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GROMOV ELLIPTICITY OF CONES OVER PROJECTIVE MANIFOLDS

S. KALIMAN AND M. ZAIDENBERG

ABSTRACT. We find classes of projective manifolds that are elliptic in the sense of Gromov and such that the affine cones over these manifolds also are elliptic off their vertices. For example, the latter holds for any generalized flag manifold of dimension $n \geq 3$ successively blown up in a finite set of points and infinitesimally near points. This also holds for any smooth projective rational surface. For the affine cones, the Gromov ellipticity is a much weaker property than the flexibility. Nonetheless, it still implies the infinite transitivity of the action of the endomorphism monoid on these cones.

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INTRODUCTION

Let \mathbb{K} be an algebraically closed field of characteristic zero and \mathbb{A}^n resp. \mathbb{P}^n be the affine resp. projective n -space over \mathbb{K} . All varieties and vector bundles in this paper are algebraic. By *spray* resp. *ellipticity* we mean algebraic spray resp. algebraic ellipticity. Let X be a smooth algebraic variety. Recall that a *spray of rank r* over X is a triple (E, p, s) where $p: E \rightarrow X$ is a vector bundle of rank r with zero section Z and $s: E \rightarrow X$ is a morphism such that $s|_Z = p|_Z$. A spray (E, p, s) is *dominating*

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at $x \in X$ if the restriction $s|_{E_x}: E_x \rightarrow X$ to the fiber $E_x = p^{-1}(x)$ is dominant at the origin $0_x \in Z \cap E_x$ of the vector space E_x . The constructive subset $s(E_x) \subset X$ is called the *s-orbit* of x . The variety X is *elliptic* if it admits a spray (E, p, s) which is dominating at each point $x \in X$. The ellipticity is equivalent to the weaker properties of local ellipticity and subellipticity, see [KZ23, Theorem 0.1]. One says that X is *locally elliptic* if for any $x \in X$ there is a local spray (E_x, p_x, s_x) defined in a neighborhood of x and dominating at x such that s_x takes values in X . The variety X is called *subelliptic* if it admits a family of sprays (E_i, p_i, s_i) defined on X which is dominating at each point $x \in X$, that is,

$$T_x X = \sum_{i=1}^n ds_i(T_{0_{i,x}} E_{i,x}) \quad \forall x \in X.$$

The ellipticity implies many other useful properties; it is important in the Oka-Grauert theory, see, e.g., [Gro89], [For17a] and [For23]. In particular, the semigroup of endomorphisms $\text{End}(X)$ of an elliptic variety X acts highly transitively on X , see Appendix A in Section 5.

In this paper we establish the ellipticity of affine cones over certain elliptic varieties, see Theorems 3.4 and 4.1. Summarizing, we prove the following theorem. Recall that a *generalized affine cone* \hat{Y} over a smooth projective variety X defined by an ample \mathbb{Q} -divisor D on X is the affine variety

$$\hat{Y} = \text{Spec} \left(\bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X([nD])) \right),$$

see, e.g., [KPZ13, Sec. 1.15]. In the case where D is a hyperplane section of $X \subset \mathbb{P}^n$ the cone \hat{Y} is the usual affine cone over X .

Theorem 0.1. *Let $X \subset \mathbb{P}^n$ be a smooth projective variety, \hat{Y} be a generalized affine cone over X , and $Y = \hat{Y} \setminus \{v\}$ where v is the vertex of \hat{Y} . Then Y is elliptic in the following cases:*

- X is a rational surface;
- X is obtained via a sequence of successive blowups of points starting from a variety X_0 of dimension $n \geq 3$ which belongs to class \mathcal{A}_0 and verifies condition (**) of Proposition 3.1. The latter holds for any generalized flag variety $X_0 = G/P$ of dimension $n \geq 3$ where G is a reductive group and P is a parabolic subgroup of G . In particular, this holds for $X_0 = \mathbb{P}^n$, $n \geq 3$, the Grassmannians, etc.

Recall that a variety X belongs to class \mathcal{A}_0 if X can be covered by open sets isomorphic to \mathbb{A}^n . A smooth projective surface belongs to class \mathcal{A}_0 if and only if it is rational. Any variety of class \mathcal{A}_0 is elliptic, see [For17a, Proposition 6.4.5] and [KZ23, Theorem 0.1].

It is known that a variety of class \mathcal{A}_0 blown up along a smooth closed subvariety is elliptic, see [LT17] and [KZ23]; cf. also [Gro89, Proposition 3.5E]. The same holds for a smooth locally stably flexible variety, see [KKT18]. The following question arises.

Question. *Is it true that the ellipticity of affine cones Y over a smooth variety X survives the blowup of a smooth closed subvariety of X ? Or, more specifically, whether the curve-orbit property (*), its enhanced version (**) (see Definition 2.7 and Proposition 3.1) and their suitable analogs survive such blowups?*

Generalizing an earlier result of Forstnerič [For17b] valid for complete elliptic varieties, Kusakabe [Kus22] established that any smooth elliptic variety admits a surjective morphism $\mathbb{A}^{\dim Y+1} \rightarrow Y$. This fact immediately implies the following corollary.

Corollary 0.2. *Let Y be a smooth elliptic variety. Then the endomorphism monoid $\text{End}(Y)$ acts highly transitively on Y , that is, for any $k \geq 1$ and any two corteges of distinct points (y_1, \dots, y_k) and (y'_1, \dots, y'_k) in Y there exists $f \in \text{End}(Y)$ such that $f(y_i) = y'_i$.*

This corollary can be applied in particular to generalized affine cones Y as in Theorem 0.1. Let us compare Theorem 0.1 with analogous facts concerning the flexibility of affine cones. Recall that a (quasi)affine variety X of dimension $n \geq 2$ is said to be *flexible* if for any smooth point $x \in X$ there exist n locally nilpotent derivations of the algebra $\mathcal{O}_X(X)$ whose vectors at x span the tangent space $T_x X$, see [AFKKZ13]. Any smooth flexible variety is elliptic. The flexibility of X implies that for any natural number m the automorphism group $\text{Aut}(X)$ acts m -transitively on the smooth locus of X , see [AFKKZ13] in the affine case and [FKZ16] in the quasiaffine case.

Let $X = X_d$ be a del Pezzo surface of degree $d \geq 3$ anticanonically embedded in \mathbb{P}^d , and let $\hat{Y} \subset \mathbb{A}^{d+1}$ be the affine cone over X . It is known that for $d \geq 4$ the cone \hat{Y} is flexible, see [Per13]; cf. also [MPS18, Section 4] and the survey article [CPPZ21] for other examples of this kind. By contrast, if $d \leq 3$ and $X_d \hookrightarrow \mathbb{P}^n$ is a plurianticanonical embedding, then the algebra $\mathcal{O}(\hat{Y})$ does not admit any nonzero locally nilpotent derivation, see [CPW16] and also (for $d = 1, 2$) [KPZ11]. In these cases \hat{Y} is not flexible. However, by Theorem 0.1 for any del Pezzo surface $X = X_d$, $d \in \{1, \dots, 9\}$ and any ample polarization D on X the smooth locus Y of the generalized affine cone \hat{Y} over X corresponding to D is elliptic. As a corollary, the smooth quasiaffine threefold Y admits a surjective morphism $\mathbb{A}^4 \rightarrow Y$ and the endomorphism monoid $\text{End}(Y)$ acts highly transitively on Y , see Section 5.

The structure of the paper is as follows. In Section 1 we recall some general facts about Gromov sprays. In particular, we need a version of the Gromov lemma on extension of sprays, see Proposition 1.1 and Appendix B (Section 6) for a proof. Besides, we show that under certain conditions a spray admits a pullback via a blowup, see Proposition 1.5. In Section 2 we notice first that the ellipticity of a projective variety X is necessary for the ellipticity of the affine cone over X , see Proposition 2.1. We do not know whether this condition alone is sufficient. However, the ellipticity of X together with an additional condition called the *curve-orbit property*, see Definition 2.7 and Proposition 3.1, guarantees that the affine cone over X is elliptic, see Corollary 2.13. Theorem 0.1 is proven in Section 3 for varieties of dimension ≥ 3 and in Section 4 for surfaces, see Theorems 3.4 and 4.1, respectively. Using results of Forstnerič [For17b]

and Kusakabe [Kus22], in Appendix A (Section 5) we deduce Corollary 0.2; see also Proposition 5.1 and Corollary 5.4 for somewhat stronger results.

1. PRELIMINARIES

In this section X is a smooth algebraic variety of positive dimension.

The following proposition is a version of the Gromov Localization Lemma adopted to our setting; see [Gro89, 3.5.B] for a sketch of the proof, [For17a, Propositions 6.4.1–6.4.2] for a rigorous proof for quasi-projective varieties and [LT17, Remark 3] for its extension to the general case.

Proposition 1.1. *Let D be a reduced effective divisor on X and (E, p, s) be a spray on $U = X \setminus \text{supp}(D)$ with values in X such that $p: E \rightarrow X$ is a trivial bundle of rank $r \geq 1$. Then there exists a spray $(\tilde{E}, \tilde{p}, \tilde{s})$ on X whose restriction to U is isomorphic to (E, p, s) and such that $\tilde{s}|_{\tilde{p}^{-1}(X \setminus U)} = \tilde{p}|_{\tilde{p}^{-1}(X \setminus U)}$ and for each $x \in U$ the \tilde{s} -orbit of x coincides with the s -orbit of x .*

For the reader's convenience we provide a proof in Appendix B, see Section 6.

Recall the notions of a locally nilpotent vector field and of its replica, see [Fre06] and [AFKKZ13].

Definition 1.2. Consider a regular vector field ν on a smooth affine variety $X = \text{Spec } A$ and the associated derivation ∂ of the algebra $A = \mathcal{O}_X(X)$. One says that ν and ∂ are *locally nilpotent* if for any $a \in A$ one has $\partial^n(a) = 0$ for some $n = n(a) \in \mathbb{N}$. Any \mathbb{G}_a -action λ on X is the flow of a locally nilpotent vector field ν . One writes in this case $\lambda(t) = \exp(t\nu)$. For any $h \in \text{Ker}(\partial)$ the vector field $h\nu$ is also locally nilpotent. The \mathbb{G}_a -action $\lambda_h(t) := \exp(th\nu)$ is called a *replica* of λ . The points $x \in h^{-1}(0)$ are fixed by the replica λ_h .

In the sequel we deal with sprays generated by \mathbb{G}_a -actions.

Definition 1.3. Let U be an affine dense open subset in X . Assume that U is equipped with an effective \mathbb{G}_a -action $\mathbb{G}_a \times U \rightarrow U$. The latter morphism defines a rank 1 spray (L, p, s) on U which we call a \mathbb{G}_a -*spray*, where $p: L \rightarrow U$ is a trivial line bundle. If (L, p, s) is a \mathbb{G}_a -spray on U associated with a \mathbb{G}_a -action λ on U and λ_h is a replica of λ , then the associated \mathbb{G}_a -spray (L, p, s_h) on U also will be called a *replica* of (L, p, s) .

Definition 1.4. Let $U = X \setminus \text{supp}(D)$ where D is an ample divisor on X and let (L, p, s) be a \mathbb{G}_a -spray on U . Due to Proposition 1.1 there exists a rank 1 spray $(\bar{L}, \bar{p}, \bar{s})$ on X whose restriction to U is isomorphic to (L, p, s) and such that $\bar{s}|_{\bar{p}^{-1}(X \setminus U)} = \bar{p}|_{\bar{p}^{-1}(X \setminus U)}$. We call $(\bar{L}, \bar{p}, \bar{s})$ an *extended \mathbb{G}_a -spray*. For a point $x \in U$ the s -orbit O_x coincides with the \mathbb{G}_a -orbit of x , while the s -orbit of $x \in X \setminus U$ is the singleton $\{x\}$. For any one-dimensional \mathbb{G}_a -orbit O in U the restriction of $(\bar{L}, \bar{p}, \bar{s})$ to O is dominating.

It is known that under certain conditions dominating sprays can be lifted via blowups, see [Gro89, 3.5E-E''], [LT17] and [KKT18]. In the next proposition we provide a simple version of such results adopted to our setup.

Proposition 1.5. *Let (E, p, s) be an extended \mathbb{G}_a -spray on X , $A \subset X$ be a smooth subvariety of codimension at least 2 such that the s -orbit of any point $x \in A$ is a singleton, and $\sigma: \tilde{X} \rightarrow X$ be the blowup of A in X . Then (E, p, s) can be lifted to an extended \mathbb{G}_a -spray $(\tilde{E}, \tilde{p}, \tilde{s})$ on \tilde{X} where $\tilde{E} = \sigma^*E$. Letting $\tilde{\sigma}: \tilde{E} \rightarrow E$ be the induced homomorphism, $\tilde{s}: \tilde{E} \rightarrow \tilde{X}$ satisfies $\sigma \circ \tilde{s} = s \circ \tilde{\sigma}$. In particular, σ sends the \tilde{s} -orbits to s -orbits. Furthermore, if (E, p, s) is dominating when restricted to a one-dimensional s -orbit O and \tilde{O} is a \tilde{s} -orbit in \tilde{X} with $\sigma(\tilde{O}) = O$ then $(\tilde{E}, \tilde{p}, \tilde{s})$ is dominating when restricted to \tilde{O} .*

Proof. Let $U \subset X$ be the affine dense open subset carrying an effective \mathbb{G}_a -action λ such that (E, p, s) is extended from the associate \mathbb{G}_a -spray on U , see Definition 1.3. Since U is affine, $D = X \setminus U$ is a divisor. The s -orbit of $x \in U$ is a singleton if and only if x is a fixed point of λ . By our assumption, each $x \in A \cap U$ is a fixed point of λ . So, the ideal $I(A \cap U)$ is stable under λ . By the universal property of blowups, see [Har04, Corollary II.7.15], λ admits a lift to a \mathbb{G}_a -action $\tilde{\lambda}$ on $\tilde{U} = \sigma^{-1}(U)$ making the morphism $\sigma|_{\tilde{U}}: \tilde{U} \rightarrow U$ equivariant. The associated \mathbb{G}_a -spray \tilde{s} on \tilde{U} satisfies $\sigma \circ \tilde{s}|_{\tilde{U}} = s \circ \tilde{\sigma}|_{\tilde{U}}$. By Proposition 1.1 \tilde{s} admits an extension to \tilde{X} denoted still by \tilde{s} such that $\sigma \circ \tilde{s} = s \circ \tilde{\sigma}$. The remaining assertions are easy consequences of the construction and the latter equality. \square

2. ELLIPTICITY OF CONES

We introduce below the so called ‘‘curve-orbit’’ property (*), see Definition 2.7. Using this property we give a criterion of ellipticity of the affine cones over projective varieties with the vertex removed.

Recall that blowing up an affine cone at its vertex yields a line bundle on the underlined projective variety. Removing the vertex of the cone results in removing the zero section of the latter line bundle. This results in a locally trivial fiber bundle whose general fiber is the punctured affine line $\mathbb{A}_*^1 = \mathbb{A}^1 \setminus \{0\}$. In subsection 2.2 we deal more generally with locally trivial \mathbb{A}_*^1 -fibrations. In subsection 2.3 we extend our results to twisted \mathbb{A}_*^1 -fibrations which are not locally trivial. In subsection 2.4 we give several examples of elliptic cones, including cones over flag varieties.

The following fact is stated in [Gro89, 3.5B’]. For the reader’s convenience we give an argument.

Proposition 2.1. *Let $\varrho: Y \rightarrow X$ be a locally trivial fiber bundle. If Y is elliptic then X is elliptic too.*

Proof. Let us show that X is locally elliptic. Let $(\hat{E}, \hat{p}, \hat{s})$ be a dominating spray on Y . For a point $x \in X$ choose a neighborhood U of x in X such that the restriction $Y|_U \rightarrow U$ is a trivial fiber bundle. Let $\xi: U \rightarrow Y$ be a section of $Y|_U \rightarrow U$, let $p: E_U := \xi^*\hat{E} \rightarrow U$ be the induced vector bundle over U and $\varphi: E_U \xrightarrow{\cong} \hat{E}|_{\xi(U)}$ be the induced isomorphism. Letting $s = \varrho \circ \hat{s} \circ \varphi: E_U \rightarrow X$ we obtain a local spray (E_U, p, s) on U with values in X . Indeed, let \hat{Z} be the zero section of $\hat{p}: \hat{E} \rightarrow Y$ and Z_U be the zero section of $p: E_U \rightarrow U$. Since $\hat{s}|_{\hat{Z}} = \hat{p}|_{\hat{Z}}$ we have $s|_{Z_U} = p|_{Z_U}$.

Let $y = \xi(x) \in \hat{U} = \varrho^{-1}(U)$. Let 0_y be the origin of the vector space $\hat{E}_y = \hat{p}^{-1}(y)$. Since $(\hat{E}, \hat{p}, \hat{s})$ is dominating, $d\hat{s}|_{T_{0_y}\hat{E}_y} : T_{0_y}\hat{E}_y \rightarrow T_y Y$ is surjective. This yields a surjection

$$ds = d\varrho \circ d\hat{s} \circ d\varphi|_{T_{0_x}E_x} : T_{0_x}E_x \xrightarrow{\cong} T_{0_y}\hat{E}_y \rightarrow T_y Y \rightarrow T_x X.$$

Thus, the spray (E_U, p, s) is dominating at x . This shows that X is locally elliptic, hence elliptic, see [KZ23, Theorem 0.1]. \square

The natural question arises whether the converse implication holds. Clearly, the product $X \times \mathbb{A}_*^1$ is not elliptic even if X is elliptic. However, we indicate below some particular settings where the answer is affirmative.

2.1. Technical lemmas. In what follows X stands for a smooth algebraic variety, $\varrho : F \rightarrow X$ stands for a line bundle on X with zero section Z_F , and $Y = F \setminus Z_F$ (except for Proposition 2.1). Thus, $\varrho|_Y : Y \rightarrow X$ is a locally trivial fiber bundle with fiber $\mathbb{A}_*^1 = \mathbb{A}^1 \setminus \{0\}$. In fact, any locally trivial fiber bundle $Y \rightarrow X$ with general fiber \mathbb{A}_*^1 arises in this way.

In Lemmas 2.2–2.4 we work as well in the $\mathbb{Z}/2\mathbb{Z}$ -equivariant setup in order to prepare tools for dealing with twisted \mathbb{A}_*^1 -fibrations, see subsection 2.3. In the sequel $\mu_2 \cong \mathbb{Z}/2\mathbb{Z}$ stands for the group of square roots of unity.

Lemma 2.2. *Let $p : E \rightarrow X$ be a vector bundle with zero section Z and $\tau_i : F_i \rightarrow E$, $i = 1, 2$ be two line bundles on E . Assume that there exists an isomorphism $\varphi_0 : F_1|_Z \rightarrow F_2|_Z$. Then φ_0 extends to an isomorphism $\varphi : F_1 \rightarrow F_2$.*

Furthermore, suppose that X, E and $F_i|_Z$ are equipped with μ_2 -actions such that p, φ_0 and $\tau_i|_Z$ for $i = 1, 2$ are μ_2 -equivariant. Then there are μ_2 -actions on F_i such that φ and τ_i for $i = 1, 2$ are μ_2 -equivariant.

Proof. The pullback yields an isomorphism $\text{Pic}(E) \cong p^* \text{Pic}(X)$, see [Mag75, Theorem 5]. Hence, $p^*(F_i|_Z) \cong F_i$ for $i = 1, 2$. Now the lemma follows. \square

Due to the following lemma a spray on X admits a pullback to Y .

Lemma 2.3. *Any spray (E, p, s) on X induces a spray $(\hat{E}, \hat{p}, \hat{s})$ on $Y = F \setminus Z_F$ such that \hat{E} fits in commutative diagrams*

$$(1) \quad \begin{array}{ccc} \hat{E} & \xrightarrow{\hat{p}} & Y \\ \downarrow \hat{e} & & \downarrow e|_Y \\ E & \xrightarrow{p} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} \hat{E} & \xrightarrow{\hat{s}} & Y \\ \downarrow \hat{e} & & \downarrow e|_Y \\ E & \xrightarrow{s} & X \end{array}$$

Furthermore, suppose there are μ_2 -actions on X, E and F (and, therefore, on Y) such that p, s and $\varrho : F \rightarrow X$ are μ_2 -equivariant. Then there is a μ_2 -action on \hat{E} such that \hat{p} and \hat{s} are μ_2 -equivariant.

Proof. Consider the line bundles $F_1 = p^*F \rightarrow E$ and $F_2 = s^*F \rightarrow E$ induced from $\varrho : F \rightarrow X$ via morphisms $p : E \rightarrow X$ and $s : E \rightarrow X$, respectively. They fit in

commutative diagrams

$$(2) \quad \begin{array}{ccc} F_1 & \xrightarrow{\hat{p}} & F \\ \downarrow & & \downarrow \varrho \\ E & \xrightarrow{p} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} F_2 & \xrightarrow{\hat{s}} & F \\ \downarrow & & \downarrow \varrho \\ E & \xrightarrow{s} & X \end{array}$$

Since $s|_Z = p|_Z$ we have $F_1|_Z = F_2|_Z \cong F$ under a natural identification of Z and X . By Lemma 2.2 there is an isomorphism $\varphi: F_1 \xrightarrow{\cong} F_2$ of the line bundles $\hat{\varrho}_1: F_1 \rightarrow E$ and $\hat{\varrho}_2: F_2 \rightarrow E$. Letting Z_i be the zero section of F_i and $\hat{Y}_i = F_i \setminus Z_i$ we get two isomorphic fiber bundles $\hat{Y}_i \rightarrow E$, $i = 1, 2$ with general fiber $\mathbb{A}_*^1 = \mathbb{A}^1 \setminus \{0\}$. Letting $\hat{E} = \hat{Y}_1$ and composing the isomorphism $\varphi|_{\hat{E}}: \hat{E} \xrightarrow{\cong} \hat{Y}_2$ with morphisms $\hat{Y}_2 \rightarrow Y$ and $\hat{Y}_2 \rightarrow E$ from the second diagram in (2) yields (1). The vector bundle $\hat{p}: \hat{E} \rightarrow Y$ in the first diagram in (1) is induced from $p: E \rightarrow X$ via the morphism $\varrho|_Y: Y \rightarrow X$. Let $\hat{Z} = \hat{\varrho}^{-1}(Z)$ be the zero section of $\hat{p}: \hat{E} \rightarrow Y$. Since $s|_Z = p|_Z$ we have $\hat{s}|_{\hat{Z}} = \hat{p}|_{\hat{Z}}$. So $(\hat{E}, \hat{p}, \hat{s})$ is a spray on Y .

For the second statement recall that the isomorphism $\varphi: F_1 \xrightarrow{\cong} F_2$ of Lemma 2.2 is μ_2 -equivariant. Hence, also the above isomorphism $\varphi|_{\hat{E}}: \hat{E} \xrightarrow{\cong} \hat{Y}_2$ is μ_2 -equivariant. The induced morphisms $\hat{p}|_{F_1}: F_1 \rightarrow F$ and $\hat{s}: F_2 \rightarrow F$ in (2) are μ_2 -equivariant too since so are p, s and ϱ by our assumption. \square

The next lemma guarantees the existence of pushforward sprays on quotients by μ_2 -actions.

Lemma 2.4. *Under the assumptions of Lemma 2.3 suppose in addition that the μ_2 -action on X is free. Letting*

$$X' = X/\mu_2, \quad Y' = Y/\mu_2, \quad E' = E/\mu_2 \quad \text{and} \quad \hat{E}' = \hat{E}/\mu_2$$

consider the induced morphisms

$$\varrho': Y' \rightarrow X', \quad p': E' \rightarrow X', \quad s': E' \rightarrow X', \quad \hat{p}': \hat{E}' \rightarrow Y' \quad \text{and} \quad \hat{s}': \hat{E}' \rightarrow Y'.$$

Then $\varrho': Y' \rightarrow X'$ is a smooth \mathbb{A}_*^1 -fibration. Suppose further that $p': E' \rightarrow X'$ and $\hat{p}': \hat{E}' \rightarrow Y'$ are vector bundles¹. Then (E', p', s') is a spray on X' and $(\hat{E}', \hat{p}', \hat{s}')$ is a spray on Y' such that $\varrho' \circ \hat{s}' = s' \circ \hat{p}'$.

The proof is immediate and we leave it to the reader.

Next we show that a spray admits pullback to an unramified μ_2 -covering.

Lemma 2.5. *Let (E, p, s) be a spray on X and $\tau: \tilde{X} \rightarrow X$ be an unramified Galois covering with Galois group μ_2 . Let also $\tilde{p}: \tilde{E} \rightarrow \tilde{X}$ be the induced vector bundle on \tilde{X} . Then there are a μ_2 -action on \tilde{E} and a spray $(\tilde{E}, \tilde{p}, \tilde{s})$ on \tilde{X} such that both \tilde{p} and \tilde{s} are μ_2 -equivariant and $s \circ \tilde{\tau} = \tau \circ \tilde{s}$ where $\tilde{\tau}: \tilde{E} \rightarrow E = \tilde{E}/\mu_2$ is the quotient morphism.*

Proof. The μ_2 -action on \tilde{X} generates a μ_2 -action on $\tilde{X} \times E$ trivial on the second factor. Recall that

$$\tilde{E} = \tilde{X} \times_X E = \{(\tilde{x}, e) \in \tilde{X} \times E \mid \tau(\tilde{x}) = p(e)\}.$$

¹A priori, these are vector bundles locally trivial in étale topology.

Clearly, the natural projection $\tilde{p}: \tilde{E} \rightarrow \tilde{X}$ is μ_2 -equivariant. Let $s' = s \circ \tilde{\tau}: \tilde{E} \rightarrow X$ and

$$E' = \tilde{X} \times_X \tilde{E} = \{(\tilde{x}, \tilde{e}) \in \tilde{X} \times \tilde{E} \mid \tau(\tilde{x}) = s'(\tilde{e})\}.$$

Since $\tau(\tilde{x}) = \tau(\tilde{x}')$ if and only if $\tilde{x}' = \lambda.\tilde{x}$ for some $\lambda \in \mu_2$, while s' is constant on the μ_2 -orbits in \tilde{E} we see that E' is an unramified double cover of \tilde{E} . Letting \tilde{Z} be the zero section of \tilde{E} consider an isomorphism $\varphi = \tilde{p}|_{\tilde{Z}}: \tilde{Z} \xrightarrow{\sim} \tilde{X}$. Notice that the preimage Z' of \tilde{Z} in E' consists of two irreducible components

$$Z'_1 = \{(\varphi(z), z) \mid z \in \tilde{Z}\} \quad \text{and} \quad Z'_2 = \{(\lambda.\varphi(z), z) \mid z \in \tilde{Z}\}$$

where $\lambda \in \mu_2$ is an element of order 2. Notice also that $\tilde{p}: \tilde{E} \rightarrow \tilde{X} \simeq \tilde{Z}$ induces a surjective morphism $E' \rightarrow Z'$. Hence, E' consists of two irreducible components E'_i , $i = 1, 2$ where E'_i contains Z'_i . Taking the component E'_1 which contains Z'_1 as the graph of \tilde{s} we get a lift $\tilde{s}: \tilde{E} \rightarrow \tilde{X}$. A natural lift to E' of the μ_2 -action interchanges E'_1 and E'_2 . By construction, $\tau \circ \tilde{s} = s \circ \tilde{\tau}$. In particular,

$$\tau \circ \tilde{s}(\lambda.\tilde{e}) = s \circ \tilde{\tau}(\lambda.\tilde{e}) = s(e) \quad \text{where} \quad e = \tilde{\tau}(\tilde{e}) = \tilde{\tau}(\lambda.\tilde{e}).$$

Thus, $\tau \circ \tilde{s}(\lambda.\tilde{e}) = s(e) = \tau \circ \tilde{s}(\tilde{e})$. That is, $\tilde{s}(\lambda.\tilde{e})$ coincides with either $\lambda.\tilde{s}(\tilde{e})$ or $\tilde{s}(\tilde{e})$. Since for $\tilde{e} \in \tilde{Z} \simeq Z'_1$ we have the former, by continuity $\tilde{s}(\lambda.\tilde{e}) = \lambda.\tilde{s}(\tilde{e})$ for each $\tilde{e} \in \tilde{E} \simeq E'_1$. This concludes the proof. \square

2.2. Sprays on untwisted \mathbb{A}_*^1 -fibrations. Let $\varrho: Y \rightarrow X$ be a locally trivial \mathbb{A}_*^1 -fibration. Our aim is to give a criterion as to when the total space Y is elliptic. Proposition 2.1 contains a necessary condition for this, whereas Proposition 2.6 and Corollary 2.9 provide sufficient conditions.

Proposition 2.6. *Given a locally trivial \mathbb{A}_*^1 -fibration $\varrho: Y \rightarrow X$ over an elliptic variety X let (E, p, s) be a spray on X and $(\hat{E}, \hat{p}, \hat{s})$ be the induced spray on Y , see Lemma 2.3. Given points $x \in X$ and $y \in \varrho^{-1}(x)$ let $(\hat{E}_i, \hat{p}_i, \hat{s}_i)$, $i = 1, 2$ be rank 1 sprays on Y such that the images $O_i = \varrho(\hat{O}_i)$ in X of the \hat{s}_i -orbits \hat{O}_i of y are smooth curves. Suppose that*

- (i) (E, p, s) is dominating at x , and
- (ii) $T_y \hat{O}_1 \neq T_y \hat{O}_2$, while $T_x O_1 = T_x O_2$, that is, O_1 and O_2 are tangent at x .

Then the family of sprays $\{(\hat{E}, \hat{p}, \hat{s}), (\hat{E}_1, \hat{p}_1, \hat{s}_1), (\hat{E}_2, \hat{p}_2, \hat{s}_2)\}$ is dominating at $y \in Y$.

Proof. If $\hat{O}_y \subset \hat{E}$ is the \hat{s} -orbit of y then $\varrho(\hat{O}_y)$ coincides with the s -orbit O_x , see the second diagram in (1). By (i) (E, p, s) is dominating at x , hence $d\varrho(T_y \hat{O}_y) = T_x X$ and so, $\text{codim}_{T_y Y} T_y \hat{O}_y = 1$. On the other hand, (ii) implies that $T_y \hat{O}_1 + T_y \hat{O}_2 \subset T_y Y$ contains a nonzero vector v such that $d\varrho(v) = 0$, that is, $v \notin T_y \hat{O}_y$. Thus, $T_y Y = \text{span}(T_y \hat{O}_y, T_y \hat{O}_1, T_y \hat{O}_2)$, as needed. \square

The following notion is inspired by Proposition 2.6.

Definition 2.7. Let X be a smooth algebraic variety, $x \in X$ and let $C_x \simeq \mathbb{P}^1$ be a smooth rational curve on X passing through x . We say that the pair (x, C_x) verifies the *two-orbit property* if

- (*) C_x is union of two orbits O_1 and O_2 of rank 1 sprays (E_1, p_1, s_1) and (E_2, p_2, s_2) on X , respectively, where $O_i \cong \mathbb{A}^1$ passes through x and the restriction of (E_i, p_i, s_i) on O_i is dominating at x .

We say that X verifies the curve-orbit property if through each point $x \in X$ passes a smooth rational curve C_x on X satisfying (*). If C_x can be chosen from a specific family \mathcal{F} of rational smooth projective curves on X then we say that X verifies the curve-orbit property with respect to \mathcal{F} .

Remark 2.8. Notice that any one-dimensional orbit O_x of a \mathbb{G}_a -spray is isomorphic to \mathbb{A}^1 and the restriction of this spray to O_x is dominating at x . In the sequel we mostly deal with \mathbb{G}_a -sprays.

Corollary 2.9. *Let X be a smooth variety, $\varrho: F \rightarrow X$ be a line bundle and $Y = F \setminus Z_F$. Suppose that*

- (i) X is elliptic and verifies the curve-orbit property with respect to a covering family of smooth projective rational curves $\{C_x\}$ on X , and
- (ii) $\varrho: F \rightarrow X$ restricts to a nontrivial line bundle on each member C_x of the family.

Then Y is elliptic.

Proof. Fix a dominating spray (E, p, s) on X . Let $(\hat{E}_i, \hat{p}_i, \hat{s}_i)$, $i = 1, 2$ and $(\hat{E}, \hat{p}, \hat{s})$ be the sprays on Y induced by the (E_i, p_i, s_i) and (E, p, s) , respectively, see Lemma 2.3. By Proposition 2.6 it suffices to verify conditions (i) and (ii) of this proposition at any point $y \in \varrho^{-1}(x) \cap Y$. Indeed, then the latter triple of sprays is dominating at y . The family of all such triples for $x \in X$ is dominating on Y , which proves that Y is subelliptic, hence elliptic by [KZ23, Theorem 0.1].

Since the spray (E, p, s) is dominating, condition (i) of Proposition 2.6 holds. To verify (ii) of Proposition 2.6 we have to show that for each $y \in \varrho^{-1}(x) \cap Y$ the \hat{s}_i -orbits \hat{O}_1 and \hat{O}_2 of y are transversal at y . By assumption (i) the s_i -orbits O_i , $i = 1, 2$ provide a trivializing cover for the line bundle $\varrho|_{C_x}: F|_{C_x} \rightarrow C_x$. Since $\varrho|_{\hat{O}_i}: \hat{O}_i \rightarrow O_i$ is an isomorphism, \hat{O}_i is a non-vanishing local section of $F|_{C_x}$ over O_i . Thus, \hat{O}_i is a constant section given by $v_i = 1$ in appropriate coordinates (u_i, v_i) in $F|_{O_i} \simeq \mathbb{A}^1 \times \mathbb{A}_*^1$.

We may consider that $x = \{u_i = 1\}$ in O_i , $i = 1, 2$. Let $z = u_1/u_2$ be an affine coordinate in $\omega = O_1 \cap O_2 \simeq \mathbb{A}_*^1$. Due to (ii) the line bundle $\varrho|_{C_x}: F|_{C_x} \rightarrow C_x$ is nontrivial. So, its transition function on ω equals $v_2/v_1 = z^k$ for some $k \neq 0$. Therefore, in coordinates (u_1, v_1) in $Y|_{O_1} \simeq \mathbb{A}^1 \times \mathbb{A}_*^1$ the curve \hat{O}_2 is given by equation $v_1 = u_1^{-k}$. Hence, \hat{O}_2 is transversal to \hat{O}_1 at the point $y = (1, 1) \in F|_{O_1}$. \square

2.3. Ellipticity of twisted \mathbb{A}_*^1 -fibrations. Let X be a smooth complete variety. In this subsection we provide an analog of Corollary 2.9 for any, not necessarily locally trivial smooth fibration $\varrho: Y \rightarrow X$ with all fibers isomorphic to $\mathbb{A}_*^1 = \mathbb{A}^1 \setminus \{0\}$. Such a fibration extends to a locally trivial fibration $\bar{\varrho}: \bar{Y} \rightarrow X$ with general fiber \mathbb{P}^1 and with a divisor $\tilde{X} = \bar{Y} \setminus Y$ being smooth and either irreducible or a union of two disjoint sections Z_0 and Z_∞ of $\bar{\varrho}$. The \mathbb{A}_*^1 -fibration $\varrho: Y \rightarrow X$ is said to be *twisted* in the former case and *untwisted* in the latter case.

Letting in the untwisted case $F = \bar{Y} \setminus Z_\infty$ we get a locally trivial line bundle $\bar{\varrho}|_F: F \rightarrow X$ with zero section Z_0 so that $Y = F \setminus Z_0$. This returns us to the setup of Corollary 2.9.

To extend Corollary 2.9 in the twisted setup, we deal with μ_2 -varieties and μ_2 -vector bundles. Indeed, a twisted \mathbb{A}_*^1 -fibration $\varrho: Y \rightarrow X$ can be untwisted as follows. Consider the unramified Galois covering $\tilde{\varrho} = \bar{\varrho}|_{\tilde{X}}: \tilde{X} \rightarrow X$ with Galois group μ_2 so that $X = \tilde{X}/\mu_2$. The induced \mathbb{P}^1 -fiber bundle $\tilde{\varrho}^*\tilde{Y} \rightarrow \tilde{X}$ over \tilde{X} admits two disjoint sections Z_0 and Z_∞ where Z_0 is the tautological section. Letting $\tilde{F} = \tilde{\varrho}^*\tilde{Y} \setminus Z_\infty$ we get a line bundle $\tilde{F} \rightarrow \tilde{X}$. Hence, the induced \mathbb{A}_*^1 -fibration $\tilde{Y} = \tilde{\varrho}^*\tilde{Y} \setminus (Z_0 \cup Z_\infty) \rightarrow \tilde{X}$ is untwisted. Furthermore, \tilde{Y} carries a free μ_2 -action making the projection $\tilde{Y} \rightarrow \tilde{X}$ equivariant and such that $Y = \tilde{Y}/\mu_2$. We can now deduce the following analog of Corollary 2.9 for twisted \mathbb{A}_*^1 -fibrations.

Proposition 2.10. *Let X be a smooth variety and $\varrho: Y \rightarrow X$ be a smooth \mathbb{A}_*^1 -fibration. Suppose that*

- (i) *X is elliptic and verifies the curve-orbit property with respect to a covering family of smooth projective rational curves $\{C_x\}$ on X ;*
- (ii) *for any member C_x of the family the restriction $\varrho|_{\varrho^{-1}(C_x)}: \varrho^{-1}(C_x) \rightarrow C_x$ is a nontrivial \mathbb{A}_*^1 -fiber bundle.*²

Then Y is elliptic.

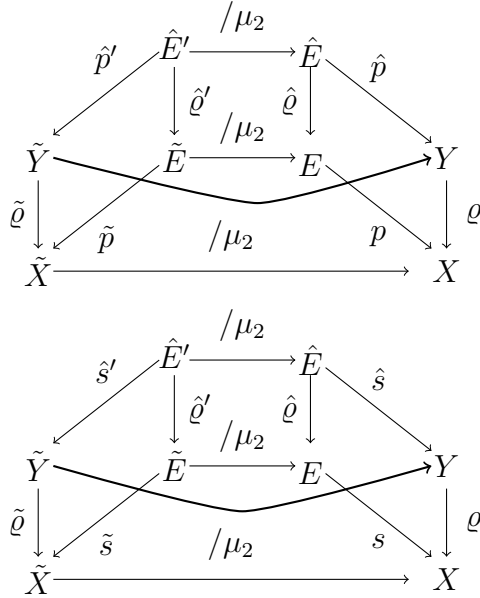


FIGURE 1.

²Condition (ii) is equivalent to the following one:

(ii') $\tilde{F} \rightarrow \tilde{X}$ restricts to a nontrivial line bundle on each of the two components of the curve $\tilde{C}_x = \tilde{\varrho}^{-1}(C_x)$ on \tilde{X} .

Proof. Since X is elliptic there exists a dominating spray (E, p, s) on X . Consider the natural μ_2 -action on \tilde{X} . Lemma 2.5 guarantees the existence of a pullback spray $(\tilde{E}, \tilde{p}, \tilde{s})$ on \tilde{X} such that the induced morphisms \tilde{p} and \tilde{s} are μ_2 -equivariant with respect to the induced μ_2 -action on \tilde{E} . By Lemma 2.3 the latter spray admits a pullback to a μ_2 -equivariant spray $(\hat{E}', \hat{p}', \hat{s}')$ on \tilde{Y} , see Figure 1.

The spray $(\hat{E}', \hat{p}', \hat{s}')$ verifies the assumptions of Lemma 2.4, with notation changed accordingly. By this lemma, the μ_2 -equivariant spray $(\hat{E}', \hat{p}', \hat{s}')$ on \tilde{Y} induces a push-forward spray $(\hat{E}, \hat{p}, \hat{s})$ on Y . Since the spray (E, p, s) on X is dominating, the resulting spray $(\hat{E}, \hat{p}, \hat{s})$ on Y is dominating in codimension 1, that is, for each $y \in Y$ and $v \in T_x X$ with $\varrho(y) = x$ the s -orbit of y is tangent to a vector $w \in T_y Y$ with $d\varrho(w) = v$. The curve-orbit property of X together with condition (ii) above guarantees the existence of a rank 1 spray (E_0, p_0, s_0) on Y such that the s_0 -orbit of y is tangent to a nonzero vector from $\text{Ker}(d\varrho)$, see the proof of Proposition 2.6. Now the conclusion follows by an argument from the latter proof. \square

2.4. Examples.

Example 2.11. Let B be a smooth complete variety and $\pi: X \rightarrow B$ be a ruling, that is, a locally trivial fiber bundle over B with general fiber \mathbb{P}^1 . Suppose that X is elliptic and let $F \rightarrow X$ be a line bundle whose restriction to a fiber of π is nontrivial. Then the assumptions (i) and (ii) of Corollary 2.9 hold with the family $\mathcal{F} = \{C_x\}_{x \in X}$ where $C_x = \pi^{-1}(\pi(x))$. According to this corollary $Y = F \setminus Z_F$ is elliptic.

Example 2.12. The criterion of Corollary 2.9 can be applied to a generalized affine cone \hat{Y} over a smooth projective variety X defined by an ample polarization D on X , see the definition in the Introduction. Indeed, blowing \hat{Y} up at the vertex $v \in \hat{Y}$ yields the line bundle $F = \mathcal{O}_X(-D) \rightarrow X$ so that $Y \setminus \{v\} = F \setminus Z_F$. For instance, if D is a hyperplane section of $X \subset \mathbb{P}^n$ then $F = \mathcal{O}_X(-1)$. Since D is ample the condition (ii) of Corollary 2.9 holds for any curve C on X . Thus, we arrive at the following criterion.

Corollary 2.13. *Let a smooth projective variety X be elliptic and verify the curve-orbit property $(*)$, see Definition 2.7. Let $\hat{Y} \rightarrow X$ be a generalized affine cone over X with smooth locus $Y = \hat{Y} \setminus \{v\}$. Then Y is elliptic.*

Example 2.14. 1. The curve-orbit property holds for $\mathbb{P}^1 = \text{SL}(2, \mathbb{K})/B_+$, where B^\pm is a Borel subgroup of $\text{SL}(2, \mathbb{K})$ consisting of upper/lower triangular matrices. Indeed, the unipotent radical U^\pm of B^\pm acts on \mathbb{P}^1 with a unique fixed point x^\pm where $x^+ \neq x^-$. By homogeneity, for any $x \in \mathbb{P}^1$ we can choose a new pair of opposite Borel subgroups \tilde{B}^\pm so that $x \notin \{\tilde{x}^+, \tilde{x}^-\}$ and so, the orbits $\tilde{U}^\pm x$ are one-dimensional and cover \mathbb{P}^1 .

2. Any generalized flag variety G/P with a reductive group G over \mathbb{K} verifies the curve-orbit property. Indeed, by homogeneity of G/P and due to the previous example it suffices to find an $\text{SL}(2, \mathbb{K})$ - or $\text{PGL}(2, \mathbb{K})$ -subgroup S of G and a point $x \in G/P$ such that the orbit $C = Sx$ is isomorphic to \mathbb{P}^1 . In fact, we are in the setup of the previous example because the isotropy subgroup of a point $x \in C$ in S is a Borel subgroup $B_S = S \cap P$ of S and the fixed point x^- of the opposite Borel subgroup B_S^- is different from x .

Fix a Cartan subalgebra \mathfrak{h} of $\mathrm{lie}(P) \subset \mathrm{lie}(G)$, let $\Delta \subset \mathfrak{h}^\vee$ be the associated root system of G , Δ^\pm be the subset of positive resp. negative roots in Δ and $\Delta_P \subset \Delta$ be the root system of P . We assume that $\Delta_P^+ = \Delta^+$. Consider the root decomposition

$$(3) \quad \mathrm{lie}(P) = \mathfrak{h} \oplus \bigoplus_{\alpha_i \in \Delta^+} \mathfrak{g}_{\alpha_i} \oplus \bigoplus_{\alpha_i \in \Delta_P^-} \mathfrak{g}_{\alpha_i}$$

where \mathfrak{g}_α is the root subspace of $\mathrm{lie}(G)$ which corresponds to a root $\alpha \in \Delta$.

Pick a negative root $\alpha \in \Delta^- \setminus \Delta_P^-$ and let S be the subgroup of G with Lie algebra

$$\mathrm{lie}(S) = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{h}_\alpha \quad \text{where} \quad \mathfrak{h}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}.$$

Thus, S is isomorphic to either $\mathrm{SL}(2, \mathbb{K})$ or $\mathrm{PGL}(2, \mathbb{K})$ and $S \cap P = B_S$ is the Borel subgroup of S with Lie algebra

$$\mathrm{lie}(B_S) = \mathfrak{g}_{-\alpha} \oplus \mathfrak{h}_\alpha.$$

Since $B_S \subset B_P$, the B_S -action on G/P fixes the distinguished point $x_0 = eP \in G/P$. Let U^- be the one-parameter unipotent subgroup of S with Lie algebra \mathfrak{g}_α . Assuming that U^- also fixes x_0 we obtain $[\mathfrak{g}_\alpha, \mathrm{lie}(P)] \subset \mathrm{lie}(P)$; in particular, $\mathfrak{g}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{h}_\alpha] \subset \mathrm{lie}(P)$. The latter contradicts (3). Therefore, $x_0 \in (G/P)^{B_S} \setminus (G/P)^S$ and so, the S -orbit $C = Sx_0$ is isomorphic to \mathbb{P}^1 .

Recall that a variety X is said to be of class \mathcal{A}_0 if X is covered by open subsets isomorphic to the affine space \mathbb{A}^n where $n = \dim X$. Such a variety is subelliptic, see [For17a, Proposition 6.4.5]. Therefore, it is elliptic by [KZ23, Theorem 0.1].

A generalized flag variety G/P where G is a reductive algebraic group and $P \subset G$ is a parabolic subgroup belongs to class \mathcal{A}_0 , see [APS14, Sec. 4(3)]; cf. also [Gro89, 3.5F]. Indeed, a maximal unipotent subgroup U of G acts on G/P with an open orbit isomorphic to an affine space, see, e.g., [Akh95, Sec. 3.1, proof of Proposition 7]. Since G acts on G/P transitively, the assertion follows. See also [APS14, Sec. 4] for further examples.

Corollary 2.15. *Let $X = G/P$ be a flag variety and \hat{Y} be a generalized affine cone over X . Then the smooth locus $Y = \hat{Y} \setminus \{v\}$ of \hat{Y} is elliptic.*

Proof. By the previous discussion and Example 2.14 X verifies the assumptions of Corollary 2.13. So, the assertion follows due to this Corollary. \square

Remark 2.16. Let \hat{Y} be the affine cone over a flag variety G/P embedded in \mathbb{P}^n as a projectively normal subvariety. Then \hat{Y} is normal and moreover, flexible, see [AKZ12, Theorem 0.2]. Hence, $Y = \hat{Y} \setminus \{v\}$ is elliptic.

3. CONES OVER BLOWN-UP VARIETIES OF CLASS \mathcal{A}_0

3.1. The enhanced curve-orbit property. Blowing up a variety X of class \mathcal{A}_0 with center at a point results again in a variety of class \mathcal{A}_0 , see, e.g., [Gro89, Sect. 3.5.D''] or [For17a, Proposition 6.4.6]. In particular, any rational smooth projective surface is of class \mathcal{A}_0 , see [For17a, Corollary 6.4.7]. The next proposition allows to apply Corollary 2.13 to varieties of class \mathcal{A}_0 blown up at a point.

Proposition 3.1. *Let X be a complete variety of class \mathcal{A}_0 and of dimension $n \geq 3$. Suppose that X verifies the following enhanced curve-orbit property:*

(**) *for each $x \in X$ and any finite subset M in $X \setminus \{x\}$ one can find a curve $C_x \simeq \mathbb{P}^1$ on X which avoids M and verifies the two-orbit property (*) of Definition 2.7.*

*Let $\sigma: \tilde{X} \rightarrow X$ be the blowup of a point $x_0 \in X$. Then \tilde{X} also verifies (**).*

Proof. Given a point $\tilde{x} \in \tilde{X}$ fix a finite subset $\tilde{M} \subset \tilde{X} \setminus \{\tilde{x}\}$. Assume first that $x := \sigma(\tilde{x}) \neq x_0$. Choose a curve $C_x \subset X$ and a pair of extended \mathbb{G}_a -sprays (E_i, p_i, s_i) on X satisfying (**) with $M = \{x_0\} \cup \sigma(\tilde{M}) \subset X$. Replace the (E_i, p_i, s_i) by suitable replicas such that the s_i -orbits of x_0 become singletons, while the s_i -orbits O_i of x do not change. By Proposition 1.5 one can lift these sprays to extended \mathbb{G}_a -sprays $(\tilde{E}_i, \tilde{p}_i, \tilde{s}_i)$ on \tilde{X} , $i = 1, 2$ such that $\sigma(\tilde{O}_i) = O_i$ where \tilde{O}_i is the \tilde{s}_i -orbit of the point \tilde{x} . It is easily seen that the curve $C_{\tilde{x}} = \sigma^{-1}(C_x)$ and the lifted sprays $(\tilde{E}_i, \tilde{p}_i, \tilde{s}_i)$ satisfy (**).

Suppose now that \tilde{x} lies on the exceptional divisor $\mathcal{E} = \sigma^{-1}(x_0)$. Choose a neighborhood $\Omega \simeq \mathbb{A}^n$ of \tilde{x} in \tilde{X} . Identifying Ω with $\mathbb{A}^n = \text{Spec } \mathbb{K}[u_1, \dots, u_n]$ we may suppose that x is the origin of \mathbb{A}^n . Then $(u_1: \dots: u_n)$ can serve as homogeneous coordinates in $\mathcal{E} \simeq \mathbb{P}^{n-1}$. Notice that \tilde{X} can be covered with n open subsets $U_i \simeq \mathbb{A}^n$ such that $U_i \cap \mathcal{E} = \mathcal{E} \setminus H_i \simeq \mathbb{A}^{n-1}$ where H_i is the coordinate hyperplane in $\mathcal{E} = \mathbb{P}^{n-1}$ given by $u_i = 0$.

Choose the affine coordinates (u_1, \dots, u_n) in $X = \mathbb{A}^n$ in such a way that $\tilde{x} = (u_1: \dots: u_n) \in \mathbb{P}^{n-1}$ has two nonzero coordinates, say u_1 and u_2 , that is, $\tilde{x} \in U_1 \cap U_2$. Since $n - 1 \geq 2$, a general projective line $C_{\tilde{x}} \subset \mathcal{E} = \mathbb{P}^{n-1}$ through \tilde{x} meets H_1 and H_2 in two distinct points and does not meet \tilde{M} . Furthermore, for $i = 1, 2$ the affine line $\bar{O}_i = C_{\tilde{x}} \setminus H_i \simeq \mathbb{A}^1$ in U_i is an orbit of a \mathbb{G}_a -action on $U_i \simeq \mathbb{A}^n$ and so, a \bar{s}_i -orbit of the associated extended \mathbb{G}_a -spray $(\bar{E}_i, \bar{p}_i, \bar{s}_i)$ on \tilde{X} . One has $C_{\tilde{x}} = \bar{O}_1 \cup \bar{O}_2$. Thus, (**) holds for \tilde{X} . \square

Remark 3.2. Assuming that \tilde{M} is empty the same construction of two extended \mathbb{G}_a -sprays on \tilde{X} with $C_{\tilde{x}} = O_1 \cup O_2$ satisfying the two-orbit property goes through for $n = 2$, while this time $\mathcal{E} \simeq \mathbb{P}^1$.

3.2. Enhanced curve-orbit property for flag varieties.

Proposition 3.3. *A generalized flag variety $X = G/P$ of dimension $n \geq 2$ verifies the enhanced curve-orbit property (**).*

Proof. The stabilizer of the distinguished point $x_0 = eP \in G/P$ coincides with the parabolic subgroup P . The construction of Example 2.14 produces a subgroup S of G isomorphic either to $\text{SL}(2, \mathbb{K})$ or to $\text{PGL}(2, \mathbb{K})$, or in other terms, a subgroup locally isomorphic to $\text{SL}(2, \mathbb{K})$, along with a pair of opposite one-parameter unipotent subgroups U^\pm of S and an S -orbit $C = S/B^+ \simeq \mathbb{P}^1$ passing through x_0 such that the orbits $U^\pm x_0$ are one-dimensional and cover C , where $B^+ = S \cap P$ is a Borel subgroup of S .

The conjugation of S by elements $a \in P$ yields a family $S_a = aSa^{-1}$ of $\text{SL}(2, \mathbb{K})$ -resp. $\text{PGL}(2, \mathbb{K})$ -subgroups of G along with a family of their orbits $C_a = aC \cong \mathbb{P}^1$

passing through x_0 . We claim that the family of \mathbb{P}^1 -curves $\mathcal{F} = \{C_a\}_{a \in P}$ has no base point different from x_0 . Assuming that \mathcal{F} contains at least two distinct curves the base point set $\text{Bs}(\mathcal{F})$ is finite. Since the parabolic subgroup P is connected each $x \in \text{Bs}(\mathcal{F})$ is fixed by P . Since P coincides with its normalizer $N_G(P)$, see [Hum95, Theorem X.29.4(c)], $x_0 = eP$ is the unique fixed point of P . Indeed, if $PgP = gP$ for some $g \in G$, then $g \in N_G(P) = P$ and so, $gP = P$.

Let us show that $\text{card}(\mathcal{F}) \geq 2$. In the notation of (3) we have

$$\dim(G/P) = \text{card}(\Delta^- \setminus \Delta_{\bar{P}}^-) \geq 2.$$

Therefore, $\Delta^- \setminus \Delta_{\bar{P}}^-$ contains two distinct negative roots α_i , $i = 1, 2$. Let U_i^+ be the root subgroup of P with $\text{lie}(U_i^+) = \mathfrak{g}_{-\alpha_i}$, S_i be the subgroup in G with

$$\text{lie}(S_i) = \text{lie}(\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{-\alpha_i}) \cong \mathfrak{sl}(2, \mathbb{K})$$

and $B_i = S_i \cap P \supset U_i^+$ be a Borel subgroup of S_i , cf. Example 2.14. The S_i -orbit of the distinguished point x_0 is a \mathbb{P}^1 -curve $C_i = S_i/B_i$ on G/P . Notice that U_1^+ and U_2^+ are two distinct subgroups of the unipotent radical $R_u(P)$. Let P^- be the opposite parabolic subgroup of G and B_i^- be the opposite Borel subgroup of S_i with unipotent radical U_i^- . Then U_1^- and U_2^- are two distinct subgroups of $R_u(P^-)$. The $R_u(P^-)$ -orbit of x_0 is open in G/P and the isotropy subgroup of x_0 in $R_u(P^-)$ is trivial. Hence the orbits $U_i^- x_0$, $i = 1, 2$ are distinct and so are the curves $C_i = U_i^- x_0$, $i = 1, 2$, as claimed.

Thus, for any finite subset $M = \{x_1, \dots, x_m\} \subset G/P \setminus \{x_0\}$ and for every $i = 1, \dots, m$ one can choose a curve $C_{a_i} = a_i C$ from our family which does not meet x_i . The set of elements $a_i \in P$ with this property is open and dense in P . Therefore, for a general $a \in P$ the curve C_a does not intersect M . Since G/P is homogeneous the choice of a point x_0 is irrelevant. It follows that G/P verifies condition (**), see Example 2.14. \square

3.3. The main result in dimensions ≥ 3 .

Theorem 3.4. *Let X be a smooth complete variety and Y be the smooth locus of a generalized affine cone over X , see the Introduction. Then Y is elliptic in each of the following cases:*

- (a) $n = \dim X \geq 3$ and X is obtained from a variety X_0 of class \mathcal{A}_0 verifying condition (**) of Proposition 3.1 (say, $X_0 = G/P$ is a flag variety of dimension $n \geq 3$, see Proposition 3.3) via a finite number of subsequent blowups;
- (b) X is obtained by blowing up a finite subset of a Hirzebruch surface \mathbb{F}_n , $n \geq 0$ or of \mathbb{P}^2 (say, X is a del Pezzo surface).

Proof. Notice that in (a) and (b) X is of class \mathcal{A}_0 , hence elliptic. Thus, it suffices to check conditions (i) and (ii) of Corollary 2.9. In all cases condition (ii) holds because of Corollary 2.13. Condition (i) is provided by Propositions 3.1 and 3.3, Remark 3.2 and Example 2.11. \square

Remark 3.5. The proof of Theorem 3.4(a) does not work for $n = 2$. Indeed, in this case the exceptional curve $\mathcal{E} = \mathbb{P}^1$ in the proof of Proposition 3.1 can contain points of \tilde{M} . However, its conclusion still holds for $n = 2$; see Theorem 4.1 in the next section.

4. ELLIPTIC CONES OVER RATIONAL SURFACES

In this section we extend Theorem 3.4 to any rational smooth projective surface X .

Theorem 4.1. *Let X be a rational smooth projective surface and $Y = \hat{Y} \setminus \{v\}$ be the smooth locus of a generalized cone \hat{Y} over X . Then Y is elliptic.*

Due to Corollary 2.13, Theorem 4.1 follows from the next proposition.

Proposition 4.2. *X verifies the curve-orbit property (*).*

The proof of Proposition 4.2 uses induction on the number of blowups when forming X . The following proposition provides the inductive step.

Proposition 4.3. *Let X be a smooth projective surface and $\pi: X \rightarrow \mathbb{P}^1$ be a fibration with general fiber \mathbb{P}^1 . Assume that for any fiber component C of π and any point $x \in C$ the following hold.*

- (i) *There exists an open subset U_x in X such that $X \setminus U_x$ has no isolated point and $O_x := U_x \cap C = C \setminus \{x\} \cong \mathbb{A}^1$;*
- (ii) *there exists a proper birational morphism $\nu_x: U_x \rightarrow \mathbb{A}^2$ such that*
 - *O_x is sent by ν_x isomorphically onto a coordinate axis L in \mathbb{A}^2 ;*
 - *the exceptional divisor of ν_x consists of some fiber components of π ;*
 - *no one of these components is contracted to a point of L ;*
- (iii) *there is a \mathbb{G}_a -action λ_x on U_x such that O_x is the λ_x -orbit of x ;*
- (iv) *for the extended \mathbb{G}_a -spray (E_x, p_x, s_x) on X associated with λ_x , O_x is the s_x -orbit of any point $x' \in O_x$.*

Let $\sigma: \tilde{X} \rightarrow X$ be the blowup of a point $x_0 \in X$ and $\tilde{\pi} = \pi \circ \sigma: \tilde{X} \rightarrow \mathbb{P}^1$ be the induced \mathbb{P}^1 -fibration. Then (i) – (iv) hold with $\pi: X \rightarrow \mathbb{P}^1$ replaced by $\tilde{\pi}: \tilde{X} \rightarrow \mathbb{P}^1$.

Proof. Let \tilde{C} be a fiber component of $\tilde{\pi}$ on \tilde{X} and $\tilde{x} \in \tilde{C}$. If \tilde{C} is different from the exceptional curve of σ then we let $C = \sigma(\tilde{C}) \subset X$ and $x = \sigma(\tilde{x}) \in C$; otherwise, we let $x = x_0$. Consider an open subset $U_x \subset X$ satisfying (i)-(iv).

The following cases can occur:

- (a) $\tilde{C} = \sigma^{-1}(x_0)$ is the exceptional curve of σ and $x = x_0$;
- (b) C is a fiber component of π and $x_0 \notin C$;
- (c) C is a fiber component of π and $x_0 \in C$.

In case (a) we choose an open neighborhood $U_0 \cong \mathbb{A}^2$ of x_0 in X . Such a neighborhood does exist since X belongs to class \mathcal{A}_0 . We also choose a coordinate system (u, v) on U_0 such that x_0 is the origin and the axis $L = \{u = 0\}$ goes in direction of $\tilde{x} \in \tilde{C} = \mathbb{P}T_{x_0}U_0$. Then $\tilde{U}_0 := \sigma^{-1}(U_0) = \tilde{U}_1 \cup \tilde{U}_2$ where the $\tilde{U}_i \simeq \mathbb{A}^2$ are equipped with coordinates (u_i, v_i) such that

(α) the restrictions $\sigma_0|_{\tilde{U}_i}: \tilde{U}_i \rightarrow U_0$ are given by

$$\sigma_0|_{\tilde{U}_1}: (u_1, v_1) \mapsto (u_1 v_1, v_1) \quad \text{and} \quad \sigma_0|_{\tilde{U}_2}: (u_2, v_2) \mapsto (u_2, u_2 v_2);$$

(β) σ_0 sends the coordinate axis $L_1 := \{u_1 = 0\}$ in \tilde{U}_1 isomorphically onto L .

Then $U_{\tilde{x}} := \tilde{U}_1$ equipped with the \mathbb{G}_a -action $\lambda_{\tilde{x}}$ by translations in direction of L_1 satisfies (i)–(iv).

If (b) or (c) takes place then the preimage $U_{\tilde{x}} = \sigma^{-1}(U_x)$ in \tilde{X} satisfies

$$O_{\tilde{x}} := U_{\tilde{x}} \cap \tilde{C} = \sigma^{-1}(C \setminus \{x\}) = \tilde{C} \setminus \{\tilde{x}\}.$$

Clearly, $U_{\tilde{x}}$ also satisfies condition (i).

Assume that case (b) takes place. It is easily seen that the composition $\nu_{\tilde{x}} = \nu_x \circ \sigma: U_{\tilde{x}} \rightarrow \mathbb{A}^2$ satisfies (ii). Replacing λ_x with an appropriate replica, if necessary, we may assume that $x_0 \notin C$ is fixed by λ_x , while $O_x = C \cap U_x = C \setminus \{x\}$ is still an orbit of λ_x . Thus, λ_x can be lifted to a \mathbb{G}_a -action $\lambda_{\tilde{x}}$ on $U_{\tilde{x}}$ such that $O_{\tilde{x}}$ is the $\lambda_{\tilde{x}}$ -orbit of \tilde{x} and so, (iii) holds for $\lambda_{\tilde{x}}$. Letting $(E_{\tilde{x}}, p_{\tilde{x}}, s_{\tilde{x}})$ be the extended \mathbb{G}_a -spray on \tilde{X} associated with $\lambda_{\tilde{x}}$ we obtain a data verifying (i)–(iv).

Case (c) splits in two sub-cases:

(c') $x = x_0$;

(c'') $x \neq x_0$.

In case (c') we have $x_0 = x \notin U_x$. Hence, letting $U_{\tilde{x}} = \sigma^{-1}(U_x)$ the restriction $\sigma|_{U_{\tilde{x}}}: U_{\tilde{x}} \rightarrow U_x$ is an isomorphism. Proceeding as before we come to the desired conclusion.

In case (c'') we have $x_0 \in C \setminus \{x\} = O_x$ and $\nu_x(x_0) \in \nu_x(O_x) = L$. We may suppose that $\nu_x(x_0) = \bar{0} \in \mathbb{A}^2$ is the origin. By (ii), $\nu_x: U_x \rightarrow \mathbb{A}^2$ is étale on $O_x \subset U_x$. Hence, ν_x is a blowup of \mathbb{A}^2 with center an ideal $I \subset \mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2)$ supported on $\mathbb{A}^2 \setminus L$. Since ν_x sends O_x isomorphically onto L , the proper birational morphism $\nu_x \circ \sigma: U_{\tilde{x}} \rightarrow \mathbb{A}^2$ also sends $O_{\tilde{x}}$ isomorphically onto L . Notice that the blowups with disjoint centers commute. Hence, $\nu_x \circ \sigma = \sigma_0 \circ \nu_0$ where $\sigma_0: \tilde{U}_0 \rightarrow \mathbb{A}^2$ is the blowup of \mathbb{A}^2 at the origin and ν_0 is the blowup of \tilde{U}_0 with center $\tilde{I} = \sigma_0^*(I) \subset \mathcal{O}_{\tilde{U}_0}(\tilde{U}_0)$. These morphisms fit in a commutative diagram

$$\begin{array}{ccc} U_{\tilde{x}} & \xrightarrow{\sigma} & U_x \\ \downarrow \nu_0 & & \downarrow \nu_x \\ \tilde{U}_0 & \xrightarrow{\sigma_0} & \mathbb{A}^2 \end{array}$$

Choose affine coordinates (u, v) on \mathbb{A}^2 such that $x_0 = (0, 0)$ is the origin and $L = \{u = 0\}$. Then $\tilde{U}_0 = \tilde{U}_1 \cup \tilde{U}_2$ where $\tilde{U}_i \simeq \mathbb{A}^2$ is equipped with coordinates (u_i, v_i) so that (α) and (β) above hold. Morphism σ_0 sends a coordinate axis L_1 in \tilde{U}_1 isomorphically onto L and ν_0 sends $O_{\tilde{x}}$ isomorphically onto L_1 .

Replace now $U_{\tilde{x}}$ by $U_{\tilde{x},1} := \nu_0^{-1}(\tilde{U}_1)$ and ν by $\nu_1 = \nu_0|_{U_{\tilde{x},1}}: U_{\tilde{x},1} \rightarrow \tilde{U}_1 \simeq \mathbb{A}^2$. We have $O_{\tilde{x}} = U_{\tilde{x},1} \cap \tilde{C}$ and $\nu_1|_{O_{\tilde{x}}}: O_{\tilde{x}} \rightarrow L_1$ is an isomorphism which sends $x_0 \in O_{\tilde{x}}$ to the origin $\bar{0} \in \tilde{U}_1 = \mathbb{A}^2$. Furthermore, ν_1 is étale on $O_{\tilde{x}}$ and so, $U_{\tilde{x},1}$ verifies (i) and (ii).

Consider now the \mathbb{G}_a -action λ on $\tilde{U}_1 = \mathbb{A}^2$ by translations in direction of L_1 so that L_1 is a λ -orbit. Replacing λ by an appropriate replica we can achieve that λ acts freely on L_1 and leaves invariant the center $\tilde{I}|_{U_{\tilde{x},1}}$ of the blowup $\nu_0|_{U_{\tilde{x},1}}$. Then λ admits a lift $\tilde{\lambda}$ to $U_{\tilde{x},1}$ such that $O_{\tilde{x}}$ is a $\tilde{\lambda}$ -orbit. Therefore, $U_{\tilde{x},1}$ satisfies (iii) and (iv). \square

Proof of Proposition 4.2. Clearly, \mathbb{P}^2 and the Hirzebruch surfaces \mathbb{F}_n with $n \geq 0$ verify the curve-orbit property. It is well known that any rational smooth projective surface X

non-isomorphic to \mathbb{P}^2 results from a sequence of blowups starting with \mathbb{F}_n for some $n \geq 0$. Moreover, properties (i)–(iv) of Proposition 3.1 hold for the \mathbb{P}^1 -fibration $\pi: \mathbb{F}_n \rightarrow \mathbb{P}^1$. By this proposition and the induction on the number of blowups these properties still hold for X . Let $x \in X$ lies on a fiber component C of the induced \mathbb{P}^1 -fibration $\pi: X \rightarrow \mathbb{P}^1$. Choose two distinct points $x_1, x_2 \in C \setminus \{x\}$. Let U_i and (E_i, p_i, s_i) verify (i)–(iv) of Proposition 3.1 with x replaced by x_i , $i = 1, 2$. Then $O_i = C \setminus \{x_i\}$ is an s_i -orbit of x . The restriction of the extended \mathbb{G}_a -spray (E_i, p_i, s_i) to O_i is dominating. Therefore, X verifies the curve-orbit property. This proves Proposition 4.2. \square

5. APPENDIX A: HIGH TRANSITIVITY OF THE $\text{End}(X)$ -ACTION

It is known that the automorphism group $\text{Aut}(X)$ of a flexible variety X acts highly transitively on the smooth locus X_{reg} , that is, it acts m -transitively for any $m \geq 1$, see [AFKKZ13, Theorem 0.1]. A similar transitivity property holds for the monoid of endomorphisms $\text{End}(X)$ provided X is a smooth elliptic variety in the Gromov' sense. Abusing the language we say that $\text{End}(X)$ acts highly transitively on X if for any two isomorphic zero-dimensional subschemes Z_1 and Z_2 of X every isomorphism $Z_1 \xrightarrow{\cong} Z_2$ extends to an endomorphism of X .

Proposition 5.1. *Let X be a smooth quasiaffine variety. If X is elliptic then $\text{End}(X)$ acts highly transitively on X .*

The above proposition is an easy consequence of the following epimorphism theorem due to Kusakabe [Kus22, Theorem 1.2], which generalizes an earlier result of Forstnerič [For17b, Theorem 1.6] valid for complete elliptic varieties. This phenomenon was expected in [Gro89, Sec. 3.4(C)]; see also [Arz22] for analogs in the case of flexible varieties.

Theorem 5.2. *For any smooth (sub)elliptic variety X of dimension n there exists a morphism $\varphi: \mathbb{A}^{n+1} \rightarrow X$ such that $\varphi(\mathbb{A}^{n+1} \setminus \text{Sing}(\varphi)) = X$ where*

$$\text{Sing}(\varphi) = \{x \in \mathbb{A}^{n+1} \mid \varphi \text{ is not smooth at } x\}$$

is the singular locus of φ . If X is complete then the same is true with \mathbb{A}^n instead of \mathbb{A}^{n+1} .

Using the fact that any morphism of a zero-dimensional scheme $Z \rightarrow X$ can be lifted through a smooth morphism $\varphi: \mathbb{A}^{n+1} \setminus \text{Sing}(\varphi) \rightarrow X$, Kusakabe derives the following interpolation result, see [Kus22, Corollary 1.5]; cf. [Gro89, Sec. 3.4(E)].

Corollary 5.3. *Let X be a smooth (sub)elliptic variety, Y be an affine variety and Z be a zero-dimensional subscheme of Y . Then for any morphism $f: Z \rightarrow X$ there exists a morphism $\tilde{f}: Y \rightarrow X$ such that $\tilde{f}|_Z = f$.*

In fact, Corollary 5.3 remains valid with the same proof if one assumes Y to be quasiaffine. Letting in Corollary 5.3 $Y = X$ and $Z_1 = Z$ we can extend the given morphism $f: Z_1 \rightarrow Z_2$ onto a zero-dimensional subscheme Z_2 of X to an endomorphism $\tilde{f}: X \rightarrow X$. This yields Proposition 5.1. Furthermore, taking in Corollary 5.3 $Y = \mathbb{A}^1$ we obtain the following corollary, cf. [Gro89, Sec. 3.4(B)] and [KZ00, p. 1661].

Corollary 5.4. *Any smooth elliptic variety X is \mathbb{A}^1 -rich, that is, through any m distinct points of X passes an \mathbb{A}^1 -curve $\mathbb{A}^1 \rightarrow X$. Moreover, given a finite collection of curve jets j_1, \dots, j_m on X with not necessarily distinct centers, there exists an \mathbb{A}^1 -curve $f: \mathbb{A}^1 \rightarrow X$ interpolating these jets.*

Notice that for a flexible smooth affine variety X one can find an orbit of a \mathbb{G}_a -action on X interpolating given curve jets, see [AFKKZ13, Theorem 4.14].

6. APPENDIX B: GROMOV'S EXTENSION LEMMA

For the reader's convenience we recall the formulation of Proposition 1.1.

Proposition 6.1. *Let D be a reduced effective divisor on X and (E, p, s) be a spray on $U = X \setminus \text{supp}(D)$ with values in X such that $p: E \rightarrow X$ is a trivial bundle of rank $r \geq 1$. Then there exists a spray $(\tilde{E}, \tilde{p}, \tilde{s})$ on X whose restriction to U is isomorphic to (E, p, s) and such that $\tilde{s}|_{\tilde{p}^{-1}(X \setminus U)} = \tilde{p}|_{\tilde{p}^{-1}(X \setminus U)}$ and for each $x \in U$ the \tilde{s} -orbit of x coincides with the s -orbit of x .*

Proof. Extend E to a trivial vector bundle on X ; abusing notation, the latter is again denoted by $p: E \rightarrow X$. Let ξ be a canonical section of $\mathcal{O}_X(D) \rightarrow X$ such that $\text{div}(\xi) = D$.

Consider the twisted vector bundle $p_n: E_n = E \otimes \mathcal{O}_X(-nD) \rightarrow X$ where $n \in \mathbb{N}$. Since ξ does not vanish on U the homomorphism $\varphi_n: E_n \rightarrow E$ induced by the tensor multiplication by ξ^n restricts to an isomorphism on U which sends $s_n := s \circ \varphi_n$ to s . We claim that s_n extends to a morphism $E_n \rightarrow X$ for a sufficiently large n . It is easily seen that in the latter case $(\tilde{E}, \tilde{p}, \tilde{s}) = (E_n, p_n, s_n)$ is a desired spray on X .

The argument is local and, thus, we restrict consideration to an affine neighborhood $\omega \subset X$ of a given point $x_0 \in \text{supp}(D)$ such that

- $D|_\omega = h^*(0)$ for some $h \in \mathcal{O}_\omega(\omega)$ and
- $E_n|_\omega \cong_\omega \omega \times \mathbb{A}^r$ is a trivial vector bundle.

Notice that over $\omega^* = \omega \setminus \text{supp}(D)$ the spray (E_n, p_n, s_n) is the h^n -homothety of (E, p, s) . The latter means that s_n is given, after trivialization, by

$$s_n: (x, v) \mapsto s(x, h^n(x)v) \quad \forall (x, v) \in \omega^* \times \mathbb{A}^r.$$

Let $\omega \hookrightarrow \mathbb{A}^m$ be a closed embedding. Then s yields a rational map

$$s: \omega \times \mathbb{A}^r \dashrightarrow \mathbb{A}^m \quad \text{of the form} \quad (x, v) \mapsto x + \psi(x, v)$$

where ψ is a rational vector function which is regular in a neighborhood of $\omega^* \times \{0\}$ and vanishes on $\omega^* \times \{0\}$. Every coordinate function of ψ can be written in the form

$$\frac{q(v)}{h^{kr}(v)} \quad \text{where} \quad v = (t_1, \dots, t_r) \in \mathbb{A}^r \quad \text{and} \quad q, r \in \mathcal{O}_\omega(\omega)[t_1, \dots, t_r]$$

are such that $q(0) = 0$ and the zero loci of r and h have no common component in $\omega \times \mathbb{A}^r$.

Furthermore, we may suppose that the divisors $r^*(0)$ and $q^*(0)$ have no common irreducible component T in $\omega \times \mathbb{A}^r$ passing through $(x_0, 0) \in \omega \times \mathbb{A}^r$. Indeed, let T be

such a component. It is well known that $\text{Pic}(\omega \times \mathbb{A}^r) = \text{pr}_1^*(\text{Pic}(\omega))$. Hence, the divisor $T - \text{pr}_1^*(T_0)$ is principal for some divisor T_0 on ω . Shrinking ω one may suppose that T_0 is principal. Hence, T is principal, i.e., $T = f^*(0)$ for some regular function f on $\omega \times \mathbb{A}^r$. Thus, dividing r and q by f we can get rid of T .

This implies that r does not vanish on $\omega^* \times \{0\}$ since otherwise ψ is not regular on $\omega^* \times \{0\}$. In particular, $r(0) = ah^l$ where a is a non-vanishing function on ω .

The h^n -homothety s_n of s yields a rational map

$$s_n: \omega \times \mathbb{A}^r \dashrightarrow \omega \subset \mathbb{A}^m, \quad (x, v) \mapsto x + \psi_n(x, v),$$

where ψ is a rational vector function which is regular in a neighborhood of $\omega^* \times \{0\}$ and vanishes on $\omega^* \times \{0\}$. Every coordinate function of $\psi_n \in$ is of the form

$$\frac{q(h^n v)}{h^k r(h^n v)}.$$

Since $q(0) = 0$ and $[r(h^n v) - r(v)]|_{v=0} = 0$ we see that

$$q(h^n v) = h^n \tilde{q}(v) \quad \text{and} \quad r(h^n v) = r(0) + h^n \tilde{r}(v) = h^l a + h^n \tilde{r}(v).$$

Hence, for $n > k + l$ the function

$$\frac{q(h^n v)}{h^k r(h^n v)} = \frac{h^{n-k-l} \tilde{q}(v)}{a + h^{n-l} \tilde{r}(v)}$$

is regular in a neighborhood Ω of $\omega \times \{0\}$ and vanishes to order $n - k - l$ on $(\text{supp}(D) \times \mathbb{A}^r) \cap \Omega$.

Consider now $s_{n+1} = s \circ \varphi_{n+1} = s_n \circ \varphi_1$ where, after trivialization,

$$\varphi_1|_{\omega \times \mathbb{A}^r}: \omega \times \mathbb{A}^r \rightarrow \omega \times \mathbb{A}^r, \quad (x, v) \mapsto (x, h(x)v).$$

Letting $\tilde{\Omega} = \varphi_1^{-1}(\Omega) \subset \omega \times \mathbb{A}^r$ we see that

$$\varphi_1(\tilde{\Omega}) \subset \Omega \quad \text{and} \quad (\text{supp}(D) \cap \omega) \times \mathbb{A}^r \subset \tilde{\Omega}.$$

Hence, s_{n+1} is regular on $\tilde{\Omega}$ and so, on $p_{n+1}^{-1}(\text{supp}(D) \cap \omega)$. This yields a spray $(E_{n+1}, p_{n+1}, s_{n+1})$ on $\omega \cup U$ with values in X . Choosing a finite cover of $\text{supp}(D)$ by affine open subsets ω_i and a sufficiently large n we get the desired extension of (E, p, s) to a spray $(\tilde{E}, \tilde{p}, \tilde{s})$ on X . \square

Remark 6.2. The same proof works also for the original Gromov's Localization Lemma, see [Gro89, 3.5.B].

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