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# Topologies on split Kac-Moody groups over valued fields

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## Abstract

Let  $G$  be a minimal split Kac-Moody group over a valued field  $\mathcal{K}$ . Motivated by the representation theory of  $G$ , we define two topologies of topological group on  $G$ , which take into account the topology on  $\mathcal{K}$ .

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# 1 Introduction

## 1.1 Motivation from representation theory

Let  $G$  be a reductive group over a nonArchimedean local field  $\mathcal{K}$ . As  $G$  is finite dimensional over  $\mathcal{K}$ ,  $G$  is naturally equipped with a topological group structure. It admits a basis of neighbourhood of the identity consisting of open compact subgroups. A complex representation  $V$  of  $G$  is called smooth if for every  $v \in V$ , the fixator of  $v$  in  $G$  is open. To every compact open subgroup  $K$  of  $G$  is associated a Hecke algebra  $\mathcal{H}_K$ , which is the space of  $K$ -bi-invariant functions from  $G$  to  $\mathbb{C}$  which have compact support. Let  $V$  be a smooth representation of  $G$ . Then the space of  $K$ -invariant vectors  $V^K$  is naturally equipped with the structure of an  $\mathcal{H}_K$ -module, and we can prove that this assignment induces a bijection between the irreducible smooth representations of  $G$  admitting a non zero  $K$ -invariant vector and the irreducible representations of  $\mathcal{H}_K$ .

Kac-Moody groups are infinite dimensional generalizations of reductive groups. For example, if  $\mathfrak{G}$  is a split reductive group and  $\mathcal{F}$  is a field, then the associated affine Kac-Moody group is a central extension of  $\mathfrak{G}(\mathcal{F}[u, u^{-1}]) \rtimes \mathcal{F}^*$ , where  $u$  is an indeterminate. Let now  $G = \mathfrak{G}(\mathcal{K})$  be a split Kac-Moody group over  $\mathcal{K}$ . Recently, Hecke algebras were associated to  $G$ . In [BK11] and [GR14], Braverman and Kazhdan (in the affine case) and Gaussent and Rousseau (in the general case) associated a spherical Hecke algebra  $\mathcal{H}_s$  to  $G$ , i.e an algebra associated to the spherical subgroup  $\mathfrak{G}(\mathcal{O})$  of  $G$ , where  $\mathcal{O}$  is the ring of integers of  $\mathcal{K}$ . In [BKP16] and [BGR16], Braverman, Kazhdan and Patnaik and Bardy-Panse, Gaussent and Rousseau defined the Iwahori-Hecke algebra  $\mathcal{H}_I$  of  $G$  (associated to the Iwahori subgroup  $K_I$  of  $G$ ). In [AH19], together with Abdellatif, we associated Hecke algebras to certain parahoric subgroups of  $G$ , which generalizes the construction of

the Iwahori-Hecke algebra of  $G$ . In [Héb22b], [Héb21a] and [Héb21b], we associated and studied principal series representations of  $\mathcal{H}_I$ .

For the moment, there is no link between the representations of  $G$  and the representations of its Hecke algebras. It seems natural to try to attach an irreducible representation of  $G$  to each irreducible representation of  $\mathcal{H}_I$ . A more modest task would be to associate to each principal series representation  $I_\tau$  of  $\mathcal{H}_I$  a principal series representation  $I(\tau)$  of  $G$ , which is irreducible when  $I_\tau$  is.

Let  $T$  be a maximal split torus of  $G$  and  $Y$  be the cocharacter lattice of  $(G, T)$ . Let  $B$  be a Borel subgroup of  $G$  containing  $T$ . Let  $T_{\mathbb{C}} = \text{Hom}_{\text{Gr}}(Y, \mathbb{C}^*)$  and  $\tau \in T_{\mathbb{C}}$ . Then  $\tau$  can be extended to a character  $\tau : B \rightarrow \mathbb{C}^*$ . Assume that  $G$  is reductive. Then the principal series representation  $I(\tau)$  of  $G$  is the induction of  $\tau\delta^{1/2}$  from  $B$  to  $G$ , where  $\delta : B \rightarrow \mathbb{R}_+^*$  is the modulus character of  $B$ . More explicitly, this is the space of locally constant functions  $f : G \rightarrow \mathbb{C}$  such that  $f(bg) = \tau\delta^{1/2}(b)f(g)$  for every  $g \in G$  and  $b \in B$ . Then  $G$  acts on  $I(\tau)$  by right translation. Then  $I_\tau := I(\tau)^{K_I}$  is a representation of  $\mathcal{H}_I$ . Assume now that  $G$  is a Kac-Moody group. Then we do not know what “locally constant” mean, but we can define the representation  $\widehat{I(\tau)}$  of  $G$  as the set of functions  $f : G \rightarrow \mathbb{C}$  such that  $f(bg) = \tau\delta^{1/2}(b)f(g)$  for every  $g \in G$  and  $b \in B$ . Let  $\mathcal{T}_G$  be a topology of topological group on  $G$  such that  $K_I$  is open. Then

$$I(\tau)_{\mathcal{T}_G} := \{f \in \widehat{I(\tau)} \mid f \text{ is locally constant for } \mathcal{T}_G\} \quad (1.1)$$

is a subrepresentation of  $G$  containing  $\widehat{I(\tau)}^{K_I}$ . Thus if we look for an irreducible representation containing  $\widehat{I(\tau)}^{K_I}$  it is natural to search it inside  $I(\tau)_{\mathcal{T}_G}$ . Moreover, the more  $\mathcal{T}_G$  is coarse, the smaller  $I(\tau)_{\mathcal{T}_G}$  is. We thus look for the coarsest topology of topological group on  $G$  for which  $K_I$  is open.

## 1.2 Topology on $G$ , measure and main results

We now assume that  $\mathcal{K}$  is any field equipped with a valuation  $\omega : \mathcal{K} \rightarrow \mathbb{R} \cup \{+\infty\}$  such that  $\omega(\mathcal{K}^*) \supset \mathbb{Z}$ . We no longer assume  $\mathcal{K}$  to be local, and  $\omega(\mathcal{K}^*)$  can be dense in  $\mathbb{R}$ . Let  $\mathcal{O}$  be its ring of integers. Let  $\mathfrak{G}$  be a split Kac-Moody group (à la Tits, as defined in [Tit87]) and  $G = \mathfrak{G}(\mathcal{K})$ . In [GR08] and [Rou16], Gaussent and Rousseau associated to  $G$  a kind of Bruhat-Tits building, called a measure, on which  $G$  acts (when  $G$  is reductive,  $\mathcal{I}$  is the usual Bruhat-Tits building). They defined the spherical subgroup  $K_s$  as the fixator of some vertex 0 in the measure (we prove in Proposition 3.1 that  $K_s = \mathfrak{G}^{\min}(\mathcal{O})$ , where  $\mathfrak{G}^{\min}$  is the minimal Kac-Moody group defined by Marquis in [Mar18]). They also define the Iwahori subgroup  $K_I$  as the fixator of some alcove  $C_0^+$  of  $\mathcal{I}$ . Then we define the topology  $\mathcal{T}_{\text{Fix}}$  on  $G$  as follows. A subset  $V$  of  $G$  is open if for every  $g \in V$ , there exists a finite subset  $F$  of  $\mathcal{I}$  such that  $G_F.g \subset V$ , where  $G_F$  is the fixator of  $F$  in  $G$ . Then we prove that  $\mathcal{T}_{\text{Fix}}$  is the coarsest topology of topological group on  $G$  for which  $K_I$  is open (see Proposition 4.14). However, it is not Hausdorff in general. Indeed, let  $\mathcal{Z} \subset T$  be the center of  $G$  and  $\mathcal{Z}_{\mathcal{O}} = \mathcal{Z} \cap \mathfrak{T}(\mathcal{O})$ . Then  $\mathcal{Z}_{\mathcal{O}}$  is the fixator of  $\mathcal{I}$  in  $G$  and when  $\mathcal{Z}_{\mathcal{O}}$  is nontrivial (which already happens for  $\text{SL}_2(\mathcal{K})$ ),  $\mathcal{T}_{\text{Fix}}$  is not Hausdorff.

To address this issue, we define an other topology,  $\mathcal{T}$ , finer than  $\mathcal{T}_{\text{Fix}}$  and Hausdorff. Let  $\mathbb{A} = Y \otimes \mathbb{R}$  be the standard apartment of  $\mathcal{I}$  and  $\Phi \subset \mathbb{A}^*$  be the set of roots of  $(G, T)$ . Then  $\mathcal{I} = \bigcup_{g \in G} g \cdot \mathbb{A}$ . Let us begin with the case where  $G = \text{SL}_2(\mathcal{K})$ . Let  $\varpi \in \mathcal{O}$  be such that  $\omega(\varpi) = 1$ . For  $n \in \mathbb{N}^*$ , let  $\pi_n : \text{SL}_2(\mathcal{O}) \rightarrow \text{SL}_2(\mathcal{O}/\varpi^n \mathcal{O})$  be the natural projection. Then a basis of the neighbourhood of the identity is given by the  $(\ker \pi_n)_{n \in \mathbb{N}^*}$ . Let  $U^+ = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$  and  $U^- = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$ . Then one can prove that  $\ker \pi_n = (U^+ \cap \ker \pi_n) \cdot (U^- \cap \ker \pi_n) \cdot (T \cap \ker \pi_n)$ . Let  $\alpha, -\alpha$  be the two roots of  $(G, T)$ . Let  $x_\alpha : a \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$  and  $x_{-\alpha} : a \mapsto \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$ . Then  $x_\alpha(\varpi^n \mathcal{O})$  fixes  $\{a \in \mathbb{A} \mid \alpha(a) \geq -n\}$  and  $x_{-\alpha}(\varpi^n \mathcal{O})$  fixes  $\{a \in \mathbb{A} \mid \alpha(a) \leq n\}$ . Therefore if  $\lambda \in \mathbb{A}$  is such that  $\alpha(\lambda) = 1$  and  $[-n\lambda, n\lambda] = \alpha^{-1}([-n, n])$ , we have

$$\ker \pi_n = (U^+ \cap \text{Fix}_G([-n\lambda, n\lambda])) \cdot (U^- \cap \text{Fix}_G([-n\lambda, n\lambda])) \cdot (T \cap \ker \pi_n).$$

We now return to the general case for  $G$ . We prove that the topology associated to  $(\ker \pi_n)_{n \in \mathbb{N}^*}$  is not a topology of topological group if  $G$  is not reductive (see Lemma 3.3). Let  $(\alpha_i)_{i \in I}$  be the set of simple roots of  $(G, T)$  and  $C_f^v = \{x \in \mathbb{A} \mid \alpha_i(x) > 0, \forall i \in I\}$ . Let  $W^v$  be the Weyl group of  $(G, T)$  and  $\lambda \in Y \cap \bigsqcup_{w \in W^v} w \cdot C_f^v$ . We define the following subset  $\mathcal{V}_{n\lambda}$  of  $G$ , for  $n \in \mathbb{N}^*$ :

$$\mathcal{V}_{n\lambda} = (U^+ \cap \text{Fix}_G([-n\lambda, n\lambda])) \cdot (U^- \cap \text{Fix}_G([-n\lambda, n\lambda])) \cdot (T \cap \ker \pi_{2N(\lambda)}),$$

where  $N(\lambda) = \min\{|\alpha(\lambda)| \mid \alpha \in \Phi_+\}$ . We prove the following theorem:

**Theorem 1.1.** (see Theorem 4.8, Lemma 4.2, Proposition 4.21 and Proposition 5.13):

1. For  $n \in \mathbb{N}^*$  and  $\lambda \in Y \cap \bigsqcup_{w \in W^v} w \cdot C_f^v$ ,  $\mathcal{V}_{n\lambda}$  is a subgroup of  $\mathfrak{G}^{\min}(\mathcal{O})$ .
2. The topology  $\mathcal{T}$  associated with  $(\mathcal{V}_{n\lambda})_{n \in \mathbb{N}^*}$  is Hausdorff, independent of the choice of  $\lambda$  and equips  $G$  with the structure of a topological group.
3. The topology  $\mathcal{T}$  is finer than  $\mathcal{T}_{\text{Fix}}$  and if  $\mathcal{K}$  is Henselian, we have  $\mathcal{T} = \mathcal{T}_{\text{Fix}}$  if and only if  $\mathcal{Z}_{\mathcal{O}} = \{1\}$ .
4. Every compact subset of  $G$  has empty interior (for  $\mathcal{T}$ ).

Note that  $\mathcal{T}$  and  $\mathcal{T}_{\text{Fix}}$  induce the same topologies on  $U^+$  and  $U^-$ . The main difference comes from what happens in  $T$ . As the elements of  $I(\tau)_{\mathcal{T}}$  and  $I(\tau)_{\mathcal{T}_{\text{Fix}}}$  are left  $\mathfrak{Z}(\mathcal{O})$ -invariant, these two spaces are actually equal (see Remark 4.22).

In [HKM13], based on works of Kac and Peterson on the topology of  $\mathfrak{G}(\mathbb{C})$ , Hartnick, Köhl and Mars defined a Kac-Peterson topology on  $\mathfrak{G}(\mathcal{F})$ , for any local field  $\mathcal{F}$  (Archimedean or not). Assume that  $\mathcal{K}$  is local and let  $\mathcal{T}_{KP}$  be the Kac-Peterson topology on  $G$ . We prove that when  $G$  is not reductive, then  $\mathcal{T}$  is strictly coarser than  $\mathcal{T}_{KP}$  (see Proposition 5.4) and thus  $\mathcal{T}$  seems more adapted for our purpose.

Assume that  $\mathfrak{G}$  is affine  $\text{SL}_2$  (with a nonfree set of simple coroots). Then  $G = \text{SL}_2(\mathcal{K}[u, u^{-1}]) \rtimes \mathcal{K}^*$ . Up to the assumption that  $\ker \pi_n \subset \begin{pmatrix} 1 + \varpi^n \mathcal{O}[u, u^{-1}] & \varpi^n \mathcal{O}[u, u^{-1}] \\ \varpi^n \mathcal{O}[u, u^{-1}] & 1 + \varpi^n \mathcal{O}[u, u^{-1}] \end{pmatrix} \rtimes$

$(1 + \varpi^n \mathcal{O})$ , for  $n \in \mathbb{N}^*$ , we prove that the topology  $\mathcal{T}$  on  $G$  is associated to the filtration  $(H_n)_{n \in \mathbb{N}^*}$ , where  $H_n = \ker(\pi_n) \cap \left( \begin{pmatrix} \mathcal{O}[(\varpi u)^n, (\varpi u^{-1})^n] & \mathcal{O}[(\varpi u)^n, (\varpi u^{-1})^n] \\ \mathcal{O}[(\varpi u)^n, (\varpi u^{-1})^n] & \mathcal{O}[(\varpi u)^n, (\varpi u^{-1})^n] \end{pmatrix}, \mathcal{K}^* \right)$ .

The paper is organized as follows. In section 2, we define Kac-Moody groups (as defined by Tits, Mathieu and Marquis) and the mesures.

In section 3, we define and study the subgroups  $\ker \pi_n$  of  $\mathfrak{G}^{\min}(\mathcal{O})$ .

In section 4, we define the topologies  $\mathcal{T}$  and  $\mathcal{T}_{\text{Fix}}$ , and compare them.

In section 5, we study the properties of  $\mathcal{T}$  and  $\mathcal{T}_{\text{Fix}}$ : we prove that  $\mathcal{T}_{KP}$  is strictly finer than  $\mathcal{T}$ , we describe the topology in the case of affine  $\text{SL}_2$ , and we prove that usual subgroups of  $G$  (i.e  $T$ ,  $N$ ,  $B$ , etc.) are closed for  $\mathcal{T}$ .

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## 2 Kac-Moody groups and mesures

In this section, we define Kac-Moody groups and mesures. Let  $\mathcal{K}$  be a field. There are several possible definitions of Kac-Moody groups and we are interested in the minimal one  $\mathfrak{G}(\mathcal{K})$ , as defined by Tits in [Tit87]. However, because of the lack of commutation relations in  $\mathfrak{G}(\mathcal{K})$ , it is convenient to embed it in its Mathieu's positive and negative completions  $\mathfrak{G}^{pma}(\mathcal{K})$  and  $\mathfrak{G}^{nma}(\mathcal{K})$ . Then one define certain subgroups of  $\mathfrak{G}(\mathcal{K})$  as the intersection of a subgroup of  $\mathfrak{G}^{pma}(\mathcal{K})$  and  $\mathfrak{G}(\mathcal{K})$ . For example if  $\mathfrak{G}$  is affine  $\text{SL}_2$  (with a nonfree set of simple roots and coroots), then  $\mathfrak{G}(\mathcal{K}) = \text{SL}_2(\mathcal{K}[u, u^{-1}])$ ,  $\mathfrak{G}^{pma}(\mathcal{K}) = \text{SL}_2(\mathcal{K}((u)))$  and  $\mathfrak{G}^{nma}(\mathcal{K}) = \text{SL}_2(\mathcal{K}((u^{-1})))$ .

As we want to define congruence subgroups in our framework, we also need to work with Kac-Moody groups over rings: if  $\mathcal{K}$  is equipped with a valuation  $\omega$  and  $\varpi$  is such that  $\omega(\varpi) = 1$ , then we want to define  $\ker \pi_n \subset \mathfrak{G}(\mathcal{O})$ , where  $\pi_n : \mathfrak{G}(\mathcal{O}) \rightarrow \mathfrak{G}(\mathcal{O}/\varpi^n \mathcal{O})$  is the natural projection. The functor defined by Tits in [Tit87] goes from the category of rings to the category of groups. However the fact that it satisfies the axioms defined by Tits is proved only for fields (see [Tit87, 3.9 Theorem 1]) and we do not know if it is “well-behaved” on rings, so we will consider it only as a functor from the category of fields to the category of groups. In [Mar18, 8.8], Marquis introduces a functor  $\mathfrak{G}^{\min}$  which goes from the category of rings to the category of groups and he proves that it has nice properties (see [Mar18, Proposition 8.128]), especially on Bézout domains. We will use its functor  $\mathfrak{G}^{\min}$ . We have  $\mathfrak{G}^{\min}(\mathcal{F}) \simeq \mathfrak{G}(\mathcal{F})$  for any field  $\mathcal{F}$ . This functor is defined as a subfunctor of  $\mathfrak{G}^{pma}$ , so we first define Tits's functor, then Mathieu's functors and then Marquis's functor.

## 2.1 Standard apartment of a masure

### 2.1.1 Root generating system

A **Kac-Moody matrix** (or generalized Cartan matrix) is a square matrix  $A = (a_{i,j})_{i,j \in I}$  indexed by a finite set  $I$ , with integral coefficients, and such that :

- (i)  $\forall i \in I, a_{i,i} = 2$ ;
- (ii)  $\forall (i, j) \in I^2, (i \neq j) \Rightarrow (a_{i,j} \leq 0)$ ;
- (iii)  $\forall (i, j) \in I^2, (a_{i,j} = 0) \Leftrightarrow (a_{j,i} = 0)$ .

A **root generating system** is a 5-tuple  $\mathcal{S} = (A, X, Y, (\alpha_i)_{i \in I}, (\alpha_i^\vee)_{i \in I})$  made of a Kac-Moody matrix  $A$  indexed by the finite set  $I$ , of two dual free  $\mathbb{Z}$ -modules  $X$  and  $Y$  of finite rank, and of a family  $(\alpha_i)_{i \in I}$  (respectively  $(\alpha_i^\vee)_{i \in I}$ ) of elements in  $X$  (resp.  $Y$ ) called **simple roots** (resp. **simple coroots**) that satisfy  $a_{i,j} = \alpha_j(\alpha_i^\vee)$  for all  $i, j$  in  $I$ . Elements of  $X$  (respectively of  $Y$ ) are called **characters** (resp. **cocharacters**).

Fix such a root generating system  $\mathcal{S} = (A, X, Y, (\alpha_i)_{i \in I}, (\alpha_i^\vee)_{i \in I})$  and set  $\mathbb{A} := Y \otimes \mathbb{R}$ . Each element of  $X$  induces a linear form on  $\mathbb{A}$ , hence  $X$  can be seen as a subset of the dual  $\mathbb{A}^*$ . In particular, the  $\alpha_i$ 's (with  $i \in I$ ) will be seen as linear forms on  $\mathbb{A}$ . This allows us to define, for any  $i \in I$ , a **simple reflection**  $r_i$  of  $\mathbb{A}$  by setting  $r_i.v := v - \alpha_i(v)\alpha_i^\vee$  for any  $v \in \mathbb{A}$ . One defines the **Weyl group of  $\mathcal{S}$**  as the subgroup  $W^v$  of  $\text{GL}(\mathbb{A})$  generated by  $\{r_i \mid i \in I\}$ . The pair  $(W^v, \{r_i \mid i \in I\})$  is a Coxeter system, hence we can consider the length  $\ell(w)$  with respect to  $\{r_i \mid i \in I\}$  of any element  $w$  of  $W^v$ .

The following formula defines an action of the Weyl group  $W^v$  on  $\mathbb{A}^*$ :

$$\forall x \in \mathbb{A}, w \in W^v, \alpha \in \mathbb{A}^*, (w.\alpha)(x) := \alpha(w^{-1}.x).$$

Let  $\Phi := \{w.\alpha_i \mid (w, i) \in W^v \times I\}$  (resp.  $\Phi^\vee = \{w.\alpha_i^\vee \mid (w, i) \in W^v \times I\}$ ) be the set of **real roots** (resp. **real coroots**): then  $\Phi$  (resp.  $\Phi^\vee$ ) is a subset of the **root lattice**  $Q := \bigoplus_{i \in I} \mathbb{Z}\alpha_i$  (resp. **coroot lattice**  $Q^\vee = \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee$ ). By [Kum02, 1.2.2 (2)], one has

$$\mathbb{R}\alpha^\vee \cap \Phi^\vee = \{\pm\alpha^\vee\} \text{ and } \mathbb{R}\alpha \cap \Phi = \{\pm\alpha\} \text{ for all } \alpha^\vee \in \Phi^\vee \text{ and } \alpha \in \Phi.$$

We define the **height**  $\text{ht} : Q \rightarrow \mathbb{Z}$  by  $\text{ht}(\sum_{i \in I} n_i \alpha_i) = \sum_{i \in I} n_i$ , for  $(n_i) \in \mathbb{Z}^I$ .

### 2.1.2 Vectorial apartment

As in the reductive case, define the **fundamental chamber** as  $C_f^v := \{v \in \mathbb{A} \mid \forall i \in I, \alpha_i(v) > 0\}$ .

Let  $\mathcal{T} := \bigcup_{w \in W^v} w.\overline{C_f^v}$  be the **Tits cone**. This is a convex cone (see [Kum02, 1.4]).

For  $J \subset I$ , set  $F^v(J) = \{x \in \mathbb{A} \mid \alpha_j(x) = 0, \forall j \in J \text{ and } \alpha_j(x) > 0, \forall j \in I \setminus J\}$ . A **positive vectorial face** (resp. **negative**) is a set of the form  $w.F^v(J)$  ( $-w.F^v(J)$ ) for some  $w \in W^v$  and  $J \subset I$ . Then by [Rém02, 5.1 Théorème (ii)], the family of positive vectorial faces of  $\mathbb{A}$  is a partition of  $\mathcal{T}$  and the stabilizer of  $F^v(J)$  is  $W_J = \langle J \rangle$ .

One sets  $Y^{++} = Y \cap \overline{C_f^v}$  and  $Y^+ = Y \cap \mathcal{T}$ . An element of  $Y^+$  is called **regular** if it does not belong to any wall, i.e if it belongs to  $\bigsqcup_{w \in W^v} w.C_f^v$ .

**Remark 2.1.** By [Kac94, §4.9] and [Kac94, § 5.8] the following conditions are equivalent:

1. the Kac-Moody matrix  $A$  is of finite type (i.e. is a Cartan matrix),
2.  $\mathbb{A} = \mathcal{T}$
3.  $W^v$  is finite.

## 2.2 Split Kac-Moody groups over fields

### 2.2.1 Minimal Kac-Moody groups over fields

Let  $\mathfrak{G} = \mathfrak{G}_{\mathcal{S}}$  be the group functor associated in [Tit87] with the root generating system  $\mathcal{S}$ , see also [Rém02, 8]. Let  $\mathcal{K}$  be a field. Let  $G = \mathfrak{G}(\mathcal{K})$  be the **split Kac-Moody group over  $\mathcal{K}$  associated with  $\mathcal{S}$** . The group  $G$  is generated by the following subgroups:

- the fundamental torus  $T = \mathfrak{T}(\mathcal{K})$ , where  $\mathfrak{T} = \text{Spec}(\mathbb{Z}[X])$ ,
- the root subgroups  $U_{\alpha} = \mathfrak{U}_{\alpha}(\mathcal{K})$ , for  $\alpha \in \Phi$ , each isomorphic to  $(\mathcal{K}, +)$  by an isomorphism  $x_{\alpha}$ .

The groups  $X$  and  $Y$  correspond to the character lattice  $\text{Hom}(\mathfrak{T}, \mathbb{G}_m)$  and cocharacter lattice  $\text{Hom}(\mathbb{G}_m, \mathfrak{T})$  of  $\mathfrak{T}$  respectively. One writes  $\mathfrak{U}^{\pm}$  the subgroup of  $\mathfrak{G}$  generated by the  $\mathfrak{U}_{\alpha}$ , for  $\alpha \in \Phi^{\pm}$  and  $U^{\pm} = \mathfrak{U}^{\pm}(\mathcal{K})$ .

By a simple computation in  $\text{SL}_2$ , we have for  $\alpha \in \Phi$  and  $a, b \in \mathcal{K}$  such that  $ab \neq -1$ :

$$\begin{aligned} x_{-\alpha}(b)x_{\alpha}(a) &= x_{\alpha}(a(1+ab)^{-1})\alpha^{\vee}(1+ab)x_{-\alpha}(b(1+ab)^{-1}) \\ &= x_{\alpha}(a(1+ab)^{-1})x_{-\alpha}(b(1+ab))\alpha^{\vee}(1+ab), \end{aligned} \quad (2.1)$$

where  $\alpha^{\vee} = w.\alpha_i^{\vee}$  if  $\alpha = w.\alpha_i$ , for  $i \in I$  and  $w \in W^v$ .

Let  $\mathfrak{N}$  be the group functor on rings such that if  $\mathcal{R}'$  is a ring,  $\mathfrak{N}(\mathcal{R}')$  is the subgroup of  $\mathfrak{G}(\mathcal{R}')$  generated by  $\mathfrak{T}(\mathcal{R}')$  and the  $\tilde{r}_i$ , for  $i \in I$ , where

$$\tilde{r}_i = x_{\alpha_i}(1)x_{-\alpha_i}(-1)x_{\alpha_i}(1). \quad (2.2)$$

Then if  $\mathcal{R}'$  is a field with at least 4 elements,  $\mathfrak{N}(\mathcal{R}')$  is the normalizer of  $\mathfrak{T}(\mathcal{R}')$  in  $\mathfrak{G}(\mathcal{R}')$ .

Let  $N = \mathfrak{N}(\mathcal{K})$  and  $\text{Aut}(\mathbb{A})$  be the group of affine automorphisms of  $\mathbb{A}$ . Then by [Rou06, 1.4 Lemme], there exists a group morphism  $\nu^v : N \rightarrow \text{GL}(\mathbb{A})$  such that:

1. for  $i \in I$ ,  $\nu^v(\tilde{r}_i)$  is the simple reflection  $r_i \in W^v$ ,
2.  $\ker \nu^v = T$ .

The aim of the next two subsections is to define Mathieu's Kac-Moody group. This group is defined by assembling three ingredients: the group  $\mathfrak{U}^{pma}$ , which corresponds to a maximal positive unipotent subgroup of  $\mathfrak{G}^{pma}$ , the torus  $\mathfrak{T}$  and copies of  $\text{SL}_2$ , one for each simple root  $\alpha_i$ ,  $i \in I$ .



## 2.2.2 The affine group scheme $\mathfrak{U}^{pma}$

In this subsection, we define  $\mathfrak{U}^{pma}$ . Let  $\mathfrak{g}$  be the Kac-Moody Lie algebra over  $\mathbb{C}$  associated with  $\mathcal{S}$  (see [Kum02, 1.2]) and  $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$  be its enveloping algebra. The group  $\mathfrak{U}^{pma}(\mathbb{C})$ , will be defined as a subgroup of a completion of  $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$ . As we want to define  $\mathfrak{U}^{pma}(\mathcal{R})$ , for any ring  $\mathcal{R}$ , we will also consider  $\mathbb{Z}$ -forms of  $\mathfrak{g}$  and  $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$ .

The Lie algebra  $\mathfrak{g}$  decomposes as  $\mathfrak{g} = \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha}$ , where  $\Delta \subset Q$  is the **set of roots** and  $\mathfrak{g}_{\alpha}$  is the proper space associated with  $\alpha$ , for  $\alpha \in \Delta$  (see [Kum02, 1.2]). We have  $\Delta = \Delta_+ \sqcup \Delta_-$ , where  $\Delta_+ = \Delta \cap Q_+$  and  $\Delta_- = -\Delta_+$ . We have  $\Phi \subset \Delta$ . The elements of  $\Phi = \Delta_{re}$  are called **real roots** and the elements of  $\Delta_{im} = \Delta \setminus \Phi$  are called **imaginary roots**.

Following [Tit87, 4] one defines  $\mathcal{U}$  as the  $\mathbb{Z}$ -subalgebra of  $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$  generated by  $e_i^{(n)} := \frac{e_i^n}{n!}$ ,  $f_i^{(n)} := \frac{f_i^n}{n!}$ ,  $\binom{h}{n}$ , for  $i \in I$  and  $h \in Y$  (where the  $e_i, f_i$  are the generators of  $\mathfrak{g}$ , see [Kum02, 1.1]). This is a  $\mathbb{Z}$ -form of  $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$ . The algebra  $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$  decomposes as  $\mathcal{U}_{\mathbb{C}}(\mathfrak{g}) = \bigoplus_{\alpha \in Q} \mathcal{U}_{\mathbb{C}}(\mathfrak{g})_{\alpha}$  where we use the standard  $Q$ -graduation on  $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$  induced by the  $Q$ -graduation of  $\mathfrak{g}$  (for  $i \in I$ ,  $\deg(e_i) = \alpha_i$ ,  $\deg(f_i) = -\alpha_i$ ,  $\deg(h) = 0$ , for  $h \in Y$ ,  $\deg(xy) = \deg(x) + \deg(y)$  for all  $x, y \in \mathcal{U}_{\mathbb{C}}(\mathfrak{g})$  which can be written as a product of nonzero elements of  $\mathfrak{g}$ ). For  $\alpha \in Q$ , one sets  $\mathcal{U}_{\alpha} = \mathcal{U}_{\mathbb{C}}(\mathfrak{g})_{\alpha} \cap \mathcal{U}$  and  $\mathcal{U}_{\alpha, \mathcal{R}} = \mathcal{U}_{\alpha} \otimes \mathcal{R}$ .

For a ring  $\mathcal{R}$ , we set  $\mathcal{U}_{\mathcal{R}} = \mathcal{U} \otimes_{\mathbb{Z}} \mathcal{R}$ . One sets  $\widehat{\mathcal{U}}^+ = \prod_{\alpha \in Q_+} \mathcal{U}_{\alpha}$  and  $\widehat{\mathcal{U}}_{\mathcal{R}}^+ = \prod_{\alpha \in Q_+} \mathcal{U}_{\alpha, \mathcal{R}}$ . This is the completion of  $\mathcal{U}^+$  with respect to the  $Q_+$ -graduation.

If  $(u_{\alpha}) \in \prod_{\alpha \in Q_+} \mathcal{U}_{\alpha, \mathcal{R}}$ , we write  $\sum_{\alpha \in Q_+} u_{\alpha}$  the corresponding element of  $\widehat{\mathcal{U}}_{\mathcal{R}}^+$ . A sequence  $(\sum_{\alpha \in Q_+} u_{\alpha}^{(n)})_{n \in \mathbb{N}}$  converges in  $\widehat{\mathcal{U}}_{\mathcal{R}}^+$  if and only if for every  $\alpha \in \Delta_+$ , the sequence  $(u_{\alpha}^{(n)})_{n \in \mathbb{N}}$  is stationary.

Let  $(E, \leq)$  be a totally ordered set. Let  $(u^{(e)}) \in (\widehat{\mathcal{U}}_{\mathcal{R}}^+)^E$ . For  $e \in E$ , write  $u = \sum_{\alpha \in Q_+} u_{\alpha}^{(e)}$ , with  $u_{\alpha}^{(e)} \in \mathcal{U}_{\alpha, \mathcal{R}}$ , for  $\alpha \in Q_+$ . We assume that for every  $\alpha \in Q_+$ ,  $\{e \in E \mid u_{\alpha}^{(e)} \neq 0\}$  is finite. Then one sets  $\prod_{e \in E} u^{(e)} = \sum_{\alpha \in Q_+} u_{\alpha}$ , where

$$u_{\alpha} = \sum_{\substack{(\beta_1, \dots, \beta_k) \in Q_+^{(N)}, \\ \beta_1 + \dots + \beta_k = \alpha}} \sum_{\substack{(\epsilon_1, \dots, \epsilon_k) \in E, \\ \epsilon_1 < \dots < \epsilon_k}} u_{\beta_1}^{(\epsilon_1)} \dots u_{\beta_k}^{(\epsilon_k)} \in \mathcal{U}_{\alpha, \mathcal{R}},$$

for  $\alpha \in \Delta_+$ . This is well-defined since in the sum defining  $u_{\alpha}$ , only finitely many nonzero terms appear.

Let  $\mathcal{A} = \bigoplus_{\alpha \in Q_+} \mathcal{U}_{\alpha}^*$ , where  $\mathcal{U}_{\alpha}^*$  denotes the dual of  $\mathcal{U}_{\alpha}$  (as a  $\mathbb{Z}$ -module). We have a natural  $\mathcal{R}$ -modules isomorphism between  $\widehat{\mathcal{U}}_{\mathcal{R}}^+$  and  $\text{Hom}_{\mathbb{Z}\text{-lin}}(\mathcal{A}, \mathcal{R})$ , for any ring  $\mathcal{R}$  (see [Mar18, (8.26)]) and we now identify these two spaces. The algebra  $\mathcal{A}$  is equipped with a Hopf algebra structure (see [Mar18, Definition 8.42]). This additional structure equips

$$\mathfrak{U}^{pma}(\mathcal{R}) := \text{Hom}_{\mathbb{Z}\text{-Alg}}(\mathcal{A}, \mathcal{R})$$

with the structure of a group (see [Mar18, Appendix A.2.2]). Otherwise said,  $\mathcal{A}$  is the representing algebra of the (infinite dimensional in general) affine group scheme  $\mathfrak{U}^{pma} : \mathbb{Z}\text{-Alg} \rightarrow \text{Grp}$ .

Let  $\alpha \in \Delta \cup \{0\}$  and  $x \in \mathfrak{g}_{\alpha, \mathbb{Z}}$ . An **exponential sequence** for  $x$  is a sequence  $(x^{[n]})_{n \in \mathbb{N}}$  of elements of  $\mathcal{U}$  such that  $x^{[0]} = 1$ ,  $x^{[1]} = x$  and  $x^{[n]} \in \mathcal{U}_{n\alpha}$  for  $n \in \mathbb{Z}_{\geq 1}$  and satisfying the conditions of [Mar18, Definition 8.45]. By [Rou16, Proposition 2.7] or [Mar18, Proposition 8.50], such a sequence exists. Note that it is not unique in general. However, if  $\alpha \in \Phi_+$ , the unique exponential sequence for  $x$  is  $(x^{[n]})_{n \in \mathbb{N}} = (\frac{1}{n!}x^n)$  by [Rou16, 2.9 2)] (this sequence is often denoted  $(x^{(n)})_{n \in \mathbb{N}}$  in the literature).

For  $r \in \mathcal{R}$ , one then sets

$$[\exp](rx) =: \sum_{n \in \mathbb{N}} x^{[n]} \otimes r^n \in \widehat{\mathcal{U}}_{\mathcal{R}}^+.$$

This is the **twisted exponential** of  $rx$  associated with the sequence  $(x^{[n]})_{n \in \mathbb{N}}$ .

We fix for every  $\alpha \in \Delta_+$  a  $\mathbb{Z}$ -basis  $\mathcal{B}_\alpha$  of  $\mathfrak{g}_{\alpha, \mathbb{Z}} := \mathfrak{g}_\alpha \cap \mathcal{U}$ . Set  $\mathcal{B} = \bigcup_{\alpha \in \Delta_+} \mathcal{B}_\alpha$ . We fix an order on each  $\mathcal{B}_\alpha$  and on  $\Delta_+$ . Let  $\alpha \in \Delta_+$ . One defines  $X_\alpha : \mathfrak{g}_{\alpha, \mathbb{Z}} \otimes \mathcal{R} \rightarrow \mathfrak{U}^{pma}(\mathcal{R})$  by  $X_\alpha(\sum_{x \in \mathcal{B}_\alpha} \lambda_x \cdot x) = \prod_{x \in \mathcal{B}_\alpha} [\exp] \lambda_x \cdot x$ , for  $(\lambda_x) \in \mathcal{R}^{\mathcal{B}_\alpha}$ . When  $\alpha \in \Phi_+$ , we have  $\mathfrak{g}_{\alpha, \mathbb{Z}} = \mathbb{Z}e_\alpha$ , where  $e_\alpha$  is defined in [Mar18, Remark 7.6]. One sets  $x_\alpha(r) = [\exp](re_\alpha)$ , for  $r \in \mathcal{R}$ . One has  $X_\alpha(\mathfrak{g}_{\alpha, \mathbb{Z}} \otimes \mathcal{R}) = x_\alpha(\mathcal{R}) := \mathfrak{U}_\alpha(\mathcal{R})$ . By [Mar18, Theorem 8.5.1], every  $g \in \mathfrak{U}^{pma}(\mathcal{R})$  can be written in a unique way as a product

$$g = \prod_{\alpha \in \Delta_+} X_\alpha(c_\alpha), \quad (2.3)$$

where  $c_\alpha \in \mathfrak{g}_{\alpha, \mathbb{Z}} \otimes \mathcal{R}$ , for  $\alpha \in \Delta_+$ , where the product is taken in the given order on  $\Delta_+$ .

Let  $\Psi \subset \Delta_+$ . We say that  $\Psi$  is **closed** if for all  $\alpha, \beta \in \Psi$ , for all  $p, q \in \mathbb{N}^*$ ,  $p\alpha + q\beta \in \Delta_+$  implies  $p\alpha + q\beta \in \Psi$ . Let  $\Psi \subset \Delta_+$  be a closed subset. One sets

$$\mathfrak{U}_\Psi^{pma}(\mathcal{R}) = \prod_{\alpha \in \Psi} X_\alpha(\mathfrak{g}_{\alpha, \mathbb{Z}} \otimes \mathcal{R}) \subset \mathfrak{U}^{pma}(\mathcal{R}).$$

This is a subgroup of  $\mathfrak{U}^{pma}$ , which does not depend on the chosen order on  $\Delta_+$  (for the product). This is not the definition given in [Rou16] or [Mar18, page 210], but it is equivalent by [Mar18, Theorem 8.51].

### 2.2.3 Mathieu's group $\mathfrak{G}^{pma}$

The **Borel subgroup** (it will be a subgroup of  $\mathfrak{G}^{pma}$ ) is  $\mathfrak{B} = \mathfrak{B}_S = \mathfrak{T}_S \ltimes \mathfrak{U}^{pma}$ , where  $\mathfrak{T}$  acts on  $\mathfrak{U}^{pma}$  as follows. Let  $\mathcal{R}$  be a ring,  $\alpha \in \Delta_+$ ,  $t \in \mathfrak{T}(\mathcal{R})$ ,  $r \in \mathcal{R}$  and  $x \in \mathfrak{g}_{\alpha, \mathcal{R}}$ ,

$$t[\exp](rx)t^{-1} = [\exp](\alpha(t)rx). \quad (2.4)$$

In particular, if  $\alpha \in \Phi$ , we have

$$tx_\alpha(r)t^{-1} = x_\alpha(\alpha(t)r).$$

For  $i \in I$ , let  $\mathfrak{U}_{\alpha_i}^Y$  be the reductive group associated with the root generating system  $((2), X, Y, \alpha_i, \alpha_i^\vee)$ . For each  $i \in I$ , Mathieu defines an (infinite dimensional) affine group

scheme  $\mathfrak{P}_i = \mathfrak{U}_{\alpha_i}^Y \times \mathfrak{U}_{\Delta_+ \setminus \{\alpha_i\}}^{ma}$  (see [Mar18, Definition 8.65] for the definition of the action of  $\mathfrak{U}_{-\alpha_i}$  on  $\mathfrak{U}_{\Delta_+ \setminus \{\alpha_i\}}^{ma}$ ), where  $\mathfrak{U}_{-\alpha_i}$  is an affine group scheme over  $\mathbb{Z}$  isomorphic to  $\mathbb{G}_a$  (see [Mar18, 7.4.3] for more details).

We do not detail the definition of  $\mathfrak{G}^{pma}$  and we refer to [Mat89], [Mar18, 8.7] or [Rou16, 3.6]. This is an ind-group scheme containing the  $\mathfrak{P}_i$  for every  $i \in I$ . Let  $w \in W^v$  and write  $w = r_{i_1} \dots r_{i_k}$ , with  $k = \ell(w)$  and  $i_1, \dots, i_k \in I$ . Then the multiplication map  $\mathfrak{P}_{i_1} \times \dots \times \mathfrak{P}_{i_k} \rightarrow \mathfrak{G}^{pma}$  is a scheme morphism, and we have  $\mathfrak{G}^{pma}(\mathcal{R}) = \bigcup_{(i_1, \dots, i_n) \in \text{Red}(W^v)} \mathfrak{P}_{i_1}(\mathcal{R}) \times \dots \times \mathfrak{P}_{i_k}(\mathcal{R})$ , where  $\text{Red}(W^v)$  is the set of reduced words of  $W^v$  (i.e  $\text{Red}(W^v) = \{(i_1, \dots, i_k) \in I^{(\mathbb{N})} \mid \ell(r_{i_1} \dots r_{i_k}) = k\}$ ).

Let  $w \in W^v$ ,  $i \in I$  and  $\alpha = w.\alpha_i$ . One sets  $\mathfrak{U}_\alpha = \tilde{w}.\mathfrak{U}_{\alpha_i}.\tilde{w}^{-1}$ , where

$$\tilde{w} = \tilde{r}_{i_1} \dots \tilde{r}_{i_k}, \quad (2.5)$$

if  $w = r_{i_1} \dots r_{i_k}$  is a reduced decomposition of  $w$ . There is an isomorphism of group schemes  $x_\alpha : \mathbb{G}_a \rightarrow \mathfrak{U}_\alpha$  (see [Mar18, page 262]). The group  $\mathfrak{G}^{pma}$  is generated by the  $\mathfrak{P}_i$ ,  $i \in I$ . Moreover, if  $i \in I$ , then  $\mathfrak{P}_i$  is generated by  $\mathfrak{T}$ ,  $\mathfrak{U}_{\pm\alpha_i}$  and  $\tilde{r}_i = x_{\alpha_i}(1)x_{-\alpha_i}(1)x_{\alpha_i}(1)$ . Thus  $\mathfrak{G}^{pma}$  is generated by  $\mathfrak{U}^{pma}$ ,  $\mathfrak{T}$ ,  $\mathfrak{U}_{-\alpha_i}$  and the  $\tilde{r}_i$ , for  $i \in I$  and thus we have:

$$\mathfrak{G}^{pma} = \langle \mathfrak{U}^{pma}, \mathfrak{T}, \mathfrak{U}_\alpha, \alpha \in \Phi_- \rangle. \quad (2.6)$$

There is a group functor morphism  $\iota : \mathfrak{G} \rightarrow \mathfrak{G}^{pma}$  such that for any ring  $\mathcal{R}$ ,  $\iota_{\mathcal{R}}$  maps  $x_\alpha(r)$  to  $x_\alpha(r)$  and  $t$  to  $t$ , for each  $\alpha \in \Phi$ ,  $r \in \mathcal{R}$ ,  $t \in \mathfrak{T}(\mathcal{R})$ . When  $\mathcal{R}$  is a field, this morphism is injective (see [Rou16, 3.12] or [Mar18, Proposition 8.117]).

**Proposition 2.2.** *Let  $\mathcal{R}$  and  $\mathcal{R}'$  be two rings and  $\varphi : \mathcal{R} \rightarrow \mathcal{R}'$  be a ring morphism.*

*Let  $f_\varphi^{\hat{U}^+} : \hat{U}_{\mathcal{R}}^+ \rightarrow \hat{U}_{\mathcal{R}'}^+$  and  $f_\varphi : \mathfrak{G}^{pma}(\mathcal{R}) \rightarrow \mathfrak{G}^{pma}(\mathcal{R}')$  be the induced morphisms. Then  $f_\varphi^{\hat{U}^+}(\mathfrak{U}^{pma}(\mathcal{R})) \subset \mathfrak{U}^{pma}(\mathcal{R}')$  and we have:*

1. *For every  $(r_x) \in \mathcal{R}^{\mathcal{B}}$ ,  $f_\varphi^{\hat{U}^+}(\prod_{x \in \mathcal{B}}[\text{exp}](r_x x)) = \prod_{x \in \mathcal{B}}[\text{exp}](\varphi(r_x)x)$ .*
2. *For  $\alpha \in \Delta_+$  and  $(\lambda_x) \in \mathcal{R}^{\mathcal{B}_\alpha}$ , we have  $f_\varphi(X_\alpha(\sum_{x \in \mathcal{B}_\alpha} \lambda_x x)) = X_\alpha(\sum_{x \in \mathcal{B}_\alpha} \varphi(\lambda_x)x)$ .*
3. *We have  $f_\varphi(u) = f_\varphi^{\hat{U}^+}(u)$  for  $u \in \mathfrak{U}^{pma}(\mathcal{R})$ ,  $f_\varphi(x_\alpha(r)) = x_\alpha(\varphi(r))$ , for  $\alpha \in \Phi$  and  $r \in \mathcal{R}$ , and  $f_\varphi(\chi(r)) = \chi(\varphi(r))$ , for  $\chi \in Y$  and  $r \in \mathcal{R}^\times$ .*
4. *If  $\varphi$  is surjective, then  $f_\varphi$  is surjective.*

*Proof.* (1), (2) By definition, we have

$$f_\varphi^{\hat{U}^+} \left( \sum_{\alpha \in Q^+} \sum_{j \in J_\alpha} u_{\alpha,j} \otimes r_j \right) = \sum_{\alpha \in Q^+} \sum_{j \in J_\alpha} u_{\alpha,j} \otimes \varphi(r_j)$$

if  $J_\alpha$  is a finite set and  $(r_j) \in \mathcal{R}^{J_\alpha}$  and  $u_{\alpha,j} \in \mathcal{U}_{\alpha, \mathcal{R}}$ , for every  $\alpha \in Q_+$ . Thus  $\varphi$  commutes with infinite sums and product, which proves (1) and (2).

(3) Let  $i \in I$ . Then the morphism  $\mathfrak{P}_i(\mathcal{R}) \rightarrow \mathfrak{P}_i(\mathcal{R}')$  induced by  $\varphi$  satisfies the formula above. Using the fact that  $x_\alpha = \tilde{w}x_{-\alpha_i}\tilde{w}^{-1}$ , for  $\alpha = -w.\alpha_i$ , with  $w \in W^v$ ,  $i \in I$  and  $\tilde{w}$  defined as in (2.5), we have (3).

(4) Assume  $\varphi$  is surjective. By (2.3) and (1), the restriction of  $f_\varphi$  to  $\mathfrak{U}^{pma}(\mathcal{R})$  is surjective. By (3), the restriction of  $f_\varphi(\mathfrak{U}^-(\mathcal{R})) = \mathfrak{U}^-(\mathcal{R}')$  and  $f_\varphi(\mathfrak{T}(\mathcal{R})) = \mathfrak{T}(\mathcal{R}')$ . We conclude by using the fact that  $\mathfrak{G}^{pma}$  is generated by  $\mathfrak{U}^{pma}$ ,  $\mathfrak{U}^-$  and  $\mathfrak{T}$  (see (2.6)).  $\square$

## 2.2.4 Minimal Kac-Moody group over rings

For  $i \in I$ , there is a natural group morphism  $\varphi_i : \mathrm{SL}_2 \rightarrow \mathfrak{U}_{\alpha_i}^Y$ .

For a ring  $\mathcal{R}$ , one sets

$$\mathfrak{G}^{\min}(\mathcal{R}) = \langle \varphi_i(\mathrm{SL}_2(\mathcal{R})), \mathfrak{T}(\mathcal{R}) \rangle \subset \mathfrak{G}^{pma}(\mathcal{R}).$$

This group is introduced by Marquis in [Mar18, Definition 8.126]. By [Mar18, Proposition 8.129], it is a nondegenerate Tits functor in the sense of [Mar18, Definition 7.83] and we have  $\mathfrak{G}^{\min}(\mathcal{R}) \simeq \mathfrak{G}(\mathcal{R})$ , for any field  $\mathcal{R}$ .

Note that if  $\varphi$  is a ring morphism between two rings  $\mathcal{R}$  and  $\mathcal{R}'$ , the induced morphism  $\mathfrak{G}^{pma}(\mathcal{R}) \rightarrow \mathfrak{G}^{pma}(\mathcal{R}')$  restricts to a morphism  $\mathfrak{G}^{\min}(\mathcal{R}) \rightarrow \mathfrak{G}^{\min}(\mathcal{R}')$ .

Let  $\mathcal{R}$  be a semilocal ring, i.e a ring with finitely many maximal ideals, then by [HO89, 4.3.9 Theorem],  $\mathrm{SL}_2(\mathcal{R})$  is generated by  $\begin{pmatrix} 1 & \mathcal{R} \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 0 \\ \mathcal{R} & 1 \end{pmatrix}$ . Therefore,

$$\mathfrak{G}^{\min}(\mathcal{R}) = \langle \mathfrak{U}_{\pm\alpha_i}(\mathcal{R}), \mathfrak{T}(\mathcal{R}) \mid i \in I \rangle \subset \mathfrak{G}^{pma}(\mathcal{R}). \quad (2.7)$$

## 2.3 Split Kac-Moody groups over valued fields and masures

We now fix a field  $\mathcal{K}$  equipped with a valuation  $\omega : \mathcal{K} \rightarrow \mathbb{R} \cup \{+\infty\}$  such that  $\Lambda := \omega(\mathcal{K}^*)$  contains  $\mathbb{Z}$ . Let  $\mathcal{O} = \{x \in \mathcal{K} \mid \omega(x) \geq 0\}$  be its ring of valuation. We defined Mathieu's positive completion  $\mathfrak{G}^{pma}$ . Replacing  $\Delta_+$  by  $w.\Delta_+$ , for  $w \in W^v$ , one can also define a group  $\mathfrak{G}^{pma,w}$ . Replacing  $\Delta_+$  by  $\Delta_-$  or by  $w.\Delta_-$ , for  $w \in W^v$ , one can also define  $\mathfrak{G}^{nma}$  or  $\mathfrak{G}^{nma,w}$ .

We set  $G = \mathfrak{G}(\mathcal{K})$ ,  $G^{pma} = \mathfrak{G}^{pma}(\mathcal{K})$  and  $G^{nma} = \mathfrak{G}^{nma}(\mathcal{K})$ .

### 2.3.1 Action of $N$ on $\mathbb{A}$

Let  $N = \mathfrak{N}(\mathcal{K})$  and  $\mathrm{Aut}(\mathbb{A})$  be the group of affine automorphism of  $\mathbb{A}$ . Then by [Rou16, 4.2], there exists a group morphism  $\nu : N \rightarrow \mathrm{Aut}(\mathbb{A})$  such that:

1. for  $i \in I$ ,  $\nu(\tilde{r}_i)$  is the simple reflection  $r_i \in W^v$ , it fixes 0,
2. for  $t \in \mathfrak{T}(\mathcal{K})$ ,  $\nu(t)$  is the translation on  $\mathbb{A}$  by the vector  $\nu(t)$  defined by  $\chi(\nu(t)) = -\omega(\chi(t))$ , for all  $\chi \in X$ .
3. we have  $\nu(N) = W^v \ltimes (Y \otimes \Lambda) := W_\Lambda$ .

### 2.3.2 Affine apartment

A **local face** in  $\mathbb{A}$  is the germ  $F(x, F^v) = \text{germ}_x(x + F^v)$  where  $x \in \mathbb{A}$  and  $F^v$  is a vectorial face (i.e,  $F(x, F^v)$  is the filter of all neighbourhoods of  $x$  in  $x + F^v$ ). It is a **local panel**, positive, or negative if  $F^v$  is. If  $F^v$  is a chamber, we call  $F(x, F^v)$  an **alcove** (or a local chamber). We denote by  $C_0^+$  the **fundamental alcove**, i.e,  $C_0^+ = \text{germ}_0(C_f^v)$ .

A **sector** in  $\mathbb{A}$  is a subset  $\mathfrak{q} = x + C^v$ , for  $x$  a point in  $\mathbb{A}$  and  $C^v$  a vectorial chamber. Its **sector germ** (at infinity) is the filter  $\mathfrak{Q} = \text{germ}_\infty(\mathfrak{q})$  of subsets of  $\mathbb{A}$  containing another sector  $x + y + C^v$ , with  $y \in C^v$ . It is entirely determined by its direction  $C^v$ . This sector or sector germ is said positive (resp. negative) if  $C^v$  has this property. We denote by  $\pm\infty$  the germ at infinity of  $\pm C_f^v$ .

For  $\alpha \in \Delta$  and  $k \in \Lambda \cup \{+\infty\}$ , we set  $D(\alpha, k) = \{x \in \mathbb{A} \mid \alpha(x) + k \geq 0\}$ . A set of the form  $D(\alpha, k)$ , for  $\alpha \in \Delta$  and  $k \in \Lambda$  is called a **half-apartment**.

### 2.3.3 Parahoric subgroups

In [Rou16] and [GR08], the measure  $\mathcal{I}$  of  $G$  is constructed as follows. To each  $x \in \mathbb{A}$  is associated a group  $\hat{P}_x = G_x$ . Then  $\mathcal{I}$  is defined in such a way that  $G_x$  is the fixator of  $x$  in  $G$  for the action on  $\mathcal{I}$ . We actually associate to each filter  $\Omega$  on  $\mathbb{A}$  a subgroup  $G_\Omega \subset G$  (with  $G_{\{x\}} = G_x$  for  $x \in \mathbb{A}$ ). Even though the measure is not yet defined, we use the terminology ‘‘fixator’’ to speak of  $G_\Omega$ , as this will be the fixator of  $\Omega$  in  $G$ . The definition of  $G_\Omega$  involves the completed groups  $G^{pma}$  and  $G^{nma}$ .

If  $\Omega$  is a non empty subset of  $\mathbb{A}$  we sometimes regard it as a filter on  $\mathbb{A}$  by identifying it with the filter consisting of the subsets of  $\mathbb{A}$  containing  $\Omega$ . Let  $\Omega \subset \mathbb{A}$  be a non empty set or filter. One defines a function  $f_\Omega : \Delta \rightarrow \mathbb{R}$  by

$$f_\Omega(\alpha) = \inf\{r \in \mathbb{R} \mid \Omega \subset D(\alpha + r)\} = \inf\{r \in \mathbb{R} \mid \alpha(\Omega) + r \subset [0, +\infty[\},$$

for  $\alpha \in \Delta$ . For  $r \in \mathbb{R}$ , one sets  $\mathcal{K}_{\omega \geq r} = \{x \in \mathcal{K} \mid \omega(x) \geq r\}$ ,  $\mathcal{K}_{\omega=r} = \{x \in \mathcal{K} \mid \omega(x) = r\}$ .

If  $\Omega$  is a set, we define the subgroup  $U_\Omega^{pma} = \prod_{\alpha \in \Delta_+} X_\alpha(\mathfrak{g}_{\alpha, \mathbb{Z}} \otimes \mathcal{K}_{\omega \geq f_\Omega(\alpha)}) \subset G^{pma}$ . Actually, for  $\alpha \in \Phi^+ = \Delta_{re}^+$ ,  $X_\alpha(\mathfrak{g}_{\alpha, \mathbb{Z}} \otimes \mathcal{K}_{\omega \geq f_\Omega(\alpha)}) = x_\alpha(\mathcal{K}_{\omega \geq f_\Omega(\alpha)}) =: U_{\alpha, \Omega}$ . We then define

$$U_\Omega^{pm+} = U_\Omega^{pma} \cap G = U_\Omega^{pma} \cap U^+,$$

see [Rou16, 4.5.2, 4.5.3 and 4.5.7]. When  $\Omega$  is a filter, we set  $U_\Omega^{pma} := \cup_{S \in \Omega} U_S^{pma}$  and  $U_\Omega^{pm+} := U_\Omega^{pma} \cap G$

We may also consider the negative completion  $G^{nma} = \mathfrak{G}^{nma}(\mathcal{K})$  of  $G$ , and define the subgroup  $U_\Omega^{ma-} = \prod_{\alpha \in \Delta_-} X_\alpha(\mathfrak{g}_{\alpha, \mathbb{Z}} \otimes \mathcal{K}_{\omega \geq f_\Omega(\alpha)})$ . For  $\alpha \in \Phi^- = \Delta_{re}^-$ ,  $X_\alpha(\mathfrak{g}_{\alpha, \mathbb{Z}} \otimes \mathcal{K}_{\omega \geq f_\Omega(\alpha)}) = x_\alpha(\mathcal{K}_{\omega \geq f_\Omega(\alpha)}) =: U_{\alpha, \Omega}$ . We then define  $U_\Omega^{nm-} = U_\Omega^{ma-} \cap G = U_\Omega^{ma-} \cap U^-$ .

Let  $\Psi$  be a closed subset of  $\Delta_+$ . One sets  $U_\Omega^{pm}(\Psi) = \mathfrak{U}_\Psi^{pma}(\mathcal{K}) \cap U_\Omega^{pm+}$ . By the uniqueness in the decomposition of the elements of  $U_\Omega^{pma}$  as a product, every element of  $U_\Omega^{pm}(\Psi)$  belongs to  $\prod_{\alpha \in \Psi} X_\alpha(\mathcal{K}_{\omega \geq f_\Omega(\alpha)})$ . If  $\Psi$  is a closed subset of  $\Delta^-$ , one sets  $U_\Omega^{nm}(\Psi) = \mathfrak{U}_\Psi^{nma}(\mathcal{K}) \cap U_\Omega^{nm-}$ . Note that  $U_\Omega^{pm+} = U_\Omega^{pm}(\Delta_+)$  and  $U_\Omega^{nm-} = U_\Omega^{nm}(\Delta_-)$ .

Let  $\Omega$  be a filter on  $\mathbb{A}$ . We denote by  $N_\Omega$  the fixator of  $\Omega$  in  $N$  (for the action of  $N$  on  $\mathbb{A}$ ). If  $\Omega$  is not a set, we have  $N_\Omega = \bigcup_{S \in \Omega} N_S$ . Note that we drop the hats used in [Rou16]. When  $\Omega$  is open one has  $N_\Omega = N_{\mathbb{A}} = \mathfrak{T}(\mathcal{O}) := \mathfrak{T}(\mathcal{K}_{\omega \geq 0}) = \mathfrak{T}(\mathcal{K}_{\omega=0})$ .

If  $x \in \mathbb{A}$ , we set  $G_x = U_x^{pm+} \cdot U_x^{nm-} \cdot N_x$ . This is a subgroup of  $G$ . If  $\Omega \subset \mathbb{A}$  is a set, we set  $G_\Omega = \bigcap_{x \in \Omega} G_x$  and if  $\Omega$  is a filter, we set  $G_\Omega = \bigcup_{S \in \Omega} G_S$ . Note that in [Rou16], the definition of  $G_x$  is much more complicated (see [Rou16, Définition 4.13]). However it is equivalent to this one by [Rou16, Proposition 4.14].

A filter is said to have a “good fixator” if it satisfies [Rou16, Définition 5.3]. There are many examples of filters with good fixators (see [Rou16, 5.7]): points, local faces, sectors, sector germs,  $\mathbb{A}$ , walls, half apartments, ... For such a filter  $\Omega$ , we have:

$$G_\Omega = U_\Omega^{pm+} \cdot U_\Omega^{nm-} \cdot N_\Omega = U_\Omega^{nm-} \cdot U_\Omega^{pm+} \cdot N_\Omega. \quad (2.8)$$

We then have:

$$U_\Omega^{pm+} = G_\Omega \cap U^+ \text{ and } U_\Omega^{nm-} = G_\Omega \cap U^-, \quad (2.9)$$

as  $U^- \cap U^+ \cdot N = U^+ \cap N = \{1\}$ , by [Rou16, Remarque 3.17] and [Rém02, 1.2.1 (RT3)].

When  $\Omega = C_0^+ = \text{germ}_0(C_f^v)$  is the (fundamental) positive local chamber in  $\mathbb{A}$ ,  $K_I := G_\Omega$  is called the (fundamental) **Iwahori subgroup**. When  $\Omega$  is a face of  $C_0^+$ ,  $G_\Omega$  is called a **parahoric subgroup**.

For  $\Omega$  a set or a filter, one defines:

$U_\Omega = \langle U_{\alpha, \Omega} \mid \alpha \in \Phi \rangle$ ,  $U_\Omega^\pm = U_\Omega \cap U^\pm$  and  $U_\Omega^{\pm\pm} = \langle U_{\alpha, \Omega} \mid \alpha \in \Phi^\pm \rangle$ . Then one has  $U_\Omega^{++} \subset U_\Omega^+ \subset U_\Omega^{pm+}$ , but these inclusions are not equalities in general, contrary to the reductive case (see [Rou16, 4.12.3 a and 5.7 3]).

**Lemma 2.3.** *Let  $(u_+, u_-, t), (u'_+, u'_-, t') \in U^+ \times U^- \times T$ . Assume that  $u_+ t u_- = u'_+ t' u'_-$  or  $u_+ u_- t = u'_+ u'_- t'$  or  $t u_+ u_- = t' u'_+ u'_-$ . Then  $u_- = u'_-$ ,  $u_+ = u'_+$  and  $t = t'$ .*

*Proof.* Assume  $u_+ t u_- = u'_+ t' u'_-$ . We have  $(u'_+)^{-1} u_+ t = t' u_- (u'_-)^{-1}$ . As  $t$  normalizes  $U^-$ , we deduce the existence of  $u''_-$  such that  $(u'_+)^{-1} u_+ t t'^{-1} = u''_-$ . By [Rou06, Proposition 1.5 (DR5)] (there is a misprint in this proposition,  $Z$  is in fact  $T$ ), we deduce  $(u'_+)^{-1} u_+ t t'^{-1} = 1$  and hence  $u'_+ = u_+$  and  $t = t'$ . Therefore  $u_- = u'_-$ . The other cases are similar.  $\square$

By [Rou16, 4.10] and (2.4), we have the following lemma.

**Lemma 2.4.** *Let  $\Omega$  be a filter on  $\mathbb{A}$ ,  $t \in T$  and  $\Psi$  be a closed subset of  $\Delta_+$  (resp.  $\Delta_-$ ). Then  $t U_\Omega^{pm+} t^{-1} = U_{t\Omega}^{pm+}$ ,  $t U_\Omega^{pm}(\Psi) t^{-1} = U_{t\Omega}^{pm}(\Psi)$  (resp.  $t U_\Omega^{nm-}(\Psi) t^{-1} = U_{t\Omega}^{nm-}(\Psi)$ ).*

### 2.3.4 Measure

We now define the measure  $\mathcal{I} = \mathcal{I}(\mathfrak{G}, \mathcal{K}, \omega)$ . As a set,  $\mathcal{I} = G \times \mathbb{A} / \sim$ , where  $\sim$  is defined as follows:

$$\forall (g, x), (h, y) \in G \times \mathbb{A}, (g, x) \sim (h, y) \Leftrightarrow \exists n \in N \mid y = \nu(n).x \text{ and } g^{-1} h n \in G_x.$$

We regard  $\mathbb{A}$  as a subset of  $\mathcal{I}$  by identifying  $x$  and  $(1, x)$ , for  $x \in \mathbb{A}$ . The group  $G$  acts on  $\mathcal{I}$  by  $g.(h, x) = (gh, x)$ , for  $g, h \in G$  and  $x \in \mathbb{A}$ . An **apartment** is a set of the form

$g.\mathbb{A}$ , for  $g \in G$ . The stabilizer of  $\mathbb{A}$  in  $G$  is  $N$  and if  $x \in \mathbb{A}$ , then the fixator of  $x$  in  $G$  is  $G_x$ . More generally, when  $\Omega \subset \mathbb{A}$ , then  $G_\Omega$  is the fixator of  $\Omega$  in  $G$  and  $G_\Omega$  permutes transitively the apartments containing  $\Omega$ . If  $A$  is an apartment, we transport all the notions that are preserved by  $W_\Lambda$  (for example segments, walls, faces, chimneys, etc.) to  $A$ . Then by [Héb22a, Corollary 3.7], if  $(\alpha_i)_{i \in I}$  and  $(\alpha_i^\vee)_{i \in I}$  are free, then  $\mathcal{I}$  satisfies the following axioms:

(MA II) : Let  $A, A'$  be two apartments. Then  $A \cap A'$  is a finite intersection of half-apartments and there exists  $g \in G$  such that  $g.A = A'$  and  $g$  fixes  $A \cap A'$ .

(MA III): if  $\mathfrak{R}$  is the germ of a splayed chimney and if  $F$  is a local face or a germ of a chimney, then there exists an apartment containing  $\mathfrak{R}$  and  $F$ .

We did not recall the definition of a chimney and we refer to [Rou11] for such a definition. We will only use the fact that a sector-germ is a particular case of a germ of a splayed chimney.

We also have:

- The stabilizer of  $\mathbb{A}$  in  $G$  is  $N$  and  $N$  acts on  $\mathbb{A} \subset \mathcal{I}$  via  $\nu$ .
- The group  $U_{\alpha,r} := \{x_\alpha(u) \mid u \in \mathcal{K}, \omega(u) \geq r\}$ , for  $\alpha \in \Phi, r \in \Lambda$ , fixes the half-apartment  $D(\alpha, r)$ . It acts simply transitively on the set of apartments in  $\mathcal{I}$  containing  $D(\alpha, r)$ .

The first point of the next proposition extends [BT72, Proposition 7.4.8] to mesures.

**Proposition 2.5.** 1. Let  $g \in G$ . Then  $\mathbb{A} \cap g^{-1}.\mathbb{A}$  is a finite intersection of half-apartments and there exists  $n \in N$  such that  $g.x = n.x$  for every  $x \in \mathbb{A} \cap g^{-1}.\mathbb{A}$ .

2. Let  $g \in G$ . Then  $\{x \in \mathbb{A} \mid g.x = x\}$  is convex. In particular if  $\Omega$  is a subset of  $\mathbb{A}$ , then  $G_\Omega = G_{\text{conv}(\Omega)}$ , where  $\text{conv}(\Omega)$  is the convex hull of  $\Omega$ .

*Proof.* 1). Let  $g \in G$ . We assume that  $\mathbb{A} \cap g^{-1}.\mathbb{A}$  is non-empty. Then it is a finite intersection of half-apartments by (MA II) and there exists  $h \in G$  such that  $hg.\mathbb{A} = \mathbb{A}$  and  $h$  fixes  $\mathbb{A} \cap g.\mathbb{A}$ . Then  $hg$  stabilizes  $\mathbb{A}$  and thus it belongs to  $N$ , by [Rou16, 5.7 5)]. We get 1) by setting  $n = hg$ .

2) Let  $g \in G$ ,  $\Omega_1 = \mathbb{A} \cap g^{-1}.\mathbb{A}$  and  $\Omega = \{x \in \mathbb{A} \mid g.x = x\}$ . We have  $\Omega \subset \Omega_1$ . Let  $n \in N$  be such that  $g.x = n.x$  for all  $x \in \Omega_1$ . Let  $f = \nu(n) : \mathbb{A} \rightarrow \mathbb{A}$ . Then  $\Omega = \Omega_1 \cap \{x \in \mathbb{A} \mid f(x) = x\}$ . As  $f$  is affine and  $\Omega_1$  is convex, we have that  $\Omega$  is convex.  $\square$

**Remark 2.6.** In 2.1.1, we did not assume the freeness of the families  $(\alpha_i)_{i \in I}$  and  $(\alpha_i^\vee)_{i \in I}$ , since there are interesting Kac-Moody groups, which do not satisfy this assumption. For example,  $G := \text{SL}_n(\mathcal{K}[u, u^{-1}]) \rtimes \mathcal{K}^*$  is naturally equipped with the structure of a Kac-Moody group associated with a root generating system  $\mathcal{S}$  having nonfree coroots. This group is particularly interesting for examples, since it is one of the only Kac-Moody groups in which we can make explicit computations. In [Héb22a], we proved that if  $(\alpha_i)_{i \in I}$  and  $(\alpha_i^\vee)_{i \in I}$  are

free families, then the measure associated with  $G$  satisfies (MA II) and (MA III). Without this assumption we do not know. In [Rou16, Théorème 5.16], Rousseau proves that  $\mathcal{I}$  satisfies the axioms (MA2) to (MA5) of [Rou11]. We did not introduce these axioms since they are more complicated and a bit less convenient. However it is easy to adapt the proofs of this paper to use the axioms of [Rou11] instead of those of [Héb22a], for example, retractions are already available in [Rou11].

### 2.3.5 Retraction centred at a sector-germ

Let  $\Omega$  be a sector-germ of  $\mathbb{A}$ . If  $x \in \mathcal{I}$ , then by (MA III), there exists an apartment  $A$  of  $\mathcal{I}$  containing  $\Omega$  and  $x$ . By (MA II), there exists  $g \in G$  such that  $g.A = \mathbb{A}$  and  $g$  fixes  $A \cap \mathbb{A}$ . One sets  $\rho_\Omega(x) = g.x \in \mathbb{A}$ . This is well-defined, independently of the choices of  $A$  and  $g$ , by (MA II). This defines the **retraction**  $\rho_\Omega : \mathcal{I} \rightarrow \mathbb{A}$  **onto  $\mathbb{A}$  and centred at  $\Omega$** . When  $\Omega = +\infty$ , we denote it  $\rho_{+\infty}$ . If  $x \in \mathcal{I}$ , then  $\rho_{+\infty}(x)$  is the unique element of  $U^+.x \cap \mathbb{A}$ .

### 2.3.6 Topology defined by a filtration

A **filtration of  $G$  by subgroups** is a sequence  $(V_n)_{n \in \mathbb{N}^*}$  of subgroups of  $G$  such that  $V_{n+1} \subset V_n$  for all  $n \in \mathbb{N}^*$ . Let  $(V_n)$  be a filtration of  $G$  by subgroups. The associated topology  $\mathcal{T}((V_n))$  is the topology on  $G$  for which a set  $V$  is open if for all  $g \in V$ , there exists  $n \in \mathbb{N}^*$  such that  $g.V_n \subset V$ .

Let  $(V_n), (\tilde{V}_n)$  be two filtrations of  $G$  by subgroups. We say that  $(V_n)$  and  $(\tilde{V}_n)$  are equivalent if for all  $n \in \mathbb{N}$ , there exist  $m, \tilde{m} \in \mathbb{N}$  such that  $V_m \subset \tilde{V}_n$  and  $\tilde{V}_{\tilde{m}} \subset V_n$ . This defines an equivalence relation on the set of filtrations of  $G$  by subgroups. Then  $(V_n)$  and  $(\tilde{V}_n)$  are equivalent filtrations, if and only if  $\mathcal{T}((V_n)) = \mathcal{T}((\tilde{V}_n))$ .

We say that  $(V_n)$  is **conjugation-invariant** if for all  $g \in G$ ,  $(gV_n g^{-1})$  is equivalent to  $(V_n)$ . Then  $\mathcal{T}((V_n))$  equips  $G$  with the structure of a topological group if and only if  $(V_n)$  is conjugation invariant, by [Mar18, Exercise 8.5].

## 3 Congruence subgroups

In this section, we define and study the congruence subgroups. They will be a key tool in order to define the topology  $\mathcal{T}$  in section 4. We prove however that the filtration  $(\ker \pi_n)_{n \in \mathbb{N}^*}$  is not conjugation-invariant. We also study how they decompose.

### 3.1 Definition of the congruence subgroup

**Proposition 3.1.** *The fixator  $G_0$  of 0 in  $G$  is the group  $\mathfrak{G}^{\min}(\mathcal{O})$ .*

*Proof.* For  $i \in I$ ,  $x_{\alpha_i}(\mathcal{O})$ ,  $x_{-\alpha_i}(\mathcal{O})$  and  $\mathfrak{I}(\mathcal{O})$  fix 0. Therefore by (2.7),  $\mathfrak{G}^{\min}(\mathcal{O}) \subset G_0$ .

By [Rou16, Proposition 4.14]

$$G_0 = U_0^{pm+} U_0^{nm-} N_0, \quad (3.1)$$



where  $N_0 = \{n \in N \mid n.0 = 0\}$ . By [BHR22, 2.4.1 2)], we have

$$U_0^{pm+} = U_0^+ := \langle x_\alpha(u) \mid \alpha \in \Phi, u \in \mathcal{O} \rangle \cap U^+ \subset \mathfrak{G}^{\min}(\mathcal{O})$$

and

$$U_0^{nm-} = U_0^- := \langle x_\alpha(u) \mid \alpha \in \Phi, u \in \mathcal{O} \rangle \cap U^- \subset \mathfrak{G}^{\min}(\mathcal{O}).$$

For  $i \in I$ , set  $\tilde{r}_i = x_{\alpha_i}(1)x_{-\alpha_i}(-1)x_{\alpha_i}(1) \in \mathfrak{G}^{\min}(\mathcal{O})$ . We have  $N = \langle \mathfrak{T}(\mathcal{K}), \tilde{r}_i \mid i \in I \rangle$ . Let  $n \in N_0$ . Write  $\nu^v(n) = w \in W^v$ , where  $\nu^v$  was defined in 2.2.1. Write  $w = r_{i_1} \dots r_{i_k}$ , with  $k = \ell(w)$  and  $i_1, \dots, i_k \in I$ . Let  $n' = \tilde{r}_{i_1} \dots \tilde{r}_{i_k} \in N_0$ . By [Rou16, 1.6 4)]  $\nu^v(n') = w$  and  $t := n'^{-1}n \in T \cap \ker(\nu)$ . By [Rou16, 4.2 3)],  $t \in \mathfrak{T}(\mathcal{O})$ . Therefore

$$N_0 = \langle \tilde{r}_i \mid i \in I \rangle \cdot \mathfrak{T}(\mathcal{O}). \quad (3.2)$$

and in particular,  $N_0 \subset \mathfrak{G}^{\min}(\mathcal{O})$ . Proposition follows.  $\square$

Recall that we assumed that  $\Lambda = \omega(\mathcal{K}^*) \supset \mathbb{Z}$ . If  $\omega(\mathcal{K}^*)$  is discrete, we can normalize  $\omega$  so that  $\Lambda = \mathbb{Z}$ . We fix  $\varpi \in \mathcal{O}$  such that  $\omega(\varpi) = 1$ .

For  $n \in \mathbb{N}^*$ , we denote by  $\pi_n^{pma} : \mathfrak{G}^{pma}(\mathcal{O}) \rightarrow \mathfrak{G}^{pma}(\mathcal{O}/\varpi^n\mathcal{O})$  and  $\pi_n^{nma} : \mathfrak{G}^{nma}(\mathcal{O}) \rightarrow \mathfrak{G}^{nma}(\mathcal{O}/\varpi^n\mathcal{O})$  the morphisms associated with the canonical projection  $\mathcal{O} \twoheadrightarrow \mathcal{O}/\varpi^n\mathcal{O}$ . We denote by  $\pi_n$  the restriction of  $\pi_n^{pma}$  to  $\mathfrak{G}^{\min}(\mathcal{O})$ . By Proposition 2.2,  $\pi_n$  is also the restriction of  $\pi_n^{nma}$  to  $\mathfrak{G}^{\min}(\mathcal{O})$  (it is also the restrictions of  $\pi_n^{pma,w} : \mathfrak{G}^{pma,w}(\mathcal{O}) \rightarrow \mathfrak{G}^{pma,w}(\mathcal{O}/\varpi^n\mathcal{O})$  and  $\pi_n^{nma,w} : \mathfrak{G}^{nma,w}(\mathcal{O}) \rightarrow \mathfrak{G}^{nma,w}(\mathcal{O}/\varpi^n\mathcal{O})$ , for  $w \in W^v$ ). By Proposition 2.2 and (2.7),  $\pi_n$ ,  $\pi_n^{pma}$  and  $\pi_n^{nma}$  are surjective. Their kernels are respectively called the  $n$ -th **congruence subgroups** of  $\mathfrak{G}^{\min}(\mathcal{O})$ ,  $\mathfrak{G}^{pma}(\mathcal{O})$  and  $\mathfrak{G}^{nma}(\mathcal{O})$ .

The family  $(\ker \pi_n)_{n \in \mathbb{N}^*}$  is a filtration of  $G$ . We prove below that it is not conjugation-invariant when  $W^v$  is infinite, which motivates the introduction of other filtrations  $(\mathcal{V}_{n\lambda})_{n \in \mathbb{N}^*}$ , for  $\lambda \in Y^+$  regular, in section 4.

**Lemma 3.2.** *Let  $x \in \mathbb{A}$  be such that  $\alpha_i(x) > 0$  for all  $i \in I$ . Suppose that  $W^v$  is infinite. Then for all  $n \in \mathbb{N}^*$ , there exists  $g \in \ker(\pi_n)$  such that  $g.x \neq x$ .*

*Proof.* Let  $n \in \mathbb{N}^*$ . As  $\Phi^+$  is infinite, there exists  $\beta \in \Phi^+$  such that  $\text{ht}(\beta) > \frac{n}{\min_{i \in I} \alpha_i(x)}$ . Then  $\beta(x) > n$ . Let  $g = x_{-\beta}(\varpi^n) \in \ker \pi_n$ . Then the subset of  $\mathbb{A}$  fixed by  $g$  is  $\{y \in \mathbb{A} \mid -\beta(y) + n \geq 0\}$ , which does not contain  $x$ .  $\square$

**Lemma 3.3.** *Assume that  $W^v$  is infinite. Then  $(\ker(\pi_n))_{n \in \mathbb{N}^*}$  is not conjugation-invariant.*

*Proof.* Suppose that  $(\ker(\pi_n))$  is conjugation-invariant. Then the topology  $\mathcal{T}((\ker(\pi_n)))$  equips  $G$  with the structure of a topological group. We have  $\ker(\pi_1) \subset \mathfrak{G}^{\min}(\mathcal{O}) = G_0$  and in particular  $G_0 = \bigcup_{g \in G_0} g \cdot \ker(\pi_1)$  is open. Let  $\lambda \in Y^+$  be such that  $\alpha_i(\lambda) = 1$  for all  $i \in I$  and  $t \in T$  be such that  $t.0 = \lambda$ . Then  $H := tG_0t^{-1}$  is open (since  $G$  is a topological group). As  $1 \in H$ , we deduce the existence of  $n \in \mathbb{N}^*$  such that  $\ker(\pi_n) \subset H$ . As  $H$  fixes  $\lambda$ , this implies that  $W^v$  is finite, by Lemma 3.2.  $\square$

### 3.2 On the decompositions of the congruence subgroups

Let  $\mathfrak{m} = \{x \in \mathcal{O} \mid \omega(x) > 0\}$  be the maximal ideal of  $\mathcal{O}$  and  $\mathbb{k} = \mathcal{O}/\mathfrak{m}$ . Let  $\pi_{\mathbb{k}} : \mathfrak{G}^{\min}(\mathcal{O}) \rightarrow \mathfrak{G}^{\min}(\mathbb{k})$  be the morphism induced by the natural projection  $\mathcal{O} \twoheadrightarrow \mathbb{k}$ . When  $\omega(\mathcal{K}^*) = \mathbb{Z}$ ,  $\pi_{\mathbb{k}} = \pi_1$ .

In this subsection we study  $\ker \pi_{\mathbb{k}}$ : we prove that it decomposes as the product of its intersections with  $U^-$ ,  $U^+$  and  $T$  (see Proposition 3.5), using the measure  $\mathcal{I}$  of  $G$ . We also describe  $U^- \cap \ker \pi_{\mathbb{k}}$  and  $U^+ \cap \ker \pi_{\mathbb{k}}$  through their actions on  $\mathcal{I}$  and we deduce that  $\ker \pi_{\mathbb{k}}$  fixes  $C_0^+ \cup C_0^-$ . It would be interesting to prove similar properties for  $\ker \pi_n$  instead of  $\ker \pi_{\mathbb{k}}$ , for  $n \in \mathbb{N}^*$ . The difficulty is that when  $\omega$  is not discrete or  $n \geq 2$ ,  $\mathcal{O}/\varpi^n \mathcal{O}$  is no longer a field and very few is known for Kac-Moody groups over rings.

Let  $C, C'$  be two alcoves of the same sign based at 0. By [Héb20, Proposition 5.17], there exists an apartment  $A$  containing  $C$  and  $C'$ . Let  $g \in G$  be such that  $g.A = \mathbb{A}$  and  $g.C = C_0^+$ . Then  $g.C'$  is an alcove of  $\mathbb{A}$  based at 0 and thus there exists  $w \in W^v$  such that  $g.C' = w.C_0^+$ . We set  $d^{W^+}(C, C') = w$ , which is well-defined, independently of the choices we made (note that in [BGR16, 1.11] the “ $W$ -distance”  $d^{W^+}$  is defined for more general pairs of alcoves). Then  $d^{W^+}$  is  $G$ -invariant.

**Lemma 3.4.** *Let  $C$  be a positive alcove of  $\mathcal{I}$  based at 0 and  $w \in W^v$ . Write  $w = r_{i_1} \dots r_{i_k}$ , with  $k = \ell(w)$  and  $i_1, \dots, i_k \in I$ . Let  $\beta_1 = \alpha_{i_1}$ ,  $\beta_2 = r_{i_1}.\alpha_{i_2}$ ,  $\dots$ ,  $\beta_k = r_{i_1} \dots r_{i_{k-1}}.\alpha_{i_k}$ . Then  $\beta_1, \dots, \beta_k \in \Phi_+$  and we have  $\rho_{+\infty}(C) = w.C_0^+$  if and only if there exists  $a_1, \dots, a_k \in \mathcal{O}$  such that  $C = x_{\beta_1}(a_1) \dots x_{\beta_k}(a_k).\tilde{r}_{i_1} \dots \tilde{r}_{i_k}.C_0^+$ .*

*Proof.* As  $x_{\beta_1}(\mathcal{O}) \dots x_{\beta_k}(\mathcal{O})$  fixes 0, an element of  $x_{\beta_1}(\mathcal{O}) \dots x_{\beta_k}(\mathcal{O}).\tilde{r}_{i_1} \dots \tilde{r}_{i_k}.C_0^+$  is a positive alcove based at 0. The fact that  $\beta_1, \dots, \beta_k \in \Phi_+$  follows from [Kum02, 1.3.14 Lemma]. Thus  $x_{\beta_1}(\mathcal{O}) \dots x_{\beta_k}(\mathcal{O}).\tilde{r}_{i_1} \dots \tilde{r}_{i_k}.C_0^+ \subset U^+.w.C_0^+$  and we have one implication.

We prove the reciprocal by induction on  $\ell(w)$ . Assume  $w = 1$ . Then  $\rho_{+\infty}(C) = C_0^+$ . Let  $A$  be an apartment containing  $+\infty$  and  $C$ . Let  $g \in G$  be such that  $g.A = \mathbb{A}$  and  $g$  fixes  $A \cap \mathbb{A}$ . Then  $C = g^{-1}.C_0^+$ , by definition of  $\rho_{+\infty}$ . Moreover,  $A$  contains 0 and  $+\infty$  and thus it contains  $\text{conv}(0, +\infty) \supset C_0^+$ . Therefore  $C = C_0^+$  and the lemma is clear in this case. Assume now that  $\ell(w) \geq 1$ .

Let  $C'_0 = C_0^+$ ,  $C'_1 = r_{i_1}.C_0^+$ ,  $\dots$ ,  $C'_k = r_{i_1} \dots r_{i_k}.C_0^+ = C'$ . Let  $C$  be a positive alcove based at 0 and such that  $\rho_{+\infty}(C) = C'_k$ . Let  $A$  be an apartment containing  $C$  and  $+\infty$ . Let  $g \in G$  be such that  $g.A = \mathbb{A}$  and  $g$  fixes  $A \cap \mathbb{A}$ . Set  $C_i = g^{-1}.C'_i$ , for  $i \in \llbracket 0, k \rrbracket$ . Then  $g$  fixes  $+\infty$  and hence  $\rho_{+\infty}(x) = g.x$  for every  $x \in \mathbb{A}$ . Therefore  $\rho_{+\infty}(C_i) = C'_i$ , for  $i \in \llbracket 0, k \rrbracket$ . In particular,  $\rho_{+\infty}(C_{k-1}) = C'_{k-1}$ . By induction, we may assume that there exist  $a_1, \dots, a_{k-1} \in \mathcal{O}$  such that  $C_{k-1} = u\tilde{v}.C_0^+$ , where  $u = x_{\beta_1}(a_1) \dots x_{\beta_k}(a_{k-1})$  and  $\tilde{v} = \tilde{r}_{i_1} \dots \tilde{r}_{i_{k-1}}$ . Moreover we have

$$d^{W^+}(C_{k-1}, C_k) = d^{W^+}(C'_{k-1}, C'_k) = r_{i_k} = d^{W^+}(u^{-1}.C_{k-1}, u^{-1}.C_k) = d^{W^+}(\tilde{v}.C_0^+, u^{-1}.C_k).$$

Let  $P$  be the panel common to  $\tilde{v}.C_0^+$  and  $u^{-1}.C_k$ . Then  $P \subset \beta_k^{-1}(\{0\})$ . Let  $D$  be the half-apartment delimited by  $\beta_k^{-1}(\{0\})$  and containing  $C_{k-1}$ . Then as  $\beta_k(C_{k-1}) = r_{i_1} \dots r_{i_{k-1}}.\alpha_{i_k}(r_{i_1} \dots r_{i_{k-1}}.C_0^+) > 0$ ,  $D$  contains  $+\infty$ . By [Rou11, Proposition 2.9], there

exists an apartment  $B$  containing  $D$  and  $u^{-1}.C_k$ . Let  $g' \in G$  be such that  $g'.B = \mathbb{A}$  and  $g'$  fixes  $\mathbb{A} \cap B$ . We have  $g'.u^{-1}.C_k = \rho_{+\infty}(C_k) = C'_k$ . By [Rou16, 5.7 7)],  $g' \in \mathfrak{T}(\mathcal{O})U_{\beta_k,0}$  and as  $\mathfrak{T}(\mathcal{O})$  fixes  $\mathbb{A}$ , we can assume  $g' \in U_{\beta_k,0} = x_{\beta_k}(\mathcal{O})$ . Write  $g' = x_{\beta_k}(-a_k)$ , with  $a_k \in \mathcal{O}$ . Then  $C_k = u.x_{\beta_k}(a_k).C_k = x_{\beta_1}(a_1) \dots x_{\beta_k}(a_k).\tilde{r}_{i_1} \dots \tilde{r}_{i_k}.C_0^+$ , which proves the lemma.  $\square$

**Proposition 3.5.** 1. We have  $U_{C_0^+}^{nm-} = U^- \cap \ker(\pi_{\mathbb{k}})$  and  $U_{-C_0^+}^{pm+} = U^+ \cap \ker(\pi_{\mathbb{k}})$ .

2. We have  $\ker(\pi_{\mathbb{k}}) = (\ker(\pi_{\mathbb{k}}) \cap U^+) \cdot (\ker(\pi_{\mathbb{k}}) \cap U^-) \cdot (\ker(\pi_{\mathbb{k}}) \cap \mathfrak{T}(\mathcal{O}))$ .

3. We have  $\ker(\pi_{\mathbb{k}}) \subset G_{C_0^+ \cup C_0^-}$ .

*Proof.* 2) Let  $u \in U_{C_0^+}^{nm-}$ . By definition, there exists  $\Omega \in C_0^+$  such that  $u \in U_{\Omega}^{nm-}$ . Let  $x \in C_f^v \cap \Omega$ . Then  $u \in \prod_{\alpha \in \Delta_-} X_{\alpha}(\mathfrak{g}_{\alpha, \mathbb{Z}} \otimes \mathcal{K}_{\omega \geq -\alpha(x)}) \cap G_0$ . As  $-\alpha(x) > 0$  for every  $\alpha \in \Delta_-$ , Proposition 2.2 implies:

$$U_{C_0^+}^{nm-} \subset \ker(\pi_{\mathbb{k}}). \quad (3.3)$$

Let  $g \in \ker(\pi_{\mathbb{k}}) \subset \mathfrak{G}^{\min}(\mathcal{O})$ . Then  $g$  fixes 0 and  $g.C_0^+$  is a positive alcove based at 0. Write  $\rho_{+\infty}(g.C_0^+) = w.C_0^+$ , with  $w \in W^v$ . Write  $w = r_{i_1} \dots r_{i_m}$ , with  $m = \ell(w)$  and  $i_1, \dots, i_m \in I$ . Let  $\tilde{n} = \tilde{r}_{i_1} \dots \tilde{r}_{i_m} \in \mathfrak{N}(\mathcal{K})$ . By Lemma 3.4, there exists  $u \in \langle x_{\beta}(\mathcal{O}) \mid \beta \in \Phi_+ \rangle$  such that  $g.C_0^+ = u\tilde{n}.C_0^+$ . Then  $g = u\tilde{n}i$ , with  $i \in G_{C_0^+}$ . As  $C_0^+$  has a good fixator ([Rou16, 5.7 2)], we have (by (2.8):

$$G_{C_0^+} = U_{C_0^+}^{pm+} U_{C_0^+}^{nm-} N_{C_0^+}.$$

As every element of  $C_0^+$  has non empty interior,  $N_{C_0^+} = \mathfrak{T}(\mathcal{O})$ . Moreover,  $U_{C_0^+}^{pm+} = U_0^{pm+}$  and  $\mathcal{T}(\mathcal{O})$  normalizes  $U_0^{pm+}$  and  $U_{C_0^+}^{nm-}$ . Therefore,

$$G_{C_0^+} = \mathfrak{T}(\mathcal{O}).U_0^{pm+}.U_{C_0^+}^{nm-}.$$

Write  $i = tu_+u_-$ , with  $t \in \mathfrak{T}(\mathcal{O})$ ,  $u_+ \in U_0^{pm+}$  and  $u_- \in U_{C_0^+}^{nm-}$ .

Therefore by (3.3), we have

$$\pi_{\mathbb{k}}(g) = 1 = \pi_{\mathbb{k}}(u\tilde{n}tu_+u_-) = \pi_{\mathbb{k}}(u)\pi_{\mathbb{k}}(\tilde{n}t)\pi_{\mathbb{k}}(u_+)\pi_{\mathbb{k}}(u_-) = \pi_{\mathbb{k}}(u)\pi_{\mathbb{k}}(\tilde{n}t)\pi_{\mathbb{k}}(u_+).$$

By [Rou16, 3.16 Proposition] or [Mar18, Theorem 8.118],

$(\mathfrak{G}^{pma}(\mathbb{k}), \mathfrak{N}(\mathbb{k}), \mathfrak{U}^{pma}(\mathbb{k}), \mathfrak{U}^-(\mathbb{k}), \mathfrak{T}(\mathbb{k}), \{r_i \mid i \in I\})$  is a refined Tits system. By [Rou16, 3.16 Remarque], we have the Birkhoff decomposition

$$\mathfrak{G}^{pma}(\mathbb{k}) = \bigsqcup_{n \in \mathfrak{N}(\mathbb{k})} \mathfrak{U}^+(\mathbb{k})n\mathfrak{U}^{pma}(\mathbb{k}).$$

As  $\pi_{\mathbb{k}}(u) \in \mathfrak{U}^+(\mathbb{k})$ ,  $\pi_{\mathbb{k}}(\tilde{n}t) \in \mathfrak{N}(\mathbb{k})$  and  $\pi_{\mathbb{k}}(u_+) \in \mathfrak{U}^{pma}(\mathbb{k})$ , we deduce  $\pi_{\mathbb{k}}(\tilde{n}t) = 1$ .

By [Rou06, 1.4 Lemme and 1.6], there exists a group morphism  $\nu_{\mathbb{k}}^v : \mathfrak{N}(\mathbb{k}) \rightarrow W^v$  such that  $\nu_{\mathbb{k}}^v(\tilde{r}_i) = r_i$  for  $i \in I$  and  $\nu_{\mathbb{k}}^v(\mathfrak{T}(\mathbb{k})) = 1$ . Then  $\nu_{\mathbb{k}}^v(\tilde{n}t) = w = 1$ . Therefore  $w = 1$  and

$g = utu_+u_- = u'tu_-$ , for some  $u' \in U_0^{pm+}$ , since  $t$  normalizes  $U_0^{pm+}$ . By Lemma 2.3 and by symmetry of the roles of  $U^{nm-}$  and  $U^{pm+}$ , we have  $u' \in U_{-C_0^+}^{pm+}$ . By (3.3) applied to  $U_{-C_0^+}^{pm+}$ , we have  $\pi_{\mathbb{k}}(g) = 1 = \pi_{\mathbb{k}}(u')\pi_{\mathbb{k}}(t)\pi_{\mathbb{k}}(u_-) = \pi_{\mathbb{k}}(t)$  and thus

$$g \in U_{-C_0^+}^{pm+} \cdot (T \cap \ker \pi_{\mathbb{k}}) \cdot U_{C_0^+}^{nm-} = U_{-C_0^+}^{pm+} \cdot U_{C_0^+}^{nm-} \cdot (T \cap \ker \pi_{\mathbb{k}}).$$

By (3.3), we deduce 2).

3) We have  $U_{-C_0^+}^{pm+} = U_{-C_0^+ \cup C_f^v}^{pm+} \subset G_{C_0^+ \cup -C_0^+}$ ,  $U_{C_0^+}^{nm-} = U_{C_0^+ \cup -C_f^v}^{nm-} \subset G_{C_0^+ \cup -C_0^+}$  and  $T \cap \ker \pi_{\mathbb{k}} \subset T \cap G_0 \subset G_{\mathbb{A}}$ , which proves 3).

1) We already proved one inclusion. Let  $u \in \ker \pi_{\mathbb{k}} \cap U^-$ . Then by what we proved above,  $u \in U_{-C_0^+}^{pm+} \cdot (T \cap \ker \pi_{\mathbb{k}}) \cdot U_{C_0^+}^{nm-}$ . By Lemma 2.3,  $u \in U_{C_0^+}^{nm-}$ , and the proposition follows.  $\square$

**Corollary 3.6.** *Let  $n \in \mathbb{N}^*$ . Then  $\ker \pi_n \subset U_{-C_0^+}^{pm+} \cdot \mathfrak{T}(\mathcal{O}) \cdot U_{C_0^+}^{nm-}$ .*

**Remark 3.7.** *Let  $u \in U^- \cap \ker \pi_{\mathbb{k}} = U_{C_0^+}^{nm-}$ . Write  $u = \prod_{\alpha \in \Delta_-} X_{\alpha}(v_{\alpha})$ , where  $v_{\alpha} \in \mathfrak{g}_{\alpha, \mathbb{Z}} \otimes \mathcal{O}$ , for every  $\alpha \in \Delta_-$ . Define  $\omega : \mathfrak{g}_{\alpha, \mathbb{Z}} \otimes \mathcal{K} \rightarrow \mathbb{R} \cup \{+\infty\}$  by  $\omega(v) = \inf\{x \in \mathbb{R} \mid v \in \mathfrak{g}_{\alpha, \mathbb{Z}} \otimes \mathcal{K}_{\omega \geq x}\}$ , for  $v \in \mathfrak{g}_{\alpha, \mathbb{Z}} \otimes \mathcal{K}$ . Let  $\lambda \in Y$  be such that  $\alpha_i(\lambda) = 1$  for every  $i \in I$ . Let  $\Omega \in C_0^+$  be such that  $u \in U_{\Omega}^{nm-}$  and  $\eta \in \mathbb{R}_+^*$  be such  $\eta\lambda \in \Omega$ . Then  $u \in U_{\Omega}^{nm-}$  implies  $\omega(v_{\alpha}) \geq |\alpha(\eta\lambda)| = \eta \text{ht}(\alpha)$  for every  $\alpha \in \Delta_-$ . In particular,  $\omega(v_{\alpha})$  goes to  $+\infty$  when  $-\text{ht}(\alpha)$  goes to  $+\infty$ .*

## 4 Definition of topologies on $G$

In this section, we define two topologies  $\mathcal{T}$  and  $\mathcal{T}_{\text{Fix}}$  on  $G$  and compare them. For the first one, we proceed as follows. We define a set  $\mathcal{V}_{\lambda}$  for every regular  $\lambda \in Y^+$ . We prove that it is actually a subgroup of  $\mathfrak{G}^{\min}(\mathcal{O})$  (Lemma 4.2) and we define  $\mathcal{T}$  as the topology associated with  $(\mathcal{V}_{n\lambda})_{n \in \mathbb{N}^*}$ . We then prove that  $\mathcal{T}$  does not depend on the choice of  $\lambda$  and that it is conjugation-invariant (Theorem 4.8) and thus that  $(G, \mathcal{T})$  is a topological group. We then introduce the topology  $\mathcal{T}_{\text{Fix}}$  associated with the fixators of finite subsets of  $\mathcal{I}$  and we end up by a comparison of  $\mathcal{T}$  and  $\mathcal{T}_{\text{Fix}}$ .

### 4.1 Subgroup $\mathcal{V}_{\lambda}$

For  $n \in \mathbb{N}^*$ , we set  $T_n = \ker \pi_n \cap T \subset \mathfrak{T}(\mathcal{O})$ . For  $\lambda \in Y^+$  regular, we set

$$N(\lambda) = \min\{|\alpha(\lambda)| \mid \alpha \in \Phi\} \in \mathbb{N}^* \text{ and } \mathcal{V}_{\lambda} = U_{[-\lambda, \lambda]}^{pm+} \cdot U_{[-\lambda, \lambda]}^{nm-} \cdot T_{2N(\lambda)}.$$

By (2.9), we have

$$\mathcal{V}_{\lambda} = (U^+ \cap G_{[-\lambda, \lambda]}). (U^- \cap G_{[-\lambda, \lambda]}). (T \cap \ker \pi_{2N(\lambda)}) \subset G_{[-\lambda, \lambda]}. \quad (4.1)$$

The  $2N(\lambda)$  appearing comes from  $x_{-\alpha}(\varpi^n \mathcal{O}). x_{\alpha}(\varpi^n \mathcal{O}) \subset x_{\alpha}(\varpi^n \mathcal{O}) x_{-\alpha}(\varpi^n \mathcal{O}) \alpha^{\vee} (1 + \varpi^{2n} \mathcal{O})$  for  $\alpha \in \Phi$  and  $n \in \mathbb{N}^*$ , which follows from (2.1). To prove that  $\mathcal{V}_{\lambda}$  is a group, the

main difficulty is to prove that it is stable by left multiplication by  $U_{[-\lambda, \lambda]}^{nm-}$ . If  $G$  is reductive, we have  $U_{[-\lambda, \lambda]}^{nm-} = U_{[-\lambda, \lambda]}^{-} := \langle x_\alpha(a) \mid \alpha \in \Phi_-, a \in \mathcal{K}, \omega(a) \geq |\alpha(\lambda)| \rangle$ . By induction, it then suffices to prove that  $x_{-\alpha}(\varpi^{|\alpha(\lambda)|} \mathcal{O}) \mathcal{V}_\lambda \subset \mathcal{V}_\lambda$ , for  $\alpha \in \Phi_+$ . When  $G$  is no longer reductive, we have  $U_{[-\lambda, \lambda]}^{-} \subsetneq U_{[-\lambda, \lambda]}^{nm-}$  in general (see [Rou16, 4.12 3]). The group  $U_{[-\lambda, \lambda]}^{nm-}$  is defined as a set of infinite products and so its seems difficult to reason by induction in our case. We could try to use the group  $U_{[-\lambda, \lambda]}^- := \langle x_\alpha(a) \mid \alpha \in \Phi, a \in \mathcal{K}, \omega \geq |\alpha(a)| \rangle \cap U^-$  since it is sometimes equal to  $U_{[-\lambda, \lambda]}^{nm-}$  (for example when  $\lambda \in C_f^v$ ,  $U_{[-\lambda, \lambda]}^{nm-} = U_{-\lambda}^{nm-} = U_\lambda^- = U_{[-\lambda, \lambda]}^-$  by [BHR22, 2.4.1 2]). However it seems difficult since if  $\alpha \in \Phi_+$ , the condition  $\omega(a) + \alpha(\lambda) \geq 0$  allows elements with a negative valuation. In order to overcome these difficulties, we use the morphisms  $\pi_n$ , for  $n \in \mathbb{N}$ .

By definition,

$$U_{[-\lambda, \lambda]}^{pm+} = G \cap \prod_{\alpha \in \Delta_+} X_\alpha(\mathfrak{g}_{\alpha, \mathbb{Z}} \otimes \mathcal{K}_{\omega \geq f_\Omega(-\lambda, \lambda)}) \subset G \cap \prod_{\alpha \in \Delta_+} X_\alpha(\mathfrak{g}_{\alpha, \mathbb{Z}} \otimes \varpi^{N(\lambda)} \mathcal{O}).$$

By Proposition 2.2 we deduce  $U_{[-\lambda, \lambda]}^{pm+} \subset \ker(\pi_{N(\lambda)})$ . Using a similar reasoning for  $U_{[-\lambda, \lambda]}^{nm-}$  we deduce

$$\mathcal{V}_\lambda \subset \ker(\pi_{N(\lambda)}). \quad (4.2)$$

For  $n \in \mathbb{N}$  and  $\alpha^\vee \in \Phi^\vee$ , one sets  $T_{\alpha^\vee, n} = \alpha^\vee(1 + \varpi^n \mathcal{O}) \subset \mathfrak{T}(\mathcal{O})$ .

**Lemma 4.1.** *Let  $\lambda \in Y^+$  be regular and  $\alpha \in \Phi$ . Then  $U_{\alpha, [-\lambda, \lambda]} \cdot U_{-\alpha, [-\lambda, \lambda]} \cdot T_{\alpha^\vee, 2N(\lambda)}$  is a subgroup of  $G$ .*

*Proof.* Set  $\Omega = [-\lambda, \lambda]$ . Set  $H = U_{\alpha, \Omega} \cdot U_{-\alpha, \Omega} \cdot T_{\alpha^\vee, 2N(\lambda)}$ . It suffices to prove that  $H$  is stable under left multiplication by  $U_{\alpha, \Omega}$ ,  $U_{-\alpha, \Omega}$  and  $T_{\alpha^\vee, 2N(\lambda)}$ . The first stability is clear and the third follows from Lemma 2.4 and the fact that  $T_{\alpha^\vee, 2N(\lambda)} \subset \mathfrak{T}(\mathcal{O})$  fixes  $\mathbb{A}$ . Let  $u_-, \tilde{u}_- \in U_{-\alpha, \Omega}$ ,  $u_+ \in U_{\alpha, \Omega}$  and  $t \in T_{\alpha^\vee, 2N(\lambda)}$ . Write  $u_- = x_{-\alpha}(a_-)$ ,  $\tilde{u}_- = x_{-\alpha}(\tilde{a}_-)$  and  $u_+ = x_\alpha(a_+)$ , for  $a_-, \tilde{a}_-, a_+ \in \mathcal{K}$ . We have  $\omega(\tilde{a}_-), \omega(a_-), \omega(a_+) \geq |\alpha(\lambda)| \geq N(\lambda)$ . Then by (2.1), we have

$$\tilde{u}_- u_+ u_- t = x_\alpha(a_+(1 + \tilde{a}_- a_+)^{-1}) x_{-\alpha}(\tilde{a}_-(1 + \tilde{a}_- a_+)^{-1} + a_-) \alpha^\vee(1 + \tilde{a}_- a_+) t.$$

As  $\omega(1 + \tilde{a}_- a_+) = 1$ , we deduce that  $\tilde{u}_- u_+ u_- t \in H$ , which proves the lemma.  $\square$

Let  $w \in W^v$  and  $\Omega$  be a filter on  $\mathbb{A}$ . Recall that  $\mathfrak{G}^{pma, w}$  and  $\mathfrak{G}^{nma, w}$  are the completions of  $\mathfrak{G}$  with respect to  $w \cdot \Delta_+$  and  $w \cdot \Delta_-$  respectively. One defines  $U_\Omega^{pm}(w \cdot \Delta_+)$  and  $U_\Omega^{nm}(w \cdot \Delta_-)$  similarly as  $U_\Omega^{pm+}$  and  $U_\Omega^{nm-}$  in these groups.

**Lemma 4.2.** *Let  $\lambda \in Y^+$  be regular. Then*

1.  $\mathcal{V}_\lambda = U_{[-\lambda, \lambda]}^{pm}(w \cdot \Delta_+) \cdot U_{[\lambda, \lambda]}^{nm}(w \cdot \Delta_-) \cdot T_{2N(\lambda)}$  for every  $w \in W^v$ ,
2.  $\mathcal{V}_\lambda$  is a subgroup of  $G$ .

*Proof.* (1) This follows the proof of [GR08, Proposition 3.4]. Set  $\Omega = [-\lambda, \lambda]$ . Let  $i \in I$  and  $\alpha = \alpha_i$ . By [GR08, 3.3.4] and Lemma 4.1,

$$\begin{aligned}\mathcal{V}_\lambda &= U_\Omega^{pm}(\Delta_+ \setminus \{\alpha\}) \cdot U_\Omega^{nm}(\Delta_- \setminus \{\alpha\}) \cdot U_{\alpha, \Omega} \cdot U_{-\alpha, \Omega} \cdot T_{2N(\lambda)} \\ &= U_\Omega^{pm}(\Delta_+ \setminus \{\alpha\}) \cdot U_\Omega^{nm}(\Delta_- \setminus \{\alpha\}) \cdot U_{-\alpha, \Omega} \cdot U_{\alpha, \Omega} \cdot T_{2N(\lambda)} \\ &= U_\Omega^{pm}(\Delta_+ \setminus \{\alpha\}) \cdot U_{-\alpha, \Omega} \cdot U_\Omega^{nm}(\Delta_- \setminus \{\alpha\}) \cdot U_{\alpha, \Omega} \cdot T_{2N(\lambda)} \\ &= U_\Omega^{pm}(r_i \cdot \Delta_+) \cdot U_\Omega^{nm}(r_i \cdot \Delta_-) \cdot T_{2N(\lambda)}\end{aligned}$$

Therefore  $\mathcal{V}_\lambda$  does not change when  $\Delta_+$  is replaced by  $w \cdot \Delta_+$ , for  $w \in W^v$ , which proves (1).

Let  $w \in W^v$  be such that  $\lambda \in w \cdot C_f^v$ . By (1) we have

$$\mathcal{V}_\lambda = U_\Omega^{pm}(w \cdot \Delta_+) \cdot U_\Omega^{nm}(w \cdot \Delta_-) \cdot T_{2N(\lambda)}.$$

Let  $t \in T$  be such that  $t \cdot 0 = \lambda$ . By Lemma 2.4, we have

$$tU_\Omega^{pm}(w \cdot \Delta_+)t^{-1} = U_{t \cdot \Omega}^{pm}(w \cdot \Delta_+) = U_{[0, 2\lambda]}^{pm}(w \cdot \Delta_+) = U_0^{pm}(w \cdot \Delta_+).$$

Similarly,  $tU_\Omega^{nm}(w \cdot \Delta_-)t^{-1} = U_{2\lambda}^{nm}(w \cdot \Delta_-)$ . As  $T$  is commutative, we also have  $tT_{2N(\lambda)}t^{-1} = T_{2N(\lambda)}$ . In order to prove (2), it suffices to prove that

$$H := tU_\Omega^{pm}(w \cdot \Delta_+)U_\Omega^{nm}(w \cdot \Delta_-)T_{2N(\lambda)}t^{-1} = U_0^{pm}(w \cdot \Delta_+)U_{2\lambda}^{nm}(w \cdot \Delta_-)T_{2N(\lambda)}$$

is a subgroup of  $G$ . It suffices to prove that  $H$  is stable under left multiplication by  $U_0^{pm}(w \cdot \Delta_+)$ ,  $U_{2\lambda}^{nm}(w \cdot \Delta_-)$  and  $T_{2N(\lambda)}$ . The first stability is clear and the third follows from Lemma 2.4.

First note that by [Rou16, 5.7 1)],

$$G_{[0, 2\lambda]} = U_0^{pm}(w \cdot \Delta_+)U_{2\lambda}^{nm}(w \cdot \Delta_-)\mathfrak{T}(\mathcal{O}) \quad (4.3)$$

is a subgroup of  $G$ .

Let  $u_-, \tilde{u}_- \in U_{2\lambda}^{nm}(w \cdot \Delta_-)$ ,  $u_+ \in U_0^{pm}(w \cdot \Delta_+)$ ,  $t \in T_{2N(\lambda)}$ . Let us prove that  $\tilde{u}_- u_+ u_- t \in H$ . We have  $\tilde{u}_- u_+ = u_+ (u_+^{-1} \tilde{u}_- u_+)$ . By Proposition 2.2,  $U_{2\lambda}^{nm}(w \cdot \Delta_-) \subset \ker \pi_{2N(\lambda)}$ . By Proposition 3.1,  $u_+ \in \mathfrak{G}^{\min}(\mathcal{O})$  and we have  $u_+^{-1} \tilde{u}_- u_+ \in G_{[0, 2\lambda]} \cap \ker \pi_{2N(\lambda)}$ . Therefore (4.3) implies that we can write  $u_+^{-1} \tilde{u}_- u_+ = u_1^+ u_1^- t_1$ , where  $u_1^+ \in U_0^{pm}(w \cdot \Delta_+)$ ,  $u_1^- \in U_{2\lambda}^{nm}(w \cdot \Delta_-)$  and  $t_1 \in \mathfrak{T}(\mathcal{O})$ . We have

$$\pi_{2N(\lambda)}(u_1^+ u_1^- t_1) = 1 = \pi_{2N(\lambda)}(u_1^+ t t_1) = \pi_{2N(\lambda)}(u_1^+) \pi_{2N(\lambda)}(t_1).$$

As  $\mathfrak{B}(\mathcal{O}/\varpi^{2N(\lambda)}\mathcal{O}) = \mathfrak{T}(\mathcal{O}/\varpi^{2N(\lambda)}\mathcal{O}) \rtimes \mathfrak{U}^{pma, w}(\mathcal{O}/\varpi^{2N(\lambda)}\mathcal{O})$  and as  $\pi_{2N(\lambda)}(\mathfrak{T}(\mathcal{O})) \subset \mathfrak{T}(\mathcal{O}/\varpi^{2N(\lambda)}\mathcal{O})$  and  $\pi_{2N(\lambda)}(\mathfrak{U}^{pma, w}(\mathcal{O})) \subset \mathfrak{U}^{pma, w}(\mathcal{O}/\varpi^{2N(\lambda)}\mathcal{O})$ , we deduce  $\pi_{2N(\lambda)}(u_1^+) = \pi_{2N(\lambda)}(t_1) = 1$  and  $t_1 \in T_{2N(\lambda)}$ . We have

$$\tilde{u}_- u_+ u_- t = u_+ u_1^+ u_1^- t_1 u_- t$$

and as  $T_{2N(\lambda)}$  normalizes  $U_{2\lambda}^{nm}(w \cdot \Delta_-)$ ,  $\tilde{u}_- u_+ u_- t \in H$ , which proves the lemma.  $\square$

**Remark 4.3.** 1. Note that if  $\lambda \in Y^+$  is regular, then  $U_{[-\lambda, \lambda]}^{pm+} U_{[-\lambda, \lambda]}^{nm-} T_{2N(\lambda)+1}$  is not a subgroup of  $G$ . Indeed, take  $\alpha \in \Phi^+$  such that  $|\alpha(\lambda)| = N(\lambda)$ . Then  $x_{-\alpha}(\varpi^{N(\lambda)})x_{\alpha}(\varpi^{N(\lambda)}) = x_{\alpha}(\varpi^{N(\lambda)}(1 + \varpi^{2N(\lambda)})^{-1})x_{-\alpha}(\varpi^{N(\lambda)}(1 + \varpi^{2N(\lambda)})^{-1})\alpha^{\vee}(1 + \varpi^{2N(\lambda)})$  and by Lemma 2.3, this does not belong to  $U_{[-\lambda, \lambda]}^{pm+} U_{[-\lambda, \lambda]}^{nm-} T_{2N(\lambda)+1}$ .

2. For every  $k \in \llbracket 0, 2N(\lambda) \rrbracket$ ,  $U_{[-\lambda, \lambda]}^{pm+} \cdot U_{[-\lambda, \lambda]}^{nm-} \cdot T_k = \mathcal{V}_{\lambda} \cdot T_k$  is a subgroup of  $G$ , since  $T_k$  normalizes  $U_{[-\lambda, \lambda]}^{pm+}$  and  $U_{[-\lambda, \lambda]}^{nm-}$  by Lemma 2.4. Note that for the definition of a topology, we could also have taken the filtration  $(U_{[-n\lambda, n\lambda]}^{pm+} \cdot U_{[-n\lambda, n\lambda]}^{nm-} \cdot T_{k(n)})_{n \in \mathbb{N}^*}$ , for any  $k(n) \in \llbracket 0, 2nN(\lambda) \rrbracket$  such that  $k(n) \xrightarrow{n \rightarrow +\infty} +\infty$ .

3. As  $\mathcal{V}_{\lambda} = \mathcal{V}_{\lambda}^{-1}$  and by Lemma 2.4, we have  $\mathcal{V}_{\lambda} = U_{[-\lambda, \lambda]}^{nm-} U_{[-\lambda, \lambda]}^{pm+} T_{2N(\lambda)}$ .

## 4.2 Filtration $(\mathcal{V}_{n\lambda})_{n \in \mathbb{N}^*}$

Let  $\Omega$  be a filter on  $\mathbb{A}$ . One defines  $\text{cl}^{\Delta}(\Omega)$  as the filter on  $\mathbb{A}$  consisting of the subsets  $\Omega'$  of  $\mathbb{A}$  for which there exists  $(k_{\alpha}) \in \prod_{\alpha \in \Delta} \Lambda_{\alpha} \cup \{+\infty\}$  such that  $\Omega' \supset \bigcap_{\alpha \in \Delta} D(\alpha, k_{\alpha}) \supset \Omega$ , where  $\Lambda_{\alpha} = \Lambda$  if  $\alpha \in \Phi$  and  $\Lambda_{\alpha} = \mathbb{R}$  otherwise. Note that  $\text{cl}^{\Delta}$  is denoted  $\text{cl}$  in [Rou11] and [Rou16]. By definition of  $U_{\Omega}^{pm+}$  and  $U_{\Omega}^{nm-}$ , we have

$$U_{\Omega}^{pm+} = U_{\Omega'}^{pm+} = U_{\text{cl}^{\Delta}(\Omega)}^{pm+} \quad \text{and} \quad U_{\Omega}^{nm-} = U_{\Omega'}^{nm-} = U_{\text{cl}^{\Delta}(\Omega)}^{nm-}, \quad (4.4)$$

for any filter  $\Omega'$  such that  $\Omega \subset \Omega' \subset \text{cl}^{\Delta}(\Omega)$ .

**Lemma 4.4.** Let  $\lambda \in C_f^v$  and  $w \in W^v$ . Then  $\text{cl}^{\Delta}([-w \cdot \lambda, w \cdot \lambda]) \supset (-w \cdot \lambda + w \cdot \overline{C_f^v}) \cap (w \cdot \lambda - w \cdot \overline{C_f^v})$ .

*Proof.* As  $\Delta$  and  $\Phi$  are  $W^v$ -invariant, we have  $w \cdot \text{cl}^{\Delta}(\Omega) = \text{cl}^{\Delta}(w \cdot \Omega)$  for every  $w \in W^v$ . Thus it suffices to determine  $\text{cl}^{\Delta}([- \lambda, \lambda])$ . Let  $(k_{\alpha}) \in \prod_{\alpha \in \Delta} \Lambda_{\alpha} \cup \{+\infty\}$  be such that  $\bigcap_{\alpha \in \Delta} D(\alpha, k_{\alpha}) \supset [- \lambda, \lambda]$ . Let  $\alpha \in \Delta_+$ . Write  $\alpha = \sum_{i \in I} n_i \alpha_i$ , with  $n_i \in \mathbb{N}$  for  $i \in I$ . Then  $k_{\alpha} \geq \alpha(\lambda) = \sum_{i \in I} n_i \alpha_i(\lambda)$ . Let  $\alpha \in \Delta_-$ . We also have  $k_{-\alpha} \geq \sum_{i \in I} n_i \alpha_i(\lambda)$ .

Let  $x \in (-\lambda + C_f^v) \cap (\lambda - C_f^v)$ . Then  $-\alpha_i(\lambda) \leq \alpha_i(x) \leq \alpha_i(\lambda)$  for every  $i \in I$ . Then  $k_{-\alpha} \leq \sum_{i \in I} n_i \alpha_i(x) \leq k_{\alpha}$  and thus  $k_{\alpha} + \alpha(x) \geq 0$  and  $k_{-\alpha} - \alpha(x) \geq 0$ . Consequently,  $x \in \bigcap_{\alpha \in \Delta} D(\alpha, k_{\alpha})$  and thus

$$(-\lambda + C_f^v) \cap (\lambda - C_f^v) \subset \bigcap_{\alpha \in \Delta} D(\alpha, k_{\alpha}),$$

which proves the lemma.  $\square$

The following lemma will be crucial throughout the paper. This is a rewriting of [BHR22, Lemma 3.2 and Lemma 3.5]. Although  $\omega$  is assumed to be discrete in [BHR22], the proofs of these lemma do not use this assumption.

**Lemma 4.5.** 1. Let  $a \in \mathbb{A}$  and  $g \in U^+$ . Then there exists  $b \in a - C_f^v$  such that  $g^{-1} U_b^{pm+} g \subset U_a^{pm+}$ .

2. Let  $y \in \mathcal{I}$ . Then there exists  $a \in \mathbb{A}$  such that  $U_a^{pm+}$  fixes  $y$ .

3. Let  $\lambda \in Y^+$  be regular and  $y \in \mathcal{I}$ . Then for  $n \in \mathbb{N}$  large enough,  $U_{[-n\lambda, n\lambda]}^{pm+}$  fixes  $y$ .

*Proof.* By [BHR22, Lemma 3.2 and 3.5], we have 1) and 2). Let  $\lambda \in Y^+$  be regular. Write  $\lambda = w.\lambda^{++}$ , with  $\lambda^{++} \in C_f^v$  and  $w \in W^v$ . Let  $a \in \mathbb{A}$  be such that  $U_a^{pm+}$  fixes  $y$ . For  $n \in \mathbb{N}^*$ , we have  $\text{cl}^\Delta([-n\lambda, n\lambda]) \supset n \left( (-w.\lambda^{++} + w.\overline{C_f^v}) \cap (w.\lambda^{++} - w.\overline{C_f^v}) \right)$ , which contains  $a$ , for  $n \gg 0$ . So for  $n \gg 0$ , we have  $U_{[-n\lambda, n\lambda]}^{pm+} \subset U_a^{pm+}$ , which proves 3).  $\square$

**Lemma 4.6.** Let  $\lambda, \mu \in Y^+$  be regular. Then  $(\mathcal{V}_{n\lambda})_{n \in \mathbb{N}^*}$  and  $(\mathcal{V}_{n\mu})_{n \in \mathbb{N}^*}$  are equivalent.

*Proof.* Write  $\mu = v.\mu^{++}$ , where  $v \in W^v$  and  $\mu^{++} \in C_f^v$ . For  $m \in \mathbb{N}$ , set

$$\Omega_{m\mu} = (-m\mu + v.\overline{C_f^v}) \cap (m\mu - v.\overline{C_f^v}).$$

By Lemma 4.4,  $\Omega_{m\mu} \subset \text{cl}^\Delta([-m\mu, m\mu])$ . Let  $n \in \mathbb{N}^*$ . As  $\Omega_{m\mu} = m\Omega_\mu$  and  $\Omega_\mu$  contains 0 in its interior, there exists  $m \in \mathbb{Z}_{\geq n}$  such that  $\Omega_{m\mu} \supset [-n\lambda, n\lambda]$ . Moreover, by (4.4),  $U_{[-m\mu, m\mu]}^{pm+} = U_{\Omega_{m\mu}}^{pm+} \subset U_{[-n\lambda, n\lambda]}^{pm+}$ , since  $\Omega' \mapsto U_{\Omega'}^{pm+}$  is decreasing for  $\subset$ . With the same reasoning for  $U^{nm-}$ , we deduce  $\mathcal{V}_{m\mu} \subset \mathcal{V}_{n\lambda}$ . By symmetry of the roles of  $\lambda$  and  $\mu$ , we deduce the lemma.  $\square$

The end of this subsection is devoted to the proof of the fact that for every  $\lambda \in Y^+$  regular,  $(\mathcal{V}_{n\lambda})_{n \in \mathbb{N}^*}$  is conjugation-invariant.

**Lemma 4.7.** Let  $\alpha \in \Phi$  and  $a \in \mathcal{K}$ . Let  $\lambda \in Y^+$  be regular. Then  $(x_\alpha(a).\mathcal{V}_{n\lambda}.x_\alpha(-a))_{n \in \mathbb{N}^*}$  is equivalent to  $(\mathcal{V}_{n\lambda})_{n \in \mathbb{N}^*}$ .

*Proof.* Let  $\epsilon \in \{-, +\}$  be such that  $\alpha \in \Phi_\epsilon$ . Let  $w \in W^v$  be such that  $\epsilon w^{-1}.\alpha$  is simple. By symmetry, we may assume that  $\epsilon = +$ . By Lemma 4.6 we may assume that  $\lambda \in w.C_f^v$ . Let  $n \in \mathbb{N}^*$ . By Lemma 4.2, we have

$$\begin{aligned} \mathcal{V}_{n\lambda} &= U_{[-n\lambda, n\lambda]}^{pm}(w.\Delta_+) U_{[-n\lambda, n\lambda]}^{nm}(w.\Delta_-) T_{2nN(\lambda)}. \\ &= U_{-n\lambda}^{pm}(w.\Delta_+) U_{n\lambda}^{nm}(w.\Delta_-) T_{2nN(\lambda)}. \end{aligned}$$

Write  $\alpha = w.\alpha_i$ , for  $i \in I$ . By Lemma 4.5,

$$x_{w.\alpha_i}(a) U_{-m\lambda}^{pm}(w.\Delta_+) x_{w.\alpha_i}(-a) \subset U_{-n\lambda}^{pm}(w.\Delta_+), \quad (4.5)$$

for  $m \gg 0$ .

By [Rou16, Lemma 3.3],  $U_{[-n\lambda, n\lambda]}^{nm}(w.\Delta_-) = U_{[-n\lambda, n\lambda]}^{nm}(w.(\Delta_- \setminus \{-\alpha_i\})). U_{-w.\alpha_i, [-n\lambda, n\lambda]}$  and  $U_{-w.\alpha_i, [-n\lambda, n\lambda]}$  normalizes  $U_{[-n\lambda, n\lambda]}^{nm}(w.(\Delta_- \setminus \{-\alpha_i\}))$ .

By [Kum02, 1.3.11 Theorem (b4)],  $r_i.(\Delta_- \setminus \{-\alpha_i\}) = \Delta_- \setminus \{-\alpha_i\}$  and thus for  $m \in \mathbb{N}$ , we have

$$\begin{aligned} U_{[-m\lambda, m\lambda]}^{nm}(w.(\Delta_- \setminus \{-\alpha_i\})) &= U_{[-m\lambda, m\lambda]}^{nm}(wr_i(\Delta_- \setminus \{-\alpha_i\})) \\ &= U^{nm}(wr_i(\Delta_- \setminus \{-\alpha_i\})) \cap U_{[-m\lambda, m\lambda]}^{nm}(wr_i.\Delta_-). \end{aligned}$$



Moreover,

$$x_{w.\alpha_i}(a)U^{nm}(wr_i(\Delta_- \setminus \{-\alpha_i\}))x_{w.\alpha_i}(-a) = U^{nm}(wr_i(\Delta_- \setminus \{-\alpha_i\})), \quad (4.6)$$

by [Rou16, Lemma 3.3] (applied to  $wr_i(\Delta_- \setminus \{-\alpha_i\}) \subset wr_i.\Delta_-$ ). By Lemma 4.5, for  $m \gg 0$ ,

$$x_{w.\alpha_i}(a)U_{[-m\lambda, m\lambda]}^{nm}(wr_i.\Delta_-)x_{w.\alpha_i}(-a) \subset U_{[-n\lambda, n\lambda]}^{nm}(wr_i.\Delta_-). \quad (4.7)$$

Combining (4.6) and (4.7), we get

$$x_{w.\alpha_i}(a)U_{[-m\lambda, m\lambda]}^{nm}(w.\Delta_- \setminus \{-\alpha_i\})x_{w.\alpha_i}(-a) \subset U_{[-n\lambda, n\lambda]}^{nm}(w.(\Delta_- \setminus \{-\alpha_i\})), \quad (4.8)$$

for  $m \gg 0$ .

For  $b \in \mathcal{K}$  such that  $1 + ab \neq 0$ , we have

$$x_{w.\alpha_i}(a)x_{-w.\alpha_i}(b)x_{w.\alpha_i}(-a) = x_{-w.\alpha_i}(b(1+ab)^{-1})\alpha^\vee(1+ab)x_{w.\alpha_i}(-a^2b(1+ab)^{-1}).$$

Therefore if  $m \gg 0$ ,  $x_{w.\alpha_i}(a)U_{-w.\alpha_i, [-m\lambda, m\lambda]}x_{w.\alpha_i}(-a) \subset \mathcal{V}_{n\lambda}$ . Combined with (4.8) we get

$$x_{w.\alpha_i}(a)U_{m\lambda}^{nm}(w.\Delta_-)x_{w.\alpha_i}(-a) \subset \mathcal{V}_{n\lambda}, \quad (4.9)$$

for  $m \gg 0$ , since  $\mathcal{V}_{n\lambda}$  is a group.

Let  $m \in \mathbb{N}^*$  and  $t \in T_{2m}$ . Then:

$$x_{w.\alpha_i}(a)tx_{w.\alpha_i}(-a) = tx_{w.\alpha_i}(a(w.\alpha_i(t^{-1}) - 1))$$

Therefore if  $m \geq nN(\lambda)$  and  $\omega(a(w.\alpha_i(t^{-1}) - 1)) \geq \omega(a) + 2mN(\lambda)$  is greater than  $|w.\alpha_i(n\lambda)|$ , then  $x_{w.\alpha_i}(a(w.\alpha_i(t^{-1}) - 1)) \in \mathcal{V}_{n\lambda}$  and  $x_{w.\alpha_i}(a)tx_{w.\alpha_i}(-a) \in \mathcal{V}_{n\lambda}$ . Combined with (4.9) and (4.5), we get  $x_\alpha(a).\mathcal{V}_{m\lambda}.x_\alpha(-a) \subset \mathcal{V}_{n\lambda}$ , for  $m \gg 0$ . Applying this to  $(x_\alpha(-a).\mathcal{V}_{k\lambda}.x_\alpha(a))_{k \in \mathbb{N}^*}$ , we get the other inclusion needed to prove that  $(\mathcal{V}_{n\lambda})$  and  $(x_\alpha(a).\mathcal{V}_{n\lambda}.x_\alpha(-a))$  are equivalent.  $\square$

**Theorem 4.8.** *Let  $\lambda \in Y^+$  be regular. For  $n \in \mathbb{N}^*$ , set  $\mathcal{V}_{n\lambda} = U_{[-n\lambda, n\lambda]}^{pm+}.U_{[-n\lambda, n\lambda]}^{nm-}.T_{2N(n\lambda)}$ . Then  $(\mathcal{V}_{n\lambda})$  is conjugation-invariant. Therefore, the associated topology  $\mathcal{T}((\mathcal{V}_{n\lambda}))$  equips  $G$  with the structure of a Hausdorff topological group.*

*Proof.* We need to prove that for every  $g \in G$ ,  $g(\mathcal{V}_{n\lambda})g^{-1}$  is equivalent to  $(\mathcal{V}_{n\lambda})$ . Using Lemma 4.6, we may assume  $\lambda \in C_f^v$ . By [Rou06, Proposition 1.5],  $G$  is generated by  $T$  and the  $x_\alpha(a)$ , for  $\alpha \in \Phi$  and  $a \in \mathcal{K}$ . By Lemma 4.7, it remains only to prove that if  $t \in T$ , then  $(t\mathcal{V}_{n\lambda}t^{-1})_{n \in \mathbb{N}^*}$  is equivalent to  $(\mathcal{V}_{n\lambda})$ . Let  $t \in T$  and  $m \in \mathbb{N}^*$ . Then by Lemma 2.4,

$$\begin{aligned} t\mathcal{V}_{m\lambda}t^{-1} &= tU_{[-m\lambda, m\lambda]}^{pm+}t^{-1}.tU_{[-m\lambda, m\lambda]}^{nm-}t^{-1}.T_{2mN(\lambda)} \\ &= U_{[t.(-m\lambda), t.m\lambda]}^{pm+}U_{[t.(-m\lambda), t.m\lambda]}^{nm-}T_{2mN(\lambda)} \end{aligned}$$

Set  $\Omega = (-\lambda + \overline{C_f^v}) \cap (\lambda - \overline{C_f^v})$ . Then by Lemma 4.4,  $\text{cl}^\Delta([-m\lambda, m\lambda]) \supset m\Omega$ . Moreover,  $\text{cl}^\Delta([t.(-m\lambda), t.m\lambda]) = t.\text{cl}^\Delta([-m\lambda, m\lambda]) \supset t.m\Omega$ . Let  $n \in \mathbb{N}^*$ . Then as  $\Omega$  contains 0 in

its interior,  $t.m\Omega \supset n\Omega$  for  $m \gg 0$ . Therefore (by (4.4))  $U_{[t,(-m\lambda),t.m\lambda]}^{pm+} \subset U_{[-n\lambda,n\lambda]}^{pm+}$  and  $U_{[t,(-m\lambda),t.m\lambda]}^{nm-} \subset U_{[-n\lambda,n\lambda]}^{nm-}$  for  $m \gg 0$ . Consequently,  $t\mathcal{V}_{n\lambda}t^{-1} \subset \mathcal{V}_{n\lambda}$  for  $m \gg 0$ , which proves that  $(\mathcal{V}_{n\lambda})$  is conjugation-invariant.

It remains to prove that  $\mathcal{S}((\mathcal{V}_{n\lambda}))$  is Hausdorff. For that it suffices to prove that  $\bigcap_{n \in \mathbb{N}^*} \mathcal{V}_{n\lambda} = \{1\}$ . Let  $g \in \bigcap_{n \in \mathbb{N}^*} \mathcal{V}_{n\lambda}$ . Let  $n \in \mathbb{N}^*$ . Then as  $[-n\lambda, n\lambda]$  has a good fixator ([Rou16, 5.7 1]) and (2.8))  $g \in G_{[-n\lambda, n\lambda]} = U_{[-n\lambda, n\lambda]}^{pm+} \cdot U_{[-n\lambda, n\lambda]}^{nm-} \cdot \mathfrak{I}(\mathcal{O})$ , so we can write  $g = u_n^+ u_n^- t_n$ , with  $(u_n^+, u_n^-, t_n) \in U_{[-n\lambda, n\lambda]}^{pm+} \times U_{[-n\lambda, n\lambda]}^{nm-} \times \mathfrak{I}(\mathcal{O})$ . By Lemma 2.3,  $u_+ := u_n^+$  does not depend on  $n$  and thus  $u^+ \in \bigcap_{n \in \mathbb{N}^*} U_{[-n\lambda, n\lambda]}^{pm+} = U_{\mathbb{A}}^{pm+} = \{1\}$ . Similarly,  $u^- := u_n^- = 1$ . Therefore  $t \in T \cap \bigcap_{n \in \mathbb{N}^*} \ker \pi_n$ . Let  $(\chi_i)_{i \in \llbracket 1, m \rrbracket}$  be a  $\mathbb{Z}$ -basis of  $Y$ . Write  $t = \prod_{i=1}^m \chi_i(a_i)$ , with  $a_i \in \mathcal{O}^*$ , for  $i \in \llbracket 1, m \rrbracket$ . Let  $n \in \mathbb{N}$ . Then  $\pi_n(t) = \prod_{i=1}^m \chi_i(\pi_n(a_i)) = 1$  and thus  $a_i \in \bigcap_{n \in \mathbb{N}^*} \varpi^n \mathcal{O} = \{0\}$ . Consequently,  $t = 1$  and  $g = 1$ . Therefore  $\bigcap_{n \in \mathbb{N}^*} \mathcal{V}_{n\lambda} = \{1\}$  and  $\mathcal{S}((\mathcal{V}_{n\lambda}))$  is Hausdorff.  $\square$

We denote by  $\mathcal{S}$  the topology  $\mathcal{S}((\mathcal{V}_{n\lambda}))$ , for any  $\lambda \in Y^+$  regular.

**Corollary 4.9.** *Let  $\lambda \in Y^+$  be regular. The filtrations  $(\ker \pi_n)_n$  and  $(\mathcal{V}_{n\lambda})_{n \in \mathbb{N}^*}$  are equivalent if and only if  $W^v$  is finite.*

*Proof.* If  $W^v$  is infinite, this follows from Lemma 3.3 and Theorem 4.8. Assume now that  $W^v$  is finite. Then by (4.2),  $\mathcal{V}_{n\lambda} \subset \ker \pi_{nN(\lambda)}$  for every  $n \in \mathbb{N}^*$ .

Let  $n \in \mathbb{N}^*$  and  $g \in \ker \pi_n$ . By Corollary 3.6,  $g \in U_{-C_0^+}^{pm+} \cdot \mathfrak{I}(\mathcal{O}) \cdot U_{C_0^+}^{nm-}$ . As  $W^v$  is finite,  $\Phi = \Delta$  (by [Kac94, Theorem 5.6]),  $U_{-C_0^+}^{pm+} = U_{-C_0^+}^+ = \prod_{\alpha \in \Phi_+} x_\alpha(\mathfrak{m})$  and  $U_{C_0^+}^{nm-} = U_{C_0^+}^- = \prod_{\alpha \in \Phi_-} x_\alpha(\mathfrak{m})$ , for any (fixed) orders on  $\Phi_-$  and  $\Phi_+$ , where  $\mathfrak{m}$  is the maximal ideal of  $\mathcal{O}$ . Write  $g = u_+ t u_-$ , with  $g \in U_{C_0^+}^-$ ,  $t \in \mathfrak{I}(\mathcal{O})$  and  $u_- \in U_{C_0^+}^-$ . Then  $\pi_n(g) = \pi_n(u_+ t) \pi_n(u_-) = 1$ . Moreover,  $\pi_n(u_+ t) \in \mathfrak{B}(\mathcal{O}/\varpi^n \mathcal{O})$  and  $\pi_n(u_-) \in \mathfrak{B}^-(\mathcal{O}/\varpi^n \mathcal{O})$ , where  $\mathfrak{B} = \mathcal{U}^+ \rtimes \mathfrak{I}$  and  $\mathfrak{B}^- = \mathcal{U}^- \rtimes \mathfrak{I}$  are two opposite Borel subgroups. Therefore  $\pi_n(u_+ t) \in \mathfrak{B}(\mathcal{O}/\varpi^n \mathcal{O}) \cap \mathfrak{B}^-(\mathcal{O}/\varpi^n \mathcal{O}) = \mathfrak{I}(\mathcal{O}/\varpi^n \mathcal{O})$ , by [Bor91, Theorem 14.1]. Therefore  $\pi_n(u_+) = \pi_n(u_-) = \pi_n(t) = 1$ . Write  $u_+ = \prod_{\alpha \in \Phi_+} x_\alpha(u_\alpha)$ , with  $u_\alpha \in \mathcal{O}$  for  $\alpha \in \Phi_+$ . Then  $\pi_n(u_+) = \prod_{\alpha \in \Phi_+} x_\alpha(\pi_n(u_\alpha)) = 1$ , by Proposition 2.2, and by [Mar18, Theorem 8.51], this implies  $\pi_n(u_\alpha) = 1$ , for every  $\alpha \in \Phi_+$ . Let  $N'(\lambda) = \max_{\alpha \in \Phi} |\alpha(\lambda)|$ . Then we have  $u_+ \in U^+ \cap G_{[-n\lambda/N'(\lambda), n\lambda/N'(\lambda)]}$ . Using a similar reasoning for  $U^-$  and (4.1), we deduce  $\ker \pi_n \subset \mathcal{V}_{n\lambda/N'(\lambda)}$ . Therefore  $(\mathcal{V}_{n\lambda})$  and  $(\ker \pi_n)$  are equivalent.  $\square$

**Remark 4.10.** 1. If  $n \in \mathbb{N}^*$ , then  $\ker \pi_n$  is open for  $\mathcal{S}$ , by (4.2).

2. The Iwahori subgroup  $G_{C_0^+} = K_I$  is open. Indeed, if  $\lambda \in Y \cap C_f^v$ , then  $\mathcal{V}_\lambda \subset K_I$  by (4.1). In particular,  $\mathfrak{G}^{\min}(\mathcal{O}) \supset K_I$  is open.

3. Assume  $\mathfrak{G}$  is reductive, i.e  $W^v$  is finite. Then by Corollary 4.9,  $\mathcal{S}$  is the usual topology on  $G$ .

## 4.3 Topology of the fixators

### 4.3.1 Definition of the topology

Recall that if  $F$  is a subset of  $\mathcal{I}$ , we denote by  $G_F$  its fixator in  $G$ . In this subsection we study the topology  $\mathcal{T}_{\text{Fix}}$  which is defined as follows. A subset  $V$  of  $G$  is open if for every  $v$  in  $V$ , there exists a finite subset  $F$  of  $\mathcal{I}$  such that  $vG_F \subset V$ . Note that  $G_0 = \mathfrak{G}^{\min}(\mathcal{O})$  is open for this topology.

We begin by constructing increasing sequences of finite sets of vertices  $(F_n) = (F_n(\lambda))_{n \in \mathbb{N}^*}$  such that  $\mathcal{T}_{\text{Fix}}$  is the topology associated with  $(G_{F_n})_{n \in \mathbb{N}}$ .

We fix  $\lambda \in Y^+$  regular. We set  $F_0 = \emptyset$ . For  $n \in \mathbb{N}^*$ , we set

$$F_n = F_n(\lambda) = \{n\lambda, -n\lambda\} \cup \{x_{\alpha_i}(\varpi^{-n}).0 \mid i \in I\} \cup F_{n-1}. \quad (4.10)$$

Let  $n \in \mathbb{N}^*$ . By [Rou16, 5.7 1)],  $[-n\lambda, n\lambda]$  has a good fixator and by Proposition 2.5, we have  $G_{\{n\lambda, -n\lambda\}} = G_{[-n\lambda, n\lambda]}$ . Therefore

$$G_{F_n} \subset G_{[-n\lambda, n\lambda]} = U_{[-n\lambda, n\lambda]}^{pm+} \cdot U_{[-n\lambda, n\lambda]}^{nm-} \cdot \mathfrak{T}(\mathcal{O}). \quad (4.11)$$

We chose  $F_n$  as above for the following reasons. We want that when  $x \in \mathcal{I}$  and  $n \gg 0$ , an element of  $G_{F_n}$  fixes  $x$ . By Lemma 4.5, if  $u \in U^+$  (resp.  $U^-$ ), and if  $u$  fixes  $-n\lambda$  (resp.  $n\lambda$ ), for some  $n$  large, then  $u$  fixes  $x$ . However, if  $t \in T$ , as  $\mathfrak{T}(\mathcal{O})$  fixes  $\mathbb{A}$ , we need to require that  $t$  fixes elements outside of  $\mathbb{A}$ , and the choice of  $x_{\alpha_i}(\varpi^n).0$  is justified by the lemma below.

**Lemma 4.11.** *Let  $i \in I$ ,  $n \in \mathbb{N}$  and  $t \in T$ . Then  $t$  fixes  $x_{\alpha_i}(\varpi^{-n}).0$  if and only if  $\omega(\alpha_i(t) - 1) \geq n$ .*

*Proof.* We have  $t.x_{\alpha_i}(\varpi^{-n}).0 = x_{\alpha_i}(\varpi^{-n}).0$  if and only if  $x_{\alpha_i}(-\varpi^{-n})x_{\alpha_i}(\alpha_i(t)\varpi^{-n}).t.0 = 0$ . We have  $\rho_{+\infty}(x_{\alpha_i}(-\varpi^{-n})x_{\alpha_i}(\alpha_i(t)\varpi^{-n}).t.0) = t.0$  and thus if  $t.x_{\alpha_i}(\varpi^{-n}).0 = x_{\alpha_i}(\varpi^{-n}).0$ , we have  $t.0 = 0$ . Thus  $t.x_{\alpha_i}(\varpi^{-n}).0 = x_{\alpha_i}(\varpi^{-n}).0$  if and only if  $x_{\alpha_i}((\alpha_i(t) - 1)\varpi^{-n}).0 = 0$  if and only if  $\omega(\alpha_i(t) - 1) \geq n$ .  $\square$

For  $n \in \mathbb{N}$ , we set

$$T_{n, \Phi} = \{t \in T \mid \omega(\alpha_i(t) - 1) \geq n, \forall i \in I\} \quad (4.12)$$

**Lemma 4.12.** *Let  $y \in \mathcal{I}$ . Then there exists  $M \in \mathbb{N}$  such that  $T_{M, \Phi}$  fixes  $y$ .*

*Proof.* By the Iwasawa decomposition ((MA III)),  $y \in U^+.z$ , where  $z = \rho_{+\infty}(y)$ . Write  $y = x_{\beta_1}(a_1) \dots x_{\beta_k}(a_k).z$ , with  $k \in \mathbb{N}$  and  $\beta_1, \dots, \beta_k \in \Phi_+$ . Let  $t \in T$ . We have  $t.y = y$  if and only if

$$\begin{aligned} z &= x_{\beta_k}(-a_k) \dots x_{\beta_1}(-a_1) t x_{\beta_1}(a_1) \dots x_{\beta_k}(a_k).z \\ &= x_{\beta_k}(-a_k) \dots x_{\beta_2}(-a_2) t x_{\beta_1}((1 - \beta_1(t^{-1}))a_1) x_{\beta_2}(a_2) \dots x_{\beta_k}(a_k).z \end{aligned}$$

Let  $M \in \mathbb{N}^*$ . Assume that  $\alpha_i(t) - 1 \in \varpi^M \mathcal{O}$  for every  $i \in I$ .

Write  $\beta_1 = \sum_{i \in I} m_i \alpha_i$ , with  $m_i \in \mathbb{N}$  for every  $i \in I$ . Then  $\beta_1(t) = \prod_{i \in I} \alpha_i^{m_i}(t)$ . Therefore  $\beta_1(t) \in 1 + \varpi^M \mathcal{O}$ . For  $M \gg 0$ ,  $x_{\beta_1}((1 - \beta_1(t^{-1}))a_1)$  fixes  $x_{\beta_2}(a_2) \dots x_{\beta_k}(a_k).z$ , by Lemma 4.5. By induction on  $k$  we deduce that  $t$  fixes  $z$  for  $M \gg 0$ .  $\square$

**Lemma 4.13.** 1. *Let  $F$  be a finite subset of  $\mathcal{I}$ . Then there exists  $n \in \mathbb{N}^*$  such that  $G_{F_n}$  fixes  $F$ .*

2. *Let  $\lambda, \mu \in Y^+$  be regular. Then the filtrations  $(G_{F_n(\lambda)})_{n \in \mathbb{N}^*}$  and  $(G_{F_n(\mu)})_{n \in \mathbb{N}^*}$  are equivalent.*

3. *The topology  $\mathcal{T}_{\text{Fix}}$  is the topology associated with  $\mathcal{T}((G_{F_n})_{n \in \mathbb{N}})$ .*

*Proof.* 1) It suffices to prove that if  $y \in \mathcal{I}$ , then  $G_{F_n}$  fixes  $y$  for  $n \gg 0$ . Let  $n \in \mathbb{N}^*$ . Then  $F_n \supset \{-n\lambda, n\lambda\}$  and by Proposition 2.5, we have  $G_{F_n} \subset G_{[-n\lambda, n\lambda]}$ .

By Lemma 4.5, there exists  $n_1 \in \mathbb{N}$  such that  $U_{[-n\lambda, n\lambda]}^{pm+}$  and  $U_{[-n\lambda, n\lambda]}^{nm-}$  fix  $y$ , for  $n \geq n_1$ . Let  $n \geq n_1$ . Let  $n_2 = n_2(n) \geq n$  be such that for every  $i \in I$ ,  $\langle U_{[-n_2\lambda, n_2\lambda]}^{pm+}, U_{[-n_2\lambda, n_2\lambda]}^{nm-} \rangle$  fixes  $x_{\alpha_i}(\varpi^{-n}).0$  for every  $i \in I$ . Let  $g \in G_{F_{n_2}}$ . Using (4.11), we write  $g = u_+ u_- t$ , with  $(u_+, u_-, t) \in U_{[-n_2\lambda, n_2\lambda]}^{pm+} \times U_{[-n_2\lambda, n_2\lambda]}^{nm-} \times \mathfrak{T}(\mathcal{O})$ . Then  $g$  fixes  $y$  if and only if  $t$  fixes  $y$ . Moreover,  $u_+ u_- t$  fixes  $F_n$  and thus  $t$  fixes  $F_n$ . By Lemma 4.11, we deduce  $\omega((\alpha_i(t) - 1) \geq n$  for every  $i \in I$ . By Lemma 4.12 we deduce that  $t$  fixes  $y$ , for  $n \gg 0$ . Thus  $G_{F_{n_2(n)}}$  fixes  $y$  for  $n \gg 0$ .

2) Follows from 1) by applying to  $F = F_m(\mu)$  and  $F_n = F_n(\lambda)$ , for  $m, n \in \mathbb{N}^*$  and by symmetry of the roles of  $\lambda$  and  $\mu$ .

3) As  $F_n$  is finite for every  $n \in \mathbb{N}^*$ ,  $\mathcal{T}((G_{F_n}))$  is coarser than  $\mathcal{T}_{\text{Fix}}$ . But by 1),  $\mathcal{T}_{\text{Fix}}$  is coarser than  $(\mathcal{T}_{G_{F_n}})$ .  $\square$

**Proposition 4.14.** *The topology  $\mathcal{T}_{\text{Fix}}$  is the coarsest topology of topological group on  $G$  such that  $\mathfrak{G}^{\min}(\mathcal{O})$  is open.*

*Proof.* For  $n \in \mathbb{N}^*$ ,  $G_{F_n} \subset \mathfrak{G}^{\min}(\mathcal{O})$  and thus  $\mathfrak{G}^{\min}(\mathcal{O})$  is open for  $\mathcal{T}_{\text{Fix}}$ .

Let now  $\mathcal{T}'$  be a topology of topological group on  $G$  such that  $\mathfrak{G}^{\min}(\mathcal{O})$  is open. Let  $n \in \mathbb{N}^*$ . Then for every element  $a$  of  $F_n$ , there exists  $g_a \in G$  such that  $g_a.0 = a$ . Then  $G_{F_n} = \bigcap_{a \in F_n} g_a.\mathfrak{G}^{\min}(\mathcal{O}).g_a^{-1}$  is open in  $G$ . Proposition follows.  $\square$

### 4.3.2 Relation between $\mathcal{T}_{\text{Fix}}$ and $\mathcal{T}$

In this subsection, we compare  $\mathcal{T}_{\text{Fix}}$  and  $\mathcal{T}$ . We prove that  $\mathcal{T}$  is finer than  $\mathcal{T}_{\text{Fix}}$ . When  $\mathcal{K}$  is Henselian, we prove that  $\mathcal{T} = \mathcal{T}_{\text{Fix}}$  if and only if the fixator of  $\mathcal{I}$  in  $G$  is  $\{1\}$  (see Proposition 4.21).

Let  $\lambda \in Y^+$  be regular. For  $n \in \mathbb{N}$ , we define  $F_n = F_n(\lambda)$  as in (4.10).

Let  $\mathcal{Z} = \bigcap_{n \in \mathbb{N}} T_{n, \Phi}$  (where the  $T_{n, \Phi}$  are defined in (4.12)) and  $\mathcal{Z}_{\mathcal{O}} = \mathcal{Z} \cap \mathfrak{T}(\mathcal{O})$ . Then  $\mathcal{Z} = \{t \in T \mid \alpha_i(t) = 1, \forall i \in I\}$  is the center of  $G$  by [Rém02, 8.4.3 Lemme].

**Lemma 4.15.** *The fixator  $G_{\mathcal{I}}$  of  $\mathcal{I}$  in  $G$  is  $\mathcal{Z}_{\mathcal{O}}$  and  $\bigcap_{n \in \mathbb{N}^*} G_{F_n} = \mathcal{Z}_{\mathcal{O}}$ .*

*Proof.* We have  $G_{\mathcal{I}} \subset G_{\mathbb{A}} = \mathfrak{T}(\mathcal{O})$ , by [Rou16, 5.7 5)]. By Lemma 4.11,  $G_{\mathcal{I}} \subset T_{n,\Phi} \cap \mathfrak{T}(\mathcal{O})$  for every  $n \in \mathbb{N}$  and thus  $G_{\mathcal{I}} \subset \mathcal{Z}_{\mathcal{O}}$ . Let  $z \in \mathcal{Z}_{\mathcal{O}}$  and  $x \in \mathcal{I}$ . Write  $x = g.a$ , with  $a \in \mathbb{A}$ . Then  $z.x = g.z.a = g.a = x$  and  $z \in G_{\mathcal{I}}$ .

Now let  $g \in \bigcap_{n \in \mathbb{N}^*} G_{F_n}$ . Then by Lemma 4.13,  $g$  fixes  $\mathcal{I}$ , which proves the lemma.  $\square$

**Lemma 4.16.** *There exists an increasing map  $M : \mathbb{N} \rightarrow \mathbb{N}$  whose limit is  $+\infty$  and such that for every  $m \in \mathbb{N}$ ,*

$$G_{F_n} \subset \mathcal{V}_{m\lambda} \cdot (T_{M(n),\Phi} \cap \mathfrak{T}(\mathcal{O})),$$

for  $m, n \in \mathbb{N}^*$  such that  $M(n) \geq m$  and  $n \geq m$ .

*Proof.* Let  $n \in \mathbb{N}^*$ . Let  $M(n) \in \mathbb{N}$  be maximum such that  $U_{[-n\lambda, n\lambda]}^{nm-}$ ,  $U_{[-n\lambda, n\lambda]}^{pm+}$ , fix  $F_{M(n)}$ . By Lemma 4.5,  $M(n) \rightarrow +\infty$ . Let  $g \in G_{F_n}$ . Using (4.11) we write  $g = u_+ u_- t$ , with  $u_+ \in U_{[-n\lambda, n\lambda]}^{pm+}$ ,  $u_- \in U_{[-n\lambda, n\lambda]}^{nm-}$  and  $t \in \mathfrak{T}(\mathcal{O})$ . Let  $m' = M(n)$ . Then  $m' \leq n$  and  $u_+, u_-$  fix  $F_{m'}$ . As  $g$  fixes  $F_{m'}$  we deduce  $t$  fixes  $F_{m'}$ . By Lemma 4.12,  $t \in T_{m',\Phi} \cap \mathfrak{T}(\mathcal{O})$ . Therefore  $g \in \mathcal{V}_{m\lambda} \cdot (T_{M(n),\Phi} \cap \mathfrak{T}(\mathcal{O}))$ , which proves the lemma.  $\square$

**Lemma 4.17.** *Let  $\mathbb{A}^* = X \otimes \mathbb{R}$ ,  $Q' = (\bigoplus_{i \in I} \mathbb{Q}\alpha_i) \cap X \subset \mathbb{A}^*$  and  $d$  be the dimension of  $Q'$  as a  $\mathbb{Q}$ -vector space. Then there exists a  $\mathbb{Z}$ -basis  $(\chi_1, \dots, \chi_\ell)$  of  $X$  such that  $(\chi_1, \dots, \chi_d)$  is a  $\mathbb{Z}$ -basis of  $Q'$ .*

*Proof.* Let  $x \in X$  and  $n \in \mathbb{Z} \setminus \{0\}$ . Assume that  $nx \in Q'$ . Then  $x \in Q'$ . Therefore  $X/Q'$  is torsion-free. Let  $(e_{d+1}, \dots, e_\ell) \in (X/Q')^{\ell-d}$  be a  $\mathbb{Z}$ -basis of  $X/Q'$ . For  $j \in \llbracket d+1, \ell \rrbracket$ , take  $\chi_j \in X$  whose reduction modulo  $Q'$  is  $e_j$ . Choose a  $\mathbb{Z}$ -basis  $(\chi_1, \dots, \chi_d)$  of  $X'$ . Then  $(\chi_1, \dots, \chi_\ell)$  satisfies the condition of the lemma.  $\square$

**Lemma 4.18.** *Assume  $\mathcal{K}$  to be Henselian. Let  $a \in \mathcal{O}$  and  $m \in \mathbb{N}^*$ . Assume  $\omega(a^m - 1) > 0$ . Then we can write  $a = b + c$ , with  $b \in \mathcal{O}$  such that  $b^m = 1$  and  $\omega(c) > 0$ .*

*Proof.* Let  $\mathbb{k} = \mathcal{O}/\mathfrak{m}$  be the residual field and  $\pi_{\mathbb{k}} : \mathcal{O} \rightarrow \mathcal{O}/\mathfrak{m}$  be the natural projection. Let  $p$  be the characteristic of  $\mathbb{k}$ . If  $p = 0$ , we set  $m' = m$  and  $k = 0$ . If  $p > 0$ , we write  $m = p^k m'$ , with  $k \in \mathbb{N}$  and  $m' \in \mathbb{N}$  prime to  $p$ . We have  $\pi_{\mathbb{k}}(a^m) = \pi_{\mathbb{k}}(a)^m = \pi_{\mathbb{k}}(1)$ . We have  $(\pi_{\mathbb{k}}(a)^{m'} - 1)^{p^k} = 0$  and thus  $\pi_{\mathbb{k}}(a)^{m'} = \pi_{\mathbb{k}}(1)$ . Let  $Z$  be an indeterminate. We have  $\overline{Z^{m'} - 1} = \overline{(Z - a)Q_{\mathbb{k}}}$ , where the bar denotes the reduction modulo  $\mathfrak{m}[Z]$  and  $Q_{\mathbb{k}} \in \mathbb{k}[Z]$  is prime to  $\overline{Z - a}$ . As  $\mathcal{O}$  is Henselian, we can write  $Z^{m'} - 1 = (Z - b)Q$ , where  $b \in \mathcal{O}$  is such that  $\pi_{\mathbb{k}}(b) = \pi_{\mathbb{k}}(a)$  and  $Q \in \mathcal{O}[Z]$  is such that  $\overline{Q} = Q_{\mathbb{k}}$ . Then  $\pi_{\mathbb{k}}(b - a) = 0$  and we get the lemma, with  $c = b - a$ .  $\square$

The following lemma was suggested to me by Guy Rousseau.

**Lemma 4.19.** *Assume  $\mathcal{K}$  to be Henselian. Let  $m \in \mathbb{N}^*$ . Then there exists  $K \in \mathbb{N}^*, K' \in \mathbb{N}$  such that for every  $n \in \mathbb{N}^*$  and  $a \in \mathcal{O}$  such that  $\omega(a^m - 1) \geq n$ , we can write  $a = b + c$ , with  $b, c \in \mathcal{O}$  such that  $b^m = 1$  and  $\omega(c) \geq n/K - K'$ .*

*Proof.* We first assume that  $\mathcal{K}$  has characteristic  $p > 0$ . Let  $n \in \mathbb{N}^*$  and  $a \in \mathcal{O}$  be such that  $\omega(a^m - 1) \geq n$ . Write  $m = m'p^k$ , with  $k \in \mathbb{N}$  and  $m' \in \mathbb{N}$  prime to  $p$ .

We have  $a^m - 1 = (a^{m'} - 1)^{p^k}$  and thus  $\omega(a^{m'} - 1) = \omega(a^m - 1)/p^k \geq n/p^k > 0$ . By Lemma 4.18, we can write  $a = b + c$ , with  $b, c \in \mathcal{O}$ ,  $b^{m'} = 1$  and  $\omega(c) > 0$ . We have  $a^{m'} - 1 = \overline{m'}b^{m'-1}c + \sum_{i=2}^{m'} \binom{m'}{i} c^i b^{m-i}$ , where  $\overline{x}$  is the image of  $x$  in  $\mathcal{K}$ , if  $x \in \mathbb{Z}$ . As  $m'$  is prime to  $p$ ,  $\overline{m'}$  is a root of 1 and thus we have  $\omega(\overline{m'}) = 0$ . As  $b^{m'} = 1$ ,  $\omega(b) = 0$ . Therefore  $\omega(c) = \omega(\overline{m'}b^{m'-1}c) < \omega(\binom{m'}{i}b^{m'-i}c^i)$  for  $i \in \llbracket 2, m' \rrbracket$ . Consequently  $\omega(a^{m'} - 1) = \omega(c)$  and  $\omega(a^m - 1) = p^k \omega(a^{m'} - 1) = p^k \omega(c) \geq n$ . This proves the lemma in this case, with  $K' = 0$  and  $K = p^k$ .

We now assume that  $\mathcal{K}$  has characteristic 0. Then by [Gre66, Theorem 1] and [Rou77, Annexe A4] (for the case where  $\omega(\mathcal{K}^*)$  is not discrete) applied with  $F = \{Z^m - 1\}$  (where  $Z$  is an indeterminate), there exist  $K \in \mathbb{N}^*$ ,  $K' \in \mathbb{N}$  such that for every  $n \in \mathbb{N}^*$ , for every  $a \in \mathcal{O}$  such that  $\omega(a^m - 1) \geq n$ , we can write  $a = b + c$ , with  $b, c \in \mathcal{O}$ ,  $b^m = 1$  and  $\omega(c) \geq n/K - K'$ , which proves the lemma in this case.  $\square$

**Lemma 4.20.** *Assume  $\mathcal{K}$  to be Henselian. There exist  $K_1 \in \mathbb{R}_+^*$ ,  $L \in \mathbb{N}$  such that for every  $n \in \mathbb{Z}_{\geq L}$ ,  $T_{n,\Phi} \cap \mathfrak{T}(\mathcal{O}) \subset \mathcal{Z}_{\mathcal{O}}.T_{n/K_1}$ .*

*Proof.* We keep the same notation as in Lemma 4.17. Let  $(\chi_1^\vee, \dots, \chi_\ell^\vee) \in Y^\ell$  be the dual basis of  $(\chi_1, \dots, \chi_\ell)$ . For  $i \in I$ , we write  $\alpha_i = \sum_{j=1}^\ell n_{j,i} \chi_j$ , with  $n_{j,i} \in \mathbb{Z}$  for all  $i, j$ . We have  $n_{j,i} = 0$  for  $j \in \llbracket d+1, \ell \rrbracket$ . Set  $\tilde{t} = \prod_{j=d+1}^\ell \chi_j^\vee(\chi_j(t)) \in \mathfrak{T}(\mathcal{O})$ . Then  $\alpha_i(\tilde{t}) = 1$  for every  $i \in I$  and thus  $\tilde{t} \in \mathcal{Z}_{\mathcal{O}}$ .

For  $j \in \llbracket 1, d \rrbracket$ , write  $\chi_j = \sum_{i \in I} m_{i,j} \alpha_i$ , with  $m_{i,j} \in \mathbb{Q}$  for every  $i \in I$ . Take  $m \in \mathbb{N}^*$  such that  $mm_{i,j} \in \mathbb{Z}$  for every  $(i, j) \in I \times \llbracket 1, d \rrbracket$ . Let  $j \in \llbracket 1, d \rrbracket$ . We have

$$\chi_j(t)^m = \prod_{i=1}^d \alpha_i(t)^{mm_{i,j}} \in 1 + \varpi^n \mathcal{O}.$$

Using Lemma 4.19 we can write  $\chi_j(t) = b_j + c_j$ , with  $b_j, c_j \in \mathcal{O}$  such that  $b_j^m = 1$  and  $\omega(c_j) \geq n/K - K'$ , with the same notation as in Lemma 4.19. Set  $c'_j = c_j b_j^{-1} \in \mathcal{O}$ . As  $b_j$  is a root of 1, we have  $\omega(b_j) = 0$  and thus  $\omega(c_j) = \omega(c'_j) \geq n/K - K'$ . We have  $b_j + c_j = b_j(1 + c'_j)$ .

Set  $t' = \prod_{j=1}^d \chi_j^\vee(b_j)$  and  $t'' = \prod_{j=1}^d \chi_j^\vee(1 + c'_j)$ . Then  $\chi_j(t') = b_j$  and  $\chi_j(t'') = 1 + c'_j$ , for  $j \in \llbracket 1, d \rrbracket$ . For  $i \in I$ , we have  $\alpha_i(t) = \alpha_i(t') \alpha_i(t'') \alpha_i(\tilde{t}) = \alpha_i(t') \alpha_i(t'')$  and  $\alpha_i(t'') \in 1 + \varpi^{n/K - K'} \mathcal{O}$  (when  $n/K - K' \notin \mathbb{N}$ ,  $\varpi^{n/K - K'} \mathcal{O}$  is just a notation for  $\mathcal{K}_{\omega \geq n/K - K'}$ ). As  $\alpha_i(t) \in 1 + \varpi^{n/K - K'} \mathcal{O}$  we deduce  $\alpha_i(t') \in 1 + \varpi^{n/K - K'} \mathcal{O}$  (replacing  $K$  by  $K+1$  if  $K \leq 1$ ).

Let  $F = \{\xi \in \mathcal{O} \mid \xi^m = 1\}$ . Then  $F$  is finite. Let  $L' = \max\{\omega(\xi - 1) \mid \xi \in F \setminus \{1\}\}$ . Let  $L \in \mathbb{N}$  be such that  $L/K - K' \geq L'$ . For  $n \in \mathbb{Z}_{\geq L}$ , we have  $n/K - K' \geq L'$ , we have  $\alpha_i(t') = 1$  for  $i \in I$  and  $t' \in \mathcal{Z}_{\mathcal{O}}$ . Maybe increasing  $L$ , we can assume that  $K_1 := 1/K - K'/L > 0$ . Then for  $n \geq L$ , we have  $n/K - K' \geq n(1/K - K'/L) \geq nK_1$ .

Consequently, for  $n \in \mathbb{Z}_{\geq L}$  and  $t \in T_{n,\Phi}$ , we have  $t = t' t''$ , with  $t' \in \mathcal{Z}_{\mathcal{O}}$  and  $t'' \in T_{n/K_1}$ , which proves the lemma.  $\square$

**Proposition 4.21.** *The topology  $\mathcal{T}$  is finer than  $\mathcal{T}_{\text{Fix}}$ . If  $\mathcal{K}$  is Henselian, then we have  $\mathcal{T} = \mathcal{T}_{\text{Fix}}$  if and only if  $\mathcal{Z}_{\mathcal{O}} = \{1\}$  if and only if  $\mathcal{T}_{\text{Fix}}$  is Hausdorff.*

*Proof.* Let  $x \in \mathcal{I}$  and  $m \in \mathbb{N}^*$ . Then by Lemma 4.5,  $U_{[-m\lambda, m\lambda]}^{pm+}$  and  $U_{[-m\lambda, m\lambda]}^{nm-}$  fix  $x$ , for  $m \gg 0$ . By Lemma 4.12,  $T_{2mN(\lambda)}$  fix  $x$  and thus  $\mathcal{V}_{m\lambda}$  fixes  $x$  for  $m \gg 0$ . Thus if  $n \in \mathbb{N}^*$ ,  $\mathcal{V}_{m\lambda} \subset G_{F_n}$  for  $m \gg 0$  and  $\mathcal{T}$  is finer than  $\mathcal{T}_{\text{Fix}}$ .

If  $\mathcal{T} = \mathcal{T}_{\text{Fix}}$ , then by Theorem 4.8,  $\mathcal{T}_{\text{Fix}}$  is Hausdorff. Therefore  $\bigcap_{n \in \mathbb{N}^*} G_{F_n} = \mathcal{Z}_{\mathcal{O}} = \{1\}$  by Lemma 4.15.

Assume  $\mathcal{K}$  is Henselian. Let  $m \in \mathbb{N}^*$ . Then by Lemma 4.16 and Lemma 4.20, there exist  $K_1 \in \mathbb{R}_+^*$  and  $L \in \mathbb{N}$  such that

$$G_{F_n} \subset \mathcal{V}_{m\lambda} \cdot \mathcal{Z}_{\mathcal{O}} \cdot T_{M(n)/K_1},$$

for  $n \geq \min(m, L)$ , with  $M(n) \xrightarrow{n \rightarrow +\infty} +\infty$ . Therefore if  $\mathcal{Z}_{\mathcal{O}} = 1$  we have

$$G_{F_n} \subset \mathcal{V}_{m\lambda} \cdot T_{M(n)/K_1} \subset \mathcal{V}_{m\lambda},$$

for  $n$  such that  $M(n)/K \geq 2mN(\lambda)$ , and thus  $(\mathcal{V}_{n\lambda})$  and  $(G_{F_n})$  are equivalent, which proves the proposition.  $\square$

**Remark 4.22.** 1. If  $(\alpha_i)_{i \in I}$  is a  $\mathbb{Z}$ -basis of  $X$ , then  $\mathcal{T} = \mathcal{T}_{\text{Fix}}$ . Indeed, assume that  $(\alpha_i)_{i \in I}$  is a  $\mathbb{Z}$ -basis of  $X$ . Let  $(\chi_i^\vee)_{i \in I}$  be the dual basis. Let  $n \in \mathbb{N}^*$  and  $t \in T_{n, \Phi} \cap \mathfrak{T}(\mathcal{O})$ . Write  $t = \prod_{i \in I} \chi_i^\vee(a_i)$ , with  $a_i \in \mathcal{O}^*$  for  $i \in I$ . Then  $\pi_n(t) = \prod_{i \in I} \chi_i^\vee(\pi_n(a_i))$  and  $\pi_n(t) = 1$  if and only if  $\pi_n(a_i) = 1$  for all  $i \in I$ . Now  $\alpha_i(t) = a_i$  and thus  $t \in T_{n, \Phi}$  if and only if  $t \in T_n$ . Therefore  $\mathcal{Z}_{\mathcal{O}} = \bigcap_{n \in \mathbb{N}} T_n = \{1\}$ .

2. Note that by Lemma 4.16, the set of left  $\mathfrak{T}(\mathcal{O})$ -invariant open subsets of  $G$  are the same for  $\mathcal{T}_{\text{Fix}}$  and  $\mathcal{T}$ . Indeed, let  $V \subset G$  be a non empty left  $\mathfrak{T}(\mathcal{O})$ -invariant open subset of  $G$  for  $\mathcal{T}_{\text{Fix}}$ . Then for every  $v \in V$ , there exists  $n \in \mathbb{N}^*$  such that  $vG_{F_n} \subset V$ . By Lemma 4.16,  $\mathfrak{T}(\mathcal{O}) \cdot G_{F_n} \subset \mathcal{V}_{n\lambda}$  and thus  $V$  is open for  $\mathcal{T}$ .

3. Assume that  $\mathcal{K}$  is local. By 2), if  $\tau \in \text{Hom}_{\text{Gr}}(Y, \mathbb{C}^*)$ , then  $I(\tau)_{\mathcal{T}} = I(\tau)_{\mathcal{T}_{\text{Fix}}}$  (see (1.1) for the definition). Indeed,  $\delta^{1/2}$  and  $\tau$  are maps from  $Y = T/\mathfrak{T}(\mathcal{O})$  to  $\mathbb{C}^*$  and thus their extensions to  $B$  are left  $\mathfrak{T}(\mathcal{O})$ -invariant. Therefore any element of  $\widehat{I(\tau)}$  is left  $\mathfrak{T}(\mathcal{O})$ -invariant.

## 5 Properties of the topologies

In this section, we study the properties of the topologies  $\mathcal{T}$  and  $\mathcal{T}_{\text{Fix}}$ . In 5.1, we prove that when  $G$  is not reductive,  $\mathcal{T}$  is strictly coarser than the Kac-Peterson topology on  $G$  (Proposition 5.4). In 5.2, we prove that certain subgroups of  $G$  are closed for  $\mathcal{T}$ . In 5.3, we prove that the compact subsets of  $G$  have empty interior. In 5.4, we describe the topology in the case of affine  $\text{SL}_2$ , under some assumption.

## 5.1 Comparison with the Kac-Peterson topology on $G$

In [KP83], Kac and Peterson defined a topology of topological group on  $\mathfrak{G}(\mathbb{C})$ . This topology was then studied in [HKM13] and generalized in [HKM13, 7]: Hartnick, Köhl and Mars define a topology of topological group on  $\mathfrak{G}(\mathcal{F})$  for  $\mathcal{F}$  a local field (Archimedean or not), taking into account the topology of  $\mathcal{F}$ . The aim of this section is to prove that the topologies we defined on  $G = \mathfrak{G}(\mathcal{K})$  are strictly coarser than the Kac-Peterson topology on  $G$ , unless  $G$  is reductive. As  $\mathcal{T}_{\text{Fix}}$  is coarser than  $\mathcal{T}$  it suffices to prove that  $\mathcal{T}$  is strictly coarser than  $\mathcal{T}_{KP}$ . To that end, we prove that  $\mathcal{T}$  is coarser than  $\mathcal{T}_{KP}$  and that the topologies induced by  $\mathcal{T}$  and  $\mathcal{T}_{KP}$  on a subset of  $B := TU^+$  differ, using the description of  $\mathcal{T}_{KP}|_B$  given in [HKM13, 7].

We assume that  $\mathcal{K}$  is local, in particular,  $\omega(\mathcal{K}^*) = \mathbb{Z}$ . We equip  $\text{SL}_2(\mathcal{K})$  with the topology associated to  $(\ker \pi_n^{\text{SL}_2})_{n \in \mathbb{N}^*}$ , where  $\pi_n^{\text{SL}_2} : \text{SL}_2(\mathcal{O}) \rightarrow \text{SL}_2(\mathcal{O}/\varpi^n \mathcal{O})$  is the natural projection. We denote by  $x_+$  (resp.  $x_-$ ) the morphism of algebraic groups  $a \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$  (resp.  $a \mapsto \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$ ) for  $a$  in a ring  $\mathcal{R}$ . Using Corollary 3.6, it is easy to check that

$$\ker \pi_n^{\text{SL}_2} = x_+(\varpi^n \mathcal{O}).x_-(\varpi^n \mathcal{O}).\left(\begin{pmatrix} 1+\varpi^n \mathcal{O} & 0 \\ 0 & 1+\varpi^n \mathcal{O} \end{pmatrix} \cap \text{SL}_2(\mathcal{K})\right),$$

and thus  $(\mathcal{V}_{n\lambda}^{\text{SL}_2})$  is equivalent to  $(\ker \pi_n^{\text{SL}_2})$  for any regular  $\lambda \in Y_{\text{SL}_2}$ .

We equip  $T$  with its usual topology  $\mathcal{T}_T$ , via the isomorphism  $T \simeq (\mathcal{K}^*)^m$ , for  $m$  the rank of  $X$ . This is the topology  $\mathcal{T}((\ker \pi_n|_T)_{n \in \mathbb{N}^*})$ . As we shall see (Proposition 5.4), this is the topology induced by  $\mathcal{T}$  on  $T$ .

For  $\alpha \in \Phi_+$ , let  $\varphi_\alpha : \text{SL}_2(\mathcal{K}) \rightarrow G$  be defined by  $\varphi_\alpha \circ x_\pm = x_{\pm\alpha}$  and let  $G_\alpha$  be its image in  $G$ . We equip  $G_\alpha$  with the quotient topology  $\mathcal{T}_{G_\alpha}$  inherited from  $\text{SL}_2(\mathcal{K})$  via  $\varphi_\alpha$ . Let  $\Sigma = \{\alpha_i \mid i \in I\}$  and  $\Sigma^{(\mathbb{N})}$  be the set of finite sequences of elements of  $\Sigma$ . For  $\underline{\alpha} = (\alpha_0, \dots, \alpha_k) \in \Sigma^{(\mathbb{N})}$ , where  $k \in \mathbb{N}^*$ , one sets  $G_{\underline{\alpha}} = G_{\alpha_0} \dots G_{\alpha_k} \subset G$ . Note that  $G_{\underline{\alpha}}$  is not a subgroup of  $G$  in general. If  $\underline{\alpha}, \underline{\beta} \in \Sigma^{(\mathbb{N})}$ , we write  $\underline{\alpha} \leq \underline{\beta}$  if  $\underline{\alpha}$  appears as an ordered subtuple of  $\underline{\beta}$ . Then  $\leq$  is a preorder on  $\Sigma^{(\mathbb{N})}$  and  $(\Sigma^{(\mathbb{N})}, \leq)$  is a directed poset.

We equip  $TG_{\underline{\alpha}}$  with the topology  $\mathcal{T}_{TG_{\underline{\alpha}}}$  obtained as the quotient topology with respect to the multiplication map

$$m_{\underline{\alpha}} : (T, \mathcal{T}_T) \times (G_{\alpha_1}, \mathcal{T}_{G_{\alpha_1}}) \times \dots \times (G_{\alpha_k}, \mathcal{T}_{G_{\alpha_k}}) \rightarrow TG_{\underline{\alpha}}. \quad (5.1)$$

In [HKM13, Definition 7.8], the authors define the Kac-Peterson topology  $\mathcal{T}_{KP}$  on  $G$  as the direct limit of the directed system  $\{(TG_{\underline{\alpha}}, \mathcal{T}_{TG_{\underline{\alpha}}}) \mid \underline{\alpha} \in \Sigma^{(\mathbb{N})}\}$ . In other words, a subset  $V$  of  $G$  is open for  $\mathcal{T}_{KP}$  if and only if  $V \cap TG_{\underline{\alpha}}$  is open for every  $\underline{\alpha} \in \Sigma^{(\mathbb{N})}$ .

**Lemma 5.1.** *Let  $w \in W^v$ . Assume that  $wr_i < w$  for every  $i \in I$  (for the Bruhat order on  $W^v$ ). Then  $W^v$  is finite.*

*Proof.* By [Kum02, 1.3.13 Lemma], we have  $w.\alpha_i \in \Phi_-$  for every  $i \in I$ . Let  $\lambda \in C_f^v$ . Then  $\alpha_i(w^{-1}.\lambda) < 0$  for every  $i \in I$  and thus  $w^{-1}.\lambda \in -C_f^v$ . Thus  $\lambda \in \mathring{\mathcal{T}} \cap -\mathring{\mathcal{T}}$ . By [Kum02, 1.4.2 Proposition] we deduce that  $\Phi$  is finite and thus  $W^v$  is finite.  $\square$



We equip  $W^v$  with the **right weak Bruhat order**  $\preceq$ : for every  $v, w \in W^v$ ,  $v \preceq w$  if  $\ell(v) + \ell(v^{-1}w) = \ell(w)$ . We assume that  $W^v$  is infinite. By Lemma 5.1, there exists a sequence  $(w_i)_{i \in \mathbb{N}} \in (W^v)^{\mathbb{N}}$  such that  $w_0 = 1$ ,  $\ell(w_{i+1}) = \ell(w_i) + 1$  and  $w_i \preceq w_{i+1}$  for every  $i \in \mathbb{N}$ .

For  $w \in W^v$ , one sets  $\text{Inv}(w) = \{\alpha \in \Phi_+ \mid w^{-1}\alpha \in \Phi_-\}$ . Let  $U_w = \langle U_\alpha \mid \alpha \in \text{Inv}(w) \rangle$ . By [CR09, Lemma 5.8], if  $w = r_{i_1} \dots r_{i_k}$ , with  $k = \ell(w)$  and  $i_1, \dots, i_k \in I$ , then  $U_w = U_{\alpha_{i_1}} \cdot U_{r_{i_1} \cdot \alpha_{i_2}} \dots U_{r_{i_1} \dots r_{i_{k-1}} \cdot \alpha_{i_k}}$  and every element of  $U_w$  admits a unique decomposition in this product. By [HKM13, Proposition 7.27], as a topological space,  $B$  is the colimit  $\lim_{\rightarrow} TU_w$  (note that  $(W^v, \preceq)$  is not directed). Let  $U' = \bigcup_{n \in \mathbb{N}} U_{w_n}$ . Then the topology induced on  $TU'$  by  $\mathcal{T}_{KP}$  is the topology of the direct limit  $\lim_{\rightarrow} TU_{w_n}$ : a subset  $V$  of  $TU'$  is open if and only if  $V \cap TU_{w_n}$  is open for every  $n \in \mathbb{N}$ .

For  $n \in \mathbb{N}$ , write  $w_{n+1} = w_n r_i$ , where  $i \in I$ . Set  $\beta[n] = w_n \cdot \alpha_i$ . Then  $\text{Inv}(w_n) = \{\beta[i] \mid i \in \llbracket 1, n \rrbracket\}$ , by [Kum02, 1.3.14 Lemma].

By [HKM13, Lemma 7.26], if  $n \in \mathbb{N}^*$  then the map  $m = m_n : T \times (\mathcal{K})^n \rightarrow TU_{w_n}$  defined by  $m(t, a_1, \dots, a_n) = t x_{\beta[1]}(a_1) \dots x_{\beta[n]}(a_n)$  is a homeomorphism, when  $TU_{w_n}$  is equipped with the restriction of  $\mathcal{T}_{KP}$ .

Recall that  $\mathcal{T} = \mathcal{T}((\mathcal{V}_{n\lambda}))$  for any  $\lambda \in Y^+ \cap C_f^v$ .

Define  $\text{ht} : Q_+ = \bigoplus_{i \in I} \mathbb{Z}\alpha_i \rightarrow \mathbb{Z}$  by  $\text{ht}(\sum_{i \in I} n_i \alpha_i) = \sum_{i \in I} n_i$ , for  $(n_i) \in \mathbb{Z}^I$ .

**Lemma 5.2.** *Assume that  $W^v$  is infinite. For  $n \in \mathbb{N}^*$ , set  $V_n = T \prod_{i=1}^n x_{\beta[i]}(\varpi^{\text{ht}(\beta[i])!} \mathcal{O})$  and set  $V = \bigcup_{n \in \mathbb{N}} V_n$ . Then  $V$  is open in  $(TU', \mathcal{T}_{KP})$  but not in  $(TU', \mathcal{T})$ . In particular,  $\mathcal{T}$  and  $\mathcal{T}_{KP}$  are different.*

*Proof.* Let  $n \in \mathbb{N}^*$ . Let  $v \in V \cap TU_{w_n}$  and choose  $k \in \mathbb{N}^*$  such that  $v \in V_k$ . If  $k \leq n$ , then  $v \in V_k \subset V_n$ . Suppose now  $k \geq n$ . Write  $v = t \prod_{i=1}^k x_{\beta[i]}(\varpi^{\text{ht}(\beta_i)!} a_i)$ , with  $a_1, \dots, a_k \in \mathcal{O}$  and  $t \in T$ . By [HKM13, Lemma 7.26], we have  $a_i = 0$  for every  $i \in \llbracket n+1, k \rrbracket$  and thus  $v \in V_n$ . Therefore  $V \cap TU_{w_n} = V_n$ . By [HKM13, Lemma 7.26],  $V_n$  is open in  $TU_{w_n}$  and thus  $V$  is open in  $(TU', \mathcal{T}_{KP})$ .

Let  $\lambda \in C_f^v \cap Y$  be such that  $\alpha_i(\lambda) = 1$  for every  $i \in I$ . Let us prove that for every  $n \in \mathbb{N}^*$ ,  $U' \cap U_{-n\lambda}^{pm+}$  is not contained in  $V$ . For  $k, n \in \mathbb{N}^*$ , set  $x_{k,n} = \prod_{i=1}^k x_{\beta[i]}(\varpi^{n \text{ht}(\beta[i])}) \in U' \cap U_{-n\lambda}^{pm+}$ . Let  $n \in \mathbb{N}^*$ . By [HKM13, Lemma 7.26], if  $x_{k,n} \in V$ , then  $n \text{ht}(\beta[i]) \geq (\text{ht}(\beta[i]))!$ , for every  $i \in \llbracket 0, k \rrbracket$ . As  $\text{ht}(\beta[i]) \xrightarrow{i \rightarrow +\infty} +\infty$ , there exists  $k \in \mathbb{N}$  such that  $x_{k,n} \in U' \cap U_{-n\lambda}^{pm+} \setminus V$  and thus,  $U' \cap U_{-n\lambda}^{pm+} \not\subset V$ . Using Lemma 2.3 we deduce that there exists no  $n \in \mathbb{N}^*$  such that  $\mathcal{V}_{n\lambda} \cap TU' \subset V$  and thus  $V$  is not open for  $\mathcal{T}$ .  $\square$

**Proposition 5.3.** *Let  $\underline{V} = (V_n)_{n \in \mathbb{N}^*}$  be a conjugation-invariant filtration of  $G$ . Let  $\mathcal{T}_{\underline{V}}$  be the associated topology on  $G$ . We assume that for every  $\alpha \in \Sigma$ , the induced topology on  $G_\alpha$  is  $\mathcal{T}_{G_\alpha}$  and that the induced topology on  $T$  is  $\mathcal{T}_T$ . Then  $\mathcal{T}_{\underline{V}}$  is coarser than  $\mathcal{T}_{KP}$ .*

*Proof.* Let  $n \in \mathbb{N}^*$ . Let us prove that  $V_n$  is open for  $\mathcal{T}_{KP}$ . Let  $\underline{\alpha} = (\alpha_k, \dots, \alpha_1) \in \Sigma^{(\mathbb{N})}$ . Let us prove that  $m_{\underline{\alpha}}^{-1}(V_n)$  is open in  $T \times G_{\alpha_k} \times \dots \times G_{\alpha_1}$ , with the notation of (5.1). Let  $v \in TG_{\underline{\alpha}} \cap V_n$  and  $(t, v_k, \dots, v_1) \in m_{\underline{\alpha}}^{-1}(\{v\})$ . We have  $v = t v_k \dots v_1$ . We set  $n_1 = n$  and we choose  $n_2, \dots, n_k, n_{k+1} \in \mathbb{N}^*$  such that for all  $j \in \llbracket 2, k \rrbracket$ , we have  $V_{n_j} v_{j-1} \dots v_1 \subset$

$v_{j-1} \dots v_1 V_n$ , which is possible since  $\underline{V}$  is conjugation-invariant. Then we have:

$$\begin{aligned} V_{n_{k+1}} v_k V_{n_k} v_{k-1} V_{n_{k-1}} \dots V_{n_3} v_2 V_{n_2} v_1 V_{n_1} &\subset V_{n_{k+1}} v_k V_{n_k} v_{k-1} V_{n_{k-1}} \dots V_{n_3} v_2 v_1 V_n \\ &\subset \dots \subset v_k \dots v_1 V_n. \end{aligned}$$

Consequently

$$(tV_{n_{k+1}} \cap T) (v_k V_{n_k} \cap G_{\alpha_k}) \dots (v_1 V_{n_1} \cap G_{\alpha_1}) \subset tv_k \dots v_1 V_n = vV_n = V_n$$

and hence

$$m_{\underline{\alpha}} \left( (tV_{n_{k+1}} \cap T) \times (v_k V_{n_k} \cap G_{\alpha_k}) \times \dots \times (v_1 V_{n_1} \cap G_{\alpha_1}) \right) \subset V_n \cap TG_{\underline{\alpha}}.$$

Therefore  $m_{\underline{\alpha}}^{-1}(V_n \cap TG_{\underline{\alpha}})$  is open or equivalently  $V_n \cap TG_{\underline{\alpha}}$  is open. As this is true for every  $\underline{\alpha} \in \Sigma^{\mathbb{N}}$ , we deduce that  $V_n$  is open for  $\mathcal{T}_{KP}$ . As  $(G, \mathcal{T}_{KP})$  is a topological group, we deduce that  $xV_n$  is open in  $\mathcal{T}_{KP}$  for every  $x \in G$  and  $n \in \mathbb{N}^*$  and we deduce that  $\mathcal{T}_{\underline{V}}$  is coarser than  $\mathcal{T}_{KP}$ .  $\square$

**Proposition 5.4.** *1. Let  $\alpha \in \Phi_+$  and  $\varphi_\alpha : \mathrm{SL}_2(\mathcal{K}) \rightarrow G$  be the group morphism defined by  $\varphi_\alpha \circ x_\pm = x_{\pm\alpha}$ . Fix a basis  $(\chi_1^\vee, \dots, \chi_\ell^\vee)$  of  $Y$  and define  $\iota : (\mathcal{K}^*)^\ell \xrightarrow{\sim} T \subset G$  by  $\iota((a_1, \dots, a_\ell)) = \chi_1^\vee(a_1) \dots \chi_\ell^\vee(a_\ell)$ , for  $a_1, \dots, a_\ell \in \mathcal{K}^*$ . Then the  $\varphi_\alpha$ ,  $\alpha \in \Phi$  and  $\iota$  are continuous when  $G$  is equipped with  $\mathcal{T}$ .*

*2. The topology induced by  $\mathcal{T}$  on  $T$  is  $\mathcal{T}_T$  and if  $\alpha \in \Phi_+$ , then the topology induced by  $\mathcal{T}$  on  $G_\alpha$  is  $\mathcal{T}_{G_\alpha}$*

*Proof.* Let  $\alpha \in \Phi$  and  $\lambda \in Y^+$  be regular. Let  $g \in \varphi_\alpha^{-1}(\mathcal{V}_\lambda)$ . By [Rou16, 3.16] and [Rém02, 1.2.4 Proposition], we have the Birkhoff decomposition in  $\mathrm{SL}_2(\mathcal{K})$  (where  $N_{\mathrm{SL}_2}$  is the set of monomial matrices with coefficient in  $\mathcal{K}^*$ ) and  $G$ :

$$\mathrm{SL}_2(\mathcal{K}) = \bigsqcup_{n \in N_{\mathrm{SL}_2}} x_+(\mathcal{K}) n x_-(\mathcal{K}) \text{ and } G = \bigsqcup_{n \in N} U^+ n U^-.$$

Let  $n \in N_{\mathrm{SL}_2}$  be such that  $g \in x_+(\mathcal{K}) n x_-(\mathcal{K})$ . If  $n \notin T_{\mathrm{SL}_2}$ , then  $\varphi_\alpha(g) \in U^+ \varphi_\alpha(n) U^-$  and  $\nu^v(\varphi_\alpha(n))$  acts as the reflection with respect to  $\alpha$  on  $\mathbb{A}$ . Then  $\varphi_\alpha(g) \notin U^+ T U^-$  which contradicts Corollary 3.6. Therefore  $n \in T$ . Write  $g = x_+(a_+) n x_-(a_-)$ , with  $a_+, a_- \in \mathcal{K}$ . Then by Lemma 2.3, we have  $x_\alpha(a_+) \in U_{[-\lambda, \lambda]}^{pm+}$ ,  $x_{-\alpha}(a_-) \in U_{[-\lambda, \lambda]}^{nm-}$  and  $\varphi_\alpha(n) \in T_{2N(\lambda)}$ . Consequently,  $\omega(a_+), \omega(a_-) \geq |\alpha(\lambda)|$  and

$$t \in T_{\mathrm{SL}_2, 2N(\lambda)} := \left( \begin{array}{cc} 1 + \varpi^{2N(\lambda)} \mathcal{O} & 0 \\ 0 & 1 + \varpi^{2N(\lambda)} \mathcal{O} \end{array} \right) \cap \mathrm{SL}_2(\mathcal{K}).$$

Therefore  $\varphi_\alpha^{-1}(\mathcal{V}_\lambda) \subset x_+(\mathcal{K}_{\omega \geq |\alpha(\lambda)|}) T_{\mathrm{SL}_2, 2N(\lambda)} x_-(\mathcal{K}_{\omega \geq |\alpha(\lambda)|})$ . Conversely,  $\varphi_\alpha(x_-(\mathcal{K}_{\omega \geq |\alpha(\lambda)|}))$ ,  $\varphi_\alpha(x_+(\mathcal{K}_{\omega \geq |\alpha(\lambda)|}))$ ,  $\varphi_\alpha(T_{\mathrm{SL}_2, 2N(\lambda)}) \subset \mathcal{V}_\lambda$  and thus  $\varphi_\alpha^{-1}(\mathcal{V}_\lambda) = x_+(\mathcal{K}_{\omega \geq |\alpha(\lambda)|}) T_{\mathrm{SL}_2, 2N(\lambda)} x_-(\mathcal{K}_{\omega \geq |\alpha(\lambda)|})$  is open in  $\mathrm{SL}_2(\mathcal{K})$ . Therefore  $\varphi_\alpha$  is continuous and  $\mathcal{V}_\lambda \cap G_\alpha$  is open.

Let  $n \in \mathbb{N}^*$ . Then  $\iota^{-1}(T_n) = (1 + \varpi^n \mathcal{O})^\ell$  and thus  $\iota$  is continuous. Moreover, we have  $T \cap \mathcal{V}_\lambda = T_{2N(\lambda)}$ , by the Birkhoff decomposition, which proves that  $\mathcal{T}$  induces  $\mathcal{T}_T$  on  $T$ .  $\square$

**Corollary 5.5.** *If  $\Phi$  is infinite, the topologies  $\mathcal{T}$  and  $\mathcal{T}_{\text{Fix}}$  are strictly coarser than  $\mathcal{T}_{KP}$ .*

*Proof.* Let  $\lambda \in Y^+$  be any regular element. By Propositions 5.4 and 5.3, applied with  $\underline{V} = (\mathcal{V}_{n\lambda})$ ,  $\mathcal{T}$  is coarser than  $\mathcal{T}_{KP}$ . By Lemma 5.2,  $\mathcal{T}$  is different from  $\mathcal{T}_{KP}$ . As  $\mathcal{T}_{\text{Fix}}$  is coarser than  $\mathcal{T}$  (by Proposition 4.21), we deduce the result.  $\square$

## 5.2 Properties of usual subgroups of $G$ for $\mathcal{T}$ and $\mathcal{T}_{\text{Fix}}$

In this subsection, we prove that many subgroups important in this theory (such as  $B$ ,  $T$ ,  $U_\alpha$ ,  $\alpha \in \Phi$ , etc.) are open or closed. We have  $\mathcal{T}_{\text{Fix}} \subset \mathcal{T}$  and thus every subset of  $G$  open or closed for  $\mathcal{T}_{\text{Fix}}$  is open or closed for  $\mathcal{T}$ . As the Kac-Peterson topology  $\mathcal{T}_{KP}$  is finer than  $\mathcal{T}$ , this improves the corresponding results of [HKM13]. Note that we consider  $B = B^+$  and  $U^+$ , but the same results hold for  $B^-$  and  $U^-$ , by symmetry.

If  $g \in G$ , we say that  $g$  **stabilizes** (resp. **pointwise fixes**)  $+\infty$  if  $g \cdot +\infty = +\infty$  (resp. if there exists  $Q \in +\infty$  such that  $g$  pointwise fixes  $Q$ ). We denote by  $\text{Stab}_G(+\infty)$  the stabilizer of  $+\infty$  in  $G$ .

By [Héb18, 3.4.1],  $\text{Stab}_G(+\infty) = B := TU^+$ . We denote by  $\text{Ch}(\partial\mathcal{I}^+)$  the set of positive sector-germs at infinity of  $\mathcal{I}$ . For  $c \in \text{Ch}(\partial\mathcal{I}^+)$  and  $x \in \mathcal{I}$ , there exists an apartment  $A$  containing  $x$  and  $c$ . We denote by

$$x + c \tag{5.2}$$

the convex hull of  $x$  and  $c$  in this apartment. This does not depend on the choice of  $A$ , by (MA II). Fix  $\lambda_0 \in C_f^v$ . For  $r \in \mathbb{R}_+$ , we set  $\mathcal{C}_r = \{c \in \text{Ch}(\partial\mathcal{I}^+) \mid [0, r \cdot \lambda_0] \subset 0 + c\}$ . This set is introduced in [CMR20, Definition 3.1] where it is denoted  $U_{0,r,c}$  or  $U_{r,c}$ .

**Proposition 5.6.** *1. The subgroup  $B$  is closed in  $G$  for  $\mathcal{T}$  and  $\mathcal{T}_{\text{Fix}}$ .*

*2. The subgroup  $U^+$  of  $G$  is closed for  $\mathcal{T}$ . It is closed for  $\mathcal{T}_{\text{Fix}}$  if and only if  $\mathcal{T} = \mathcal{T}_{\text{Fix}}$ .*

*Proof.* 1) Let  $g \in G \setminus B$ . Then  $g \cdot (+\infty) \neq +\infty$ . By [CMR20, Lemma 7.6],  $\bigcap_{r \in \mathbb{R}_+} \mathcal{C}_r = \{+\infty\}$ . Thus there exists  $n \in \mathbb{N}^*$  such that  $g^{-1} \cdot +\infty \notin \mathcal{C}_n$ . Then  $\mathcal{V}_{n\lambda_0}$  fix  $[0, n\lambda_0]$ . Let  $v \in \mathcal{V}_{n\lambda_0}$ . Then

$$v \cdot (0 + \infty) = v \cdot 0 + (v \cdot +\infty) = 0 + (v \cdot +\infty) \supset v \cdot [0, n\lambda_0] = [0, n\lambda_0].$$

Therefore  $\mathcal{V}_{n\lambda_0} \cdot (+\infty) \subset \mathcal{C}_n$ . Consequently,  $g^{-1} \cdot (+\infty) \notin \mathcal{V}_{n\lambda_0} \cdot +\infty$  and thus  $g\mathcal{V}_{n\lambda_0} \cdot +\infty \not\subset +\infty$ . Thus  $g\mathcal{V}_{n\lambda_0} \subset G \setminus B$ , which proves that  $G \setminus B$  is open for  $\mathcal{T}_{\text{Fix}}$ .

2) Let  $g \in G \setminus U^+$ . If  $g \in G \setminus U^+T$ , then by 1), there exists  $V \in \mathcal{T}$  such that  $gV \subset G \setminus U^+T \subset G \setminus U^+$ . We now assume  $g \in U^+T \setminus U$ . Write  $g = u_+t$ , with  $u_+ \in U^+$  and  $t \in T \setminus \{1\}$ . Let  $\lambda \in Y \cap C_f^v$  and assume that  $g\mathcal{V}_\lambda \cap U^+ \neq \emptyset$ . Then there exists  $(u'_+, u'_-, t') \in U_{[-\lambda, \lambda]}^{pm+} \times U_{[-\lambda, \lambda]}^{nm-} \times T_{2N(\lambda)}$  such that  $u_+tu'_+u'_-t' = u''_+$ , where  $u''_+ \in U^+$ . As  $t$  normalizes  $U^+$  and  $U^-$ , we can write  $tu'_+u'_- = u_+^{(3)}u_-^{(3)}t$ , for some  $u_+^{(3)} \in U^+$ ,  $u_-^{(3)} \in U^-$ . Then we have

$$u_+^{''-1}u_+u_+^{(3)}u_-^{(3)}tt' = 1.$$

By Lemma 2.3 we deduce  $tt' = 1$ . Therefore  $t \in T_{2N(\lambda)}$ . Thus if  $\lambda$  is sufficiently dominant,  $g\mathcal{V}_\lambda \cap U^+ = \emptyset$  and  $g\mathcal{V}_\lambda \subset G \setminus U^+$ . We deduce that  $G \setminus U^+$  is closed for  $\mathcal{T}$ .

Suppose now  $\mathcal{T} \neq \mathcal{T}_{\text{Fix}}$ . Then by Proposition 4.21,  $\mathcal{Z}_\mathcal{O} \neq \{1\}$ . Then every non empty open subset of  $G$  for  $\mathcal{T}_{\text{Fix}}$  contains  $\mathcal{Z}_\mathcal{O}$ . Take  $z \in \mathcal{Z}_\mathcal{O} \setminus \{1\}$ . As  $z \in \mathfrak{T}(\mathcal{O})$ ,  $z \in G \setminus U^+$ . Moreover, for any non empty open subset  $V$  of  $G$ ,  $zV \ni 1 \in U^+$ . Therefore  $G \setminus U^+$  is not open, which completes the proof of the proposition.  $\square$

**Proposition 5.7.** 1. Let  $x \in \mathcal{I}$ . Then the fixator  $G_x$  of  $x$  in  $G$  is open (for  $\mathcal{T}_{\text{Fix}}$  and  $\mathcal{T}$ ). In particular,  $\mathfrak{G}^{\min}(\mathcal{O})$  is open in  $G$ .

2. Let  $E \subset \mathcal{I}$ . Then the fixator and the stabilizer of  $E$  in  $G$  are closed for  $\mathcal{T}_{\text{Fix}}$  and  $\mathcal{T}$ .

*Proof.* 1) By Lemma 4.13,  $G_{F_n} \subset G_x$  for  $n \gg 0$ , which proves 1).

2) Let  $g \in G \setminus G_E$ . Let  $x \in E$  be such that  $g.x \neq x$ . Then  $g.G_x \subset G \setminus G_E$  and hence  $G_E$  is open. Let  $g \in G \setminus \text{Stab}_G(E)$ . Let  $x \in E$  be such that  $g.x \notin E$ . Then  $g.G_x \subset G \setminus \text{Stab}_G(E)$  and hence  $\text{Stab}_G(E)$  is closed.  $\square$

**Corollary 5.8.** The subgroups  $N$  and  $T$  are closed in  $G$  for  $\mathcal{T}_{\text{Fix}}$  and  $\mathcal{T}$ .

*Proof.* By Proposition 5.7,  $N = \text{Stab}_G(\mathbb{A})$  is closed. We have  $T = \text{Stab}_G(+\infty) \cap N$ . Indeed, it is clear that  $T \subset \text{Stab}_G(+\infty) \cap N$ . Conversely, let  $g \in \text{Stab}_G(+\infty) \cap N$ . Let  $w \in W^v$  and  $\lambda \in \Lambda$  be such that  $g.x = \lambda + w.x$  for every  $x \in \mathbb{A}$ . Then  $w.C_f^v = C_f^v$  and thus  $w = 1$ . Therefore  $g$  acts by translation on  $\mathbb{A}$  and hence  $g \in T$ . This proves that  $T = \text{Stab}_G(+\infty) \cap N$  and we conclude with Proposition 5.6.  $\square$

**Remark 5.9.** 1. The fixator  $K_I$  of  $C_0^+$  is open for  $\mathcal{T}_{\text{Fix}}$ . Indeed, let  $\lambda \in C_f^v \cap Y$ . Then  $G_{[0,\lambda]} = G_0 \cap G_\lambda$  is open for  $\mathcal{T}_{\text{Fix}}$  and  $G_{[0,\lambda]} \subset K_I$ .

2. For  $x, y \in \mathcal{I}$ , one writes  $x \leq y$  if there exists  $g \in G$  such that  $g.x, g.y \in \mathbb{A}$  and  $g.y - g.x \in \mathcal{T} = \bigcup_{w \in W^v} w.C_f^v$ . By  $W^v$ -invariance of  $\mathcal{T}$ , this does not depend on the choice of  $g$  and by [Rou11, Théorème 5.9],  $\leq$  is a preorder on  $\mathcal{I}$ . One sets

$$G^+ = \{g \in G \mid g.0 \geq 0\}.$$

This is a subsemigroup of  $G$  which is crucial for the definition of the Hecke algebras associated with  $G$  (when  $\mathcal{K}$  is local), see [BK11], [BKP16], [GR14] or [BGR16]. Then  $G^+ \supset G_0 = \mathfrak{G}^{\min}(\mathcal{O})$  and thus  $G^+$  is open in  $G$ .

**Lemma 5.10.** Let  $g \in G$ . Then there exists  $n \in N$  such that  $g.a = n.a$  for every  $a \in \mathbb{A} \cap g^{-1}.\mathbb{A}$ .

*Proof.* Let  $h \in G$  be such that  $hg.\mathbb{A} = \mathbb{A}$  and  $h$  fixes  $\mathbb{A} \cap g.\mathbb{A}$ , which exists by (MA II). Then  $n := hg$  stabilizes  $\mathbb{A}$  and thus it belongs to  $N$ . Moreover,  $hg.a = n.a = g.a$  for every  $a \in \mathbb{A} \cap g^{-1}.\mathbb{A}$ , which proves the lemma.  $\square$

**Lemma 5.11.** *Let  $\alpha \in \Phi$ . Write  $\alpha = \epsilon w \alpha_i$ , for  $w \in W^v$ ,  $\epsilon \in \{-, +\}$  and  $i \in I$ . Let  $\mathfrak{Q}$  be the sector-germ at infinity of  $-\epsilon w r_i(C_f^v)$ . Then  $U_\alpha T = \text{Stab}_G(w.\epsilon\infty) \cap \text{Stab}_G(\mathfrak{Q})$ .*

*Proof.* There is no loss of generality in assuming that  $w = 1$  and  $\epsilon = +$ . Let  $u \in U_{\alpha_i}$ . Then  $u$  fixes a translate of  $\alpha_i^{-1}(\mathbb{R}_+)$ . Therefore  $TU_{\alpha_i}$  stabilizes  $\mathfrak{Q}$  and  $+\infty$ . Conversely, let  $g \in \text{Stab}_G(+\infty) \cap \text{Stab}_G(\mathfrak{Q})$ . Then there exist  $x, x' \in \mathbb{A}$  such that  $g.(x+C_f^v) = x'+C_f^v$ . Then by Lemma 5.10, there exists  $n \in N$  such that  $g.x'' = n.x''$  for every  $x'' \in \mathbb{A} \cap g^{-1}.\mathbb{A}$ . Then  $n$  fixes  $+\infty$  and thus  $n \in T$  (by the proof of Corollary 5.8). Then  $n^{-1}g.x = x$ . Considering  $n^{-1}g$  instead of  $g$ , we may assume that  $g$  pointwise fixes  $+\infty$ . Therefore  $g$  pointwise fixes  $\mathfrak{Q}$ . There exists  $a, a' \in \mathbb{A}$  such that  $g$  fixes  $a + C_f^v$  and  $g$  fixes  $a' - r_i(C_f^v)$ . Let  $A = g.\mathbb{A}$ . Then  $A \cap \mathbb{A}$  is a finite intersection of half-apartments by (MA II) and thus either  $A = \mathbb{A}$  or  $A \cap \mathbb{A}$  is a translate of  $\alpha_i^{-1}(\mathbb{R}_+)$ . Moreover,  $g$  fixes  $A \cap \mathbb{A}$  since it fixes an open subset of  $A \cap \mathbb{A}$ . By [Rou16, 5.7 3)],  $g \in U_{\alpha_i} \mathfrak{T}(\mathcal{O})$ . Consequently  $\text{Stab}_G(+\infty) \cap \text{Stab}_G(\mathfrak{Q}) \subset U_{\alpha_i} T$ , and the lemma follows.  $\square$

**Proposition 5.12.** *Let  $\alpha \in \Phi$ .*

1. *The group  $U_\alpha T$  is closed for  $\mathcal{T}$  and  $\mathcal{T}_{\text{Fix}}$ .*
2. *The group  $U_\alpha$  is closed for  $\mathcal{T}$ .*
3. *If  $\mathcal{T} \neq \mathcal{T}_{\text{Fix}}$ , then  $U_\alpha$  is not closed for  $\mathcal{T}_{\text{Fix}}$ .*

*Proof.* 1) Let  $\mathfrak{Q}$  be a sector-germ of  $\mathcal{I}$  (positive or negative). Then by Proposition 5.6 (or the similar proposition for  $B^-$  if  $\mathfrak{Q}$  is negative),  $\text{Stab}_G(\mathfrak{Q})$  is closed in  $G$  for  $\mathcal{T}_{\text{Fix}}$ . By Lemma 5.11 and Proposition 5.7, we have 1).

2) We have  $U_\alpha = U_\alpha T \cap U^+$ , by Lemma 2.3 and thus 2) follows from 1) and Proposition 5.6. The proof of 3) is similar to the proof of the corresponding result of Proposition 5.6.  $\square$

### 5.3 Compact subsets have empty interior

By [AH19, Theorem 3.1], for any topology of topological group on  $G$ ,  $G_{C_0^+}$  or  $G_0$  are not compact and open. In particular,  $G_0$  and  $G_{C_0^+}$  are not compact for  $\mathcal{T}$ . With a similar reasoning, we have the following.

**Proposition 5.13.** *Assume that  $W^v$  is infinite.*

1. *Let  $n \in \mathbb{N}^*$  and  $\lambda \in Y^+$  regular. Then  $\mathcal{V}_{n\lambda}/\mathcal{V}_{(n+1)\lambda}$  is infinite.*
2. *Every compact subset of  $(G, \mathcal{T})$  has empty interior.*

*Proof.* 1) Set  $H = G_{[-n\lambda, (n+1)\lambda]} \subset G$ . Then  $H \supset \mathcal{V}_{(n+1)\lambda}$ , by (4.1). Thus  $|\mathcal{V}_{n\lambda}/\mathcal{V}_{(n+1)\lambda}| \geq |\mathcal{V}_{n\lambda}/(H \cap \mathcal{V}_{n\lambda})|$  and it suffices to prove that  $\mathcal{V}_{n\lambda}/(H \cap \mathcal{V}_{n\lambda})$  is infinite. We have  $\mathcal{V}_{n\lambda} = \bigsqcup_{v \in \mathcal{V}_{n\lambda}/(H \cap \mathcal{V}_{n\lambda})} v.(H \cap \mathcal{V}_{n\lambda})$ . Moreover if  $v, v' \in \mathcal{V}_{n\lambda}$ , then  $v.((n+1)\lambda) = v'.((n+1)\lambda)$  if and only if  $v'.(G_{(n+1)\lambda} \cap \mathcal{V}_{n\lambda}) = v.(G_{(n+1)\lambda} \cap \mathcal{V}_{n\lambda})$ .

Let us prove that  $G_{(n+1)\lambda} \cap \mathcal{V}_{n\lambda} = H \cap \mathcal{V}_{n\lambda}$ . Let  $g \in G_{(n+1)\lambda} \cap \mathcal{V}_{n\lambda}$ . Then by (4.1),  $g$  fixes  $[-n\lambda, n\lambda]$  and  $(n+1)\lambda$ . Then  $g.\mathbb{A}$  is an apartment containing  $[-n\lambda, n\lambda] \cup \{(n+1)\lambda\}$ . As  $g.\mathbb{A} \cap \mathbb{A}$  is convex,  $g.\mathbb{A}$  contains  $[-n\lambda, (n+1)\lambda]$ . By (MA II), there exists  $h \in G$  such that  $g.\mathbb{A} = h.\mathbb{A}$  and  $h$  fixes  $\mathbb{A} \cap g.\mathbb{A}$ . Then  $h^{-1}g.\mathbb{A} = \mathbb{A}$  and  $h^{-1}g$  acts on  $\mathbb{A}$  by an affine map. As  $h^{-1}g$  fixes  $[-n\lambda, n\lambda]$ , it fixes  $[-n\lambda, (n+1)\lambda]$ . Therefore  $g$  fixes  $[-n\lambda, (n+1)\lambda]$  and thus  $g \in H$ . Therefore  $G_{(n+1)\lambda} \cap \mathcal{V}_{n\lambda} = H \cap \mathcal{V}_{n\lambda}$ . Consequently,

$$\mathcal{V}_{n\lambda} \cdot ((n+1)\lambda) = \bigsqcup_{v \in \mathcal{V}_{n\lambda} / (H \cap \mathcal{V}_{n\lambda})} \{v \cdot (n+1)\lambda\} \text{ and } |\mathcal{V}_{n\lambda} / (H \cap \mathcal{V}_{n\lambda})| = |\mathcal{V}_{n\lambda} \cdot ((n+1)\lambda)|.$$

Let  $(\beta_\ell) \in (\Phi_+)^{\mathbb{N}}$  be an injective sequence. Write  $\beta_\ell = \sum_{i \in I} m_i^{(\ell)} \alpha_i$ , with  $m_i^{(\ell)} \in \mathbb{N}$  for  $\ell \in \mathbb{N}$ . Then  $\beta_\ell(\lambda) \geq (\sum_{i \in I} m_i^{(\ell)}) (\min_{i \in I} \alpha_i(\lambda)) \xrightarrow{\ell \rightarrow \infty} +\infty$ . Let  $\ell \in \mathbb{N}$ . For  $k \in \llbracket \beta_\ell(n\lambda), \beta_\ell((n+1)\lambda) - 1 \rrbracket$ ,  $x_{-\beta_\ell}(\varpi^k) \in U_{[-n\lambda, n\lambda]}^{nm^-} \subset \mathcal{V}_{n\lambda}$ . Set

$$x_k = x_{-\beta_\ell}(\varpi^k) \cdot ((n+1)\lambda) \in \mathcal{V}_{n\lambda} \cdot ((n+1)\lambda).$$

Let  $k' \in \llbracket \beta_\ell(n\lambda), \beta_\ell((n+1)\lambda) - 1 \rrbracket$ . Then  $x_k = x_{k'}$  if and only if  $x_{-\beta_\ell}(\varpi^k) \cdot ((n+1)\lambda) = x_{-\beta_\ell}(\varpi^{k'}) \cdot ((n+1)\lambda)$  if and only if  $x_{-\beta_\ell}(\varpi^k - \varpi^{k'}) \cdot ((n+1)\lambda) = (n+1)\lambda$  if and only if  $\omega(\varpi^k - \varpi^{k'}) \geq (n+1)\beta_\ell(\lambda)$  if and only if  $k = k'$ . Therefore  $|\mathcal{V}_{n\lambda} \cdot ((n+1)\lambda)| \geq \beta_\ell(\lambda)$ . As this is true for every  $\ell \in \mathbb{N}$ ,  $|\mathcal{V}_{n\lambda} \cdot ((n+1)\lambda)|$  is infinite, which proves 1).

2) Let  $V$  be a compact subset of  $G$  and assume that  $V$  has non empty interior. Considering  $v^{-1}.V$  instead of  $V$ , we may assume  $1 \in V$ . Then there exists  $\lambda \in Y^+ \cap C_f^v$  such that  $\mathcal{V}_\lambda \subset V$ , and we have  $\mathcal{V}_{2\lambda} \subset \mathcal{V}_\lambda$ . As  $\mathcal{V}_\lambda$  is closed, it is compact. By 1),  $\mathcal{V}_\lambda / \mathcal{V}_{2\lambda}$  is infinite. Therefore  $\mathcal{V}_\lambda = \bigsqcup_{v \in \mathcal{V}_\lambda / \mathcal{V}_{2\lambda}} v \cdot \mathcal{V}_{2\lambda}$  is a cover of  $\mathcal{V}_\lambda$  by open subsets from which we can not extract a finite subcover: we reach a contradiction. Thus every compact subset of  $G$  has empty interior.  $\square$

## 5.4 Example of affine $\text{SL}_2$

In this subsection, we determine an explicit filtration equivalent to  $(\mathcal{V}_{n\lambda})$  in the case of affine  $\text{SL}_2$  (quotiented by the central extension).

Let  $Y = \mathbb{Z}\hat{\alpha}^\vee \oplus \mathbb{Z}d$ , where  $\hat{\alpha}^\vee, d$  are some symbols, corresponding to the positive root of  $\text{SL}_2(\mathcal{K})$  and to the semi-direct extension by  $\mathcal{K}^*$  respectively. Let  $X = \mathbb{Z}\hat{\alpha} \oplus \mathbb{Z}\delta$ , where  $\hat{\alpha}, \delta : Y \rightarrow \mathbb{Z}$  are the  $\mathbb{Z}$ -module morphisms defined by  $\hat{\alpha}(\hat{\alpha}^\vee) = 2$ ,  $\hat{\alpha}(d) = 0$ ,  $\delta(\hat{\alpha}^\vee) = 0$  and  $\delta(d) = 1$ . Let  $\alpha_0 = \delta - \hat{\alpha}$ ,  $\alpha_1 = \hat{\alpha}$ ,  $\alpha_0^\vee = -\hat{\alpha}^\vee$  and  $\alpha_1^\vee = \hat{\alpha}^\vee$ . Then  $\mathcal{S} = ((\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}), X, Y, \{\alpha_0, \alpha_1\}, \{\alpha_0^\vee, \alpha_1^\vee\})$  is a root generating system. Let  $\mathfrak{G}$  be the Kac-Moody group associated with  $\mathcal{S}$  and  $G = \mathfrak{G}(\mathcal{K})$ . Then by [Kum02, 13] and [Mar18, 7.6],  $G = \text{SL}_2(\mathcal{K}[u, u^{-1}]) \rtimes \mathcal{K}^*$ , where  $u$  is an indeterminate and if  $(M, z), (M_1, z_1) \in G$ , with  $M = \begin{pmatrix} a(\varpi, u) & b(\varpi, u) \\ c(\varpi, u) & d(\varpi, u) \end{pmatrix}$ ,  $M_1 = \begin{pmatrix} a_1(\varpi, u) & b_1(\varpi, u) \\ c_1(\varpi, u) & d_1(\varpi, u) \end{pmatrix}$ , we have

$$(M, z) \cdot (M_1, z_1) = (M \begin{pmatrix} a_1(\varpi, zu) & b_1(\varpi, zu) \\ c_1(\varpi, zu) & d_1(\varpi, zu) \end{pmatrix}, zz_1). \quad (5.3)$$

Note that the family  $(\alpha_0^\vee, \alpha_1^\vee)$  is not free. We have  $\Phi = \{\alpha + k\delta \mid \alpha \in \{\pm\dot{\alpha}\}, k \in \mathbb{Z}\}$  and  $(\alpha_0, \alpha_1)$  is a basis of this root system. We denote by  $\Phi^+$  (resp.  $\Phi^-$ ) the set  $\Phi \cap (\mathbb{N}\alpha_0 + \mathbb{N}\alpha_1)$  (resp.  $-\Phi_+$ ). For  $k \in \mathbb{Z}$  and  $y \in \mathcal{K}$ , we set  $x_{\dot{\alpha}+k\delta}(y) = \left(\begin{pmatrix} 1 & u^k y \\ 0 & 1 \end{pmatrix}, 1\right) \in G$  and  $x_{-\dot{\alpha}+k\delta}(y) = \left(\begin{pmatrix} 1 & 0 \\ u^k y & 1 \end{pmatrix}, 1\right) \in G$ .

Let  $f, g \in \mathcal{K}$  be such that  $\omega(f) = \omega(g) = 0$ . Let  $\ell, n \in \mathbb{Z}$ . Then  $\left(\begin{pmatrix} f\varpi^\ell & 0 \\ 0 & f^{-1}\varpi^{-\ell} \end{pmatrix}, g\varpi^n\right)$  acts on  $\mathbb{A}$  by the translation of vector  $-\ell\dot{\alpha}^\vee - nd$ . For  $\mu = \ell\dot{\alpha}^\vee + nd \in Y$ , we set  $t_\mu = \left(\begin{pmatrix} \varpi^{-\ell} & 0 \\ 0 & \varpi^\ell \end{pmatrix}, \varpi^{-n}\right)$ , which acts by the translation of vector  $\mu$  on  $\mathbb{A}$ . We set  $\lambda = \dot{\alpha}^\vee + 3d$ . We have  $\alpha_0(\lambda) = 1$ ,  $\alpha_1(\lambda) = 2$  and thus  $\lambda \in C_f^v$ .

By [Rou16, 4.12 3 b],  $U_0^{pm+} = \left(\begin{pmatrix} 1+u\mathcal{O}[u] & \mathcal{O}[u] \\ u\mathcal{O}[u] & 1+u\mathcal{O}[u] \end{pmatrix}, 1\right) \cap G$  and similarly

$$U_0^{nm-} = \left(\begin{pmatrix} 1+u^{-1}\mathcal{O}[u^{-1}] & u^{-1}\mathcal{O}[u^{-1}] \\ \mathcal{O}[u^{-1}] & 1+u^{-1}\mathcal{O}[u^{-1}] \end{pmatrix}, 1\right) \cap G.$$

We make the following assumption:

$$\forall n \in \mathbb{N}^*, \ker \pi_n \subset \left(\begin{pmatrix} 1+\varpi^n\mathcal{O}[u, u^{-1}] & 1+\varpi^n\mathcal{O}[u, u^{-1}] \\ 1+\varpi^n\mathcal{O}[u, u^{-1}] & 1+\varpi^n\mathcal{O}[u, u^{-1}] \end{pmatrix} \times (1 + \varpi^n\mathcal{O})\right). \quad (5.4)$$

If for any  $n \in \mathbb{N}^*$ , we have  $\mathfrak{G}(\mathcal{O}/\varpi^n\mathcal{O}) \simeq \left(\begin{pmatrix} (\mathcal{O}/\varpi^n\mathcal{O})[u, u^{-1}] & (\mathcal{O}/\varpi^n\mathcal{O})[u, u^{-1}] \\ (\mathcal{O}/\varpi^n\mathcal{O})[u, u^{-1}] & (\mathcal{O}/\varpi^n\mathcal{O})[u, u^{-1}] \end{pmatrix} \times (\mathcal{O}/\varpi^n\mathcal{O})^\times\right)$  and  $\pi_n$  is the canonical projection, then the assumption is satisfied. However we do not know if it is true. In [Mar18, 7.6],  $\mathfrak{G}$  is described only on fields and in [Kum02, 13], only on  $\mathbb{C}$ .

For  $n \in \mathbb{N}^*$ , we set  $H_n = \ker(\pi_n) \cap \left(\begin{pmatrix} \mathcal{O}[(\varpi u)^n, (\varpi u^{-1})^n] & \mathcal{O}[(\varpi u)^n, (\varpi u^{-1})^n] \\ \mathcal{O}[(\varpi u)^n, (\varpi u^{-1})^n] & \mathcal{O}[(\varpi u)^n, (\varpi u^{-1})^n] \end{pmatrix}, \mathcal{K}^*\right)$ .

**Proposition 5.14.** *If (5.4) is true, then the filtrations  $(H_n)_{n \in \mathbb{N}^*}$  and  $(\mathcal{V}_{n\lambda})_{n \in \mathbb{N}^*}$  are equivalent.*

*Proof.* Let  $n \in \mathbb{N}^*$ . By Lemma 2.4, we have  $U_{[-n\lambda, n\lambda]}^{pm+} = U_{-n\lambda}^{pm+} = t_{-n\lambda}U_0^{pm+}t_{n\lambda}$ . We have  $t_{n\lambda} = t_{n\dot{\alpha}^\vee}t_{3nd}$ . We have

$$t_{-3nd}U_0^{pm+}t_{3nd} \subset \left(\begin{pmatrix} 1+(\varpi^{3n}u)\mathcal{O}[\varpi^{3n}u] & \mathcal{O}[\varpi^{3n}u] \\ (\varpi^{3n}u)\mathcal{O}[\varpi^{3n}u] & 1+(\varpi^{3n}u)\mathcal{O}[\varpi^{3n}u] \end{pmatrix} \times \{1\}\right).$$

We have

$$\begin{aligned} & t_{-n\dot{\alpha}^\vee} \left(\begin{pmatrix} 1+(\varpi^{3n}u)\mathcal{O}[\varpi^{3n}u] & \mathcal{O}[\varpi^{3n}u] \\ (\varpi^{3n}u)\mathcal{O}[\varpi^{3n}u] & 1+(\varpi^{3n}u)\mathcal{O}[\varpi^{3n}u] \end{pmatrix} \times \{1\}\right) t_{n\dot{\alpha}^\vee} \\ & \subset \left(\begin{pmatrix} 1+\varpi^{3n}u\mathcal{O}[\varpi^{3n}u] & \varpi^{2n}\mathcal{O}[\varpi^{3n}u] \\ \varpi^{-2n}(\varpi^{3n}u)\mathcal{O}[\varpi^{3n}u] & 1+\varpi^{3n}u\mathcal{O}[\varpi^{3n}u] \end{pmatrix} \times \{1\}\right) \\ & \subset \left(\begin{pmatrix} 1+(\varpi^n u)\mathcal{O}[\varpi^n u] & \mathcal{O}[\varpi^n u] \\ \varpi^n u\mathcal{O}[\varpi^n u] & 1+\varpi^n u\mathcal{O}[\varpi^n u] \end{pmatrix} \times \{1\}\right) \\ & \subset \mathrm{SL}_2(\mathcal{O}[\varpi^n u]) \times \{1\} \subset \mathrm{SL}_2(\mathcal{O}[\varpi^n u, \varpi^n u^{-1}]) \times \{1\}. \end{aligned}$$

Similarly,  $U_{[-n\lambda, n\lambda]}^{nm-} \subset \mathrm{SL}_2(\mathcal{O}[\varpi^n u, \varpi^n u^{-1}]) \times \{1\}$ .

As  $T_{2n} \subset \mathrm{SL}_2(\mathcal{O}[\varpi^n u, \varpi^n u]) \times \{1 + \varpi^n\mathcal{O}\}$ , we deduce  $\mathcal{V}_{n\lambda} \subset H_n$ , since  $\mathcal{V}_{n\lambda} \subset \ker(\pi_n)$ . Now let  $M \in \mathrm{SL}_2(\mathcal{O}[\varpi^{2n}u, \varpi^{2n}u^{-1}]) \cap \ker(\pi_{2n})$  and  $a \in \mathcal{K}^*$ . Using (5.4), we write

$$M = \begin{pmatrix} 1 + a_0\varpi^{2n} + \sum_{|i| \geq 1} a_i\varpi^{2n|i|}u^i & b_0\varpi^{2n} + \sum_{|i| \geq 1} a_i\varpi^{2n|i|}u^i \\ c_0\varpi^{2n} + \sum_{|i| \geq 1} c_i\varpi^{2n|i|}u^i & 1 + d_0\varpi^{2n} + \sum_{|i| \geq 1} d_i\varpi^{2n|i|}u^i \end{pmatrix},$$

with  $a_i, b_i, c_i, d_i \in \mathcal{O}$ , for all  $i$ . Then

$$t_{-nd}(M, a)t_{nd} = \left( \begin{pmatrix} 1 + a_0\varpi^{2n} + \sum_{|i| \geq 1} a_i \varpi^{n(2|i|-i)} u^i & b_0\varpi^{2n} + \sum_{|i| \geq 1} a_i \varpi^{n(2|i|-i)} u^i \\ c_0\varpi^{2n} + \sum_{|i| \geq 1} c_i \varpi^{n(2|i|-i)} u^i & 1 + d_0\varpi^{2n} + \sum_{|i| \geq 1} d_i \varpi^{n(2|i|-i)} u^i \end{pmatrix}, a \right).$$

Therefore

$$\begin{aligned} & t_{-n\hat{\alpha}^\vee - nd}(M, a)t_{n(\hat{\alpha}^\vee + d)} \\ &= \left( \begin{pmatrix} 1 + a_0\varpi^{2n} + \sum_{|i| \geq 1} a_i \varpi^{n(2|i|-i)} u^i & \varpi^{2n}(b_0\varpi^{2n} + \sum_{|i| \geq 1} a_i \varpi^{n(2|i|-i)} u^i) \\ \varpi^{-2n}(c_0\varpi^{2n} + \sum_{|i| \geq 1} c_i \varpi^{n(2|i|-i)} u^i) & 1 + d_0\varpi^{2n} + \sum_{|i| \geq 1} d_i \varpi^{n(2|i|-i)} u^i \end{pmatrix}, a \right) \\ &\in \mathrm{SL}_2(\mathcal{O}[u, u^{-1}]) \ltimes \mathcal{O}^*. \end{aligned}$$

By [BHR22, Lemma 6.10],  $H_{2n}$  fixes  $n\lambda'$ , where  $\lambda' = \hat{\alpha}^\vee + d$ . Similarly, it fixes  $-n\lambda'$ . Therefore  $H_{2n} \subset G_{[-n\lambda', n\lambda']} \cap \ker \pi_{2n}$ . We have  $G_{[-n\lambda', n\lambda']} = U_{[-n\lambda', n\lambda']}^{pm+} \cdot U_{[-n\lambda', n\lambda']}^{nm-} \cdot \mathfrak{I}(\mathcal{O})$ , by (2.8). Using the inclusion  $\langle U_{[-n\lambda', n\lambda']}^{pm+}, U_{[-n\lambda', n\lambda']}^{nm-} \rangle \subset \ker \pi_n$ , we deduce that  $G_{[-n\lambda', n\lambda']} \cap \ker \pi_n \subset \mathcal{V}_{n\lambda'}$ . As  $(\mathcal{V}_{m\lambda})$  and  $(\mathcal{V}_{m\lambda'})$  are equivalent, we deduce that  $(H_m)$  and  $(\mathcal{V}_{n\lambda})$  are equivalent.  $\square$



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$\mathfrak{G}^{\text{pma}}$ , 9	$\mathcal{S}$ , 5	$\mathcal{V}_\lambda$ , 18
$\mathfrak{G}$ , 6	$\Sigma$ , 29	$W^v$ , 5
$G_x, G_\Omega$ , 11	$T$ , 6	$W_\Lambda$ , 10
$G_\alpha$ , 29	$T_n$ , 18	$X_\alpha$ , 8
$\mathfrak{g}$ , 7	$T_{\alpha^\vee, n}$ , 19	$x_\alpha$ , 6
$\mathfrak{g}_{\alpha, \mathbb{Z}}$ , 8	$\mathcal{T}$ , 5	$Y$ , 5
ht, 5	$\mathfrak{T}(\mathcal{O})$ , 11	$Y^+, Y^{++}$ , 6
$\mathcal{I}$ , 12	$\mathcal{T}$ , 23	$\mathcal{Z}$ , 26
$\text{Inv}(w)$ , 30	$\mathcal{T}((V_n))$ , 14	$\mathcal{Z}_{\mathcal{O}}$ , 26

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