$\tau_\eta := \frac{1}{\sqrt{1 + |\partial_{x_1}\eta|^2}} \begin{bmatrix} 1 \\ \partial_{x_1}\eta \end{bmatrix}, \quad \nu_\eta := \frac{1}{\sqrt{1 + |\partial_{x_1}\eta|^2}} \begin{bmatrix} -\partial_{x_1}\eta \\ 1 \end{bmatrix} \quad \text{on } \Gamma_\eta, \tag{1.4}$$

$$\tau := -e_1, \quad \nu := -e_2 \quad \text{on } \Gamma_{\rm b}.$$

The equations in (1.2) correspond to the Navier slip boundary conditions and in order to write them, we have used the normal and tangential components of a vector $w \in \mathbb{R}^2$:

$$w_\nu := w \cdot \nu, \quad w_\tau := w \cdot \tau \quad \text{on } \partial\Omega_\eta.$$

The system (1.1)–(1.3) contains a boundary control $U_{\rm cont}$ localized on a non empty open subset $\Gamma_{\rm cont}$ of $\Gamma_{\rm b}$. Our aim is to write $U_{\rm cont}$ as a feedback control such that $(V, \eta, \partial_t\eta)$ converges for $t \to \infty$ to a stationary state $(v^{\rm S}, \eta^{\rm S}, 0)$, solution of the system

$$\left\{\begin{array}{rl} \left(v^{\rm S}\cdot\nabla\right)v^{\rm S} - \operatorname{div}\mathbb{T}(v^{\rm S},p^{\rm S}) = f^{\rm S} & \text{in } \Omega_{\eta^{\rm S}},\\ \operatorname{div} v^{\rm S} = 0 & \text{in } \Omega_{\eta^{\rm S}},\\ \partial_{x_1}^4\eta^{\rm S} = \widetilde{\mathbb{H}}_{\eta^{\rm S}}(v^{\rm S},p^{\rm S}) + h^{\rm S} & \text{in } \mathcal{I},\\ v_\nu^{\rm S} = 0 \quad \text{and} \quad \left[2\mathbb{D}(v^{\rm S})\nu + v^{\rm S}\right]_\tau = 0 & \text{on } \partial\Omega_{\eta^{\rm S}}. \end{array}\right. \tag{1.5}$$

Here $v^{\rm S}$, $p^{\rm S}$, $\eta^{\rm S}$, $f^{\rm S}$ and $h^{\rm S}$ are time independent functions. Note that in the above system, ν and τ stand for the normal and the tangential vectors on $\partial\Omega_{\eta^{\rm S}}$, i.e: $\nu = \nu_{\eta^{\rm S}}$, $\tau = \tau_{\eta^{\rm S}}$ on $\partial\Omega_{\eta^{\rm S}}$ (see (1.4)) and can be different from the normal and the tangential vectors in (1.2). If needed, we will make explicit this dependence by keeping the indices.

To be more precise in our objective, our feedback control is finite-dimensional: we are going to show that for a chosen exponential decay rate $-\sigma < 0$, there exist $N_\sigma \in \mathbb{N}$ and a family

$$\left(w^{(j)}\right)_{j\in\{1,\dots,N_\sigma\}} \subset L^2(\Gamma_{\rm b})^2, \quad \operatorname{supp} w^{(j)} \subset \Gamma_{\rm cont}, \quad \int_{\Gamma_{\rm b}} w^{(j)}\cdot\nu\, dx_1 = 0 \quad (j \in \{1,\dots,N_\sigma\}), \tag{1.6}$$

and such that we can stabilize (1.1)–(1.3) with a control of the form

$$U_{\rm cont} = \mathbb{M}u := \sum_{j=1}^{N_\sigma} u_j w^{(j)}. \tag{1.7}$$

Before stating our main result, let us point out some remarks and introduce some notation. First, if there is no control in (1.1)–(1.3), the incompressibility condition and the boundary conditions imply

$$\frac{d}{dt}\int_{\mathcal{I}} \eta(t,x_1)\, dx_1 = 0.$$

In particular, assuming that $\eta_1^0 \in L_0^2(\mathcal{I})$, where

$$L_0^2(\mathcal{I}) := \left\{\xi \in L^2(\mathcal{I}) \ : \ \int_{\mathcal{I}} \xi(x_1)\, dx_1 = 0\right\},$$

we deduce that $\eta(t,\cdot) \in L_0^2(\mathcal{I})$ for all $t \geqslant 0$. We denote by $\operatorname{proj}_{L_0^2} : L^2(\mathcal{I}) \to L_0^2(\mathcal{I})$ the orthogonal projection and we set

$$\mathbb{H}_\eta(V,P) = -\operatorname{proj}_{L_0^2}\sqrt{1+|\partial_{x_1}\eta|^2}\,(\mathbb{T}(V,P)\nu\cdot e_2)\,.$$

We also introduce the operators $(A_1, \mathcal{D}(A_1))$ and $(A_2, \mathcal{D}(A_2))$ on $L_0^2(\mathcal{I})$ defined by

$$\mathcal{D}(A_1) = H^4(\mathcal{I}) \cap L_0^2(\mathcal{I}), \quad A_1\eta = \partial_{x_1}^4\eta, \quad \mathcal{D}(A_2) = H^2(\mathcal{I}) \cap L_0^2(\mathcal{I}), \quad A_2\eta = -\partial_{x_1}^2\eta. \tag{1.8}$$

Then we can replace the beam equation in (1.1) and in (1.5) by its projection on $L_0^2(\mathcal{I})$:

$$\partial_{tt}\eta + A_1\eta + A_2\partial_t\eta = \mathbb{H}_\eta(V,P) + \operatorname{proj}_{L_0^2} h^{\rm S}, \quad A_1\eta^{\rm S} = \mathbb{H}_{\eta^{\rm S}}(v^{\rm S},p^{\rm S}) + \operatorname{proj}_{L_0^2} h^{\rm S}.$$

In that case, the pressures P and p^{S} are determined up to a constant, and we can determine these constants by using the projections of the beam equations on $L^2_0(\mathcal{I})^\perp$.

A standard difficulty in the study of fluid-structure interaction system comes from the fact that the fluid velocity and pressure (V, P) are written in a moving domain $\Omega_{\eta(t)}$ and since we want to compare them with the stationary state $(v^{\mathrm{S}}, p^{\mathrm{S}})$ written in $\Omega_{\eta^{\mathrm{S}}}$, we need to use a change of variables. We thus set

$$\Omega := \Omega_{\eta^{\mathrm{S}}}, \quad \Gamma := \Gamma_{\eta^{\mathrm{S}}},$$

and we consider the following diffeomorphisms

$$X(t,\cdot) : \Omega \longrightarrow \Omega_{\eta(t)}, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \longmapsto \begin{bmatrix} y_1 \\ \dfrac{1+\eta(t,y_1)}{1+\eta^{\mathrm{S}}(y_1)} y_2 \end{bmatrix}, \tag{1.9}$$

$$Y(t,\cdot) : \Omega_{\eta(t)} \longrightarrow \Omega, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto \begin{bmatrix} x_1 \\ \dfrac{1+\eta^{\mathrm{S}}(x_1)}{1+\eta(t,x_1)} x_2 \end{bmatrix}. \tag{1.10}$$

Then, we can transform (V, P) into

$$\widetilde{v}(t,y) := (\mathrm{Cof}\, \nabla X(t,y))^* V(t, X(t,y)), \quad \widetilde{p}(t,y) := P(t, X(t,y)) \quad (t \geqslant 0,\ y \in \Omega). \tag{1.11}$$

We also transform the initial condition for the fluid velocity:

$$\widetilde{v}^0 := (\mathrm{Cof}\, \nabla X^0)^*\ V^0 \circ X^0, \quad \text{where } X^0 := X(0,\cdot).$$

Then, applying the change of variables on (1.1)–(1.3) (see Section 2.1 for the details and the definitions of the operators a, b, $\mathbf{D}$ and $\mathbf{N}$), we obtain a system of the form

$$\begin{aligned} (\nabla X)\partial_t \widetilde{v} + (\nabla X)(\nabla \widetilde{v})\partial_t Y(X) + \det(\nabla X)\partial_t a(X)\widetilde{v} \\ - \operatorname{div}(\mathbf{D}\widetilde{v}) + b\nabla \widetilde{p} + \mathbf{N}(\widetilde{v}) = (\det \nabla X) f^{\mathrm{S}}(X) \quad \text{in } (0,\infty)\times\Omega, \end{aligned} \tag{1.12}$$

$$\begin{aligned} \operatorname{div} \widetilde{v} = 0 \quad &\text{in } (0,\infty)\times\Omega, \\ \partial_t^2 \eta + \partial_{x_1}^4 \eta - \partial_t \partial_{x_1}^2 \eta = -\mathcal{T}^* \left(\mathbf{D}(\widetilde{v})_{|\Gamma} \nu_{\eta^{\mathrm{S}}} - \widetilde{p}_{|\Gamma} \nu_{\eta^{\mathrm{S}}} \right) + \mathrm{proj}_{L^2_0} h^{\mathrm{S}} \quad &\text{in } (0,\infty)\times\mathcal{I}, \end{aligned} \tag{1.13}$$

$$\begin{cases} [\widetilde{v} - \mathcal{T}\partial_t \eta]_\nu = (\mathbb{M}u)_\nu \quad \text{on } (0,\infty)\times\partial\Omega, \\ \mathbf{D}(\widetilde{v})_{1,2} - \dfrac{1+\eta^{\mathrm{S}}}{1+\eta}\widetilde{v}_1 + (\mathbb{M}u)_1 = 0 \quad \text{on} \quad (0,\infty)\times\Gamma_{\mathrm{b}}, \\ \mathbf{D}(\widetilde{v})\nu_{\eta^{\mathrm{S}}} \cdot \tau_\eta(X) \sqrt{\dfrac{1+|\partial_{y_1}\eta^{\mathrm{S}}|^2}{1+|\partial_{y_1}\eta|^2}} + a(X)\widetilde{v}\cdot\tau_\eta(X) - \partial_t \eta e_2 \cdot \tau_\eta(X) = 0 \quad \text{on} \quad (0,\infty)\times\Gamma, \end{cases} \tag{1.14}$$

$$\eta(0,\cdot) = \eta_1^0 \text{ in } \mathcal{I}, \quad \partial_t\eta(0,\cdot) = \eta_2^0 \text{ in } \mathcal{I}, \quad \widetilde{v}(0,\cdot) = \widetilde{v}^0 \text{ in } \Omega. \tag{1.15}$$

In order to write the boundary conditions, we have used the operator $\mathcal{T} \in \mathcal{L}(L^2_0(\mathcal{I}), L^2(\partial\Omega)^2)$ defined by

$$(\mathcal{T}\xi)(x) = \begin{cases} 0 & \text{if } x \in \Gamma_{\mathrm{b}}, \\ \xi(x_1)e_2 & \text{if } (x_1, 1+\eta^{\mathrm{S}}(x_1)) \in \Gamma. \end{cases} \tag{1.16}$$

We can check the adjoint of $\mathcal{T}$ satisfies for $\zeta \in L^2(\partial\Omega)^2$,

$$(\mathcal{T}^*\zeta)(x_1) = \mathrm{proj}_{L^2_0} \sqrt{1+|\partial_{x_1}\eta^{\mathrm{S}}(x_1)|^2}\zeta(x_1, 1+\eta^{\mathrm{S}}(x_1))\cdot e_2 \quad (x_1 \in \mathcal{I}). \tag{1.17}$$

Now, $(\widetilde{v}, \widetilde{p})$ and $(v^{\mathrm{S}}, p^{\mathrm{S}})$ are defined on the same spatial domain Ω and we can consider their differences:

$$v := \widetilde{v} - v^{\mathrm{S}}, \quad p := \widetilde{p} - p^{\mathrm{S}}, \quad \xi = \eta - \eta^{\mathrm{S}}, \tag{1.18}$$

and the differences of the initial conditions:

$$\xi_1^0 := \eta_1^0 - \eta^{\mathrm{S}}, \quad \xi_2^0 := \eta_2^0, \quad v^0 := \widetilde{v}^0 - v^{\mathrm{S}}. \tag{1.19}$$

Then subtracting (1.5) to (1.12)–(1.15) yield the following system where we separate the linear part, that is the operators $\mathbf{L}^1$, $\mathbf{L}^2$, $\mathbf{L}^3$, $\mathbf{L}^4$, and the nonlinear parts, that is $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{H}$ (see Section 2.3):

$$\left\{\begin{array}{rl} \partial_t v - \nabla\cdot\mathbb{T}(v,p) + (v^{\mathrm{S}}\cdot\nabla)v + (v\cdot\nabla)v^{\mathrm{S}} - \nabla\cdot\mathbf{L}^1(\xi) + \mathbf{L}^2(\xi) + \mathbf{L}^3(\partial_t\xi) = \mathbf{F}(v,p,\xi) & \text{in } (0,\infty)\times\Omega, \\ \nabla\cdot v = 0 & \text{in } (0,\infty)\times\Omega, \\ \partial_{tt}\xi + A_1\xi + A_2\partial_t\xi + \mathcal{T}^*(\mathbf{L}^1(\xi)\nu) = -\mathcal{T}^*(\mathbb{T}(v,p)\nu) + \mathbf{H}(v,\xi) & \text{in } (0,\infty)\times\mathcal{I}, \end{array}\right. \tag{1.20}$$

$$\left\{\begin{array}{rl} [v - \mathcal{T}\partial_t\xi]_\nu = (\mathbb{M}u)_\nu & \text{on } (0,\infty)\times\partial\Omega, \\ \left[2\mathbb{D}(v)\nu + (v - \mathcal{T}\partial_t\xi) + \mathbf{L}^1(\xi)\nu + \mathbf{L}^4(\xi)\right]_\tau = (\mathbb{M}u)_\tau + \mathbf{G}(v,\xi) & \text{on } (0,\infty)\times\partial\Omega, \end{array}\right. \tag{1.21}$$

$$\xi(0,\cdot) = \xi_1^0 \text{ in } \mathcal{I}, \quad \partial_t\xi(0,\cdot) = \xi_2^0 \text{ in } \mathcal{I}, \quad v(0,\cdot) = v^0 \text{ in } \Omega. \tag{1.22}$$

Let us introduce some notation for the functional framework. For a Banach space $\mathfrak{X}$ and $\sigma > 0$, we define for $p \in [1,+\infty]$ the space

$$L^p_\sigma(\mathfrak{X}) := \{f \in L^p(0,\infty;\mathfrak{X}) \ : \ t \mapsto e^{\sigma t} f(t) \in L^p(0,\infty;\mathfrak{X})\}, \quad \|f\|_{L^p_\sigma(\mathfrak{X})} := \left\|t \mapsto e^{\sigma t} f(t)\right\|_{L^p(0,\infty;\mathfrak{X})}. \tag{1.23}$$

We define similarly $H^s_\sigma(\mathfrak{X})$ for $s \in \mathbb{R}$. We also define

$$\mathcal{H} := \left\{(v,\xi_1,\xi_2) \in [L^2(\Omega)]^2 \times \mathcal{D}(A_1^{1/2}) \times L^2_0(\mathcal{I}) \ : \ \nabla\cdot v = 0 \text{ in } \Omega, \ [v - \mathcal{T}\xi_2]_\nu = 0 \text{ on } \partial\Omega\right\}, \tag{1.24}$$

$$\mathcal{V} := \mathcal{H} \cap \left(\left[H^1(\Omega)\right]^2 \times \mathcal{D}\left(A_1^{3/4}\right) \times \mathcal{D}\left(A_1^{1/4}\right)\right), \tag{1.25}$$

with their canonical inner product.

In this article, we are interested by the stabilization of strong solutions. In this context, we are going to consider initial conditions regular enough, in particular $v^0 \in H^1(\Omega)^2$ and this leads to compatibility conditions between u and v^0 that are not always satisfied for a general feedback control. This difficulty already appears for the Navier-Stokes system without any structure, see [47] or [2]. In order to deal with this difficulty, we use the strategy considered in [4] (see also [5]) by considering dynamical controllers. This means that u is obtained as a solution of a differential equation and that the feedback operators act on this additional equation. More precisely, we will show the existence of

$$\mathcal{R}_\sigma^{(1)} \in \mathcal{L}(\mathcal{H},\mathbb{R}^{N_\sigma}), \quad \mathcal{R}_\sigma^{(2)} \in \mathcal{L}(\mathbb{R}^{N_\sigma}), \tag{1.26}$$

such that (1.20)–(1.22) combined with

$$u' = \mathcal{R}_\sigma^{(1)}(v,\xi,\partial_t\xi) + \mathcal{R}_\sigma^{(2)} u \quad \text{in } (0,\infty), \quad u(0) = 0, \tag{1.27}$$

is exponentially stable. This is the statement of the main result of this article.

Theorem 1.1. *Assume*

$$(v^{\mathrm{S}},p^{\mathrm{S}},\eta^{\mathrm{S}}) \in [W^{2,\infty}(\Omega)]^2 \times W^{1,\infty}(\Omega)/\mathbb{R} \times W^{3,\infty}(\mathcal{I}), \quad 1 + \eta^{\mathrm{S}} > 0, \quad f^{\mathrm{S}} \in [W^{3,\infty}(\Omega)]^2, \tag{1.28}$$

such that the system (1.5) *is verified. Then, there exist* $N_\sigma \in \mathbb{N}^*$, *a family* $\left(w^{(j)}\right)_{j\in\{1,\dots,N_\sigma\}}$ *verifying* (1.6), $\mathcal{R}_\sigma^{(1)}$ *and* $\mathcal{R}_\sigma^{(2)}$ *satisfying* (1.26) *and* $C_0 > 0$ *such that if*

$$(v^0,\xi_1^0,\xi_2^0) \in \mathcal{V}, \quad \left\|(v^0,\xi_1^0,\xi_2^0)\right\|_{\mathcal{V}} \leqslant C_0, \tag{1.29}$$

then there exists a unique strong solution

$$v \in L^2_\sigma(H^2(\Omega)) \cap H^1_\sigma(L^2(\Omega)), \quad p \in L^2_\sigma(H^1(\Omega)/\mathbb{R}), \quad \eta \in L^2_\sigma(\mathcal{D}(A_1)) \cap H^2_\sigma(L^2_0(\mathcal{I})), \quad u \in H^1_\sigma(\mathbb{R}^{N_\sigma}),$$

of (1.20)–(1.22) *together with* (1.27). *Moreover, there exists a constant* $C > 0$ *such that*

$$\|(v(t),\xi(t),\partial_t\xi(t))\|_{\mathcal{V}} \leqslant Ce^{-\sigma t}\left\|(v^0,\xi_1^0,\xi_2^0)\right\|_{\mathcal{V}} \quad (t > 0).$$

Note that systems similar to (1.1)–(1.3) have already been studied by several authors: [18] and [17] (existence of weak solutions), [11], [33], [24], [25], [37] and [16] (existence of strong solutions). Without the damping term $-\partial_t\partial_{x_1}^2\eta$, the existence of strong solutions is more difficult to achieve. It is obtained with an additional term of inertia of rotation ($-\partial_t^2\partial_{y_1}^2\eta$) in [25]. Without any additional term, the corresponding fluid-structure system is studied in [7] and [8], where the existence and uniqueness of strong solutions is done by using Gevrey semigroups. In the framework of weak solutions, there have been some works studying the existence of weak solutions for a beam equation without dissipation (or a similar structure equation): [23], [54], etc. We can also mention some works for different similar models: [9] (linear wave equation), [32, 31] (linear elastic Koiter shell), [39] (dynamic pressure boundary conditions), [40, 41, 38] (nonlinear elastic Koiter shell models), [52] and [53] (nonlinear elastic and thermoelastic plate equations), [35], [36] (compressible fluids), etc. There are only few works where the no-slip boundary conditions are not assumed: [16] and [15] where the author uses boundary conditions involving the pressure and [19] where we show the existence and the uniqueness of strong solutions for the same system as here. Note also that in the case where the structures are rigid bodies, some works are dealing with the Navier slip boundary conditions: [21], [13], [43], [45], etc.

In the context of control theory, there are few results for this kind of system: the first result of stabilization was obtained in [48] for strong solutions and with a distributed control acting on the structure. Then in [6], the authors show the boundary stabilization of weak solutions around a stationary state. Both previous results are obtained with the no-slip boundary conditions. With respect to these previous works, we are considering here the Navier boundary conditions and we are stabilizing strong solutions around a stationary state. As explained above, one of the difficulties corresponds to the fact that for such a problem there are compatibility conditions between the initial state and the boundary feedback control. We overcome this issue by using dynamical controls. Another difficulty comes from the transformation of the Navier slip boundary conditions: we need to estimate the nonlinear term $\mathbf{G}$ in $H^{1/4}_\sigma(L^2(\partial\Omega)^2)\cap L^2_\sigma(H^{1/2}(\partial\Omega)^2)$. In the case of the Navier-Stokes system, such a term can be estimated by using an adequate lifting. This is done for instance in [3] in order to deal with nonlinear Neumann boundary conditions. Nevertheless with the method used in [3], one need that $\mathbf{G}$ in $H^{1/4+\varepsilon/2}_\sigma(L^2(\partial\Omega)^2)\cap L^2_\sigma(H^{1/2+\varepsilon}(\partial\Omega)^2)$ for $\varepsilon>0$ arbitrary small. Here our nonlinear term does not satisfy this assumption and we thus adapt a strategy proposed by [50], see Proposition 3.8 below.

The outline of the article is as follows: in the next section, we apply the change of variables on (1.1)–(1.3) and obtain, after the separation between linear and nonlinear terms, a system of the form on (1.20)–(1.22). We show in Section 3 that the corresponding linearized system is exponentially stabilizable with a finite number of dynamical controllers. Then using this result and the estimates of the nonlinear terms obtained in Section 2, we can prove Theorem 1.1 in Section 4.

2 Change of variables

2.1 Transformation of the system

We set

$$a=(\mathrm{Cof}(\nabla Y))^*,\quad b=\mathrm{Cof}(\nabla X),\tag{2.1}$$

where $\mathrm{Cof}(M)$ stands for the cofactor matrix of the matrix M. We also define

$$[\mathbf{D}\widetilde{v}]_{i,j}=\sum_{m,k}\left(\frac{\partial a_{i,k}}{\partial x_m}(X)+\frac{\partial a_{m,k}}{\partial x_i}(X)\right)\widetilde{v}_k b_{m,j}+\sum_{m,k,l}\left(a_{i,k}(X)\frac{\partial Y_l}{\partial x_m}(X)+a_{m,k}(X)\frac{\partial Y_l}{\partial x_i}(X)\right)\frac{\partial\widetilde{v}_k}{\partial y_l}b_{m,j},\tag{2.2}$$

and

$$[\mathbf{N}(\widetilde{v})]_i=\sum_{k,j,m}\frac{\partial a_{i,k}}{\partial x_j}(X)\frac{\partial X_j}{\partial y_m}\widetilde{v}_k\widetilde{v}_m+\sum_{k,l}a_{i,k}(X)\frac{\partial\widetilde{v}_k}{\partial y_l}\widetilde{v}_l.\tag{2.3}$$

From (1.11), we can write

$$V(t,x)=a(t,x)\widetilde{v}(t,Y(t,x)),\quad P(t,x)=\widetilde{p}(t,Y(t,x)),\tag{2.4}$$

and after standard computations, we transform the terms appearing in (1.1)–(1.3):

$$(\det \nabla X)\, \partial_t V(X) = (\nabla X)\partial_t \widetilde{v} + (\nabla X)(\nabla \widetilde{v})\partial_t Y(X) + \det(\nabla X)\partial_t a(X)\widetilde{v}, \tag{2.5}$$

$$(\det \nabla X)\,(\Delta V)\,(X) = \operatorname{div}(\mathbf{D}\widetilde{v}), \quad (\det \nabla X)\,\nabla P(X) = b\nabla \widetilde{p}, \quad (\det \nabla X)\,[(V\cdot\nabla)V]\,(X) = \mathbf{N}(\widetilde{v}), \tag{2.6}$$

and

$$\widetilde{\mathbb{H}}_\eta(V,P) = -\sqrt{1+|\partial_{y_1}\eta^{\mathrm{S}}|^2}\mathbf{D}(\widetilde{v})\nu_{\eta^{\mathrm{s}}}\cdot e_2 + \widetilde{p}.$$

Then, we can also transform the Navier slip boundary conditions (1.2) into the boundary conditions (1.14) and we deduce that $(\widetilde{v},\widetilde{p},\eta)$ satisfies (1.12)–(1.15).

2.2 Linearization after the change of variables

Next we can consider the differences between the controlled state and the stationary state, that is we set (1.18) and (1.19) and we are going to make the difference between (1.12)–(1.15) and (1.5). Our aim is to linearize the corresponding system.

Let us set

$$\theta := \frac{1+\eta}{1+\eta^{\mathrm{S}}} = 1 + \frac{\xi}{1+\eta^{\mathrm{S}}}. \tag{2.7}$$

We have

$$\partial_{y_1}\theta = \frac{\partial_{y_1}\xi}{1+\eta^{\mathrm{S}}} - \frac{\partial_{y_1}\eta^{\mathrm{S}}}{(1+\eta^{\mathrm{S}})^2}\xi, \quad \frac{1}{\theta} = 1 - \frac{\xi}{1+\eta^{\mathrm{S}}+\xi} = 1 - \frac{\xi}{1+\eta^{\mathrm{S}}} + \frac{\xi^2}{(1+\eta^{\mathrm{S}})(1+\eta^{\mathrm{S}}+\xi)}, \tag{2.8}$$

$$\partial^2_{y_1}\theta = \frac{\partial^2_{y_1}\xi}{1+\eta^{\mathrm{S}}} - 2\frac{\partial_{y_1}\eta^{\mathrm{S}}}{(1+\eta^{\mathrm{S}})^2}\partial_{y_1}\xi + \left(-\frac{\partial^2_{y_1}\eta^{\mathrm{S}}}{(1+\eta^{\mathrm{S}})^2} + 2\frac{(\partial_{y_1}\eta^{\mathrm{S}})^2}{(1+\eta^{\mathrm{S}})^3}\right)\xi, \tag{2.9}$$

$$\partial_t\theta = \frac{\partial_t\xi}{1+\eta^{\mathrm{S}}}, \quad \partial_{t,y_1}\theta = \frac{\partial_{t,y_1}\xi}{1+\eta^{\mathrm{S}}} - \frac{\partial_{y_1}\eta^{\mathrm{S}}}{(1+\eta^{\mathrm{S}})^2}\partial_t\xi. \tag{2.10}$$

We use the above computation to split the coefficients coming from the change of variables into a linear part $\ell^{(i)}$ and a nonlinear part $\varepsilon^{(i)}$ with respect to the variables ξ, v, and p. In these expressions, we have coefficients of the form

$$c_\beta := \frac{y_2^{\beta_1}\left(\partial_{y_1}\eta^{\mathrm{S}}\right)^{\beta_2}\left(\partial^2_{y_1}\eta^{\mathrm{S}}\right)^{\beta_3}}{(1+\eta^{\mathrm{S}})^{\beta_4}\left(\sqrt{1+|\partial_{y_1}\eta^{\mathrm{S}}|^2}\right)^{\beta_5}}\left(v^{\mathrm{S}}_{i_1}\right)^{\beta_6}\left(v^{\mathrm{S}}_{i_2}\right)^{\beta_7}\left(\frac{\partial v^{\mathrm{S}}_{i_3}}{\partial y_{i_4}}\right)^{\beta_8}\left(f^{\mathrm{S}}_{i_6}\right)^{\beta_9}\left(\frac{\partial f^{\mathrm{S}}_{i_7}}{\partial y_{i_8}}\right)^{\beta_{10}} \qquad \beta_j\in\mathbb{N}. \tag{2.11}$$

We don't need the exact expressions of the linear and nonlinear parts, so that we use the same notation c_β for different values of β during the computation below. Note that with the regularity (1.28), $c_\beta \in W^{1,\infty}(\Omega)$.

We recall that X and Y are given by (1.9) and (1.10), a and b are given by (2.1). Then, we have

$$\nabla X = \begin{bmatrix} 1 & 0 \\ y_2\partial_{y_1}\theta & \theta \end{bmatrix}, \quad b = \begin{bmatrix} \theta & -y_2\partial_{y_1}\theta \\ 0 & 1 \end{bmatrix}, \quad \nabla Y(X) = \begin{bmatrix} 1 & 0 \\ -y_2\partial_{y_1}\theta/\theta & 1/\theta \end{bmatrix}, \quad a(X) = \begin{bmatrix} 1/\theta & 0 \\ y_2\partial_{y_1}\theta/\theta & 1 \end{bmatrix},$$

so that

$$\nabla X = I_2 + \ell^{(1)}, \quad b = I_2 + \ell^{(2)}, \quad \nabla Y(X) = I_2 + \ell^{(3)} + \varepsilon^{(3)}, \quad a(X) = I_2 + \ell^{(4)} + \varepsilon^{(4)}, \tag{2.12}$$

where the coefficients of the linear parts $\ell^{(1)},\ell^{(2)},\ell^{(3)},\ell^{(4)}$ are of the form $\dfrac{\xi}{1+\eta^{\mathrm{S}}}$ or $\partial_{y_1}\theta$, and where the coefficients of the nonlinear parts $\varepsilon^{(3)},\varepsilon^{(4)}$ are of the form

$$c_\beta\frac{\xi^{\delta_1}\left(\partial_{y_1}\xi\right)^{\delta_2}}{\left(1+\eta^{\mathrm{S}}+\xi\right)^{\delta_0}}, \quad \delta_i\in\mathbb{N}, \quad \delta_1+\delta_2\geqslant 2. \tag{2.13}$$

We also obtain

$$(\partial_{x_1} a)(X) = \begin{bmatrix} -\partial_{y_1}\theta/\theta^2 & 0 \\ y_2\left(\partial_{y_1}^2\theta/\theta - 2\left(\partial_{y_1}\theta/\theta\right)^2\right) & 0 \end{bmatrix}, \quad (\partial_{x_2} a)(X) = \begin{bmatrix} 0 & 0 \\ \partial_{y_1}\theta/\theta^2 & 0 \end{bmatrix}$$

so that

$$\frac{\partial a_{i,k}}{\partial x_m}(X) = \ell^{(5)}_{i,k,m} + \varepsilon^{(5)}_{i,k,m}, \tag{2.14}$$

where the coefficients of the linear parts $\ell^{(5)}$ are of the form $\partial_{y_1}\theta$ and $y_2\partial_{y_1}^2\theta$ and where coefficients of the nonlinear parts $\varepsilon^{(5)}$ are of the form (2.13) or

$$c_\beta \frac{\xi\partial_{y_1}^2\xi}{(1+\eta^{\mathrm{S}}+\xi)}$$

We also have

$$\partial_t Y(X) = -y_2\frac{\partial_t\theta}{\theta}e_2, \quad \partial_t a(X) = \begin{bmatrix} -\partial_t\theta/\theta^2 & 0 \\ y_2\left(\partial_{t,y_1}\theta/\theta - 2\partial_{y_1}\theta\partial_t\theta/\theta^2\right) & 0 \end{bmatrix}, \quad \det\nabla X = \theta,$$

so that

$$\partial_t Y(X) = \ell^{(6)} + \varepsilon^{(6)}, \quad (\det\nabla X)\,\partial_t a(X) = \ell^{(7)} + \varepsilon^{(7)}, \tag{2.15}$$

where

$$\ell^{(6)} := -\frac{y_2}{1+\eta^{\mathrm{S}}}\partial_t\xi e_2, \quad \varepsilon^{(6)} := y_2\frac{\xi\partial_t\xi}{(1+\eta^{\mathrm{S}})(1+\eta^{\mathrm{S}}+\xi)}e_2,$$

where the coefficients of $\ell^{(7)}$ are of the form $\partial_t\theta$ and $y_2\partial_{t,y_1}\theta$ and where the coefficients of $\varepsilon^{(7)}$ are of the form

$$c_\beta\frac{\xi\partial_t\xi}{(1+\eta^{\mathrm{S}}+\xi)}, \quad c_\beta\frac{\partial_{y_1}\xi\partial_t\xi}{(1+\eta^{\mathrm{S}}+\xi)}.$$

From these first linearization, we can handle the terms appearing in (1.12). First, combining (2.5), (2.12) and (2.15), we deduce

$$(\det\nabla X)\,\partial_t V(X) = \partial_t v + \ell^{(8)} + \varepsilon^{(8)}, \tag{2.16}$$

where $\ell^{(8)}$ is a linear combination of terms of the form

$$c_\beta\partial_t\xi, \quad c_\beta\partial_{t,y_1}\xi,$$

and where $\varepsilon^{(8)}$ is a linear combination of terms of the form

$$c_\beta\xi^{\delta_1}\left(\partial_{y_1}\xi\right)^{\delta_2}\partial_t v \quad |\delta| \geqslant 1, \quad c_\beta\frac{\xi^{\delta_1}\left(\partial_{y_1}\xi\right)^{\delta_2}\left(\partial_t\xi\right)^{\delta_3}}{1+\eta^{\mathrm{S}}+\xi}\left(\frac{\partial v_{i_1}}{\partial y_{i_2}}\right)^{\delta_4} \quad |\delta| \geqslant 2,\ \delta_3,\ \delta_4 \leqslant 1,$$
$$c_\beta\frac{\xi^{\delta_1}\left(\partial_{y_1}\xi\right)^{\delta_2}\left(\partial_t\xi\right)^{\delta_3}}{1+\eta^{\mathrm{S}}+\xi}\left(v_{i_1}\right)^{\delta_4} \quad |\delta| \geqslant 2,\ \delta_3,\ \delta_4 \leqslant 1, \quad c_\beta\partial_{t,y_1}\xi v_{i_1}, \tag{2.17}$$

where $i_j \in \{1,2\}$.

From (2.2), (2.12), and (2.14),

$$\mathbf{D}\widetilde{v} = 2\mathbb{D}(v^{\mathrm{S}}) + \ell^{(9)} + 2\mathbb{D}(v) + \varepsilon^{(9)}, \tag{2.18}$$

where $\ell^{(9)}$ is a linear combination of terms of the form $c_\beta\xi$, $c_\beta\partial_{y_1}\xi$, $c_\beta\partial_{y_1}^2\xi$, and where $\varepsilon^{(9)}$ is a linear combination of terms of the form

$$c_\beta\frac{\xi^{\delta_1}\left(\partial_{y_1}\xi\right)^{\delta_2}}{\left(1+\eta^{\mathrm{S}}+\xi\right)^{\delta_0}}\left(\frac{\partial v_{i_1}}{\partial y_{i_2}}\right)^{\delta_3} \quad |\delta| \geqslant 2,\ \delta_3 \leqslant 1, \quad c_\beta\frac{\xi^{\delta_1}\left(\partial_{y_1}\xi\right)^{\delta_2}\left(\partial_{y_1}^2\xi\right)^{\delta_3}}{\left(1+\eta^{\mathrm{S}}+\xi\right)^{\delta_0}}\left(v_{i_1}\right)^{\delta_4} \quad |\delta| \geqslant 2,\ \delta_3,\ \delta_4 \leqslant 1, \tag{2.19}$$

where $i_j \in \{1,2\}$. We thus deduce from (2.6) that

$$(\det \nabla X)\,(\Delta V)\,(X) = \operatorname{div}(\mathbf{D}\widetilde{v}) = \Delta v^{\mathrm{S}} + \operatorname{div}\ell^{(9)} + \Delta v + \operatorname{div}\varepsilon^{(9)}. \tag{2.20}$$

From (2.6), (2.3), (2.12) and (2.14),

$$(\det \nabla X)\,[(V\cdot\nabla)V]\,(X) = \left(v^{\mathrm{S}}\cdot\nabla\right)v^{\mathrm{S}} + (v\cdot\nabla)\,v^{\mathrm{S}} + \left(v^{\mathrm{S}}\cdot\nabla\right)v + \ell^{(10)} + \varepsilon^{(10)}, \tag{2.21}$$

where $\ell^{(10)}$ is a linear combination of terms of the form $c_\beta\xi$, $c_\beta\partial_{y_1}\xi$, $c_\beta\partial_{y_1}^2\xi$, and where $\varepsilon^{(10)}$ is a linear combination of terms of the form

$$c_\beta \frac{\xi^{\delta_1}\left(\partial_{y_1}\xi\right)^{\delta_2}\left(\partial_{y_1}^2\xi\right)^{\delta_3}}{\left(1+\eta^{\mathrm{S}}+\xi\right)^{\delta_0}}\left(v_{i_1}\right)^{\delta_4}\left(v_{i_2}\right)^{\delta_5} \quad |\delta| \geqslant 2,\ \delta_3,\ \delta_4,\ \delta_5 \leqslant 1,$$

$$c_\beta \frac{\xi^{\delta_1}\left(\partial_{y_1}\xi\right)^{\delta_2}}{\left(1+\eta^{\mathrm{S}}+\xi\right)^{\delta_0}}\left(v_{i_1}\right)^{\delta_3}\left(\frac{\partial v_{i_2}}{\partial y_{i_3}}\right)^{\delta_4} \quad |\delta| \geqslant 2,\ \delta_3,\ \delta_4 \leqslant 1, \tag{2.22}$$

where $i_j \in \{1,2\}$. Finally, from (2.6) and (2.12),

$$(\det \nabla X)\,\nabla P(X) = \nabla p^{\mathrm{S}} + \nabla p + \ell^{(11)}(\xi,\partial_{y_1}\xi) + \varepsilon^{(11)}(\xi,\partial_{y_1}\xi,\nabla p), \tag{2.23}$$

where $\ell^{(11)}$ is a linear combination of terms of the form

$$c_\beta\left(\frac{\partial p^{\mathrm{S}}}{\partial y_i}\right)\xi, \quad c_\beta\left(\frac{\partial p^{\mathrm{S}}}{\partial y_i}\right)\partial_{y_1}\xi, \quad i=1,2,$$

and where $\varepsilon^{(11)}$ is a linear combination of terms of the form

$$c_\beta\xi^{\delta_1}\left(\partial_{y_1}\xi\right)^{\delta_2}\frac{\partial p}{\partial y_i} \quad |\delta| \geqslant 1, \tag{2.24}$$

where $i \in \{1,2\}$.

We recall that $\det \nabla X = \theta$ where θ is defined by (2.7) so that

$$(\det \nabla X) f^{\mathrm{S}}(X) = f^{\mathrm{S}} + \theta\left(f^{\mathrm{S}}(X) - f^{\mathrm{S}}\right) + \frac{\xi}{1+\eta^{\mathrm{S}}}f^{\mathrm{S}}. \tag{2.25}$$

Applying the Taylor-Lagrange theorem, for any $y \in \Omega$,

$$f^{\mathrm{S}}(X(y)) = f^{\mathrm{S}}(y) + \nabla f^{\mathrm{S}}(y)\cdot(X(y)-y) + \int_0^1 (1-s)\,\nabla^2 f^{\mathrm{S}}((1-s)y + sX(y))\,(X(y)-y)\cdot(X(y)-y)\ ds$$

and since

$$X(y) - y = \frac{\xi(y_1)}{1+\eta^{\mathrm{S}}(y_1)}y_2 e_2,$$

we deduce that

$$(\det \nabla X) f^{\mathrm{S}}(X) = f^{\mathrm{S}} + \ell^{(12)}(\xi) + \varepsilon^{(12)}(\xi), \tag{2.26}$$

where $\ell^{(12)}$ is a linear combination of terms of the form $c_\beta\xi$, and where $\varepsilon^{(12)}$ is a linear combination of $c_\beta\xi^2$ and of

$$\theta\int_0^1 (1-s)\,\frac{\partial^2 f^{\mathrm{S}}}{\partial y_2^2}\left(y + \frac{s y_2\xi(y_1)}{1+\eta^{\mathrm{S}}(y_1)}e_2\right)\ ds\left(\frac{y_2\xi(y_1)}{1+\eta^{\mathrm{S}}(y_1)}\right)^2. \tag{2.27}$$

Assume that (v,p,ξ) are given by (2.4) and (1.18). Then, gathering (2.16), (2.20), (2.21), and (2.23), we deduce

$$\begin{aligned}(\det \nabla X)\left(\partial_t V + (V\cdot\nabla)V - \operatorname{div}\mathbb{T}(V,P) - f^{\mathrm{S}}\right)(X) &= -\Delta v^{\mathrm{S}} + \nabla p^{\mathrm{S}} + \left(v^{\mathrm{S}}\cdot\nabla\right)v^{\mathrm{S}} - f^{\mathrm{S}} \\ &+ \partial_t v - \Delta v + \nabla p + (v\cdot\nabla)\,v^{\mathrm{S}} + \left(v^{\mathrm{S}}\cdot\nabla\right)v + \ell^{(8)} - \operatorname{div}\ell^{(9)} + \ell^{(13)} - \operatorname{div}\varepsilon^{(9)} + \varepsilon^{(13)},\end{aligned} \tag{2.28}$$

where $\ell^{(13)}$ is a linear combination of terms of the form

$$\widetilde{c}_\beta \xi, \quad \widetilde{c}_\beta \partial_{y_1} \xi, \quad \widetilde{c}_\beta \partial_{y_1}^2 \xi, \tag{2.29}$$

where $\widetilde{c}_\beta = c_\beta$ or $c_\beta \left(\dfrac{\partial p^{\mathrm{S}}}{\partial y_i} \right)$ for some $i = 1, 2$ and where $\varepsilon^{(13)}$ is a linear combination of terms of the form (2.17), (2.22), (2.24) and (2.27).

Let us now linearize the terms coming from the change of variables that appear on the boundary. We keep the notation (2.11) but with y_2 replaced by 0 on Γ_{b} and by $1+\eta^{\mathrm{S}}$ on Γ. First on the bottom of the fluid domain, we have from (1.14) and (2.18):

$$\begin{aligned} &\mathbf{D}(\widetilde{v})_{1,2} - \frac{1+\eta^{\mathrm{S}}}{1+\eta} \widetilde{v}_1 + (\mathbb{M}u)_1 \\ &\quad = 2\mathbb{D}(v^{\mathrm{S}})_{1,2} - v_1^{\mathrm{S}} + 2\mathbb{D}(v)_{1,2} - (v - \mathbb{M}u)_1 + \ell_{1,2}^{(9)} + \ell^{(14)} + \varepsilon_{1,2}^{(9)} + \varepsilon^{(14)} = 0 \quad \text{on} \quad (0,\infty) \times \Gamma_{\mathrm{b}}, \end{aligned} \tag{2.30}$$

where $\ell^{(14)}$ is of the form $c_\beta \xi$ and where $\varepsilon^{(14)}$ is a linear combination of terms of the form

$$c_\beta \frac{\xi^{\delta_1}}{\left(1+\eta^{\mathrm{S}}+\xi\right)^{\delta_0}} \left(v_1\right)^{\delta_2} \quad |\delta| \geqslant 2, \ \delta_2 \leqslant 1. \tag{2.31}$$

On the top boundary of the fluid domain, that is on $(0,\infty) \times \Gamma$, we deduce from (1.4) that

$$\tau_\eta(X) = \frac{1}{\sqrt{1+|\partial_{y_1}\eta|^2}} \left(\begin{bmatrix} 1 \\ \partial_{y_1}\eta^{\mathrm{S}} \end{bmatrix} + \begin{bmatrix} 0 \\ \partial_{y_1}\xi \end{bmatrix} \right) = \sqrt{\frac{1+|\partial_{y_1}\eta^{\mathrm{S}}|^2}{1+|\partial_{y_1}\eta|^2}} \left(\tau_{\eta^{\mathrm{S}}} + \frac{\partial_{y_1}\xi e_2}{\sqrt{1+|\partial_{y_1}\eta^{\mathrm{S}}|^2}} \right). \tag{2.32}$$

Moreover, we have

$$\sqrt{\frac{1+|\partial_{y_1}\eta^{\mathrm{S}}|^2}{1+|\partial_{y_1}\eta|^2}} = 1 + \ell^{(15)} + \varepsilon^{(15)}, \tag{2.33}$$

where $\ell^{(15)}$ is of the form $c_\beta \partial_{y_1}\xi$ and where $\varepsilon^{(15)}$ is a linear combination of terms of the form

$$c_\beta \frac{\left(\partial_{y_1}\xi\right)^{\delta_1}}{\langle \partial_{y_1}\eta^{\mathrm{S}} + \partial_{y_1}\xi \rangle \left(\langle \partial_{y_1}\eta^{\mathrm{S}} \rangle + \langle \partial_{y_1}\eta^{\mathrm{S}} + \partial_{y_1}\xi \rangle \right)^{\delta_0}} \quad \delta_1 \geqslant 2,$$

where

$$\langle x \rangle := \sqrt{1+x^2} \quad (x \in \mathbb{R}).$$

From (2.32), (1.14), (2.12) and (2.18), on $(0,\infty) \times \Gamma$,

$$\begin{aligned} &\sqrt{\frac{1+|\partial_{y_1}\eta|^2}{1+|\partial_{y_1}\eta^{\mathrm{S}}|^2}} \left(\mathbf{D}(\widetilde{v})\nu_{\eta^{\mathrm{S}}} \cdot \tau_\eta(X) \sqrt{\frac{1+|\partial_{y_1}\eta^{\mathrm{S}}|^2}{1+|\partial_{y_1}\eta|^2}} + \widetilde{v} \cdot \mathrm{Cof}(\nabla Y(X)) \tau_\eta(X) - \partial_t \eta e_2 \cdot \tau_\eta(X) \right) \\ &\quad = \left[2\mathbb{D}(v^{\mathrm{S}})\nu_{\eta^{\mathrm{S}}} + v^{\mathrm{S}} \right] \cdot \tau_{\eta^{\mathrm{S}}} + \left[2\mathbb{D}(v)\nu_{\eta^{\mathrm{S}}} + (v - \partial_t \xi e_2) \right] \cdot \tau_{\eta^{\mathrm{S}}} + \ell^{(9)} \nu_{\eta^{\mathrm{S}}} \cdot \tau_{\eta^{\mathrm{S}}} + \ell^{(16)} + \varepsilon^{(16)} = 0, \end{aligned} \tag{2.34}$$

where $\ell^{(16)}$ is a linear combination of terms of the form $c_\beta \xi$, $c_\beta \partial_{y_1}\xi$, and where $\varepsilon^{(16)}$ is a linear combination of terms of the form $c_\beta \partial_{y_1}\xi \partial_t \xi$,

$$c_\beta \frac{\xi^{\delta_1} \left(\partial_{y_1}\xi\right)^{\delta_2} \left(\dfrac{\partial v_{i_1}}{\partial y_{i_2}} \right)^{\delta_3}}{\left(1+\eta^{\mathrm{S}}+\xi\right)^{\delta_0} \langle \partial_{y_1}\eta^{\mathrm{S}} + \partial_{y_1}\xi \rangle^{\delta_{-1}} \left(\langle \partial_{y_1}\eta^{\mathrm{S}} \rangle + \langle \partial_{y_1}\eta^{\mathrm{S}} + \partial_{y_1}\xi \rangle \right)^{\delta_{-2}}} \quad |\delta| \geqslant 2, \ \delta_3 \leqslant 1, \tag{2.35}$$

and

$$c_\beta \frac{\xi^{\delta_1} \left(\partial_{y_1}\xi\right)^{\delta_2} \left(\partial_{y_1}^2\xi\right)^{\delta_3} \left(v_{i_1}\right)^{\delta_4}}{(1+\eta^{\mathrm{S}}+\xi)^{\delta_0} \left\langle \partial_{y_1}\eta^{\mathrm{S}} + \partial_{y_1}\xi \right\rangle^{\delta_{-1}} \left(\left\langle \partial_{y_1}\eta^{\mathrm{S}} \right\rangle + \left\langle \partial_{y_1}\eta^{\mathrm{S}} + \partial_{y_1}\xi \right\rangle \right)^{\delta_{-2}}} \quad |\delta| \geqslant 2,\ \delta_3, \delta_4 \leqslant 1. \tag{2.36}$$

Finally, from (2.18)

$$-\operatorname{proj}_{L^2_0} \sqrt{1+|\partial_{y_1}\eta^{\mathrm{S}}|^2}\mathbf{D}(\widetilde{v})\nu_{\eta^{\mathrm{S}}} \cdot e_2 = -\mathcal{T}^* \left(\left(2\mathbb{D}(v^{\mathrm{S}}) + \ell^{(9)} + 2\mathbb{D}(v) + \varepsilon^{(9)} \right) \nu_{\eta^{\mathrm{S}}} \right). \tag{2.37}$$

2.3 Estimates on the linear and nonlinear parts

We now set

$$\mathbf{L}^1(\xi) := \ell^{(9)}, \quad \mathbf{L}^2(\xi) := \ell^{(13)}, \quad \mathbf{L}^3(\partial_t\xi) := \ell^{(8)}, \quad \mathbf{L}^4(\xi) := \begin{cases} \ell^{(9)}_{1,2} + \ell^{(14)} & \text{on } \Gamma_{\mathrm{b}} \\ \ell^{(9)}_{1,2} + \ell^{(16)} & \text{on } \Gamma \end{cases}, \tag{2.38}$$

$$\mathbf{F} := -\operatorname{div}\varepsilon^{(9)} + \varepsilon^{(13)}, \quad \mathbf{H}(v,\xi) := -\mathcal{T}^*\left(\varepsilon^{(9)}\nu_{\eta^{\mathrm{S}}}\right), \quad \mathbf{G}(v,\xi) := \begin{cases} \varepsilon^{(9)}_{1,2} + \varepsilon^{(14)} & \text{on } (0,\infty)\times\Gamma_{\mathrm{b}} \\ \varepsilon^{(16)} & \text{on } (0,\infty)\times\Gamma \end{cases}. \tag{2.39}$$

Then, from (2.28), (2.30), (2.34), (2.37), (1.5) and (1.12)–(1.15) we deduce that (v,p,ξ) satisfies a system of the form (1.20)–(1.22).

Moreover, from the regularity hypothesis (1.28) and the definition (2.11) of c_β, we can check that

$$\mathbf{L}^1 \in \mathcal{L}\left(\mathcal{D}(A_1^{1/2}), L^2(\Omega)^4\right) \cap \mathcal{L}\left(\mathcal{D}(A_1^{3/4}), H^1(\Omega)^4\right), \tag{2.40}$$

$$\mathbf{L}^2 \in \mathcal{L}\left(\mathcal{D}(A_1^{1/2}), L^2(\Omega)^2\right), \quad \mathbf{L}^3 \in \mathcal{L}\left(\mathcal{D}(A_2^{1/2}), L^2(\Omega)^2\right), \tag{2.41}$$

$$\mathbf{L}^4 \in \mathcal{L}\left(\mathcal{D}(A_1^{1/2}), L^2(\partial\Omega)^2\right) \cap \mathcal{L}\left(\mathcal{D}(A_1^{3/4}), H^1(\partial\Omega)^2\right). \tag{2.42}$$

We also introduce the adjoint operators

$$\left(\mathbf{L}^1\right)^* \in \mathcal{L}\left(L^2(\Omega)^4, \mathcal{D}(A_1^{1/2})\right), \quad \left(\mathbf{L}^2\right)^* \in \mathcal{L}\left(L^2(\Omega)^2, \mathcal{D}(A_1^{1/2})\right), \quad \left(\mathbf{L}^4\right)^* \in \mathcal{L}\left(L^2(\partial\Omega)^2, \mathcal{D}(A_1^{1/2})\right). \tag{2.43}$$

For $\mathbf{L}^3$, we recall that it is a linear combination of terms of the form $c_\beta\xi_2$ and $c_\beta\partial_{y_1}\xi_2$. Assume $\varphi \in H^1(\Omega)^2$, we can integrate by parts in y_1:

$$\int_\Omega \partial_{y_1}\xi_2(y_1) c_\beta(y_1,y_2)\cdot\varphi(y_1,y_2)\, dy_1\, dy_2 = -\int_{\mathcal{I}} \xi_2(y_1)\partial_{y_1}\left(\int_0^{1+\eta^{\mathrm{S}}(y_1)} c_\beta(y_1,y_2)\cdot\varphi(y_1,y_2)\, dy_2 \right) dy_1,$$

and since

$$y_1 \mapsto \partial_{y_1}\left(\int_0^{1+\eta^{\mathrm{S}}(y_1)} c_\beta(y_1,y_2)\cdot\varphi(y_1,y_2)\, dy_2 \right)$$

is in $L^2_0(\mathcal{I})$, this allows us to define the adjoint of $\mathbf{L}^3$ as

$$\left(\mathbf{L}^3\right)^* \in \mathcal{L}\left(H^1(\Omega)^2, L^2_0(\mathcal{I})\right). \tag{2.44}$$

Concerning the nonlinear terms, we define for $R>0$ and $\sigma>0$

$$B_{R,\sigma} := \Big\{ (v,p,\xi) \in \left[L^2_\sigma(H^2(\Omega)) \cap H^1_\sigma(L^2(\Omega))\right]^2 \times L^2_\sigma(H^1(\Omega)/\mathbb{R}) \times \left[L^2_\sigma(\mathcal{D}(A_1)) \cap H^2_\sigma(L^2_0(\mathcal{I}))\right] : \|v\|_{L^2_\sigma(H^2(\Omega)^2)\cap H^1_\sigma(L^2(\Omega)^2)} + \|\nabla p\|_{L^2_\sigma(L^2(\Omega))} + \|\xi\|_{L^2_\sigma(\mathcal{D}(A_1))\cap H^2_\sigma(L^2_0(\mathcal{I}))} \leqslant R \Big\}. \tag{2.45}$$

In what follows, we set

$$\|(v,p,\xi)\|_\star := \|v\|_{L^2_\sigma(H^2(\Omega)^2)\cap H^1_\sigma(L^2(\Omega)^2)} + \|\nabla p\|_{L^2_\sigma(L^2(\Omega))} + \|\xi\|_{L^2_\sigma(\mathcal{D}(A_1))\cap H^2_\sigma(L^2_0(\mathcal{I}))}.$$

Then, we can show the following result

Proposition 2.1. *For $R>0$ small enough, the functions $\mathbf{F},\mathbf{G}$ and $\mathbf{H}$ defined by (2.39) satisfy*

$$\mathbf{F}:B_{R,\sigma}\to L^2_\sigma(L^2(\Omega)^2),\quad \mathbf{G}:B_{R,\sigma}\to L^2_\sigma(H^{1/2}(\partial\Omega)^2)\cap H^{1/4}_\sigma(L^2(\partial\Omega)^2),\quad \mathbf{H}:B_{R,\sigma}\to L^2_\sigma(L^2_0(\mathcal{I})). \tag{2.46}$$

Moreover, for $R\in(0,1)$ small enough, there exists a constant $C>0$ such that for any

$$(v,p,\xi),\left(v^{(1)},p^{(1)},\xi^{(1)}\right),\left(v^{(2)},p^{(2)},\xi^{(2)}\right)\in B_{R,\sigma}, \tag{2.47}$$

the triplet $(\mathbf{F},\mathbf{G},\mathbf{H})$ satisfies

$$\|\mathbf{F}(v,p,\xi)\|_{L^2_\sigma(L^2(\Omega)^2)}\leqslant CR^2, \tag{2.48}$$

$$\left\|\mathbf{F}\left(v^{(1)},p^{(1)},\xi^{(1)}\right)-\mathbf{F}\left(v^{(2)},p^{(2)},\xi^{(2)}\right)\right\|_{L^2_\sigma(L^2(\Omega)^2)}\leqslant CR\left\|\left(v^{(1)},p^{(1)},\xi^{(1)}\right)-\left(v^{(2)},p^{(2)},\xi^{(2)}\right)\right\|_\star, \tag{2.49}$$

$$\|\mathbf{G}(v,p,\xi)\|_{L^2_\sigma(H^{1/2}(\partial\Omega)^2)\cap H^{1/4}_\sigma(L^2(\partial\Omega)^2)}\leqslant CR^2, \tag{2.50}$$

$$\begin{aligned}\left\|\mathbf{G}\left(v^{(1)},p^{(1)},\xi^{(1)}\right)-\mathbf{G}\left(v^{(2)},p^{(2)},\xi^{(2)}\right)\right\|_{L^2_\sigma(H^{1/2}(\partial\Omega)^2)\cap H^{1/4}_\sigma(L^2(\partial\Omega)^2)}\\ \leqslant CR\left\|\left(v^{(1)},p^{(1)},\xi^{(1)}\right)-\left(v^{(2)},p^{(2)},\xi^{(2)}\right)\right\|_\star,\end{aligned} \tag{2.51}$$

$$\|\mathbf{H}(v,\xi)\|_{L^2_\sigma(L^2(\mathcal{I}))}\leqslant CR^2, \tag{2.52}$$

$$\left\|\mathbf{H}\left(v^{(1)},\xi^{(1)}\right)-\mathbf{H}\left(v^{(2)},\xi^{(2)}\right)\right\|_{L^2_\sigma(L^2(\mathcal{I}))}\leqslant CR\left\|\left(v^{(1)},p^{(1)},\xi^{(1)}\right)-\left(v^{(2)},p^{(2)},\xi^{(2)}\right)\right\|_\star. \tag{2.53}$$

Proof. From (2.45), (2.47) and Sobolev embeddings, there exists a constant $C>0$ such that

$$\begin{aligned}\|\xi\|_{L^\infty_\sigma(C^2(\mathcal{I}))}+\|\partial_t\xi\|_{L^\infty_\sigma(L^\infty(\mathcal{I}))}+\left\|\partial^3_{y_1}\xi\right\|_{L^2_\sigma(L^\infty(\mathcal{I}))}+\|v\|_{L^2_\sigma(L^\infty(\Omega)^2)}\\ \leqslant C\left(\|v\|_{L^2_\sigma(H^2(\Omega)^2)\cap H^1_\sigma(L^2(\Omega)^2)}+\|\xi\|_{L^2_\sigma(\mathcal{D}(A_1))\cap H^2_\sigma(L^2_0(\mathcal{I}))}\right),\end{aligned} \tag{2.54}$$

and the same relation for $\left(v^{(1)},p^{(1)},\xi^{(1)}\right),\left(v^{(2)},p^{(2)},\xi^{(2)}\right)$. For R small enough, there exists a constant $C>0$ such that

$$\left\|\frac{1}{1+\eta^{\mathrm{S}}+\xi}\right\|_{L^\infty(L^\infty(\mathcal{I}))}\leqslant C, \tag{2.55}$$

and

$$\left\|\frac{1}{1+\eta^{\mathrm{S}}+\xi^{(1)}}-\frac{1}{1+\eta^{\mathrm{S}}+\xi^{(2)}}\right\|_{L^\infty_\sigma(L^\infty(\mathcal{I}))}\leqslant C\left\|\xi^{(1)}-\xi^{(2)}\right\|_{L^\infty_\sigma(L^\infty(\mathcal{I}))}. \tag{2.56}$$

In particular,

$$\left\|\frac{\xi^{\delta_1}\left(\partial_{y_1}\xi\right)^{\delta_2}}{\left(1+\eta^{\mathrm{S}}+\xi\right)^{\delta_0}}\right\|_{H^1_\sigma(H^1(\mathcal{I}))}\leqslant CR^{\delta_1+\delta_2},$$

$$\left\|\frac{\left(\xi^{(1)}\right)^{\delta_1}\left(\partial_{y_1}\xi^{(1)}\right)^{\delta_2}}{\left(1+\eta^{\mathrm{S}}+\xi^{(1)}\right)^{\delta_0}}-\frac{\left(\xi^{(2)}\right)^{\delta_1}\left(\partial_{y_1}\xi^{(2)}\right)^{\delta_2}}{\left(1+\eta^{\mathrm{S}}+\xi^{(2)}\right)^{\delta_0}}\right\|_{H^1_\sigma(H^1(\mathcal{I}))}\leqslant CR^{\delta_1+\delta_2-1}\left\|\xi^{(1)}-\xi^{(2)}\right\|_{H^1_\sigma(H^2(\mathcal{I}))}.$$

We have similarly

$$\left\|\frac{1}{\langle\partial_{y_1}\eta^{\mathrm{S}}+\partial_{y_1}\xi\rangle}\right\|_{H^1_\sigma(H^1(\mathcal{I}))}\leqslant C,\quad \left\|\frac{1}{\langle\partial_{y_1}\eta^{\mathrm{S}}\rangle+\langle\partial_{y_1}\eta^{\mathrm{S}}+\partial_{y_1}\xi\rangle}\right\|_{H^1_\sigma(H^1(\mathcal{I}))}\leqslant C,$$

$$\left\| \frac{1}{\langle \partial_{y_1}\eta^{\mathrm{S}} + \partial_{y_1}\xi^{(1)} \rangle} - \frac{1}{\langle \partial_{y_1}\eta^{\mathrm{S}} + \partial_{y_1}\xi^{(2)} \rangle} \right\|_{H^1_\sigma(H^1(\mathcal{I}))} \leqslant C \left\| \xi^{(1)} - \xi^{(2)} \right\|_{H^1_\sigma(H^2(\mathcal{I}))},$$

$$\left\| \frac{1}{\langle \partial_{y_1}\eta^{\mathrm{S}} \rangle + \langle \partial_{y_1}\eta^{\mathrm{S}} + \partial_{y_1}\xi^{(1)} \rangle} - \frac{1}{\langle \partial_{y_1}\eta^{\mathrm{S}} \rangle + \langle \partial_{y_1}\eta^{\mathrm{S}} + \partial_{y_1}\xi^{(2)} \rangle} \right\|_{H^1_\sigma(H^1(\mathcal{I}))} \leqslant C \left\| \xi^{(1)} - \xi^{(2)} \right\|_{H^1_\sigma(H^2(\mathcal{I}))}.$$

The above relations allow us to estimate terms of the form (2.17), (2.22), (2.24) and (2.27) and the space derivatives of terms of the form (2.19). We can thus deduce that $\mathbf{F}$ satisfies (2.48) and (2.49). Similarly, we can obtain that $\mathbf{H}$ satisfies (2.52) and (2.53).

Finally, $\mathbf{G}$ is a linear combination of terms of the form (2.35) and (2.36). Note that with the above estimates

$$c_\beta \frac{\xi^{\delta_1} \left(\partial_{y_1}\xi\right)^{\delta_2}}{\left(1+\eta^{\mathrm{S}}+\xi\right)^{\delta_0} \langle \partial_{y_1}\eta^{\mathrm{S}} + \partial_{y_1}\xi \rangle^{\delta_{-1}} \left(\langle \partial_{y_1}\eta^{\mathrm{S}} \rangle + \langle \partial_{y_1}\eta^{\mathrm{S}} + \partial_{y_1}\xi \rangle\right)^{\delta_{-2}}} \in H^1_\sigma(H^1(\partial\Omega)),$$

and by using a trace theorem (see for instance [34, Theorem 2.1, p.9])

$$\frac{\partial v_{i_1}}{\partial y_{i_2}} \in L^2_\sigma(H^{1/2}(\partial\Omega)) \cap H^{1/4}_\sigma(L^2(\partial\Omega)).$$

Now, using a standard result on the product in Sobolev spaces (see, for instance, [14, Lemma 6]), we deduce that terms of the form (2.35) are in $L^2_\sigma(H^{1/2}(\partial\Omega)) \cap H^{1/4}_\sigma(L^2(\partial\Omega))$. By a trace theorem (see [34, Theorem 2.1, p.9]), we also have

$$v_{i_1} \in L^2_\sigma(H^{3/2}(\partial\Omega)) \cap H^{3/4}_\sigma(L^2(\partial\Omega)),$$

and

$$\partial^2_{y_1}\xi \in L^\infty_\sigma(H^1(\mathcal{I})) \cap H^{1/4}_\sigma(H^{3/2}(\mathcal{I})).$$

Using again the result on the product in Sobolev spaces (see, for instance, [14, Lemma 6]), we deduce that $\mathbf{G}$ satisfies (2.50) and (2.51). □

3 Stabilization of the linear system

We show in this section that the linear system corresponding to (1.20), (1.21), (1.22) where the nonlinearities $\mathbf{F}, \mathbf{G}, \mathbf{H}$ are replaced by given function (f, g, h) (see (3.39), (3.40), (3.41) below) is stabilizable with dynamic controllers.

3.1 Functional Framework

Let us consider $m \in C^2(\partial\Omega)$, $\operatorname{supp} m \subset \Gamma_{\mathrm{cont}}$, $\displaystyle\int_{\Gamma_{\mathrm{cont}}} m \, ds = 1$ and let us set

$$\mathcal{M}U := mU - \left(\int_{\Gamma_{\mathrm{cont}}} mU \cdot \nu \, dx' \right) m\nu. \tag{3.1}$$

We can check that $\mathcal{M} \in \mathcal{L}(\mathcal{U}, L^2(\partial\Omega))$, where $\mathcal{U} := L^2(\Gamma_{\mathrm{cont}})$ and that we have the compatibility condition

$$\int_{\partial\Omega} (\mathcal{M}U)_\nu \, ds = 0. \tag{3.2}$$

We can thus consider the following system, where the control is $U \in L^2(0,T;\mathcal{U})$:

$$\begin{cases} \partial_t v - \nabla \cdot \mathbb{T}(v,p) + (v^{\mathrm{S}} \cdot \nabla)v + (v \cdot \nabla)v^{\mathrm{S}} - \nabla \cdot \mathbf{L}^1(\xi) + \mathbf{L}^2(\xi) + \mathbf{L}^3(\partial_t \xi) = f & \text{in } (0,\infty)\times\Omega, \\ \nabla \cdot v = 0 & \text{in } (0,\infty)\times\Omega, \\ \partial_{tt}\xi + A_1\xi + A_2\partial_t\xi + \mathcal{T}^*(\mathbf{L}^1(\xi)\nu) = -\mathcal{T}^*(\mathbb{T}(v,p)\nu) + h & \text{in } (0,\infty)\times\mathcal{I}, \end{cases} \tag{3.3}$$

$$\left\{\begin{array}{rl} [v-\mathcal{T}\partial_t\xi]_\nu = (\mathcal{M}U)_\nu & \text{on } (0,\infty)\times\partial\Omega,\\ \left[2\mathbb{D}(v)\nu+(v-\mathcal{T}\partial_t\xi)+\mathbf{L}^1(\xi)\nu+\mathbf{L}^4(\xi)\right]_\tau = (\mathcal{M}U)_\tau & \text{on } (0,\infty)\times\partial\Omega,\end{array}\right. \tag{3.4}$$

$$\xi(0,\cdot)=\xi_1^0 \text{ in } \mathcal{I},\quad \partial_t\xi(0,\cdot)=\xi_2^0 \text{ in } \mathcal{I},\quad v(0,\cdot)=v^0 \text{ in } \Omega. \tag{3.5}$$

We are going to write the above system as

$$\left\{\begin{array}{l}\mathbb{P}W' = \mathcal{A}\mathbb{P}W+\mathcal{B}U+\mathbb{P}F\\ \mathbb{P}W(0)=\mathbb{P}W^0\end{array}\right. \quad\text{and}\quad (\mathrm{Id}-\mathbb{P})W=(\mathrm{Id}-\mathbb{P})\mathbb{E}U, \tag{3.6}$$

where

$$W=\begin{bmatrix} v\\ \xi\\ \partial_t\xi\end{bmatrix},\quad W^0=\begin{bmatrix} v^0\\ \xi_1^0\\ \xi_2^0\end{bmatrix},\quad F=\begin{bmatrix} f\\ 0\\ h\end{bmatrix}. \tag{3.7}$$

Such a decomposition into a dynamical system and a stationary system is standard for the controllability of Stokes type systems and was introduced in [49]). We are going to show that $\mathcal{A}$ is the infinitesimal generator of an analytic semigroup with compact resolvent and satisfies a Fattorini-Hautus test so that it is exponentially stabilizable with a finite number of controls (see below for details). Then, we will be able to replace $\mathcal{M}U$ by $\mathbb{M}u$ as in (1.20), (1.21), (1.22).

First, we denote by $\mathbb{P}$ the orthogonal projection

$$\mathbb{P}:[L^2(\Omega)]^2\times\mathcal{D}(A_1^{1/2})\times L^2_0(\mathcal{I})\longrightarrow\mathcal{H},$$

where $\mathcal{H}$ is defined by (1.24). We then define the operator $\mathcal{A}:\mathcal{D}(\mathcal{A})\to\mathcal{H}$ by

$$\begin{aligned}\mathcal{D}(\mathcal{A}):=\Big\{(v,\xi_1,\xi_2)\in\Big[H^2(\Omega)^2\times\mathcal{D}(A_1)\times\mathcal{D}(A_1^{1/2})\Big]\cap\mathcal{H}\\ \text{such that } \left[2\mathbb{D}(v)\nu+(v-\mathcal{T}\xi_2)+\mathbf{L}^1(\xi_1)\nu+\mathbf{L}^4(\xi_1)\right]_\tau=0 \quad\text{on } \partial\Omega\Big\},\end{aligned} \tag{3.8}$$

$$\mathcal{A}\begin{bmatrix} v\\ \xi_1\\ \xi_2\end{bmatrix}:=\mathbb{P}\begin{bmatrix}\Delta v-(v^{\mathrm{S}}\cdot\nabla)v-(v\cdot\nabla)v^{\mathrm{S}}+\nabla\cdot\mathbf{L}^1(\xi_1)-\mathbf{L}^2(\xi_1)-\mathbf{L}^3(\xi_2)\\ \xi_2\\ -A_1\xi_1-A_2\xi_2-\mathcal{T}^*(\mathbf{L}^1(\xi_1)\nu)-\mathcal{T}^*(2\mathbb{D}(v)\nu)\end{bmatrix}. \tag{3.9}$$

We have the following result, similar to results obtained in [6, Section 3] and [19, Section 5]. We give a sketch of the corresponding proof for completeness.

Theorem 3.1. *The operator $(\mathcal{D}(\mathcal{A}),\mathcal{A})$ defined by* (3.8) *and* (3.9) *has compact resolvent and is the infinitesimal generator of an analytical semigroup on $\mathcal{H}$. Moreover, its adjoint is given by*

$$\mathcal{D}(\mathcal{A}^*)=\Big\{(\varphi,\zeta_1,\zeta_2)\in\Big([H^2(\Omega)]^3\times\mathcal{D}(A_1)\times\mathcal{D}(A_1^{1/2})\Big)\cap\mathcal{H}\ :\ [2\mathbb{D}(\varphi)\nu+(\varphi-\mathcal{T}\zeta_2)]_\tau=0\quad \textit{on } \partial\Omega\Big\}, \tag{3.10}$$

$$\mathcal{A}^*\begin{bmatrix}\varphi\\ \zeta_1\\ \zeta_2\end{bmatrix}=\mathbb{P}\begin{bmatrix}\Delta\varphi+(v^{\mathrm{S}}\cdot\nabla)\varphi-\left(\nabla v^{\mathrm{S}}\right)^*\varphi\\ -\zeta_2-\left(\mathbf{L}^1\right)^*(\nabla\varphi)-\left(\mathbf{L}^2\right)^*(\varphi)+\left(\mathbf{L}^4\right)^*(\varphi-\mathcal{T}\zeta_2)\\ A_1\zeta_1-A_2\zeta_2-\mathcal{T}^*\left(2\left(\mathbb{D}\varphi\right)\nu\right)-\left(\mathbf{L}^3\right)^*(\varphi)\end{bmatrix}. \tag{3.11}$$

Moreover, for any $\alpha\in[0,1]$

$$\mathcal{D}((\lambda_0\,\mathrm{Id}-\mathcal{A})^\alpha)=[\mathcal{D}(\mathcal{A}),\mathcal{H}]_{1-\alpha},\quad \mathcal{D}((\lambda_0\,\mathrm{Id}-\mathcal{A}^*)^\alpha)=[\mathcal{D}(\mathcal{A}^*),\mathcal{H}]_{1-\alpha}, \tag{3.12}$$

where $[\cdot,\cdot]_\theta$ denotes the complex interpolation.

Proof. First, if $(v,\xi_1,\xi_2)\in\mathcal{D}(\mathcal{A})$, then after some integrations by parts, we obtain

$$\left\langle \mathcal{A}\begin{bmatrix} v\\ \xi_1\\ \xi_2\end{bmatrix}, \begin{bmatrix} v\\ \xi_1\\ \xi_2\end{bmatrix}\right\rangle_{\mathcal{H}} = -\int_\Omega 2\left|\mathbb{D}v\right|^2\,dy - \int_\Omega (v\cdot\nabla)\,v^{\mathrm{S}}\cdot v\,dy - \int_\Omega \mathbf{L}^1(\xi_1):\nabla v\,dy - \int_\Omega \mathbf{L}^2(\xi_1)\cdot v\,dy - \int_\Omega \mathbf{L}^3(\xi_2)\cdot v\,dy$$
$$- \left\|A_2^{1/2}\xi_2\right\|^2_{L^2(\mathcal{I})} - \int_{\partial\Omega}\left|(v-\mathcal{T}\xi_2)_\tau\right|^2\,ds - \int_{\partial\Omega}\mathbf{L}^4(\xi_1)_\tau\,(v-\mathcal{T}\xi_2)_\tau\,ds \quad (3.13)$$

so that using Proposition 2.1 and the Korn inequality, we deduce that for $\lambda>0$ large enough, $\mathcal{A}-\lambda\,\mathrm{Id}$ is dissipative. We can then show that for $\lambda>0$ large enough, $\mathcal{A}-\lambda\,\mathrm{Id}:\mathcal{D}(\mathcal{A})\to\mathcal{H}$ is invertible by following the proof of Proposition 5.1 in [19]. Consequently, there exists $\lambda_0>0$ such that for $\lambda\geqslant\lambda_0$, $\mathcal{A}-\lambda\,\mathrm{Id}$ is m-dissipative and (3.12) is a consequence of [12, Proposition 6.1, p.171].

The adjoint of $\mathcal{A}$ can be obtained by integration by parts: assume $(v,\xi_1,\xi_2)\in\mathcal{D}(\mathcal{A})$ and $(\varphi,\zeta_1,\zeta_2)\in\mathcal{D}(\mathcal{A}^*)$. Then

$$\left\langle \mathcal{A}\begin{bmatrix} v\\ \xi_1\\ \xi_2\end{bmatrix}, \begin{bmatrix} \varphi\\ \zeta_1\\ \zeta_2\end{bmatrix}\right\rangle_{\mathcal{H}} = \int_\Omega v\cdot\left(\Delta\varphi+(v^{\mathrm{S}}\cdot\nabla)\varphi-\left(\nabla v^{\mathrm{S}}\right)^*\varphi\right)\,dy + \int_{\partial\Omega}\mathbf{L}^4(\xi_1)\cdot(\varphi-\mathcal{T}\zeta_2)\,ds$$
$$- \int_\Omega \mathbf{L}^1(\xi_1):\nabla\varphi\,dy - \int_\Omega \mathbf{L}^2(\xi_1)\cdot\varphi\,dy - \int_\Omega \mathbf{L}^3(\xi_2)\cdot\varphi\,dy$$
$$+\left\langle A_1^{1/2}\xi_1, -A_1^{1/2}\zeta_2\right\rangle + \left\langle \xi_2, A_1\zeta_1 - A_2\zeta_2 - \mathcal{T}^*\left(2\left(\mathbb{D}\varphi\right)\nu\right)\right\rangle. \quad (3.14)$$

Moreover, we can write the operator $\mathcal{A}^*$ as $\mathcal{A}^*=\mathcal{A}_0+\mathcal{A}_1$ with

$$\mathcal{D}(\mathcal{A}_0)=\mathcal{D}(\mathcal{A}^*),\quad \mathcal{A}_0\begin{bmatrix}\varphi\\ \zeta_1\\ \zeta_2\end{bmatrix} := \mathbb{P}\begin{bmatrix}\Delta\varphi\\ -\zeta_2\\ A_1\zeta_1 - A_2\zeta_2 - \mathcal{T}^*\left(2\left(\mathbb{D}\varphi\right)\nu\right)\end{bmatrix}, \quad (3.15)$$

$$\mathcal{D}(\mathcal{A}_1)=\mathcal{V},\quad \mathcal{A}_1\begin{bmatrix}\varphi\\ \zeta_1\\ \zeta_2\end{bmatrix} := \mathbb{P}\begin{bmatrix}(v^{\mathrm{S}}\cdot\nabla)\varphi-\left(\nabla v^{\mathrm{S}}\right)^*\varphi\\ -\left(\mathbf{L}^1\right)^*(\nabla\varphi)-\left(\mathbf{L}^2\right)^*(\varphi)+\left(\mathbf{L}^4\right)^*(\varphi-\mathcal{T}\zeta_2)\\ -\left(\mathbf{L}^3\right)^*(\varphi)\end{bmatrix}. \quad (3.16)$$

We have shown in [19, Proposition 5.3] that $\mathcal{A}_0$ is the infinitesimal generator of an analytical semigroup on $\mathcal{H}$ and that $\mathcal{D}\left(\left(\lambda_0\,\mathrm{Id}-\mathcal{A}_0\right)^{1/2}\right)=\mathcal{V}$. Note that $\mathcal{A}_1$ is well-defined since $\mathbf{L}^1$, $\mathbf{L}^2$, $\mathbf{L}^3$ and $\mathbf{L}^4$ satisfy (2.43) and (2.44). Then, using [44, Corollary 2.4, p.81], we deduce that $\mathcal{A}^*$ and thus $\mathcal{A}$ are infinitesimal generators of analytical semigroups on $\mathcal{H}$. □

Before defining the control operator, let us state some useful lemmas:

Lemma 3.2. *The operator $(\mathcal{D}(\mathcal{A}),\mathcal{A})$ defined by* (3.8) *and* (3.9) *satisfies*

$$\mathcal{D}((\lambda_0\,\mathrm{Id}-\mathcal{A})^\alpha)=\mathcal{D}((\lambda_0\,\mathrm{Id}-\mathcal{A}^*)^\alpha)=\left[H^{2\alpha}(\Omega)^2\times\mathcal{D}\left(A_1^{1/2+\alpha/2}\right)\times\mathcal{D}\left(A_1^{\alpha/2}\right)\right]\cap\mathcal{H}$$
$$(\alpha\in[0,1/4)\cup(1/4,3/4)),$$

and for $\alpha\in(3/4,1)$,

$$\mathcal{D}((\lambda_0\,\mathrm{Id}-\mathcal{A})^\alpha)=\Big\{(v,\xi_1,\xi_2)\in\left[H^{2\alpha}(\Omega)^2\times\mathcal{D}\left(A_1^{1/2+\alpha/2}\right)\times\mathcal{D}\left(A_1^{\alpha/2}\right)\right]\cap\mathcal{H}$$
$$\text{such that }\left[2\mathbb{D}(v)\nu+(v-\mathcal{T}\xi_2)+\mathbf{L}^1(\xi_1)\nu+\mathbf{L}^4(\xi_1)\right]_\tau=0\quad\text{on }\partial\Omega\Big\},$$

$$\mathcal{D}((\lambda_0\,\mathrm{Id}-\mathcal{A}^*)^\alpha)=\Big\{(\varphi,\zeta_1,\zeta_2)\in\left[H^{2\alpha}(\Omega)^2\times\mathcal{D}\left(A_1^{1/2+\alpha/2}\right)\times\mathcal{D}\left(A_1^{\alpha/2}\right)\right]\cap\mathcal{H}$$
$$\text{such that }\left[2\mathbb{D}(\varphi)\nu+(\varphi-\mathcal{T}\zeta_2)\right]_\tau=0\quad\text{on }\partial\Omega\Big\}.$$

Proof. We introduce the spaces

$$L^2_\nu(\Omega) := \left\{ f \in L^2(\Omega)^2 \ : \ \operatorname{div} f = 0, \quad f_\nu = 0 \right\}, \quad H^2_\nu(\Omega) := \left\{ f \in H^2(\Omega)^2 \cap L^2_\nu(\Omega) \ : \ [2(\mathbb{D}f)\nu + f]_\tau = 0 \right\}.$$

It is shown in [51] that

$$\left[H^2_\nu(\Omega), L^2_\nu(\Omega)\right]_{1-\alpha} = L^2_\nu(\Omega) \cap H^{2\alpha}(\Omega)^2 \quad (\alpha \in [0, 1/4) \cup (1/4, 3/4)),$$

$$\left[H^2_\nu(\Omega), L^2_\nu(\Omega)\right]_{1-\alpha} = \left\{ f \in H^{2\alpha}(\Omega)^2 \cap L^2_\nu(\Omega) \ : \ [2(\mathbb{D}f)\nu + f]_\tau = 0 \right\}. \quad (\alpha \in (3/4, 1)).$$

Then, we consider the operator $\mathbf{E}$ defined by $\mathbf{E}(\xi_1, \xi_2) := \widehat{v}$ is the solution of

$$\left\{ \begin{array}{rl} \lambda_0 \widehat{v} - \nabla \cdot \mathbb{T}(\widehat{v}, \widehat{p}) - \nabla \cdot \mathbf{L}^1(\xi_1) = 0 & \text{in } \Omega, \\ \operatorname{div} \widehat{v} = 0 & \text{in } \Omega, \\ [\widehat{v} - \mathcal{T}\xi_2]_\nu = 0 & \text{on } \partial\Omega, \\ \left[2\mathbb{D}(\widehat{v})\nu + (\widehat{v} - \mathcal{T}\xi_2) + \mathbf{L}^1(\xi_1)\nu + \mathbf{L}^4(\xi_1)\right]_\tau & \text{on } \partial\Omega. \end{array} \right. \tag{3.17}$$

If $(\xi_1, \xi_2) \in \mathcal{D}(A_1) \times \mathcal{D}(A_1^{1/2})$, then $\mathcal{T}\xi_2 \in H^2(\partial\Omega)^2$ and from Proposition 2.1, $\mathbf{L}^1(\xi_1) \in H^1(\Omega)^4$, $\mathbf{L}^4(\xi_1) \in H^{1/2}(\partial\Omega)^2$. Thus, from [1] or [10], there exists a unique strong solution $(\widehat{v}, \widehat{p}) \in H^2(\Omega)^2 \times H^1(\Omega)/\mathbb{R}$ of the above system and we deduce that $\mathbf{E} \in \mathcal{L}\left(\mathcal{D}(A_1) \times \mathcal{D}(A_1^{1/2}), H^2(\Omega)^2\right)$. If $(\xi_1, \xi_2) \in \mathcal{D}(A_1^{1/2}) \times L^2_0(\mathcal{I})$, then $\mathcal{T}\xi_2 \in L^2(\partial\Omega)^2$ and from Proposition 2.1, $\mathbf{L}^1(\xi_1) \in L^2(\Omega)^4$, $\mathbf{L}^4(\xi_1) \in L^2(\partial\Omega)^2$. Therefore, there exists a unique solution $\widehat{v} \in L^2(\Omega)^2$ of (3.17) by using the transposition method. We thus obtain $\mathbf{E} \in \mathcal{L}\left(\mathcal{D}(A_1^{1/2}) \times L^2_0(\mathcal{I}), L^2(\Omega)^2\right)$.

Now, we can see that (1.24) and (3.8) can be written as follows

$$\mathcal{H} = \left\{ (v, \xi_1, \xi_2) \in L^2(\Omega)^2 \times \mathcal{D}(A_1^{1/2}) \times L^2_0(\mathcal{I}) \ : \ v - \mathbf{E}(\xi_1, \xi_2) \in L^2_\nu(\Omega) \right\},$$

$$\mathcal{D}(\mathcal{A}) = \left\{ (v, \xi_1, \xi_2) \in H^2(\Omega)^2 \times \mathcal{D}(A_1) \times \mathcal{D}(A_1^{1/2}) \ v - \mathbf{E}(\xi_1, \xi_2) \in H^2_\nu(\Omega) \right\},$$

and from (3.12), we deduce that

$$\mathcal{D}((\lambda_0 \operatorname{Id} - \mathcal{A})^\alpha) = \left\{ (v, \xi_1, \xi_2) \in H^{2\alpha}(\Omega)^2 \times \mathcal{D}\left(A_1^{1/2+\alpha/2}\right) \times \mathcal{D}\left(A_1^{\alpha/2}\right) \ : \ v - \mathbf{E}(\xi_1, \xi_2) \in \left[H^2_\nu(\Omega), L^2_\nu(\Omega)\right]_{1-\alpha} \right\},$$

which gives the result for $\mathcal{D}((\lambda_0 \operatorname{Id} - \mathcal{A})^\alpha)$. The proof for $\mathcal{D}((\lambda_0 \operatorname{Id} - \mathcal{A}^*)^\alpha)$ is similar. □

Lemma 3.3. *The orthogonal projection*

$$\mathbb{P} : L^2(\Omega)^2 \times \mathcal{D}(A_1^{1/2}) \times L^2_0(\mathcal{I}) \longrightarrow \mathcal{H},$$

satisfies for $\alpha \in [0, 2]$

$$\mathbb{P} \in \mathcal{L}\left(H^\alpha(\Omega)^2 \times \mathcal{D}\left(A_1^{1/2+\alpha/4}\right) \times \mathcal{D}\left(A_1^{\alpha/4}\right)\right). \tag{3.18}$$

Proof. The proof is already done in [6] for $\alpha \in [0, 1]$. Here our domain Ω is assumed to be more regular so that we can obtain the result for $\alpha = 2$ and proceed by interpolation to deduce the lemma. Assume

$$(v, \xi_1, \xi_2) \in H^2(\Omega)^2 \times \mathcal{D}(A_1) \times \mathcal{D}\left(A_1^{1/2}\right).$$

It is obtained in the proof of Proposition 3.2 in [6] that

$$\mathbb{P} \begin{bmatrix} v \\ \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} v - \nabla p \\ \xi_1 \\ \xi_2 + \mathcal{T}^*(p\nu) \end{bmatrix},$$

where p is the solution of the Neumann problem

$$\left\{\begin{array}{rl} \Delta p = \operatorname{div} v & \text{in } \Omega, \\ \dfrac{\partial p}{\partial \nu} + \mathcal{T}\mathcal{T}^*(p\nu)\cdot\nu = (v - \mathcal{T}\xi_2)\cdot\nu & \text{on } \partial\Omega, \end{array}\right. \tag{3.19}$$

where we recall that (see (1.16) and (1.17)),

$$\mathcal{T}\mathcal{T}^*(p\nu)\cdot\nu = \left\{\begin{array}{rl} 0 & \text{in } \Gamma_{\mathrm{b}}, \\ \dfrac{p(y_1, 1+\eta^{\mathrm{S}}(y_1))}{\sqrt{1+|\partial_{y_1}\eta^{\mathrm{S}}(y_1)|^2}} & \text{if } (y_1, 1+\eta^{\mathrm{S}}(y_1)) \in \Gamma. \end{array}\right.$$

Using standard results on the elliptic regularity of the Laplace equation with Robin equations (see, for instance, [26, Theorem 2.5.1.1, p.128]) and that $\partial\Omega$ is $C^{2,1}$, we deduce that $p \in H^3(\Omega)$ and thus (3.18) for $\alpha = 2$. This concludes the proof of the result. □

In order to define the operator $\mathcal{B}$, we consider $\lambda_0 > 0$ large enough such that $\lambda_0 \in \rho(\mathcal{A})$ and we consider the solution $\mathbb{E}U := (v^U, \xi_1^U, \xi_2^U)$ of the system

$$\left\{\begin{array}{rl} \lambda_0 v^U - \nabla\cdot\mathbb{T}(v^U, p^U) + (v^{\mathrm{S}}\cdot\nabla)v^U + (v^U\cdot\nabla)v^{\mathrm{S}} - \nabla\cdot\mathbf{L}^1(\xi_1^U) + \mathbf{L}^2(\xi_1^U) + \mathbf{L}^3(\xi_2^U) = 0 & \text{in } \Omega, \\ \operatorname{div} v^U = 0 & \text{in } \Omega, \\ \lambda_0\xi_1^U = \xi_2^U & \text{in } \mathcal{I}, \\ \lambda_0\xi_2^U + A_1\xi_1^U + A_2\xi_2^U + \mathcal{T}^*(\mathbf{L}^1(\xi_1^U)\nu) = -\mathcal{T}^*(\mathbb{T}(v^U, p^U)\nu) & \text{in } \mathcal{I}, \end{array}\right. \tag{3.20}$$

$$\left\{\begin{array}{rl} \left[v^U - \mathcal{T}\xi_2^U\right]_\nu = (\mathcal{M}U)_\nu & \text{on } \partial\Omega, \\ \left[2\mathbb{D}(v^U)\nu + (v^U - \mathcal{T}\xi_2^U) + \mathbf{L}^1(\xi_1^U)\nu + \mathbf{L}^4(\xi_2^U)\right]_\tau = (\mathcal{M}U)_\tau & \text{on } \partial\Omega. \end{array}\right. \tag{3.21}$$

We have the following result on the above lifting

Lemma 3.4. *The operator $\mathbb{E}$ defined above satisfies for $s \in \left[-\frac{1}{2}, \frac{3}{2}\right]$*

$$\mathbb{E} \in \mathcal{L}\left(H^s(\partial\Omega), [H^{s+1/2}(\Omega)]^2 \times \mathcal{D}(A_1^{s/4+5/8}) \times \mathcal{D}(A_1^{s/4+1/8})\right).$$

Proof. We prove the above result by interpolation: for $s = 3/2$, we note that if $U \in H^{3/2}(\partial\Omega)^2$, then $\mathcal{M}U$ defined by (3.1) is in $H^{3/2}(\partial\Omega)^2$ and satisfies (3.2). As a consequence (see [1] or [10]), there exists a unique strong solution $(\widehat{v}, \widehat{p}) \in H^2(\Omega)^2 \times H^1(\Omega)/\mathbb{R}$ of

$$\left\{\begin{array}{rl} \lambda_0\widehat{v} - \nabla\cdot\mathbb{T}(\widehat{v}, \widehat{p}) = 0 & \text{in } \Omega, \\ \operatorname{div}\widehat{v} = 0 & \text{in } \Omega, \\ [\widehat{v}]_\nu = (\mathcal{M}U)_\nu & \text{on } \partial\Omega, \\ [2\mathbb{D}(\widehat{v})\nu + \widehat{v}]_\tau = (\mathcal{M}U)_\tau & \text{on } \partial\Omega. \end{array}\right.$$

Then, we consider the solution $(\widehat{v}^U, \xi_1^U, \xi_2^U) \in \mathcal{D}(\mathcal{A})$ of

$$(\lambda_0\,\mathrm{Id} - \mathcal{A})\begin{bmatrix}\widehat{v}^U \\ \xi_1^U \\ \xi_2^U\end{bmatrix} := \mathbb{P}\begin{bmatrix} -(v^{\mathrm{S}}\cdot\nabla)\widehat{v} - (\widehat{v}\cdot\nabla)v^{\mathrm{S}} \\ 0 \\ -\mathcal{T}^*(2\mathbb{D}(\widehat{v}^U)\nu)\end{bmatrix} \in \mathcal{H}.$$

Setting $v^U := \widehat{v}^U + \widehat{v} \in H^2(\Omega)^2$ we can check that (v^U, ξ_1^U, ξ_2^U) is the solution of (3.20)–(3.21).

For $s = -1/2$, we define the solution by transposition: (v^U, ξ_1^U, ξ_2^U) is the unique element of $\mathcal{H}$ such that for any $F \in \mathcal{H}$,

$$\left\langle \begin{bmatrix} v^U \\ \xi_1^U \\ \xi_2^U \end{bmatrix}, F \right\rangle_{\mathcal{H}} = -\left\langle U, \mathcal{M}\left(\mathbb{T}(\varphi, \pi_\varphi)\nu\right)\right\rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}, \quad \text{where} \quad \begin{bmatrix}\varphi \\ \zeta_1 \\ \zeta_2\end{bmatrix} = \left(\lambda_0\,\mathrm{Id} - \mathcal{A}^*\right)^{-1} F, \tag{3.22}$$

and where $\pi_\varphi \in H^1(\Omega)$ is such that

$$\begin{bmatrix} \nabla\pi_\varphi \\ 0 \\ -\mathcal{T}^*(\pi_\varphi\nu) \end{bmatrix} = (\mathrm{Id} - \mathbb{P}) \begin{bmatrix} \Delta\varphi + (v^{\mathrm{S}}\cdot\nabla)\varphi - \left(\nabla v^{\mathrm{S}}\right)^*\varphi \\ -\zeta_2 - \left(\mathbf{L}^1\right)^*(\nabla\varphi) - \left(\mathbf{L}^2\right)^*(\varphi) + \left(\mathbf{L}^4\right)^*(\varphi - \mathcal{T}\zeta_2) \\ A_1\zeta_1 - A_2\zeta_2 - \mathcal{T}^*\left(2\left(\mathbb{D}\varphi\right)\nu\right) - \left(\mathbf{L}^3\right)^*(\varphi) \end{bmatrix}. \tag{3.23}$$

□

Then, we can define the control operator $\mathcal{B} \in \mathcal{L}\left(\mathcal{U}, (\mathcal{D}(\mathcal{A}^*)')\right)$ by

$$\mathcal{B} := (\lambda_0 \,\mathrm{Id} - \mathcal{A})\,\mathbb{P}\mathbb{E}. \tag{3.24}$$

Combining Theorem 3.1, Lemma 3.2, Lemma 3.3 and Lemma 3.4, we deduce the following result:

Proposition 3.5. *For any $\varepsilon \in (0, 1/4)$,*

$$(\lambda_0 \,\mathrm{Id} - \mathcal{A})^{-1+\varepsilon}\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{H}).$$

Moreover, the adjoint of $\mathcal{B}$ is given by

$$\mathcal{B}^* \begin{bmatrix} \varphi \\ \zeta_1 \\ \zeta_2 \end{bmatrix} = -\mathcal{M}\left(\mathbb{T}(\varphi, \pi_\varphi)\nu\right), \tag{3.25}$$

where π_φ is given by (3.23).

Using the operator $\mathcal{A}$ and $\mathcal{B}$ defined by (3.8), (3.9) and (3.24), we can check that (3.3)–(3.5) writes as (3.6), (3.7). Indeed, taking $\Phi := (\varphi, \zeta_1, \zeta_2) \in \mathcal{D}(\mathcal{A}^*)$ and multiplying the first equation of (3.3) by φ and the third equation by ζ_2, we obtain after some integrations by parts, that

$$\left\langle \frac{d}{dt}W, \Phi \right\rangle_{\mathcal{H}} = \langle W, \mathcal{A}^*\Phi\rangle_{\mathcal{H}} + \langle U, \mathcal{B}^*\Phi\rangle_{\mathcal{U}} + \langle F, \Phi\rangle_{\mathcal{H}}.$$

Moreover, we have $W - \mathbb{E}U \in \mathcal{H}$ so that $(\mathrm{Id} - \mathbb{P})(W - \mathbb{E}U) = 0$.

3.2 Stabilization of the linear system

We can now check that $(\mathcal{A}, \mathcal{B})$ satisfies the Fattorini-Hautus stabilization criterion (see, [4] and [5, Theorem 1.6]); it is stabilizable if the following condition hold

$$\mathcal{A}^* \begin{bmatrix} \varphi \\ \zeta_1 \\ \zeta_2 \end{bmatrix} = \lambda \begin{bmatrix} \varphi \\ \zeta_1 \\ \zeta_2 \end{bmatrix}, \quad \mathcal{B}^* \begin{bmatrix} \varphi \\ \zeta_1 \\ \zeta_2 \end{bmatrix} = 0 \implies \begin{bmatrix} \varphi \\ \zeta_1 \\ \zeta_2 \end{bmatrix} = 0. \tag{3.26}$$

We thus consider $(\varphi, \zeta_1, \zeta_2) \in [H^2(\Omega)]^2 \times \mathcal{D}(A_1) \times \mathcal{D}(A_1^{1/2})$ satisfying

$$\begin{cases} \lambda\varphi - \nabla\cdot\mathbb{T}(\varphi, \pi_\varphi) - (v^{\mathrm{S}}\cdot\nabla)\varphi + \left(\nabla v^{\mathrm{S}}\right)^*\varphi = 0 & \text{in } \Omega, \\ \operatorname{div}\varphi = 0 & \text{in } \Omega, \\ \lambda\zeta_1 + \zeta_2 + \left(\mathbf{L}^1\right)^*(\nabla\varphi) + \left(\mathbf{L}^2\right)^*(\varphi) - \left(\mathbf{L}^4\right)^*(\varphi - \mathcal{T}\zeta_2) = 0 & \text{in } \mathcal{I}, \\ \lambda\zeta_2 - A_1\zeta_1 + A_2\zeta_2 + \mathcal{T}^*\left(\mathbb{T}(\varphi, \pi_\varphi)\nu\right) + \left(\mathbf{L}^3\right)^*(\varphi) = 0 & \text{in } \mathcal{I}, \end{cases} \tag{3.27}$$

$$\begin{cases} [\varphi - \mathcal{T}\zeta_2]_\nu = 0 & \text{on } \partial\Omega, \\ [2\mathbb{D}(\varphi)\nu + (\varphi - \mathcal{T}\zeta_2)]_\tau = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.28}$$

and

$$\mathcal{M}\left(\mathbb{T}(\varphi, \pi_\varphi)\nu\right) = 0. \tag{3.29}$$

We set

$$c_\varphi := \int_{\Gamma_{\rm cont}} m\mathbb{T}(\varphi,\pi_\varphi)\nu\cdot\nu\ dy_1,$$

and we deduce from (3.1) that

$$\mathbb{T}(\varphi,\pi_\varphi + c_\varphi)\nu = 0 \quad \text{on } \operatorname{supp} m \subset \Gamma_{\rm b}. \tag{3.30}$$

Combining this with (3.28), this yields that $\varphi = 0$ on $\operatorname{supp} m$. We can thus apply a classical unique continuation result [20] for the Stokes system (see [5, Appendix A] for more details) and we deduce $(\varphi,\pi_\varphi) = 0$ in Ω and then (3.26) by using (3.27)–(3.28).

Applying [4, Proposition 17] or [5, Theorem 1.6], we can construct a feedback stabilization with finite dimensional controllers. More precisely, for a finite family $\left(\widetilde{w}^{(j)}\right)_{j=1,\dots,N_\sigma} \subset \mathcal{U}$ we define the operator

$$\mathbb{B} : \mathbb{R}^{N_\sigma} \to \mathcal{D}(\mathcal{A}^*)', \quad u \mapsto \sum_{j=1}^{N_\sigma} u_j \mathcal{B}\widetilde{w}^{(j)}. \tag{3.31}$$

Then we deduce the following result from (3.26):

Proposition 3.6. *For $\sigma > 0$, there exist $N_\sigma \in \mathbb{N}^*$, a family $\left(\widetilde{w}^{(j)}\right)_{j=1,\dots,N_\sigma} \subset H^{3/2}(\Gamma_{\rm b})^2$ and operators $\mathcal{R}_\sigma^{(1)} \in \mathcal{L}(\mathcal{H},\mathbb{R}^{N_\sigma})$, $\mathcal{R}_\sigma^{(2)} \in \mathcal{L}(\mathbb{R}^{N_\sigma})$ such that the operator*

$$\mathcal{D}\left(\mathcal{A}_\sigma\right) := \left\{[v,\xi_1,\xi_2,u] \in \mathcal{H}\times\mathbb{R}^{N_\sigma} \ :\ \mathcal{A}[v,\xi_1,\xi_2] + \mathbb{B}u \in \mathcal{H}\right\}, \quad \mathcal{A}_\sigma := \begin{bmatrix} \mathcal{A} & \mathbb{B} \\ \mathcal{R}_\sigma^{(1)} & \mathcal{R}_\sigma^{(2)} \end{bmatrix} \tag{3.32}$$

is the infinitesimal generator of an analytic and exponentially stable semigroup on $\mathcal{H}\times\mathbb{R}^{N_\sigma}$ of type lower that $-\sigma$.

Proof. Since the proof is already done in [4] and in [5], we only recall briefly the main arguments for the exponential stabilization. From (3.26) and from [5, Theorem 1.6], there exists a family $\left(\widetilde{w}^{(j)}\right)_{j=1,\dots,N_\sigma} \subset H^{3/2}(\Gamma_{\rm b})^2$ such that the system $Z' = \mathcal{A}Z + \mathbb{B}u$ is exponentially stabilizable with a decay rate of order $-\sigma$. Moreover, we have (see [5, Theorem 1.6, item 3.]),

$$\forall\lambda\in\mathbb{C},\ \operatorname{Re}\lambda \geqslant -\sigma, \quad \forall\Phi\in\operatorname{Ker}(\lambda I - \mathcal{A}^*), \quad \begin{pmatrix} \langle\widetilde{w}^{(1)},\mathcal{B}^*\Phi\rangle_{\mathcal{U}} \\ \vdots \\ \langle\widetilde{w}^{(N_\sigma)},\mathcal{B}^*\Phi\rangle_{\mathcal{U}} \end{pmatrix} = 0 \implies \Phi = 0. \tag{3.33}$$

Then, we can consider the operators

$$\mathcal{D}\left(\mathcal{A}_\flat\right) := \left\{[v,\xi_1,\xi_2,u] \in \mathcal{H}\times\mathbb{R}^{N_\sigma} \ :\ \mathcal{A}[v,\xi_1,\xi_2] + \mathbb{B}u \in \mathcal{H}\right\}, \quad \mathcal{A}_\flat := \begin{bmatrix} \mathcal{A} & \mathbb{B} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{B}_\flat := \begin{bmatrix} 0 \\ I_N \end{bmatrix}. \tag{3.34}$$

They corresponds to the control problem

$$Z' = \mathcal{A}Z + \mathbb{B}u, \quad u' = u^\star, \tag{3.35}$$

where u is now included in the state (Z,u) and the new control is $u^\star$. The Fattorini-Hautus criterion applied on $(\mathcal{A}_\flat^*,\mathcal{B}_\flat^*)$ leads to the condition

$$\forall\lambda\in\mathbb{C},\ \operatorname{Re}\lambda \geqslant -\sigma,\ \mathcal{A}^*\Phi = \lambda\Phi, \quad \mathbb{B}^*\Phi = 0 \implies \Phi = 0. \tag{3.36}$$

However, from (3.31)

$$\mathbb{B}^*\Phi = \begin{pmatrix} \langle\widetilde{w}^{(1)},\mathcal{B}^*\Phi\rangle_{\mathcal{U}} \\ \vdots \\ \langle\widetilde{w}^{(N_\sigma)},\mathcal{B}^*\Phi\rangle_{\mathcal{U}} \end{pmatrix},$$

so that (3.33) implies (3.36). Hence, using again [5, Theorem 1.6], we deduce the existence of $\mathcal{R}_\sigma^{(1)} \in \mathcal{L}(\mathcal{H}, \mathbb{R}^{N_\sigma})$ and $\mathcal{R}_\sigma^{(2)} \in \mathcal{L}(\mathbb{R}^{N_\sigma})$ such that the control

$$u^\star = \mathcal{R}_\sigma^{(1)} Z + \mathcal{R}_\sigma^{(2)} u$$

stabilizes exponentially the system (3.35) with a decay rate of order $-\sigma$. □

We define

$$w^{(j)} := \mathcal{M}\widetilde{w}^{(j)}, \tag{3.37}$$

that satisfies $w^{(j)} \in H^{3/2}(\partial\Omega)^2$ with $\operatorname{supp} w^{(j)} \in \Gamma_{\mathrm{cont}}$. The operator $\mathbb{M}$ is defined by (1.7) that is

$$\mathbb{M} : \mathbb{R}^{N_\sigma} \to L^2(\partial\Omega)^2, \quad u \mapsto \sum_{j=1}^{N_\sigma} u_j \mathcal{M}\widetilde{w}^{(j)}. \tag{3.38}$$

Using Proposition 3.6, we are going to show that the following linear system is exponentially stable:

$$\begin{cases} \partial_t v - \nabla\cdot\mathbb{T}(v,p) + (v^{\mathrm{S}}\cdot\nabla)v + (v\cdot\nabla)v^{\mathrm{S}} - \nabla\cdot\mathbf{L}^1(\xi) + \mathbf{L}^2(\xi) + \mathbf{L}^3(\partial_t\xi) = f & \text{in } (0,\infty)\times\Omega, \\ \nabla\cdot v = 0 & \text{in } (0,\infty)\times\Omega, \\ \partial_{tt}\xi + A_1\xi + A_2\partial_t\xi + \mathcal{T}^*(\mathbf{L}^1(\xi)\nu) = -\mathcal{T}^*(\mathbb{T}(v,p)\nu) + h & \text{in } (0,\infty)\times\mathcal{I}, \\ u' = \mathcal{R}_\sigma^{(1)}(v,\xi,\partial_t\xi) + \mathcal{R}_\sigma^{(2)}u + \gamma & \text{in } (0,\infty), \end{cases} \tag{3.39}$$

$$\begin{cases} [v - \mathcal{T}\partial_t\xi]_\nu = (\mathbb{M}u)_\nu & \text{on } (0,\infty)\times\partial\Omega, \\ \left[2\mathbb{D}(v)\nu + (v - \mathcal{T}\partial_t\xi) + \mathbf{L}^1(\xi)\nu + \mathbf{L}^4(\xi)\right]_\tau = (\mathbb{M}u)_\tau + g & \text{on } (0,\infty)\times\partial\Omega, \end{cases} \tag{3.40}$$

$$\xi(0,\cdot) = \xi_1^0 \text{ in } \mathcal{I}, \quad \partial_t\xi(0,\cdot) = \xi_2^0 \text{ in } \mathcal{I}, \quad v(0,\cdot) = v^0 \text{ in } \Omega, \quad u(0) = 0. \tag{3.41}$$

First we consider the case $g = 0$:

Proposition 3.7. *Assume*

$$[v^0, \xi_1^0, \xi_2^0] \in \mathcal{V}, \quad g = 0, \quad f \in L^2_\sigma(L^2(\Omega)^2), \quad h \in L^2_\sigma(L^2_0(\mathcal{I})), \quad \gamma \in L^2_\sigma(\mathbb{R}^{N_\sigma}).$$

Then (3.39)–(3.41) *admits a unique solution*

$$v \in L^2_\sigma\left(H^2(\Omega)^2\right) \cap H^1_\sigma\left(L^2(\Omega)^2\right), \quad \xi \in H^2_\sigma\left(L^2_0(\mathcal{I})\right) \cap H^1_\sigma\left(\mathcal{D}(A_1^{1/2})\right) \cap L^2_\sigma\left(\mathcal{D}(A_1)\right), \quad u \in H^1_\sigma(\mathbb{R}^{N_\sigma}),$$

and there exists a constant $C > 0$ such that

$$\begin{aligned} \|v\|_{L^2_\sigma(H^2(\Omega)^2)\cap H^1_\sigma(L^2(\Omega)^2)} + \|\nabla p\|_{L^2_\sigma(L^2(\Omega)^2)} + \|\xi\|_{H^2_\sigma\left(L^2_0(\mathcal{I})\right)\cap H^1_\sigma\left(\mathcal{D}(A_1^{1/2})\right)\cap L^2_\sigma(\mathcal{D}(A_1))} + \|u\|_{H^1_\sigma(\mathbb{R}^{N_\sigma})} \\ \leqslant C\left(\left\|[v^0,\xi_1^0,\xi_2^0]\right\|_{\mathcal{V}} + \|f\|_{L^2_\sigma(L^2(\Omega)^2)} + \|h\|_{L^2_\sigma(L^2_0(\mathcal{I}))} + \|\gamma\|_{L^2_\sigma(\mathbb{R}^{N_\sigma})}\right). \end{aligned} \tag{3.42}$$

Proof. We deduce from (3.6) and (3.7) that (3.39)–(3.41) can be written as

$$\begin{bmatrix} \mathbb{P}W' \\ u' \end{bmatrix} = \mathcal{A}_\sigma \begin{bmatrix} \mathbb{P}W \\ u \end{bmatrix} + \begin{bmatrix} \mathbb{P}F \\ \gamma \end{bmatrix}, \quad \begin{bmatrix} \mathbb{P}W \\ u \end{bmatrix}(0) = \begin{bmatrix} W^0 \\ 0 \end{bmatrix}, \tag{3.43}$$

$$(\mathrm{Id} - \mathbb{P})W = (\mathrm{Id} - \mathbb{P})\mathbb{E}\mathbb{M}u. \tag{3.44}$$

Using Proposition 3.6, we deduce that

$$\begin{bmatrix} \mathbb{P}W \\ u \end{bmatrix} \in L^2_\sigma\left(\mathcal{D}(\mathcal{A}) \times \mathbb{R}^{N_\sigma}\right) \cap H^1_\sigma\left(\mathcal{H} \times \mathbb{R}^{N_\sigma}\right).$$

We then deduce from Lemma 3.4 that

$$(\mathrm{Id} - \mathbb{P})W \in H^1_\sigma(H^2(\Omega)^2 \times \mathcal{D}(A_1) \times \mathcal{D}(A_1^{1/2})).$$

□

Let us now handle the case

$$f = 0, \quad h = 0, \quad \gamma = 0, \quad \xi_1^0 = \xi_2^0 = 0, \quad v^0 = 0, \tag{3.45}$$

and

$$g \in H^{1/4}_\sigma([L^2(\partial\Omega)]^2) \cap L^2_\sigma([H^{1/2}(\partial\Omega)]^2). \tag{3.46}$$

To obtain the regularity of the solutions, we adapt the proof done in [50].

Proposition 3.8. *Assume* (3.45), (3.46). *Then* (3.39)–(3.41) *admits a unique solution*

$$v \in L^2_\sigma\left(H^2(\Omega)^2\right) \cap H^1_\sigma\left(L^2(\Omega)^2\right), \quad p \in L^2_\sigma\left(H^1(\Omega)/\mathbb{R}\right),$$
$$\xi \in H^2_\sigma\left(L^2_0(\mathcal{I})\right) \cap H^1_\sigma\left(\mathcal{D}(A_1^{1/2})\right) \cap L^2_\sigma\left(\mathcal{D}(A_1)\right), \quad u \in H^1_\sigma(\mathbb{R}^{N_\sigma}),$$

and there exists a constant $C > 0$ such that

$$\|v\|_{L^2_\sigma(H^2(\Omega)^2)\cap H^1_\sigma(L^2(\Omega)^2)} + \|\nabla p\|_{L^2_\sigma(L^2(\Omega)^2)} + \|\xi\|_{H^2_\sigma\left(L^2_0(\mathcal{I})\right)\cap H^1_\sigma\left(\mathcal{D}(A_1^{1/2})\right)\cap L^2_\sigma(\mathcal{D}(A_1))} + \|u\|_{H^1_\sigma(\mathbb{R}^{N_\sigma})}$$
$$\leqslant C\,\|g\|_{H^{1/4}_\sigma([L^2(\partial\Omega)]^2)\cap L^2_\sigma([H^{1/2}(\partial\Omega)]^2)}. \tag{3.47}$$

Proof. By using a change of variable $g \mapsto e^{\sigma t} g$, it is sufficient to show the proposition in the case $\sigma = 0$. Assume (3.45) and let us construct by density the linear mapping

$$g \mapsto (v, \xi, u),$$

from $H^{1/4}(L^2(\partial\Omega)^2) \cap L^2(H^{1/2}(\partial\Omega)^2)$ into

$$\left[L^2\left(H^2(\Omega)^2\right) \cap H^1\left(L^2(\Omega)^2\right)\right] \times \left[H^2\left(L^2_0(\mathcal{I})\right) \cap H^1\left(\mathcal{D}(A_1^{1/2})\right) \cap L^2\left(\mathcal{D}(A_1)\right)\right] \times H^1(\mathbb{R}^{N_\sigma}).$$

More precisely, we first assume $g \in C^\infty(\mathbb{R}; H^{1/2}(\partial\Omega)^2)$ with compact support in $\mathbb{R}_+$. In that case, for $\lambda > 0$, from [10] and [1], there exists a unique solution

$$(v_g, p_g) \in C^\infty(\mathbb{R}; H^2(\Omega)^2 \times H^1(\Omega)/\mathbb{R}),$$

of

$$\begin{cases} \lambda v_g - \nabla \cdot \mathbb{T}(v_g, p_g) = 0 & \text{in } (0,\infty)\times\Omega, \\ \nabla \cdot v_g = 0 & \text{in } (0,\infty)\times\Omega, \\ [v_g]_\nu = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ [2\mathbb{D}(v_g)\nu + v_g]_\tau = g & \text{on } (0,\infty)\times\partial\Omega, \end{cases} \tag{3.48}$$

and there exists a constant $C > 0$ such that

$$\|(v_g, p_g)\|_{L^2(H^2(\Omega)^2\times H^1(\Omega)/\mathbb{R})} \leqslant C\,\|g\|_{L^2(H^{1/2}(\partial\Omega))}.$$

Using this lifting in (3.39)–(3.41) and applying Proposition 3.7, we deduce that (3.39)–(3.41) admits a unique solution with

$$(v, \xi, u) \in C^\infty\left(\mathbb{R}_+; H^2(\Omega)^2 \times \mathcal{D}(A_1) \times \mathbb{R}^{N_\sigma}\right).$$

Let us estimate this solution. In order to do that, we differentiate (3.39)–(3.41) with respect to time:

$$\begin{cases} \partial_t^2 v - \nabla \cdot \mathbb{T}(\partial_t v, \partial_t p) + (v^{\mathrm{S}} \cdot \nabla)\partial_t v + (\partial_t v \cdot \nabla) v^{\mathrm{S}} - \nabla \cdot \mathbf{L}^1(\partial_t \xi) + \mathbf{L}^2(\partial_t \xi) + \mathbf{L}^3(\partial_t^2 \xi) = 0 & \text{in } (0,\infty)\times\Omega, \\ \operatorname{div} \partial_t v = 0 & \text{in } (0,\infty)\times\Omega, \\ \partial_t^3 \xi + A_1 \partial_t \xi + A_2 \partial_t^2 \xi + \mathcal{T}^*(\mathbf{L}^1(\partial_t \xi)\nu) = -\mathcal{T}^*(\mathbb{T}(\partial_t v, \partial_t p)\nu) & \text{in } (0,\infty)\times\mathcal{I}, \\ u'' = \mathcal{R}^{(1)}_\sigma(\partial_t v, \partial_t \xi, \partial_t^2 \xi) + \mathcal{R}^{(2)}_\sigma u' & \text{in } (0,\infty), \end{cases} \tag{3.49}$$

$$\begin{cases} \left[\partial_t v - \mathcal{T}\partial_t^2 \xi\right]_\nu = (\mathbb{M}u')_\nu & \text{on } (0,\infty)\times\partial\Omega, \\ \left[2\mathbb{D}(\partial_t v)\nu + (\partial_t v - \mathcal{T}\partial_t^2\xi) + \mathbf{L}^1(\partial_t\xi)\nu + \mathbf{L}^4(\partial_t\xi)\right]_\tau = (\mathbb{M}u')_\tau + \partial_t g & \text{on } (0,\infty)\times\partial\Omega, \end{cases} \tag{3.50}$$

$$\partial_t\xi(0,\cdot) = 0 \text{ in } \mathcal{I}, \quad \partial_t^2\xi(0,\cdot) = 0 \text{ in } \mathcal{I}, \quad \partial_t v(0,\cdot) = 0 \text{ in } \Omega, \quad u'(0) = 0. \tag{3.51}$$

Let us consider

$$(\Psi, d) \in C^\infty\left(\mathbb{R}; \mathcal{H}\times\mathbb{R}^{N_\sigma}\right),$$

with compact support in $\mathbb{R}_+$. There exists $T>0$ such that $\operatorname{supp}(\Psi,d)\subset[0,T]$ and we consider the unique solution

$$(\Phi,\mu) \in L^2\left((-\infty,T]; (\mathcal{D}(\mathcal{A}^*)\times\mathbb{R}^{N_\sigma}\right) \cap H^1\left((-\infty,T]; \mathcal{H}\times\mathbb{R}^{N_\sigma}\right), \tag{3.52}$$

of

$$\begin{bmatrix} -\partial_t\Phi \\ -\mu' \end{bmatrix} = \mathcal{A}_\sigma^* \begin{bmatrix} \Phi \\ \mu \end{bmatrix} + \begin{bmatrix} \Psi \\ d \end{bmatrix}, \quad \begin{bmatrix} \Phi \\ \mu \end{bmatrix}(T) = 0, \tag{3.53}$$

where $\mathcal{A}_\sigma$ is defined by (3.32). Setting

$$\Phi = \begin{bmatrix} \varphi \\ \zeta_1 \\ \zeta_2 \end{bmatrix}, \quad \Psi = \begin{bmatrix} \psi \\ \kappa_1 \\ \kappa_2 \end{bmatrix}, \quad \left(\mathcal{R}_\sigma^{(1)}\right)^* \mu = \begin{bmatrix} \mathcal{R}_\sigma^{(3)}\mu \\ \mathcal{R}_\sigma^{(4)}\mu \\ \mathcal{R}_\sigma^{(5)}\mu \end{bmatrix},$$

the system (3.53) can be written as

$$\begin{cases} -\partial_t\varphi - \nabla\cdot\mathbb{T}(\varphi,\pi_\varphi) - (v^{\mathrm{S}}\cdot\nabla)\varphi + \left(\nabla v^{\mathrm{S}}\right)^*\varphi - \mathcal{R}_\sigma^{(3)}\mu = \psi & \text{in } (-\infty,T)\times\Omega, \\ \operatorname{div}\varphi = 0 & \text{in } (-\infty,T)\times\Omega, \\ -\partial_t\zeta_1 + \zeta_2 + \left(\mathbf{L}^1\right)^*(\nabla\varphi) + \left(\mathbf{L}^2\right)^*(\varphi) - \left(\mathbf{L}^4\right)^*(\varphi - \mathcal{T}\zeta_2) - \mathcal{R}_\sigma^{(4)}\mu = \kappa_1 & \text{in } (-\infty,T)\times\mathcal{I}, \\ -\partial_t\zeta_2 - A_1\zeta_1 + A_2\zeta_2 + \mathcal{T}^*\left(\mathbb{T}(\varphi,\pi_\varphi)\nu\right) + \left(\mathbf{L}^3\right)^*(\varphi) - \mathcal{R}_\sigma^{(5)}\mu = \kappa_2 & \text{in } (-\infty,T)\times\mathcal{I}, \\ -\mu' = \mathbb{B}^*(\varphi,\zeta_1,\zeta_2) + \left(\mathcal{R}_\sigma^{(2)}\right)^*\mu + d & \text{in } (-\infty,T), \end{cases} \tag{3.54}$$

$$\begin{cases} [\varphi - \mathcal{T}\zeta_2]_\nu = 0 & \text{on } (-\infty,T)\times\partial\Omega, \\ [2\mathbb{D}(\varphi)\nu + (\varphi - \mathcal{T}\zeta_2)]_\tau = 0 & \text{on } (-\infty,T)\times\partial\Omega, \end{cases} \tag{3.55}$$

$$\zeta_1(T,\cdot) = 0 \text{ in } \mathcal{I}, \quad \zeta_2(T,\cdot) = 0 \text{ in } \mathcal{I}, \quad \varphi(T,\cdot) = 0 \text{ in } \Omega, \quad \mu(T) = 0. \tag{3.56}$$

Note that from (3.25), (3.31) and (3.38)

$$\mathbb{B}^*(\varphi,\zeta_1,\zeta_2)\cdot u = -\int_{\partial\Omega} \mathbb{M}u\cdot\mathbb{T}(\varphi,\pi_\varphi)\nu \, ds.$$

Let us multiply the first equation of (3.49) by φ and integrate in $[0,T]$. After several integrations by parts and using the above relation, (3.49)–(3.51) and (3.54)–(3.56), we obtain

$$\int_0^T \left\langle \left[\partial_t v, \partial_t\xi, \partial_t^2\xi, u'\right], [\Psi, d]\right\rangle_{\mathcal{H}\times\mathbb{R}^{N_\sigma}} = \int_0^T\int_{\partial\Omega} (\partial_t g)\left[\varphi - \mathcal{T}\zeta_2\right]_\tau \, ds \, dt. \tag{3.57}$$

From (3.52), we have

$$[\varphi - \mathcal{T}\zeta_2]_\tau \in H^{3/4}(-\infty,T; L^2(\partial\Omega)).$$

and cancels at $t=T$. We can thus extend it by 0 as a function in $H^{3/4}(\mathbb{R}; L^2(\partial\Omega))$ and we deduce the existence of a constant $C>0$ such that

$$\begin{aligned} \left|\int_0^T\int_{\partial\Omega} (\partial_t g)\left[\varphi - \mathcal{T}\zeta_2\right]_\tau \, ds\, dt\right| &\leqslant C\,\|g\|_{H^{1/4}(\mathbb{R};L^2(\partial\Omega))}\,\|[\varphi - \mathcal{T}\zeta_2]_\tau\|_{H^{3/4}(\mathbb{R};L^2(\partial\Omega))} \\ &\leqslant C\,\|g\|_{H^{1/4}(\mathbb{R}_+;L^2(\partial\Omega))}\,\|(\Psi,d)\|_{L^2(\mathbb{R}_+;\mathcal{H}\times\mathbb{R}^{N_\sigma})}. \end{aligned}$$

The above estimate and (3.57) imply that

$$\left\| \left[\partial_t v, \partial_t \xi, \partial_t^2 \xi, u' \right] \right\|_{L^2(\mathcal{H}\times\mathbb{R}^{N_\sigma})} \leqslant C \left\| g \right\|_{H^{1/4}(\mathbb{R}_+;L^2(\partial\Omega))}. \tag{3.58}$$

We deduce similarly that

$$\left\| [v, \xi, \partial_t \xi, u] \right\|_{L^2(\mathcal{H}\times\mathbb{R}^{N_\sigma})} \leqslant C \left\| g \right\|_{H^{1/4}(\mathbb{R}_+;L^2(\partial\Omega))}. \tag{3.59}$$

Using the lifting (v_g, p_g) defined by (3.48) and some similar lifting for $\mathbb{M}u$ in (3.39)–(3.41) and Theorem 3.1, we deduce that

$$\left\| [v, \xi, \partial_t \xi, u] \right\|_{L^2\left(H^2(\Omega)^2\times\mathcal{D}(A_1)\times\mathcal{D}(A_1^{1/2})\times\mathbb{R}^{N_\sigma}\right)} \leqslant C \left(\left\| [v, \xi, \partial_t \xi, u] \right\|_{H^1(\mathcal{H}\times\mathbb{R}^{N_\sigma})} + \left\| g \right\|_{L^2(H^{1/2}(\partial\Omega))} \right).$$

Combining this result with (3.58) and (3.59), we deduce the result for $g \in C^\infty(\mathbb{R}; H^{1/2}(\partial\Omega)^2)$ with compact support in $\mathbb{R}_+$. Then, by a density argument we conclude the proof of the proposition. □

4 Proof of the main result

We are now in a position to prove Theorem 1.1. We consider $N_\sigma \in \mathbb{N}^*$, the family $\left(w^{(j)}\right)_{j\in\{1,\dots,N_\sigma\}}$ and the operators

$$\mathcal{R}_\sigma^{(1)} \in \mathcal{L}(\mathcal{H}, \mathbb{R}^{N_\sigma}), \quad \mathcal{R}_\sigma^{(2)} \in \mathcal{L}(\mathbb{R}^{N_\sigma})$$

from Proposition 3.6 and (3.37). Then, we assume $(v^0, \xi_1^0, \xi_2^0) \in \mathcal{V}$ and we define

$$R := \left\| (v^0, \xi_1^0, \xi_2^0) \right\|_{\mathcal{V}},$$

and we consider

$$\mathfrak{B}_R := \Big\{ (f,g,h) \in L^2_\sigma(L^2(\Omega)^2) \times \left[H^{1/4}_\sigma([L^2(\partial\Omega)]^2) \cap L^2_\sigma([H^{1/2}(\partial\Omega)]^2) \right] \times L^2_\sigma(L^2_0(\mathcal{I})),$$
$$\left\| f \right\|_{L^2_\sigma(L^2(\Omega)^2)} + \left\| g \right\|_{H^{1/4}_\sigma([L^2(\partial\Omega)]^2)\cap L^2_\sigma([H^{1/2}(\partial\Omega)]^2)} + \left\| h \right\|_{L^2_\sigma(L^2_0(\mathcal{I}))} \leqslant R \Big\}.$$

If $(f,g,h) \in \mathfrak{B}_R$, then from Proposition 3.7 and Proposition 3.8, (3.39)–(3.41) admits a unique solution

$$v \in L^2_\sigma\left(H^2(\Omega)^2\right) \cap H^1_\sigma\left(L^2(\Omega)^2\right), \quad \xi \in H^2_\sigma\left(L^2_0(\mathcal{I})\right) \cap H^1_\sigma\left(\mathcal{D}(A_1^{1/2})\right) \cap L^2_\sigma\left(\mathcal{D}(A_1)\right), \quad u \in H^1_\sigma(\mathbb{R}^{N_\sigma}),$$

and there exists a constant $C > 0$ such that

$$\left\| v \right\|_{L^2_\sigma(H^2(\Omega)^2)\cap H^1_\sigma(L^2(\Omega)^2)} + \left\| \nabla p \right\|_{L^2_\sigma(L^2(\Omega)^2)} + \left\| \xi \right\|_{H^2_\sigma\left(L^2_0(\mathcal{I})\right)\cap H^1_\sigma\left(\mathcal{D}(A_1^{1/2})\right)\cap L^2_\sigma(\mathcal{D}(A_1))} + \left\| u \right\|_{H^1_\sigma(\mathbb{R}^{N_\sigma})} \leqslant CR. \tag{4.1}$$

Applying Proposition 2.1, for $R > 0$ small enough, the triplet $(\mathbf{F}(v,p,\xi), \mathbf{G}(v,\xi), \mathbf{H}(v,\xi))$ is well-defined and satisfies (2.48)–(2.53). In particular, for $R > 0$ possibly smaller, the mapping

$$\begin{array}{rccl} \mathcal{Z}: & \mathfrak{B}_R & \to & \mathfrak{B}_R, \\ & (f,g,h) & \mapsto & (\mathbf{F}(v,p,\xi), \mathbf{G}(v,\xi), \mathbf{H}(v,\xi)), \end{array}$$

is well-defined and is a strict contraction. From the Banach fixed point argument, we deduce that if $R > 0$ small enough, (1.20)–(1.22) admits a strong solution. This concludes the proof of the main result of this article.

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