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# GROMOV ELLIPTICITY OF CONES OVER PROJECTIVE MANIFOLDS

S. KALIMAN AND M. ZAIDENBERG

ABSTRACT. We find classes of projective manifolds that are elliptic in the sense of Gromov and such that the affine cones over these manifolds also are elliptic off their vertices. For example, the latter holds for any generalized flag manifold of dimension  $n \geq 3$  successively blown up in a finite set of points and infinitesimally near points. This also holds for any smooth projective rational surface. For the affine cones, the Gromov ellipticity is a much weaker property than the flexibility. Nonetheless, it still implies the infinite transitivity of the action of the endomorphism monoid on these cones.

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## INTRODUCTION

Let  $\mathbb{K}$  be an algebraically closed field of characteristic zero and  $\mathbb{A}^n$  resp.  $\mathbb{P}^n$  be the affine resp. projective  $n$ -space over  $\mathbb{K}$ . All varieties and vector bundles in this paper are algebraic. By *spray* resp. *ellipticity* we mean algebraic spray resp. algebraic ellipticity. Let  $X$  be a smooth algebraic variety. Recall that a *spray of rank  $r$*  over  $X$  is a triple  $(E, p, s)$  where  $p: E \rightarrow X$  is a vector bundle of rank  $r$  with zero section  $Z$  and  $s: E \rightarrow X$  is a morphism such that  $s|_Z = p|_Z$ . A spray  $(E, p, s)$  is *dominating*

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at  $x \in X$  if the restriction  $s|_{E_x}: E_x \rightarrow X$  to the fiber  $E_x = p^{-1}(x)$  is dominant at the origin  $0_x \in Z \cap E_x$  of the vector space  $E_x$ . The constructive subset  $s(E_x) \subset X$  is called the *s-orbit* of  $x$ . The variety  $X$  is *elliptic* if it admits a spray  $(E, p, s)$  which is dominating at each point  $x \in X$ . The ellipticity is equivalent to the weaker properties of local ellipticity and subellipticity, see [KZ23, Theorem 0.1]. One says that  $X$  is *locally elliptic* if for any  $x \in X$  there is a local spray  $(E_x, p_x, s_x)$  defined in a neighborhood of  $x$  and dominating at  $x$  such that  $s_x$  takes values in  $X$ . The variety  $X$  is called *subelliptic* if it admits a family of sprays  $(E_i, p_i, s_i)$  defined on  $X$  which is dominating at each point  $x \in X$ , that is,

$$T_x X = \sum_{i=1}^n ds_i(T_{0_{i,x}} E_{i,x}) \quad \forall x \in X.$$

The ellipticity implies many other useful properties; it is important in the Oka-Grauert theory, see, e.g., [Gro89], [For17a] and [For23]. In particular, the semigroup of endomorphisms  $\text{End}(X)$  of an elliptic variety  $X$  acts highly transitively on  $X$ , see Appendix A in Section 5.

In this paper we establish the ellipticity of affine cones over certain elliptic varieties, see Theorems 3.4 and 4.1. Summarizing, we prove the following theorem. Recall that a *generalized affine cone*  $\hat{Y}$  over a smooth projective variety  $X$  defined by an ample  $\mathbb{Q}$ -divisor  $D$  on  $X$  is the affine variety

$$\hat{Y} = \text{Spec} \left( \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X(\lfloor nD \rfloor)) \right),$$

see, e.g., [KPZ13, Sec. 1.15]. In the case where  $D$  is a hyperplane section of  $X \subset \mathbb{P}^n$  the cone  $\hat{Y}$  is the usual affine cone over  $X$ .

**Theorem 0.1.** *Let  $X \subset \mathbb{P}^n$  be a smooth projective variety,  $\hat{Y}$  be a generalized affine cone over  $X$ , and  $Y = \hat{Y} \setminus \{v\}$  where  $v$  is the vertex of  $\hat{Y}$ . Then  $Y$  is elliptic in the following cases:*

- $X$  is a rational surface;
- $X$  is obtained via a sequence of successive blowups of points starting from a variety  $X_0$  of dimension  $n \geq 3$  which belongs to class  $\mathcal{A}_0$  and verifies condition  $(**)$  of Proposition 3.1. The latter holds for any generalized flag variety  $X_0 = G/P$  of dimension  $n \geq 3$  where  $G$  is a reductive group and  $P$  is a parabolic subgroup of  $G$ . In particular, this holds for  $X_0 = \mathbb{P}^n$ ,  $n \geq 3$ , the Grassmannians, etc.

Recall that a variety  $X$  belongs to class  $\mathcal{A}_0$  if  $X$  can be covered by open sets isomorphic to  $\mathbb{A}^n$ . A smooth projective surface belongs to class  $\mathcal{A}_0$  if and only if it is rational. Any variety of class  $\mathcal{A}_0$  is elliptic, see [For17a, Proposition 6.4.5] and [KZ23, Theorem 0.1].

It is known that a variety of class  $\mathcal{A}_0$  blown up along a smooth closed subvariety is elliptic, see [LT17] and [KZ23]; cf. also [Gro89, Proposition 3.5E]. The same holds for a smooth locally stably flexible variety, see [KKT18]. The following question arises.

**Question.** *Is it true that the ellipticity of affine cones  $Y$  over a smooth variety  $X$  survives the blowup of a smooth closed subvariety of  $X$ ? Or, more specifically, whether the curve-orbit property  $(*)$ , its enhanced version  $(**)$  (see Definition 2.7 and Proposition 3.1) and their suitable analogs survive such blowups?*

Generalizing an earlier result of Forstnerič [For17b] valid for complete elliptic varieties, Kusakabe [Kus22] established that any smooth elliptic variety admits a surjective morphism  $\mathbb{A}^{\dim Y + 1} \rightarrow Y$ . This fact immediately implies the following corollary.

**Corollary 0.2.** *Let  $Y$  be a smooth elliptic variety. Then the endomorphism monoid  $\text{End}(Y)$  acts highly transitively on  $Y$ , that is, for any  $k \geq 1$  and any two corteges of distinct points  $(y_1, \dots, y_k)$  and  $(y'_1, \dots, y'_k)$  in  $Y$  there exists  $f \in \text{End}(Y)$  such that  $f(y_i) = y'_i$ .*

This corollary can be applied in particular to generalized affine cones  $Y$  as in Theorem 0.1. Let us compare Theorem 0.1 with analogous facts concerning the flexibility of affine cones. Recall that a (quasi)affine variety  $X$  of dimension  $n \geq 2$  is said to be *flexible* if for any smooth point  $x \in X$  there exist  $n$  locally nilpotent derivations of the algebra  $\mathcal{O}_X(X)$  whose vectors at  $x$  span the tangent space  $T_x X$ , see [AFKKZ13]. Any smooth flexible variety is elliptic. The flexibility of  $X$  implies that for any natural number  $m$  the automorphism group  $\text{Aut}(X)$  acts  $m$ -transitively on the smooth locus of  $X$ , see [AFKKZ13] in the affine case and [FKZ16] in the quas affine case.

Let  $X = X_d$  be a del Pezzo surface of degree  $d \geq 3$  anticanonically embedded in  $\mathbb{P}^d$ , and let  $\hat{Y} \subset \mathbb{A}^{d+1}$  be the affine cone over  $X$ . It is known that for  $d \geq 4$  the cone  $\hat{Y}$  is flexible, see [Per13]; cf. also [MPS18, Section 4] and the survey article [CPPZ21] for other examples of this kind. By contrast, if  $d \leq 3$  and  $X_d \hookrightarrow \mathbb{P}^n$  is a plurianticanonical embedding, then the algebra  $\mathcal{O}(\hat{Y})$  does not admit any nonzero locally nilpotent derivation, see [CPW16] and also (for  $d = 1, 2$ ) [KPZ11]. In these cases  $\hat{Y}$  is not flexible. However, by Theorem 0.1 for any del Pezzo surface  $X = X_d$ ,  $d \in \{1, \dots, 9\}$  and any ample polarization  $D$  on  $X$  the smooth locus  $Y$  of the generalized affine cone  $\hat{Y}$  over  $X$  corresponding to  $D$  is elliptic. As a corollary, the smooth quas affine threefold  $Y$  admits a surjective morphism  $\mathbb{A}^4 \rightarrow Y$  and the endomorphism monoid  $\text{End}(Y)$  acts highly transitively on  $Y$ , see Section 5.

The structure of the paper is as follows. In Section 1 we recall some general facts about Gromov sprays. In particular, we need a version of the Gromov lemma on extension of sprays, see Proposition 1.1 and Appendix B (Section 6) for a proof. Besides, we show that under certain conditions a spray admits a pullback via a blowup, see Proposition 1.5. In Section 2 we notice first that the ellipticity of a projective variety  $X$  is necessary for the ellipticity of the affine cone over  $X$ , see Proposition 2.1. We do not know whether this condition alone is sufficient. However, the ellipticity of  $X$  together with an additional condition called the *curve-orbit property*, see Definition 2.7 and Proposition 3.1, guarantees that the affine cone over  $X$  is elliptic, see Corollary 2.13. Theorem 0.1 is proven in Section 3 for varieties of dimension  $\geq 3$  and in Section 4 for surfaces, see Theorems 3.4 and 4.1, respectively. Using results of Forstnerič [For17b]

and Kusakabe [Kus22], in Appendix A (Section 5) we deduce Corollary 0.2; see also Proposition 5.1 and Corollary 5.4 for somewhat stronger results.

## 1. PRELIMINARIES

In this section  $X$  is a smooth algebraic variety of positive dimension.

The following proposition is a version of the Gromov Localization Lemma adopted to our setting; see [Gro89, 3.5.B] for a sketch of the proof, [For17a, Propositions 6.4.1–6.4.2] for a rigorous proof for quasi-projective varieties and [LT17, Remark 3] for its extension to the general case.

**Proposition 1.1.** *Let  $D$  be a reduced effective divisor on  $X$  and  $(E, p, s)$  be a spray on  $U = X \setminus \text{supp}(D)$  with values in  $X$  such that  $p: E \rightarrow X$  is a trivial bundle of rank  $r \geq 1$ . Then there exists a spray  $(\tilde{E}, \tilde{p}, \tilde{s})$  on  $X$  whose restriction to  $U$  is isomorphic to  $(E, p, s)$  and such that  $\tilde{s}|_{\tilde{p}^{-1}(X \setminus U)} = \tilde{p}|_{\tilde{p}^{-1}(X \setminus U)}$  and for each  $x \in U$  the  $\tilde{s}$ -orbit of  $x$  coincides with the  $s$ -orbit of  $x$ .*

For the reader's convenience we provide a proof in Appendix B, see Section 6.

Recall the notions of a locally nilpotent vector field and of its replica, see [Fre06] and [AFKKZ13].

**Definition 1.2.** Consider a regular vector field  $\nu$  on a smooth affine variety  $X = \text{Spec } A$  and the associated derivation  $\partial$  of the algebra  $A = \mathcal{O}_X(X)$ . One says that  $\nu$  and  $\partial$  are *locally nilpotent* if for any  $a \in A$  one has  $\partial^n(a) = 0$  for some  $n = n(a) \in \mathbb{N}$ . Any  $\mathbb{G}_a$ -action  $\lambda$  on  $X$  is the flow of a locally nilpotent vector field  $\nu$ . One writes in this case  $\lambda(t) = \exp(t\nu)$ . For any  $h \in \text{Ker}(\partial)$  the vector field  $h\nu$  is also locally nilpotent. The  $\mathbb{G}_a$ -action  $\lambda_h(t) := \exp(th\nu)$  is called a *replica* of  $\lambda$ . The points  $x \in h^{-1}(0)$  are fixed by the replica  $\lambda_h$ .

In the sequel we deal with sprays generated by  $\mathbb{G}_a$ -actions.

**Definition 1.3.** Let  $U$  be an affine dense open subset in  $X$ . Assume that  $U$  is equipped with an effective  $\mathbb{G}_a$ -action  $\mathbb{G}_a \times U \rightarrow U$ . The latter morphism defines a rank 1 spray  $(L, p, s)$  on  $U$  which we call a  $\mathbb{G}_a$ -*spray*, where  $p: L \rightarrow U$  is a trivial line bundle. If  $(L, p, s)$  is a  $\mathbb{G}_a$ -spray on  $U$  associated with a  $\mathbb{G}_a$ -action  $\lambda$  on  $U$  and  $\lambda_h$  is a replica of  $\lambda$ , then the associated  $\mathbb{G}_a$ -spray  $(L, p, s_h)$  on  $U$  also will be called a *replica* of  $(L, p, s)$ .

**Definition 1.4.** Let  $U = X \setminus \text{supp}(D)$  where  $D$  is an ample divisor on  $X$  and let  $(L, p, s)$  be a  $\mathbb{G}_a$ -spray on  $U$ . Due to Proposition 1.1 there exists a rank 1 spray  $(\bar{L}, \bar{p}, \bar{s})$  on  $X$  whose restriction to  $U$  is isomorphic to  $(L, p, s)$  and such that  $\bar{s}|_{\bar{p}^{-1}(X \setminus U)} = \bar{p}|_{\bar{p}^{-1}(X \setminus U)}$ . We call  $(\bar{L}, \bar{p}, \bar{s})$  an *extended  $\mathbb{G}_a$ -spray*. For a point  $x \in U$  the  $s$ -orbit  $O_x$  coincides with the  $\mathbb{G}_a$ -orbit of  $x$ , while the  $s$ -orbit of  $x \in X \setminus U$  is the singleton  $\{x\}$ . For any one-dimensional  $\mathbb{G}_a$ -orbit  $O$  in  $U$  the restriction of  $(\bar{L}, \bar{p}, \bar{s})$  to  $O$  is dominating.

It is known that under certain conditions dominating sprays can be lifted via blowups, see [Gro89, 3.5E-E''], [LT17] and [KKT18]. In the next proposition we provide a simple version of such results adopted to our setup.

**Proposition 1.5.** *Let  $(E, p, s)$  be an extended  $\mathbb{G}_a$ -spray on  $X$ ,  $A \subset X$  be a smooth subvariety of codimension at least 2 such that the  $s$ -orbit of any point  $x \in A$  is a singleton, and  $\sigma: \tilde{X} \rightarrow X$  be the blowup of  $A$  in  $X$ . Then  $(E, p, s)$  can be lifted to an extended  $\mathbb{G}_a$ -spray  $(\tilde{E}, \tilde{p}, \tilde{s})$  on  $\tilde{X}$  where  $\tilde{E} = \sigma^*E$ . Letting  $\tilde{\sigma}: \tilde{E} \rightarrow E$  be the induced homomorphism,  $\tilde{s}: \tilde{E} \rightarrow \tilde{X}$  satisfies  $\sigma \circ \tilde{s} = s \circ \tilde{\sigma}$ . In particular,  $\sigma$  sends the  $\tilde{s}$ -orbits to  $s$ -orbits. Furthermore, if  $(E, p, s)$  is dominating when restricted to a one-dimensional  $s$ -orbit  $O$  and  $\tilde{O}$  is a  $\tilde{s}$ -orbit in  $\tilde{X}$  with  $\sigma(\tilde{O}) = O$  then  $(\tilde{E}, \tilde{p}, \tilde{s})$  is dominating when restricted to  $\tilde{O}$ .*

*Proof.* Let  $U \subset X$  be the affine dense open subset carrying an effective  $\mathbb{G}_a$ -action  $\lambda$  such that  $(E, p, s)$  is extended from the associate  $\mathbb{G}_a$ -spray on  $U$ , see Definition 1.3. Since  $U$  is affine,  $D = X \setminus U$  is a divisor. The  $s$ -orbit of  $x \in U$  is a singleton if and only if  $x$  is a fixed point of  $\lambda$ . By our assumption, each  $x \in A \cap U$  is a fixed point of  $\lambda$ . So, the ideal  $I(A \cap U)$  is stable under  $\lambda$ . By the universal property of blowups, see [Har04, Corollary II.7.15],  $\lambda$  admits a lift to a  $\mathbb{G}_a$ -action  $\tilde{\lambda}$  on  $\tilde{U} = \sigma^{-1}(U)$  making the morphism  $\sigma|_{\tilde{U}}: \tilde{U} \rightarrow U$  equivariant. The associated  $\mathbb{G}_a$ -spray  $\tilde{s}$  on  $\tilde{U}$  satisfies  $\sigma \circ \tilde{s}|_{\tilde{U}} = s \circ \tilde{\sigma}|_{\tilde{U}}$ . By Proposition 1.1  $\tilde{s}$  admits an extension to  $\tilde{X}$  denoted still by  $\tilde{s}$  such that  $\sigma \circ \tilde{s} = s \circ \tilde{\sigma}$ . The remaining assertions are easy consequences of the construction and the latter equality.  $\square$

## 2. ELLIPTICITY OF CONES

We introduce below the so called “curve-orbit” property (\*), see Definition 2.7. Using this property we give a criterion of ellipticity of the affine cones over projective varieties with the vertex removed.

Recall that blowing up an affine cone at its vertex yields a line bundle on the underlined projective variety. Removing the vertex of the cone results in removing the zero section of the latter line bundle. This results in a locally trivial fiber bundle whose general fiber is the punctured affine line  $\mathbb{A}_*^1 = \mathbb{A}^1 \setminus \{0\}$ . In subsection 2.2 we deal more generally with locally trivial  $\mathbb{A}_*^1$ -fibrations. In subsection 2.3 we extend our results to twisted  $\mathbb{A}_*^1$ -fibrations which are not locally trivial. In subsection 2.4 we give several examples of elliptic cones, including cones over flag varieties.

The following fact is stated in [Gro89, 3.5B’]. For the reader’s convenience we give an argument.

**Proposition 2.1.** *Let  $\varrho: Y \rightarrow X$  be a locally trivial fiber bundle. If  $Y$  is elliptic then  $X$  is elliptic too.*

*Proof.* Let us show that  $X$  is locally elliptic. Let  $(\hat{E}, \hat{p}, \hat{s})$  be a dominating spray on  $Y$ . For a point  $x \in X$  choose a neighborhood  $U$  of  $x$  in  $X$  such that the restriction  $Y|_U \rightarrow U$  is a trivial fiber bundle. Let  $\xi: U \rightarrow Y$  be a section of  $Y|_U \rightarrow U$ , let  $p: E_U := \xi^*\hat{E} \rightarrow U$  be the induced vector bundle over  $U$  and  $\varphi: E_U \xrightarrow{\cong} \hat{E}|_{\xi(U)}$  be the induced isomorphism. Letting  $s = \varrho \circ \hat{s} \circ \varphi: E_U \rightarrow X$  we obtain a local spray  $(E_U, p, s)$  on  $U$  with values in  $X$ . Indeed, let  $\hat{Z}$  be the zero section of  $\hat{p}: \hat{E} \rightarrow Y$  and  $Z_U$  be the zero section of  $p: E_U \rightarrow U$ . Since  $\hat{s}|_{\hat{Z}} = \hat{p}|_{\hat{Z}}$  we have  $s|_{Z_U} = p|_{Z_U}$ .

Let  $y = \xi(x) \in \hat{U} = \varrho^{-1}(U)$ . Let  $0_y$  be the origin of the vector space  $\hat{E}_y = \hat{p}^{-1}(y)$ . Since  $(\hat{E}, \hat{p}, \hat{s})$  is dominating,  $d\hat{s}|_{T_{0_y}\hat{E}_y} : T_{0_y}\hat{E}_y \rightarrow T_yY$  is surjective. This yields a surjection

$$ds = d\varrho \circ d\hat{s} \circ d\varphi|_{T_{0_x}E_x} : T_{0_x}E_x \xrightarrow{\cong} T_{0_y}\hat{E}_y \rightarrow T_yY \rightarrow T_xX.$$

Thus, the spray  $(E_U, p, s)$  is dominating at  $x$ . This shows that  $X$  is locally elliptic, hence elliptic, see [KZ23, Theorem 0.1].  $\square$

The natural question arises whether the converse implication holds. Clearly, the product  $X \times \mathbb{A}_*^1$  is not elliptic even if  $X$  is elliptic. However, we indicate below some particular settings where the answer is affirmative.

**2.1. Technical lemmas.** In what follows  $X$  stands for a smooth algebraic variety,  $\varrho : F \rightarrow X$  stands for a line bundle on  $X$  with zero section  $Z_F$ , and  $Y = F \setminus Z_F$  (except for Proposition 2.1). Thus,  $\varrho|_Y : Y \rightarrow X$  is a locally trivial fiber bundle with fiber  $\mathbb{A}_*^1 = \mathbb{A}^1 \setminus \{0\}$ . In fact, any locally trivial fiber bundle  $Y \rightarrow X$  with general fiber  $\mathbb{A}_*^1$  arises in this way.

In Lemmas 2.2–2.4 we work as well in the  $\mathbb{Z}/2\mathbb{Z}$ -equivariant setup in order to prepare tools for dealing with twisted  $\mathbb{A}_*^1$ -fibrations, see subsection 2.3. In the sequel  $\mu_2 \cong \mathbb{Z}/2\mathbb{Z}$  stands for the group of square roots of unity.

**Lemma 2.2.** *Let  $p : E \rightarrow X$  be a vector bundle with zero section  $Z$  and  $\tau_i : F_i \rightarrow E$ ,  $i = 1, 2$  be two line bundles on  $E$ . Assume that there exists an isomorphism  $\varphi_0 : F_1|_Z \rightarrow F_2|_Z$ . Then  $\varphi_0$  extends to an isomorphism  $\varphi : F_1 \rightarrow F_2$ .*

*Furthermore, suppose that  $X, E$  and  $F_i|_Z$  are equipped with  $\mu_2$ -actions such that  $p, \varphi_0$  and  $\tau_i|_Z$  for  $i = 1, 2$  are  $\mu_2$ -equivariant. Then there are  $\mu_2$ -actions on  $F_i$  such that  $\varphi$  and  $\tau_i$  for  $i = 1, 2$  are  $\mu_2$ -equivariant.*

*Proof.* The pullback yields an isomorphism  $\text{Pic}(E) \cong p^* \text{Pic}(X)$ , see [Mag75, Theorem 5]. Hence,  $p^*(F_i|_Z) \cong F_i$  for  $i = 1, 2$ . Now the lemma follows.  $\square$

Due to the following lemma a spray on  $X$  admits a pullback to  $Y$ .

**Lemma 2.3.** *Any spray  $(E, p, s)$  on  $X$  induces a spray  $(\hat{E}, \hat{p}, \hat{s})$  on  $Y = F \setminus Z_F$  such that  $\hat{E}$  fits in commutative diagrams*

$$(1) \quad \begin{array}{ccc} \hat{E} & \xrightarrow{\hat{p}} & Y \\ \downarrow \hat{e} & & \downarrow e|_Y \\ E & \xrightarrow{p} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} \hat{E} & \xrightarrow{\hat{s}} & Y \\ \downarrow \hat{e} & & \downarrow e|_Y \\ E & \xrightarrow{s} & X \end{array}$$

*Furthermore, suppose there are  $\mu_2$ -actions on  $X, E$  and  $F$  (and, therefore, on  $Y$ ) such that  $p, s$  and  $\varrho : F \rightarrow X$  are  $\mu_2$ -equivariant. Then there is a  $\mu_2$ -action on  $\hat{E}$  such that  $\hat{p}$  and  $\hat{s}$  are  $\mu_2$ -equivariant.*

*Proof.* Consider the line bundles  $F_1 = p^*F \rightarrow E$  and  $F_2 = s^*F \rightarrow E$  induced from  $\varrho : F \rightarrow X$  via morphisms  $p : E \rightarrow X$  and  $s : E \rightarrow X$ , respectively. They fit in



commutative diagrams

$$(2) \quad \begin{array}{ccc} F_1 & \xrightarrow{\hat{p}} & F \\ \downarrow & & \downarrow \varrho \\ E & \xrightarrow{p} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} F_2 & \xrightarrow{\hat{s}} & F \\ \downarrow & & \downarrow \varrho \\ E & \xrightarrow{s} & X \end{array}$$

Since  $s|_Z = p|_Z$  we have  $F_1|_Z = F_2|_Z \cong F$  under a natural identification of  $Z$  and  $X$ . By Lemma 2.2 there is an isomorphism  $\varphi: F_1 \xrightarrow{\cong} F_2$  of the line bundles  $\hat{\varrho}_1: F_1 \rightarrow E$  and  $\hat{\varrho}_2: F_2 \rightarrow E$ . Letting  $Z_i$  be the zero section of  $F_i$  and  $\hat{Y}_i = F_i \setminus Z_i$  we get two isomorphic fiber bundles  $\hat{Y}_i \rightarrow E$ ,  $i = 1, 2$  with general fiber  $\mathbb{A}_*^1 = \mathbb{A}^1 \setminus \{0\}$ . Letting  $\hat{E} = \hat{Y}_1$  and composing the isomorphism  $\varphi|_{\hat{E}}: \hat{E} \xrightarrow{\cong} \hat{Y}_2$  with morphisms  $\hat{Y}_2 \rightarrow Y$  and  $\hat{Y}_2 \rightarrow E$  from the second diagram in (2) yields (1). The vector bundle  $\hat{p}: \hat{E} \rightarrow Y$  in the first diagram in (1) is induced from  $p: E \rightarrow X$  via the morphism  $\varrho|_Y: Y \rightarrow X$ . Let  $\hat{Z} = \hat{\varrho}^{-1}(Z)$  be the zero section of  $\hat{p}: \hat{E} \rightarrow Y$ . Since  $s|_Z = p|_Z$  we have  $\hat{s}|_{\hat{Z}} = \hat{p}|_{\hat{Z}}$ . So  $(\hat{E}, \hat{p}, \hat{s})$  is a spray on  $Y$ .

For the second statement recall that the isomorphism  $\varphi: F_1 \xrightarrow{\cong} F_2$  of Lemma 2.2 is  $\mu_2$ -equivariant. Hence, also the above isomorphism  $\varphi|_{\hat{E}}: \hat{E} \xrightarrow{\cong} \hat{Y}_2$  is  $\mu_2$ -equivariant. The induced morphisms  $\hat{p}|_{F_1}: F_1 \rightarrow F$  and  $\hat{s}: F_2 \rightarrow F$  in (2) are  $\mu_2$ -equivariant too since so are  $p, s$  and  $\varrho$  by our assumption.  $\square$

The next lemma guarantees the existence of pushforward sprays on quotients by  $\mu_2$ -actions.

**Lemma 2.4.** *Under the assumptions of Lemma 2.3 suppose in addition that the  $\mu_2$ -action on  $X$  is free. Letting*

$$X' = X/\mu_2, \quad Y' = Y/\mu_2, \quad E' = E/\mu_2 \quad \text{and} \quad \hat{E}' = \hat{E}/\mu_2$$

*consider the induced morphisms*

$$\varrho': Y' \rightarrow X', \quad p': E' \rightarrow X', \quad s': E' \rightarrow X', \quad \hat{p}': \hat{E}' \rightarrow Y' \quad \text{and} \quad \hat{s}': \hat{E}' \rightarrow Y'.$$

*Then  $\varrho': Y' \rightarrow X'$  is a smooth  $\mathbb{A}_*^1$ -fibration. Suppose further that  $p': E' \rightarrow X'$  and  $\hat{p}': \hat{E}' \rightarrow Y'$  are vector bundles<sup>1</sup>. Then  $(E', p', s')$  is a spray on  $X'$  and  $(\hat{E}', \hat{p}', \hat{s}')$  is a spray on  $Y'$  such that  $\varrho' \circ \hat{s}' = s' \circ \hat{p}'$ .*

The proof is immediate and we leave it to the reader.

Next we show that a spray admits pullback to an unramified  $\mu_2$ -covering.

**Lemma 2.5.** *Let  $(E, p, s)$  be a spray on  $X$  and  $\tau: \tilde{X} \rightarrow X$  be an unramified Galois covering with Galois group  $\mu_2$ . Let also  $\tilde{p}: \tilde{E} \rightarrow \tilde{X}$  be the induced vector bundle on  $\tilde{X}$ . Then there are a  $\mu_2$ -action on  $\tilde{E}$  and a spray  $(\tilde{E}, \tilde{p}, \tilde{s})$  on  $\tilde{X}$  such that both  $\tilde{p}$  and  $\tilde{s}$  are  $\mu_2$ -equivariant and  $s \circ \tilde{\tau} = \tau \circ \tilde{s}$  where  $\tilde{\tau}: \tilde{E} \rightarrow E = \tilde{E}/\mu_2$  is the quotient morphism.*

*Proof.* The  $\mu_2$ -action on  $\tilde{X}$  generates a  $\mu_2$ -action on  $\tilde{X} \times E$  trivial on the second factor. Recall that

$$\tilde{E} = \tilde{X} \times_X E = \{(\tilde{x}, e) \in \tilde{X} \times E \mid \tau(\tilde{x}) = p(e)\}.$$

<sup>1</sup>A priori, these are vector bundles locally trivial in étale topology.



Clearly, the natural projection  $\tilde{p}: \tilde{E} \rightarrow \tilde{X}$  is  $\mu_2$ -equivariant. Let  $s' = s \circ \tilde{\tau}: \tilde{E} \rightarrow X$  and

$$E' = \tilde{X} \times_X \tilde{E} = \{(\tilde{x}, \tilde{e}) \in \tilde{X} \times \tilde{E} \mid \tau(\tilde{x}) = s'(\tilde{e})\}.$$

Since  $\tau(\tilde{x}) = \tau(\tilde{x}')$  if and only if  $\tilde{x}' = \lambda \cdot \tilde{x}$  for some  $\lambda \in \mu_2$ , while  $s'$  is constant on the  $\mu_2$ -orbits in  $\tilde{E}$  we see that  $E'$  is an unramified double cover of  $\tilde{E}$ . Letting  $\tilde{Z}$  be the zero section of  $\tilde{E}$  consider an isomorphism  $\varphi = \tilde{p}|_{\tilde{Z}}: \tilde{Z} \xrightarrow{\sim} \tilde{X}$ . Notice that the preimage  $Z'$  of  $\tilde{Z}$  in  $E'$  consists of two irreducible components

$$Z'_1 = \{(\varphi(z), z) \mid z \in \tilde{Z}\} \quad \text{and} \quad Z'_2 = \{(\lambda \cdot \varphi(z), z) \mid z \in \tilde{Z}\}$$

where  $\lambda \in \mu_2$  is an element of order 2. Notice also that  $\tilde{p}: \tilde{E} \rightarrow \tilde{X} \simeq \tilde{Z}$  induces a surjective morphism  $E' \rightarrow Z'$ . Hence,  $E'$  consists of two irreducible components  $E'_i$ ,  $i = 1, 2$  where  $E'_i$  contains  $Z'_i$ . Taking the component  $E'_1$  which contains  $Z'_1$  as the graph of  $\tilde{s}$  we get a lift  $\tilde{s}: \tilde{E} \rightarrow \tilde{X}$ . A natural lift to  $E'$  of the  $\mu_2$ -action interchanges  $E'_1$  and  $E'_2$ . By construction,  $\tau \circ \tilde{s} = s \circ \tilde{\tau}$ . In particular,

$$\tau \circ \tilde{s}(\lambda \cdot \tilde{e}) = s \circ \tilde{\tau}(\lambda \cdot \tilde{e}) = s(e) \quad \text{where} \quad e = \tilde{\tau}(\tilde{e}) = \tilde{\tau}(\lambda \cdot \tilde{e}).$$

Thus,  $\tau \circ \tilde{s}(\lambda \cdot \tilde{e}) = s(e) = \tau \circ \tilde{s}(\tilde{e})$ . That is,  $\tilde{s}(\lambda \cdot \tilde{e})$  coincides with either  $\lambda \cdot \tilde{s}(\tilde{e})$  or  $\tilde{s}(\tilde{e})$ . Since for  $\tilde{e} \in \tilde{Z} \simeq Z'_1$  we have the former, by continuity  $\tilde{s}(\lambda \cdot \tilde{e}) = \lambda \cdot \tilde{s}(\tilde{e})$  for each  $\tilde{e} \in \tilde{E} \simeq E'_1$ . This concludes the proof.  $\square$

**2.2. Sprays on untwisted  $\mathbb{A}_*^1$ -fibrations.** Let  $\varrho: Y \rightarrow X$  be a locally trivial  $\mathbb{A}_*^1$ -fibration. Our aim is to give a criterion as to when the total space  $Y$  is elliptic. Proposition 2.1 contains a necessary condition for this, whereas Proposition 2.6 and Corollary 2.9 provide sufficient conditions.

**Proposition 2.6.** *Given a locally trivial  $\mathbb{A}_*^1$ -fibration  $\varrho: Y \rightarrow X$  over an elliptic variety  $X$  let  $(E, p, s)$  be a spray on  $X$  and  $(\hat{E}, \hat{p}, \hat{s})$  be the induced spray on  $Y$ , see Lemma 2.3. Given points  $x \in X$  and  $y \in \varrho^{-1}(x)$  let  $(\hat{E}_i, \hat{p}_i, \hat{s}_i)$ ,  $i = 1, 2$  be rank 1 sprays on  $Y$  such that the images  $O_i = \varrho(\hat{O}_i)$  in  $X$  of the  $\hat{s}_i$ -orbits  $\hat{O}_i$  of  $y$  are smooth curves. Suppose that*

- (i)  $(E, p, s)$  is dominating at  $x$ , and
- (ii)  $T_y \hat{O}_1 \neq T_y \hat{O}_2$ , while  $T_x O_1 = T_x O_2$ , that is,  $O_1$  and  $O_2$  are tangent at  $x$ .

*Then the family of sprays  $\{(\hat{E}, \hat{p}, \hat{s}), (\hat{E}_1, \hat{p}_1, \hat{s}_1), (\hat{E}_2, \hat{p}_2, \hat{s}_2)\}$  is dominating at  $y \in Y$ .*

*Proof.* If  $\hat{O}_y \subset \hat{E}$  is the  $\hat{s}$ -orbit of  $y$  then  $\varrho(\hat{O}_y)$  coincides with the  $s$ -orbit  $O_x$ , see the second diagram in (1). By (i)  $(E, p, s)$  is dominating at  $x$ , hence  $d\varrho(T_y \hat{O}_y) = T_x X$  and so,  $\text{codim}_{T_y Y} T_y \hat{O}_y = 1$ . On the other hand, (ii) implies that  $T_y \hat{O}_1 + T_y \hat{O}_2 \subset T_y Y$  contains a nonzero vector  $v$  such that  $d\varrho(v) = 0$ , that is,  $v \notin T_y \hat{O}_y$ . Thus,  $T_y Y = \text{span}(T_y \hat{O}_y, T_y \hat{O}_1, T_y \hat{O}_2)$ , as needed.  $\square$

The following notion is inspired by Proposition 2.6.

**Definition 2.7.** Let  $X$  be a smooth algebraic variety,  $x \in X$  and let  $C_x \simeq \mathbb{P}^1$  be a smooth rational curve on  $X$  passing through  $x$ . We say that the pair  $(x, C_x)$  verifies the *two-orbit property* if

- (\*)  $C_x$  is union of two orbits  $O_1$  and  $O_2$  of rank 1 sprays  $(E_1, p_1, s_1)$  and  $(E_2, p_2, s_2)$  on  $X$ , respectively, where  $O_i \cong \mathbb{A}^1$  passes through  $x$  and the restriction of  $(E_i, p_i, s_i)$  on  $O_i$  is dominating at  $x$ .

We say that  $X$  *verifies the curve-orbit property* if through each point  $x \in X$  passes a smooth rational curve  $C_x$  on  $X$  satisfying (\*). If  $C_x$  can be chosen from a specific family  $\mathcal{F}$  of rational smooth projective curves on  $X$  then we say that  $X$  *verifies the curve-orbit property with respect to  $\mathcal{F}$* .

**Remark 2.8.** Notice that any one-dimensional orbit  $O_x$  of a  $\mathbb{G}_a$ -spray is isomorphic to  $\mathbb{A}^1$  and the restriction of this spray to  $O_x$  is dominating at  $x$ . In the sequel we mostly deal with  $\mathbb{G}_a$ -sprays.

**Corollary 2.9.** *Let  $X$  be a smooth variety,  $\varrho: F \rightarrow X$  be a line bundle and  $Y = F \setminus Z_F$ . Suppose that*

- (i)  *$X$  is elliptic and verifies the curve-orbit property with respect to a covering family of smooth projective rational curves  $\{C_x\}$  on  $X$ , and*
- (ii)  *$\varrho: F \rightarrow X$  restricts to a nontrivial line bundle on each member  $C_x$  of the family.*

*Then  $Y$  is elliptic.*

*Proof.* Fix a dominating spray  $(E, p, s)$  on  $X$ . Let  $(\hat{E}_i, \hat{p}_i, \hat{s}_i)$ ,  $i = 1, 2$  and  $(\hat{E}, \hat{p}, \hat{s})$  be the sprays on  $Y$  induced by the  $(E_i, p_i, s_i)$  and  $(E, p, s)$ , respectively, see Lemma 2.3. By Proposition 2.6 it suffices to verify conditions (i) and (ii) of this proposition at any point  $y \in \varrho^{-1}(x) \cap Y$ . Indeed, then the latter triple of sprays is dominating at  $y$ . The family of all such triples for  $x \in X$  is dominating on  $Y$ , which proves that  $Y$  is subelliptic, hence elliptic by [KZ23, Theorem 0.1].

Since the spray  $(E, p, s)$  is dominating, condition (i) of Proposition 2.6 holds. To verify (ii) of Proposition 2.6 we have to show that for each  $y \in \varrho^{-1}(x) \cap Y$  the  $\hat{s}_i$ -orbits  $\hat{O}_1$  and  $\hat{O}_2$  of  $y$  are transversal at  $y$ . By assumption (i) the  $s_i$ -orbits  $O_i$ ,  $i = 1, 2$  provide a trivializing cover for the line bundle  $\varrho|_{C_x}: F|_{C_x} \rightarrow C_x$ . Since  $\varrho|_{\hat{O}_i}: \hat{O}_i \rightarrow O_i$  is an isomorphism,  $\hat{O}_i$  is a non-vanishing local section of  $F|_{C_x}$  over  $O_i$ . Thus,  $\hat{O}_i$  is a constant section given by  $v_i = 1$  in appropriate coordinates  $(u_i, v_i)$  in  $F|_{O_i} \simeq \mathbb{A}^1 \times \mathbb{A}_*^1$ .

We may consider that  $x = \{u_i = 1\}$  in  $O_i$ ,  $i = 1, 2$ . Let  $z = u_1/u_2$  be an affine coordinate in  $\omega = O_1 \cap O_2 \simeq \mathbb{A}_*^1$ . Due to (ii) the line bundle  $\varrho|_{C_x}: F|_{C_x} \rightarrow C_x$  is nontrivial. So, its transition function on  $\omega$  equals  $v_2/v_1 = z^k$  for some  $k \neq 0$ . Therefore, in coordinates  $(u_1, v_1)$  in  $Y|_{O_1} \simeq \mathbb{A}^1 \times \mathbb{A}_*^1$  the curve  $\hat{O}_2$  is given by equation  $v_1 = u_1^{-k}$ . Hence,  $\hat{O}_2$  is transversal to  $\hat{O}_1$  at the point  $y = (1, 1) \in F|_{O_1}$ .  $\square$

**2.3. Ellipticity of twisted  $\mathbb{A}_*^1$ -fibrations.** Let  $X$  be a smooth complete variety. In this subsection we provide an analog of Corollary 2.9 for any, not necessarily locally trivial smooth fibration  $\varrho: Y \rightarrow X$  with all fibers isomorphic to  $\mathbb{A}_*^1 = \mathbb{A}^1 \setminus \{0\}$ . Such a fibration extends to a locally trivial fibration  $\bar{\varrho}: \bar{Y} \rightarrow X$  with general fiber  $\mathbb{P}^1$  and with a divisor  $\tilde{X} = \bar{Y} \setminus Y$  being smooth and either irreducible or a union of two disjoint sections  $Z_0$  and  $Z_\infty$  of  $\bar{\varrho}$ . The  $\mathbb{A}_*^1$ -fibration  $\varrho: Y \rightarrow X$  is said to be *twisted* in the former case and *untwisted* in the latter case.

Letting in the untwisted case  $F = \bar{Y} \setminus Z_\infty$  we get a locally trivial line bundle  $\bar{\varrho}|_F: F \rightarrow X$  with zero section  $Z_0$  so that  $Y = F \setminus Z_0$ . This returns us to the setup of Corollary 2.9.

To extend Corollary 2.9 in the twisted setup, we deal with  $\mu_2$ -varieties and  $\mu_2$ -vector bundles. Indeed, a twisted  $\mathbb{A}_*^1$ -fibration  $\varrho: Y \rightarrow X$  can be untwisted as follows. Consider the unramified Galois covering  $\tilde{\varrho} = \bar{\varrho}|_{\tilde{X}}: \tilde{X} \rightarrow X$  with Galois group  $\mu_2$  so that  $X = \tilde{X}/\mu_2$ . The induced  $\mathbb{P}^1$ -fiber bundle  $\tilde{\varrho}^*\tilde{Y} \rightarrow \tilde{X}$  over  $\tilde{X}$  admits two disjoint sections  $Z_0$  and  $Z_\infty$  where  $Z_0$  is the tautological section. Letting  $\tilde{F} = \tilde{\varrho}^*\tilde{Y} \setminus Z_\infty$  we get a line bundle  $\tilde{F} \rightarrow \tilde{X}$ . Hence, the induced  $\mathbb{A}_*^1$ -fibration  $\tilde{Y} = \tilde{\varrho}^*\tilde{Y} \setminus (Z_0 \cup Z_\infty) \rightarrow \tilde{X}$  is untwisted. Furthermore,  $\tilde{Y}$  carries a free  $\mu_2$ -action making the projection  $\tilde{Y} \rightarrow \tilde{X}$  equivariant and such that  $Y = \tilde{Y}/\mu_2$ . We can now deduce the following analog of Corollary 2.9 for twisted  $\mathbb{A}_*^1$ -fibrations.

**Proposition 2.10.** *Let  $X$  be a smooth variety and  $\varrho: Y \rightarrow X$  be a smooth  $\mathbb{A}_*^1$ -fibration. Suppose that*

- (i)  *$X$  is elliptic and verifies the curve-orbit property with respect to a covering family of smooth projective rational curves  $\{C_x\}$  on  $X$ ;*
- (ii) *for any member  $C_x$  of the family the restriction  $\varrho|_{\varrho^{-1}(C_x)}: \varrho^{-1}(C_x) \rightarrow C_x$  is a nontrivial  $\mathbb{A}_*^1$ -fiber bundle.*<sup>2</sup>

*Then  $Y$  is elliptic.*

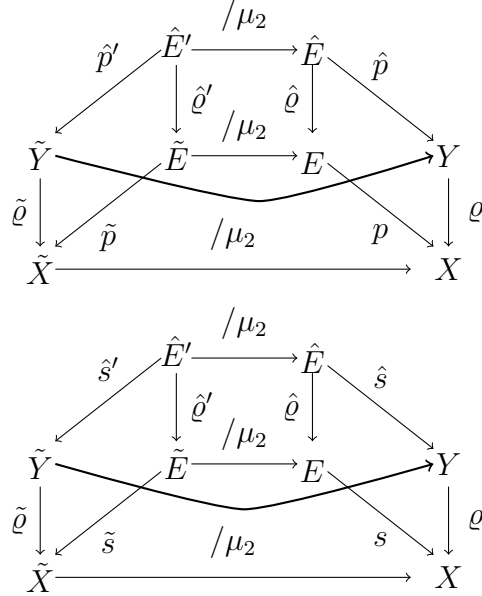


FIGURE 1.

<sup>2</sup>Condition (ii) is equivalent to the following one:

(ii')  $\tilde{F} \rightarrow \tilde{X}$  restricts to a nontrivial line bundle on each of the two components of the curve  $\tilde{C}_x = \tilde{\varrho}^{-1}(C_x)$  on  $\tilde{X}$ .

*Proof.* Since  $X$  is elliptic there exists a dominating spray  $(E, p, s)$  on  $X$ . Consider the natural  $\mu_2$ -action on  $\tilde{X}$ . Lemma 2.5 guarantees the existence of a pullback spray  $(\tilde{E}, \tilde{p}, \tilde{s})$  on  $\tilde{X}$  such that the induced morphisms  $\tilde{p}$  and  $\tilde{s}$  are  $\mu_2$ -equivariant with respect to the induced  $\mu_2$ -action on  $\tilde{E}$ . By Lemma 2.3 the latter spray admits a pullback to a  $\mu_2$ -equivariant spray  $(\hat{E}', \hat{p}', \hat{s}')$  on  $\tilde{Y}$ , see Figure 1.

The spray  $(\hat{E}', \hat{p}', \hat{s}')$  verifies the assumptions of Lemma 2.4, with notation changed accordingly. By this lemma, the  $\mu_2$ -equivariant spray  $(\hat{E}', \hat{p}', \hat{s}')$  on  $\tilde{Y}$  induces a push-forward spray  $(\hat{E}, \hat{p}, \hat{s})$  on  $Y$ . Since the spray  $(E, p, s)$  on  $X$  is dominating, the resulting spray  $(\hat{E}, \hat{p}, \hat{s})$  on  $Y$  is dominating in codimension 1, that is, for each  $y \in Y$  and  $v \in T_x X$  with  $\varrho(y) = x$  the  $s$ -orbit of  $y$  is tangent to a vector  $w \in T_y Y$  with  $d\varrho(w) = v$ . The curve-orbit property of  $X$  together with condition (ii) above guarantees the existence of a rank 1 spray  $(E_0, p_0, s_0)$  on  $Y$  such that the  $s_0$ -orbit of  $y$  is tangent to a nonzero vector from  $\text{Ker}(d\varrho)$ , see the proof of Proposition 2.6. Now the conclusion follows by an argument from the latter proof.  $\square$

## 2.4. Examples.

**Example 2.11.** Let  $B$  be a smooth complete variety and  $\pi: X \rightarrow B$  be a ruling, that is, a locally trivial fiber bundle over  $B$  with general fiber  $\mathbb{P}^1$ . Suppose that  $X$  is elliptic and let  $F \rightarrow X$  be a line bundle whose restriction to a fiber of  $\pi$  is nontrivial. Then the assumptions (i) and (ii) of Corollary 2.9 hold with the family  $\mathcal{F} = \{C_x\}_{x \in X}$  where  $C_x = \pi^{-1}(\pi(x))$ . According to this corollary  $Y = F \setminus Z_F$  is elliptic.

**Example 2.12.** The criterion of Corollary 2.9 can be applied to a generalized affine cone  $\hat{Y}$  over a smooth projective variety  $X$  defined by an ample polarization  $D$  on  $X$ , see the definition in the Introduction. Indeed, blowing  $\hat{Y}$  up at the vertex  $v \in \hat{Y}$  yields the line bundle  $F = \mathcal{O}_X(-D) \rightarrow X$  so that  $Y \setminus \{v\} = F \setminus Z_F$ . For instance, if  $D$  is a hyperplane section of  $X \subset \mathbb{P}^n$  then  $F = \mathcal{O}_X(-1)$ . Since  $D$  is ample the condition (ii) of Corollary 2.9 holds for any curve  $C$  on  $X$ . Thus, we arrive at the following criterion.

**Corollary 2.13.** *Let a smooth projective variety  $X$  be elliptic and verify the curve-orbit property  $(*)$ , see Definition 2.7. Let  $\hat{Y} \rightarrow X$  be a generalized affine cone over  $X$  with smooth locus  $Y = \hat{Y} \setminus \{v\}$ . Then  $Y$  is elliptic.*

**Example 2.14.** 1. The curve-orbit property holds for  $X = \mathbb{P}^1$ . Indeed, given a point  $x \in \mathbb{P}^1$  we can choose two opposite Borel subgroups  $B^\pm$  of  $\text{PGL}(2, \mathbb{K}) = \text{Aut}(\mathbb{P}^1)$  in such a way that the unique fixed point  $x^\pm$  of the unipotent radical  $U^\pm$  of  $B^\pm$  is different from  $x$ . Then the orbits  $U^\pm x$  are one-dimensional and cover  $\mathbb{P}^1$ , that is,  $(*)$  holds.

2. Any generalized flag variety  $G/P$  with a reductive group  $G$  over  $\mathbb{K}$  verifies the curve-orbit property. Indeed, by homogeneity of  $G/P$  and due to the previous example it suffices to find an  $\text{SL}(2, \mathbb{K})$ - or  $\text{PGL}(2, \mathbb{K})$ -subgroup  $S$  of  $G$  and a point  $x \in G/P$  such that the orbit  $C = Sx$  is isomorphic to  $\mathbb{P}^1$ .

Fix a Cartan subalgebra  $\mathfrak{h}$  of  $\text{lie}(P) \subset \text{lie}(G)$ , let  $\Delta \subset \mathfrak{h}^\vee$  be the associated root system of  $G$ ,  $\Delta^\pm$  be the subset of positive resp. negative roots in  $\Delta$  and  $\Delta_P \subset \Delta$  be

the root system of  $P$ . We assume that  $\Delta_P^+ = \Delta^+$ . Consider the root decomposition

$$(3) \quad \mathrm{lie}(P) = \mathfrak{h} \oplus \bigoplus_{\alpha_i \in \Delta^+} \mathfrak{g}_{\alpha_i} \oplus \bigoplus_{\alpha_i \in \Delta_P^-} \mathfrak{g}_{\alpha_i}$$

where  $\mathfrak{g}_\alpha$  is the root subspace of  $\mathrm{lie}(G)$  which corresponds to a root  $\alpha \in \Delta$ .

Pick a negative root  $\alpha \in \Delta^- \setminus \Delta_P^-$  and let  $S$  be the subgroup of  $G$  with Lie algebra

$$\mathrm{lie}(S) = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{h}_\alpha \quad \text{where} \quad \mathfrak{h}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}.$$

Thus,  $S$  is isomorphic to either  $\mathrm{SL}(2, \mathbb{K})$  or  $\mathrm{PGL}(2, \mathbb{K})$  and  $S \cap P = B_S$  is the Borel subgroup of  $S$  with Lie algebra

$$\mathrm{lie}(B_S) = \mathfrak{g}_{-\alpha} \oplus \mathfrak{h}_\alpha.$$

Since  $B_S \subset B_P$ , the  $B_S$ -action on  $G/P$  fixes the distinguished point  $x_0 = eP \in G/P$ . Let  $U^-$  be the one-parameter unipotent subgroup of  $S$  with Lie algebra  $\mathfrak{g}_\alpha$ . Assuming that  $U^-$  also fixes  $x_0$  we obtain  $[\mathfrak{g}_\alpha, \mathrm{lie}(P)] \subset \mathrm{lie}(P)$ ; in particular,  $\mathfrak{g}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{h}_\alpha] \subset \mathrm{lie}(P)$ . The latter contradicts (3). Therefore,  $x_0 \in (G/P)^{B_S} \setminus (G/P)^S$  and so, the  $S$ -orbit  $C = Sx_0$  is isomorphic to  $\mathbb{P}^1$ .

Recall that a variety  $X$  is said to be *of class  $\mathcal{A}_0$*  if  $X$  is covered by open subsets isomorphic to the affine space  $\mathbb{A}^n$  where  $n = \dim X$ . Such a variety is subelliptic, see [For17a, Proposition 6.4.5]. Therefore, it is elliptic by [KZ23, Theorem 0.1].

A generalized flag variety  $G/P$  where  $G$  is a reductive algebraic group and  $P \subset G$  is a parabolic subgroup belongs to class  $\mathcal{A}_0$ , see [APS14, Sec. 4(3)]; cf. also [Gro89, 3.5F]. Indeed, a maximal unipotent subgroup  $U$  of  $G$  acts on  $G/P$  with an open orbit isomorphic to an affine space, see, e.g., [Akh95, Sec. 3.1, proof of Proposition 7]. Since  $G$  acts on  $G/P$  transitively, the assertion follows. See also [APS14, Sec. 4] for further examples.

**Corollary 2.15.** *Let  $X = G/P$  be a flag variety and  $\hat{Y}$  be a generalized affine cone over  $X$ . Then the smooth locus  $Y = \hat{Y} \setminus \{v\}$  of  $\hat{Y}$  is elliptic.*

*Proof.* By the previous discussion and Example 2.14  $X$  verifies the assumptions of Corollary 2.13. So, the assertion follows due to this Corollary.  $\square$

**Remark 2.16.** Let  $\hat{Y}$  be the affine cone over a flag variety  $G/P$  embedded in  $\mathbb{P}^n$  as a projectively normal subvariety. Then  $\hat{Y}$  is normal and moreover, flexible, see [AKZ12, Theorem 0.2]. Hence,  $Y = \hat{Y} \setminus \{v\}$  is elliptic.

### 3. CONES OVER BLOWN-UP VARIETIES OF CLASS $\mathcal{A}_0$

**3.1. The enhanced curve-orbit property.** Blowing up a variety  $X$  of class  $\mathcal{A}_0$  with center at a point results again in a variety of class  $\mathcal{A}_0$ , see, e.g., [Gro89, Sect. 3.5.D''] or [For17a, Proposition 6.4.6]. In particular, any rational smooth projective surface is of class  $\mathcal{A}_0$ , see [For17a, Corollary 6.4.7]. The next proposition allows to apply Corollary 2.13 to varieties of class  $\mathcal{A}_0$  blown up at a point.

**Proposition 3.1.** *Let  $X$  be a complete variety of class  $\mathcal{A}_0$  and of dimension  $n \geq 3$ . Suppose that  $X$  verifies the following enhanced curve-orbit property:*

(\*\*) for each  $x \in X$  and any finite subset  $M$  in  $X \setminus \{x\}$  one can find a curve  $C_x \simeq \mathbb{P}^1$  on  $X$  which avoids  $M$  and verifies the two-orbit property (\*) of Definition 2.7 with pairs of  $\mathbb{G}_a$ -sprays  $(E_i, p_i, s_i)$ ,  $i = 1, 2$ .

Let  $\sigma: \tilde{X} \rightarrow X$  be the blowup of a point  $x_0 \in X$ . Then  $\tilde{X}$  also verifies (\*\*).

*Proof.* Given a point  $\tilde{x} \in \tilde{X}$  fix a finite subset  $\tilde{M} \subset \tilde{X} \setminus \{\tilde{x}\}$ . Assume first that  $x := \sigma(\tilde{x}) \neq x_0$ . Choose a curve  $C_x \subset X$  and a pair of extended  $\mathbb{G}_a$ -sprays  $(E_i, p_i, s_i)$  on  $X$  satisfying (\*\*) with  $M = \{x_0\} \cup \sigma(\tilde{M}) \subset X$ . Replace the  $(E_i, p_i, s_i)$  by suitable replicas such that the  $s_i$ -orbits of  $x_0$  become singletons, while the  $s_i$ -orbits  $O_i$  of  $x$  do not change. By Proposition 1.5 one can lift these sprays to extended  $\mathbb{G}_a$ -sprays  $(\tilde{E}_i, \tilde{p}_i, \tilde{s}_i)$  on  $\tilde{X}$ ,  $i = 1, 2$  such that  $\sigma(\tilde{O}_i) = O_i$  where  $\tilde{O}_i$  is the  $\tilde{s}_i$ -orbit of the point  $\tilde{x}$ . It is easily seen that the curve  $C_{\tilde{x}} = \sigma^{-1}(C_x)$  and the lifted sprays  $(\tilde{E}_i, \tilde{p}_i, \tilde{s}_i)$  satisfy (\*\*).

Suppose now that  $\tilde{x}$  lies on the exceptional divisor  $\mathcal{E} = \sigma^{-1}(x_0)$ . Choose a neighborhood  $\Omega \simeq \mathbb{A}^n$  of  $\tilde{x}$  in  $\tilde{X}$ . Identifying  $\Omega$  with  $\mathbb{A}^n = \text{Spec } \mathbb{K}[u_1, \dots, u_n]$  we may suppose that  $x$  is the origin of  $\mathbb{A}^n$ . Then  $(u_1: \dots: u_n)$  can serve as homogeneous coordinates in  $\mathcal{E} \simeq \mathbb{P}^{n-1}$ . Notice that  $\tilde{X}$  can be covered with  $n$  open subsets  $U_i \simeq \mathbb{A}^n$  such that  $U_i \cap \mathcal{E} = \mathcal{E} \setminus H_i \simeq \mathbb{A}^{n-1}$  where  $H_i$  is the coordinate hyperplane in  $\mathcal{E} = \mathbb{P}^{n-1}$  given by  $u_i = 0$ .

Choose the affine coordinates  $(u_1, \dots, u_n)$  in  $X = \mathbb{A}^n$  in such a way that  $\tilde{x} = (u_1: \dots: u_n) \in \mathbb{P}^{n-1}$  has two nonzero coordinates, say  $u_1$  and  $u_2$ , that is,  $\tilde{x} \in U_1 \cap U_2$ . Since  $n - 1 \geq 2$ , a general projective line  $C_{\tilde{x}} \subset \mathcal{E} = \mathbb{P}^{n-1}$  through  $\tilde{x}$  meets  $H_1$  and  $H_2$  in two distinct points and does not meet  $\tilde{M}$ . Furthermore, for  $i = 1, 2$  the affine line  $\bar{O}_i = C_{\tilde{x}} \setminus H_i \simeq \mathbb{A}^1$  in  $U_i$  is an orbit of a  $\mathbb{G}_a$ -action on  $U_i \simeq \mathbb{A}^n$  and so, a  $\bar{s}_i$ -orbit of the associated extended  $\mathbb{G}_a$ -spray  $(\bar{E}_i, \bar{p}_i, \bar{s}_i)$  on  $\tilde{X}$ . One has  $C_{\tilde{x}} = \bar{O}_1 \cup \bar{O}_2$ . Thus, (\*\*) holds for  $\tilde{X}$ .  $\square$

**Remark 3.2.** Assuming that  $\tilde{M}$  is empty the same construction of two extended  $\mathbb{G}_a$ -sprays on  $\tilde{X}$  with  $C_{\tilde{x}} = O_1 \cup O_2$  satisfying the two-orbit property goes through for  $n = 2$ , while this time  $\mathcal{E} \simeq \mathbb{P}^1$ .

### 3.2. Enhanced curve-orbit property for flag varieties.

**Proposition 3.3.** *A generalized flag variety  $X = G/P$  of dimension  $n \geq 2$  verifies the enhanced curve-orbit property (\*\*).*

*Proof.* The stabilizer of the distinguished point  $x_0 = eP \in G/P$  coincides with the parabolic subgroup  $P$ . The construction of Example 2.14 produces a subgroup  $S$  of  $G$  isomorphic either to  $\text{SL}(2, \mathbb{K})$  or to  $\text{PGL}(2, \mathbb{K})$ , or in other terms, a subgroup locally isomorphic to  $\text{SL}(2, \mathbb{K})$ , along with a pair of opposite one-parameter unipotent subgroups  $U^\pm$  of  $S$  and an  $S$ -orbit  $C = S/B^+ \simeq \mathbb{P}^1$  passing through  $x_0$  such that the orbits  $U^\pm x_0$  are one-dimensional and cover  $C$ , where  $B^+ = S \cap P$  is a Borel subgroup of  $S$ .

The conjugation of  $S$  by elements  $a \in P$  yields a family  $S_a = aSa^{-1}$  of  $\text{SL}(2, \mathbb{K})$ -resp.  $\text{PGL}(2, \mathbb{K})$ -subgroups of  $G$  along with a family of their orbits  $C_a = aC \simeq \mathbb{P}^1$  passing through  $x_0$ . We claim that the family of  $\mathbb{P}^1$ -curves  $\mathcal{F} = \{C_a\}_{a \in P}$  has no base



point different from  $x_0$ . Assuming that  $\mathcal{F}$  contains at least two distinct curves the base point set  $\text{Bs}(\mathcal{F})$  is finite. Since the parabolic subgroup  $P$  is connected each  $x \in \text{Bs}(\mathcal{F})$  is fixed by  $P$ . Since  $P$  coincides with its normalizer  $N_G(P)$ , see [Hum95, Theorem X.29.4(c)],  $x_0 = eP$  is the unique fixed point of  $P$ . Indeed, if  $PgP = gP$  for some  $g \in G$ , then  $g \in N_G(P) = P$  and so,  $gP = P$ .

Let us show that  $\text{card}(\mathcal{F}) \geq 2$ . In the notation of (3) we have

$$\dim(G/P) = \text{card}(\Delta^- \setminus \Delta_P^-) \geq 2.$$

Therefore,  $\Delta^- \setminus \Delta_P^-$  contains two distinct negative roots  $\alpha_i$ ,  $i = 1, 2$ . Let  $U_i^+$  be the root subgroup of  $P$  with  $\text{lie}(U_i^+) = \mathfrak{g}_{-\alpha_i}$ ,  $S_i$  be the subgroup in  $G$  with

$$\text{lie}(S_i) = \text{lie}(\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{-\alpha_i}) \cong \mathfrak{sl}(2, \mathbb{K})$$

and  $B_i = S_i \cap P \supset U_i^+$  be a Borel subgroup of  $S_i$ , cf. Example 2.14. The  $S_i$ -orbit of the distinguished point  $x_0$  is a  $\mathbb{P}^1$ -curve  $C_i = S_i/B_i$  on  $G/P$ . Notice that  $U_1^+$  and  $U_2^+$  are two distinct subgroups of the unipotent radical  $R_u(P)$ . Let  $P^-$  be the opposite parabolic subgroup of  $G$  and  $B_i^-$  be the opposite Borel subgroup of  $S_i$  with unipotent radical  $U_i^-$ . Then  $U_1^-$  and  $U_2^-$  are two distinct subgroups of  $R_u(P^-)$ . The  $R_u(P^-)$ -orbit of  $x_0$  is open in  $G/P$  and the isotropy subgroup of  $x_0$  in  $R_u(P^-)$  is trivial. Hence the orbits  $U_i^- x_0$ ,  $i = 1, 2$  are distinct and so are the curves  $C_i = \overline{U_i^- x_0}$ ,  $i = 1, 2$ , as claimed.

Thus, for any finite subset  $M = \{x_1, \dots, x_m\} \subset G/P \setminus \{x_0\}$  and for every  $i = 1, \dots, m$  one can choose a curve  $C_{a_i} = a_i C$  from our family which does not meet  $x_i$ . The set of elements  $a_i \in P$  with this property is open and dense in  $P$ . Therefore, for a general  $a \in P$  the curve  $C_a$  does not intersect  $M$ . Since  $G/P$  is homogeneous the choice of a point  $x_0$  is irrelevant. It follows that  $G/P$  verifies condition (\*\*), see Example 2.14.  $\square$

### 3.3. The main result in dimensions $\geq 3$ .

**Theorem 3.4.** *Let  $X$  be a smooth complete variety and  $Y$  be the smooth locus of a generalized affine cone over  $X$ , see the Introduction. Then  $Y$  is elliptic in each of the following cases:*

- (a)  $n = \dim X \geq 3$  and  $X$  is obtained from a variety  $X_0$  of class  $\mathcal{A}_0$  verifying condition (\*\*) of Proposition 3.1 (say,  $X_0 = G/P$  is a flag variety of dimension  $n \geq 3$ , see Proposition 3.3) via a finite number of subsequent blowups;
- (b)  $X$  is obtained by blowing up a finite subset of a Hirzebruch surface  $\mathbb{F}_n$ ,  $n \geq 0$  or of  $\mathbb{P}^2$  (say,  $X$  is a del Pezzo surface).

*Proof.* Notice that in (a) and (b)  $X$  is of class  $\mathcal{A}_0$ , hence elliptic. Thus, it suffices to check conditions (i) and (ii) of Corollary 2.9. In all cases condition (ii) holds because of Corollary 2.13. Condition (i) is provided by Propositions 3.1 and 3.3, Remark 3.2 and Example 2.11.  $\square$

**Remark 3.5.** The proof of Theorem 3.4(a) does not work for  $n = 2$ . Indeed, in this case the exceptional curve  $\mathcal{E} = \mathbb{P}^1$  in the proof of Proposition 3.1 can contain points of  $\tilde{M}$ . However, its conclusion still holds for  $n = 2$ ; see Theorem 4.1 in the next section.



## 4. ELLIPTIC CONES OVER RATIONAL SURFACES

In this section we extend Theorem 3.4 to any rational smooth projective surface  $X$ .

**Theorem 4.1.** *Let  $X$  be a rational smooth projective surface and  $Y = \hat{Y} \setminus \{v\}$  be the smooth locus of a generalized cone  $\hat{Y}$  over  $X$ . Then  $Y$  is elliptic.*

Due to Corollary 2.13, Theorem 4.1 follows from the next proposition.

**Proposition 4.2.**  *$X$  verifies the curve-orbit property  $(*)$ .*

The proof of Proposition 4.2 uses induction on the number of blowups when forming  $X$ . The following proposition provides the inductive step.

**Proposition 4.3.** *Let  $X$  be a smooth projective surface and  $\pi: X \rightarrow \mathbb{P}^1$  be a fibration with general fiber  $\mathbb{P}^1$ . Assume that for any fiber component  $C$  of  $\pi$  and any point  $x \in C$  the following hold.*

- (i) *There exists an open subset  $U_x$  in  $X$  such that  $X \setminus U_x$  has no isolated point and  $O_x := U_x \cap C = C \setminus \{x\} \cong \mathbb{A}^1$ ;*
- (ii) *there exists a proper birational morphism  $\nu_x: U_x \rightarrow \mathbb{A}^2$  such that*
  - *$O_x$  is sent by  $\nu_x$  isomorphically onto a coordinate axis  $L$  in  $\mathbb{A}^2$ ;*
  - *the exceptional divisor of  $\nu_x$  consists of some fiber components of  $\pi$ ;*
  - *no one of these components is contracted to a point of  $L$ ;*
- (iii) *there is a  $\mathbb{G}_a$ -action  $\lambda_x$  on  $U_x$  such that  $O_x$  is the  $\lambda_x$ -orbit of  $x$ ;*
- (iv) *for the extended  $\mathbb{G}_a$ -spray  $(E_x, p_x, s_x)$  on  $X$  associated with  $\lambda_x$ ,  $O_x$  is the  $s_x$ -orbit of any point  $x' \in O_x$ .*

Let  $\sigma: \tilde{X} \rightarrow X$  be the blowup of a point  $x_0 \in X$  and  $\tilde{\pi} = \pi \circ \sigma: \tilde{X} \rightarrow \mathbb{P}^1$  be the induced  $\mathbb{P}^1$ -fibration. Then (i) – (iv) hold with  $\pi: X \rightarrow \mathbb{P}^1$  replaced by  $\tilde{\pi}: \tilde{X} \rightarrow \mathbb{P}^1$ .

*Proof.* Let  $\tilde{C}$  be a fiber component of  $\tilde{\pi}$  on  $\tilde{X}$  and  $\tilde{x} \in \tilde{C}$ . If  $\tilde{C}$  is different from the exceptional curve of  $\sigma$  then we let  $C = \sigma(\tilde{C}) \subset X$  and  $x = \sigma(\tilde{x}) \in C$ ; otherwise, we let  $x = x_0$ . Consider an open subset  $U_x \subset X$  satisfying (i)-(iv).

The following cases can occur:

- (a)  $\tilde{C} = \sigma^{-1}(x_0)$  is the exceptional curve of  $\sigma$  and  $x = x_0$ ;
- (b)  $C$  is a fiber component of  $\pi$  and  $x_0 \notin C$ ;
- (c)  $C$  is a fiber component of  $\pi$  and  $x_0 \in C$ .

In case (a) we choose an open neighborhood  $U_0 \cong \mathbb{A}^2$  of  $x_0$  in  $X$ . Such a neighborhood does exist since  $X$  belongs to class  $\mathcal{A}_0$ . We also choose a coordinate system  $(u, v)$  on  $U_0$  such that  $x_0$  is the origin and the axis  $L = \{u = 0\}$  goes in direction of  $\tilde{x} \in \tilde{C} = \mathbb{P}T_{x_0}U_0$ . Then  $\tilde{U}_0 := \sigma^{-1}(U_0) = \tilde{U}_1 \cup \tilde{U}_2$  where the  $\tilde{U}_i \simeq \mathbb{A}^2$  are equipped with coordinates  $(u_i, v_i)$  such that

( $\alpha$ ) the restrictions  $\sigma_0|_{\tilde{U}_i}: \tilde{U}_i \rightarrow U_0$  are given by

$$\sigma_0|_{\tilde{U}_1}: (u_1, v_1) \mapsto (u_1 v_1, v_1) \quad \text{and} \quad \sigma_0|_{\tilde{U}_2}: (u_2, v_2) \mapsto (u_2, u_2 v_2);$$

( $\beta$ )  $\sigma_0$  sends the coordinate axis  $L_1 := \{u_1 = 0\}$  in  $\tilde{U}_1$  isomorphically onto  $L$ .

Then  $U_{\tilde{x}} := \tilde{U}_1$  equipped with the  $\mathbb{G}_a$ -action  $\lambda_{\tilde{x}}$  by translations in direction of  $L_1$  satisfies (i)–(iv).

If (b) or (c) takes place then the preimage  $U_{\tilde{x}} = \sigma^{-1}(U_x)$  in  $\tilde{X}$  satisfies

$$O_{\tilde{x}} := U_{\tilde{x}} \cap \tilde{C} = \sigma^{-1}(C \setminus \{x\}) = \tilde{C} \setminus \{\tilde{x}\}.$$

Clearly,  $U_{\tilde{x}}$  also satisfies condition (i).

Assume that case (b) takes place. It is easily seen that the composition  $\nu_{\tilde{x}} = \nu_x \circ \sigma: U_{\tilde{x}} \rightarrow \mathbb{A}^2$  satisfies (ii). Replacing  $\lambda_x$  with an appropriate replica, if necessary, we may assume that  $x_0 \notin C$  is fixed by  $\lambda_x$ , while  $O_x = C \cap U_x = C \setminus \{x\}$  is still an orbit of  $\lambda_x$ . Thus,  $\lambda_x$  can be lifted to a  $\mathbb{G}_a$ -action  $\lambda_{\tilde{x}}$  on  $U_{\tilde{x}}$  such that  $O_{\tilde{x}}$  is the  $\lambda_{\tilde{x}}$ -orbit of  $\tilde{x}$  and so, (iii) holds for  $\lambda_{\tilde{x}}$ . Letting  $(E_{\tilde{x}}, p_{\tilde{x}}, s_{\tilde{x}})$  be the extended  $\mathbb{G}_a$ -spray on  $\tilde{X}$  associated with  $\lambda_{\tilde{x}}$  we obtain a data verifying (i)–(iv).

Case (c) splits in two sub-cases:

(c')  $x = x_0$ ;

(c'')  $x \neq x_0$ .

In case (c') we have  $x_0 = x \notin U_x$ . Hence, letting  $U_{\tilde{x}} = \sigma^{-1}(U_x)$  the restriction  $\sigma|_{U_{\tilde{x}}}: U_{\tilde{x}} \rightarrow U_x$  is an isomorphism. Proceeding as before we come to the desired conclusion.

In case (c'') we have  $x_0 \in C \setminus \{x\} = O_x$  and  $\nu_x(x_0) \in \nu_x(O_x) = L$ . We may suppose that  $\nu_x(x_0) = \bar{0} \in \mathbb{A}^2$  is the origin. By (ii),  $\nu_x: U_x \rightarrow \mathbb{A}^2$  is étale on  $O_x \subset U_x$ . Hence,  $\nu_x$  is a blowup of  $\mathbb{A}^2$  with center an ideal  $I \subset \mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2)$  supported on  $\mathbb{A}^2 \setminus L$ . Since  $\nu_x$  sends  $O_x$  isomorphically onto  $L$ , the proper birational morphism  $\nu_x \circ \sigma: U_{\tilde{x}} \rightarrow \mathbb{A}^2$  also sends  $O_{\tilde{x}}$  isomorphically onto  $L$ . Notice that the blowups with disjoint centers commute. Hence,  $\nu_x \circ \sigma = \sigma_0 \circ \nu_0$  where  $\sigma_0: \tilde{U}_0 \rightarrow \mathbb{A}^2$  is the blowup of  $\mathbb{A}^2$  at the origin and  $\nu_0$  is the blowup of  $\tilde{U}_0$  with center  $\tilde{I} = \sigma_0^*(I) \subset \mathcal{O}_{\tilde{U}_0}(\tilde{U}_0)$ . These morphisms fit in a commutative diagram

$$\begin{array}{ccc} U_{\tilde{x}} & \xrightarrow{\sigma} & U_x \\ \downarrow \nu_0 & & \downarrow \nu_x \\ \tilde{U}_0 & \xrightarrow{\sigma_0} & \mathbb{A}^2 \end{array}$$

Choose affine coordinates  $(u, v)$  on  $\mathbb{A}^2$  such that  $x_0 = (0, 0)$  is the origin and  $L = \{u = 0\}$ . Then  $\tilde{U}_0 = \tilde{U}_1 \cup \tilde{U}_2$  where  $\tilde{U}_i \simeq \mathbb{A}^2$  is equipped with coordinates  $(u_i, v_i)$  so that  $(\alpha)$  and  $(\beta)$  above hold. Morphism  $\sigma_0$  sends a coordinate axis  $L_1$  in  $\tilde{U}_1$  isomorphically onto  $L$  and  $\nu_0$  sends  $O_{\tilde{x}}$  isomorphically onto  $L_1$ .

Replace now  $U_{\tilde{x}}$  by  $U_{\tilde{x},1} := \nu_0^{-1}(\tilde{U}_1)$  and  $\nu$  by  $\nu_1 = \nu_0|_{U_{\tilde{x},1}}: U_{\tilde{x},1} \rightarrow \tilde{U}_1 \simeq \mathbb{A}^2$ . We have  $O_{\tilde{x}} = U_{\tilde{x},1} \cap \tilde{C}$  and  $\nu_1|_{O_{\tilde{x}}}: O_{\tilde{x}} \rightarrow L_1$  is an isomorphism which sends  $x_0 \in O_{\tilde{x}}$  to the origin  $\bar{0} \in \tilde{U}_1 = \mathbb{A}^2$ . Furthermore,  $\nu_1$  is étale on  $O_{\tilde{x}}$  and so,  $U_{\tilde{x},1}$  verifies (i) and (ii).

Consider now the  $\mathbb{G}_a$ -action  $\lambda$  on  $\tilde{U}_1 = \mathbb{A}^2$  by translations in direction of  $L_1$  so that  $L_1$  is a  $\lambda$ -orbit. Replacing  $\lambda$  by an appropriate replica we can achieve that  $\lambda$  acts freely on  $L_1$  and leaves invariant the center  $\tilde{I}|_{U_{\tilde{x},1}}$  of the blowup  $\nu_0|_{U_{\tilde{x},1}}$ . Then  $\lambda$  admits a lift  $\tilde{\lambda}$  to  $U_{\tilde{x},1}$  such that  $O_{\tilde{x}}$  is a  $\tilde{\lambda}$ -orbit. Therefore,  $U_{\tilde{x},1}$  satisfies (iii) and (iv).  $\square$

*Proof of Proposition 4.2.* Clearly,  $\mathbb{P}^2$  and the Hirzebruch surfaces  $\mathbb{F}_n$  with  $n \geq 0$  verify the curve-orbit property. It is well known that any rational smooth projective surface  $X$

non-isomorphic to  $\mathbb{P}^2$  results from a sequence of blowups starting with  $\mathbb{F}_n$  for some  $n \geq 0$ . Moreover, properties (i)–(iv) of Proposition 3.1 hold for the  $\mathbb{P}^1$ -fibration  $\pi: \mathbb{F}_n \rightarrow \mathbb{P}^1$ . By this proposition and the induction on the number of blowups these properties still hold for  $X$ . Let  $x \in X$  lies on a fiber component  $C$  of the induced  $\mathbb{P}^1$ -fibration  $\pi: X \rightarrow \mathbb{P}^1$ . Choose two distinct points  $x_1, x_2 \in C \setminus \{x\}$ . Let  $U_i$  and  $(E_i, p_i, s_i)$  verify (i)–(iv) of Proposition 3.1 with  $x$  replaced by  $x_i$ ,  $i = 1, 2$ . Then  $O_i = C \setminus \{x_i\}$  is an  $s_i$ -orbit of  $x$ . The restriction of the extended  $\mathbb{G}_a$ -spray  $(E_i, p_i, s_i)$  to  $O_i$  is dominating. Therefore,  $X$  verifies the curve-orbit property. This proves Proposition 4.2.  $\square$

## 5. APPENDIX A: HIGH TRANSITIVITY OF THE $\text{End}(X)$ -ACTION

It is known that the automorphism group  $\text{Aut}(X)$  of a flexible variety  $X$  acts highly transitively on the smooth locus  $X_{\text{reg}}$ , that is, it acts  $m$ -transitively for any  $m \geq 1$ , see [AFKKZ13, Theorem 0.1]. A similar transitivity property holds for the monoid of endomorphisms  $\text{End}(X)$  provided  $X$  is a smooth elliptic variety in the Gromov' sense. Abusing the language we say that  $\text{End}(X)$  *acts highly transitively on  $X$*  if for any two isomorphic zero-dimensional subschemes  $Z_1$  and  $Z_2$  of  $X$  every isomorphism  $Z_1 \xrightarrow{\sim} Z_2$  extends to an endomorphism of  $X$ .

**Proposition 5.1.** *Let  $X$  be a smooth quasiaffine variety. If  $X$  is elliptic then  $\text{End}(X)$  acts highly transitively on  $X$ .*

The above proposition is an easy consequence of the following epimorphism theorem due to Kusakabe [Kus22, Theorem 1.2], which generalizes an earlier result of Forstnerič [For17b, Theorem 1.6] valid for complete elliptic varieties. This phenomenon was expected in [Gro89, Sec. 3.4(C)]; see also [Arz22] for analogs in the case of flexible varieties.

**Theorem 5.2.** *For any smooth (sub)elliptic variety  $X$  of dimension  $n$  there exists a morphism  $\varphi: \mathbb{A}^{n+1} \rightarrow X$  such that  $\varphi(\mathbb{A}^{n+1} \setminus \text{Sing}(\varphi)) = X$  where*

$$\text{Sing}(\varphi) = \{x \in \mathbb{A}^{n+1} \mid \varphi \text{ is not smooth at } x\}$$

*is the singular locus of  $\varphi$ . If  $X$  is complete then the same is true with  $\mathbb{A}^n$  instead of  $\mathbb{A}^{n+1}$ .*

Using the fact that any morphism of a zero-dimensional scheme  $Z \rightarrow X$  can be lifted through a smooth morphism  $\varphi: \mathbb{A}^{n+1} \setminus \text{Sing}(\varphi) \rightarrow X$ , Kusakabe derives the following interpolation result, see [Kus22, Corollary 1.5]; cf. [Gro89, Sec. 3.4(E)].

**Corollary 5.3.** *Let  $X$  be a smooth (sub)elliptic variety,  $Y$  be an affine variety and  $Z$  be a zero-dimensional subscheme of  $Y$ . Then for any morphism  $f: Z \rightarrow X$  there exists a morphism  $\tilde{f}: Y \rightarrow X$  such that  $\tilde{f}|_Z = f$ .*

In fact, Corollary 5.3 remains valid with the same proof if one assumes  $Y$  to be quasiaffine. Letting in Corollary 5.3  $Y = X$  and  $Z_1 = Z$  we can extend the given morphism  $f: Z_1 \rightarrow Z_2$  onto a zero-dimensional subscheme  $Z_2$  of  $X$  to an endomorphism  $\tilde{f}: X \rightarrow X$ . This yields Proposition 5.1. Furthermore, taking in Corollary 5.3  $Y = \mathbb{A}^1$  we obtain the following corollary, cf. [Gro89, Sec. 3.4(B)] and [KZ00, p. 1661].

**Corollary 5.4.** *Any smooth elliptic variety  $X$  is  $\mathbb{A}^1$ -rich, that is, through any  $m$  distinct points of  $X$  passes an  $\mathbb{A}^1$ -curve  $\mathbb{A}^1 \rightarrow X$ . Moreover, given a finite collection of curve jets  $j_1, \dots, j_m$  on  $X$  with not necessarily distinct centers, there exists an  $\mathbb{A}^1$ -curve  $f: \mathbb{A}^1 \rightarrow X$  interpolating these jets.*

Notice that for a flexible smooth affine variety  $X$  one can find an orbit of a  $\mathbb{G}_a$ -action on  $X$  interpolating given curve jets, see [AFKKZ13, Theorem 4.14].

## 6. APPENDIX B: GROMOV'S EXTENSION LEMMA

For the reader's convenience we recall the formulation of Proposition 1.1.

**Proposition 6.1.** *Let  $D$  be a reduced effective divisor on  $X$  and  $(E, p, s)$  be a spray on  $U = X \setminus \text{supp}(D)$  with values in  $X$  such that  $p: E \rightarrow X$  is a trivial bundle of rank  $r \geq 1$ . Then there exists a spray  $(\tilde{E}, \tilde{p}, \tilde{s})$  on  $X$  whose restriction to  $U$  is isomorphic to  $(E, p, s)$  and such that  $\tilde{s}|_{\tilde{p}^{-1}(X \setminus U)} = \tilde{p}|_{\tilde{p}^{-1}(X \setminus U)}$  and for each  $x \in U$  the  $\tilde{s}$ -orbit of  $x$  coincides with the  $s$ -orbit of  $x$ .*

*Proof.* Extend  $E$  to a trivial vector bundle on  $X$ ; abusing notation, the latter is again denoted by  $p: E \rightarrow X$ . Let  $\xi$  be a canonical section of  $\mathcal{O}_X(D) \rightarrow X$  such that  $\text{div}(\xi) = D$ .

Consider the twisted vector bundle  $p_n: E_n = E \otimes \mathcal{O}_X(-nD) \rightarrow X$  where  $n \in \mathbb{N}$ . Since  $\xi$  does not vanish on  $U$  the homomorphism  $\varphi_n: E_n \rightarrow E$  induced by the tensor multiplication by  $\xi^n$  restricts to an isomorphism on  $U$  which sends  $s_n := s \circ \varphi_n$  to  $s$ . We claim that  $s_n$  extends to a morphism  $E_n \rightarrow X$  for a sufficiently large  $n$ . It is easily seen that in the latter case  $(\tilde{E}, \tilde{p}, \tilde{s}) = (E_n, p_n, s_n)$  is a desired spray on  $X$ .

The argument is local and, thus, we restrict consideration to an affine neighborhood  $\omega \subset X$  of a given point  $x_0 \in \text{supp}(D)$  such that

- $D|_\omega = h^*(0)$  for some  $h \in \mathcal{O}_\omega(\omega)$  and
- $E_n|_\omega \cong_\omega \omega \times \mathbb{A}^r$  is a trivial vector bundle.

Notice that over  $\omega^* = \omega \setminus \text{supp}(D)$  the spray  $(E_n, p_n, s_n)$  is the  $h^n$ -homothety of  $(E, p, s)$ . The latter means that  $s_n$  is given, after trivialization, by

$$s_n: (x, v) \mapsto s(x, h^n(x)v) \quad \forall (x, v) \in \omega^* \times \mathbb{A}^r.$$

Let  $\omega \hookrightarrow \mathbb{A}^m$  be a closed embedding. Then  $s$  yields a rational map

$$s: \omega \times \mathbb{A}^r \dashrightarrow \mathbb{A}^m \quad \text{of the form} \quad (x, v) \mapsto x + \psi(x, v)$$

where  $\psi$  is a rational vector function which is regular in a neighborhood of  $\omega^* \times \{0\}$  and vanishes on  $\omega^* \times \{0\}$ . Every coordinate function of  $\psi$  can be written in the form

$$\frac{q(v)}{h^k r(v)} \quad \text{where} \quad v = (t_1, \dots, t_r) \in \mathbb{A}^r \quad \text{and} \quad q, r \in \mathcal{O}_\omega(\omega)[t_1, \dots, t_r]$$

are such that  $q(0) = 0$  and the zero loci of  $r$  and  $h$  have no common component in  $\omega \times \mathbb{A}^r$ .

Furthermore, we may suppose that the divisors  $r^*(0)$  and  $q^*(0)$  have no common irreducible component  $T$  in  $\omega \times \mathbb{A}^r$  passing through  $(x_0, 0) \in \omega \times \mathbb{A}^r$ . Indeed, let  $T$  be

such a component. It is well known that  $\text{Pic}(\omega \times \mathbb{A}^r) = \text{pr}_1^*(\text{Pic}(\omega))$ . Hence, the divisor  $T - \text{pr}_1^*(T_0)$  is principal for some divisor  $T_0$  on  $\omega$ . Shrinking  $\omega$  one may suppose that  $T_0$  is principal. Hence,  $T$  is principal, i.e.,  $T = f^*(0)$  for some regular function  $f$  on  $\omega \times \mathbb{A}^r$ . Thus, dividing  $r$  and  $q$  by  $f$  we can get rid of  $T$ .

This implies that  $r$  does not vanish on  $\omega^* \times \{0\}$  since otherwise  $\psi$  is not regular on  $\omega^* \times \{0\}$ . In particular,  $r(0) = ah^l$  where  $a$  is a non-vanishing function on  $\omega$ .

The  $h^n$ -homothety  $s_n$  of  $s$  yields a rational map

$$s_n: \omega \times \mathbb{A}^r \dashrightarrow \omega \subset \mathbb{A}^m, \quad (x, v) \mapsto x + \psi_n(x, v),$$

where  $\psi$  is a rational vector function which is regular in a neighborhood of  $\omega^* \times \{0\}$  and vanishes on  $\omega^* \times \{0\}$ . Every coordinate function of  $\psi_n \in$  is of the form

$$\frac{q(h^n v)}{h^k r(h^n v)}.$$

Since  $q(0) = 0$  and  $[r(h^n v) - r(v)]|_{v=0} = 0$  we see that

$$q(h^n v) = h^n \tilde{q}(v) \quad \text{and} \quad r(h^n v) = r(0) + h^n \tilde{r}(v) = h^l a + h^n \tilde{r}(v).$$

Hence, for  $n > k + l$  the function

$$\frac{q(h^n v)}{h^k r(h^n v)} = \frac{h^{n-k-l} \tilde{q}(v)}{a + h^{n-l} \tilde{r}(v)}$$

is regular in a neighborhood  $\Omega$  of  $\omega \times \{0\}$  and vanishes to order  $n - k - l$  on  $(\text{supp}(D) \times \mathbb{A}^r) \cap \Omega$ .

Consider now  $s_{n+1} = s \circ \varphi_{n+1} = s_n \circ \varphi_1$  where, after trivialization,

$$\varphi_1|_{\omega \times \mathbb{A}^r}: \omega \times \mathbb{A}^r \rightarrow \omega \times \mathbb{A}^r, \quad (x, v) \mapsto (x, h(x)v).$$

Letting  $\tilde{\Omega} = \varphi_1^{-1}(\Omega) \subset \omega \times \mathbb{A}^r$  we see that

$$\varphi_1(\tilde{\Omega}) \subset \Omega \quad \text{and} \quad (\text{supp}(D) \cap \omega) \times \mathbb{A}^r \subset \tilde{\Omega}.$$

Hence,  $s_{n+1}$  is regular on  $\tilde{\Omega}$  and so, on  $p_{n+1}^{-1}(\text{supp}(D) \cap \omega)$ . This yields a spray  $(E_{n+1}, p_{n+1}, s_{n+1})$  on  $\omega \cup U$  with values in  $X$ . Choosing a finite cover of  $\text{supp}(D)$  by affine open subsets  $\omega_i$  and a sufficiently large  $n$  we get the desired extension of  $(E, p, s)$  to a spray  $(\tilde{E}, \tilde{p}, \tilde{s})$  on  $X$ .  $\square$

**Remark 6.2.** The same proof works also for the original Gromov's Localization Lemma, see [Gro89, 3.5.B].

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