# Topologies on split Kac-Moody groups over valued fields

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**Abstract**

Let $G$ be a minimal split Kac-Moody group over a valued field $\mathcal{K}$. Motivated by the representation theory of $G$, we define two topologies of topological group on $G$, which take into account the topology on $\mathcal{K}$.

## Contents

# 1 Introduction

## 1.1 Motivation from representation theory

Let $G$ be a reductive group over a nonArchimedean local field $\mathcal{K}$. As $G$ is finite dimensional over $\mathcal{K}$, $G$ is naturally equipped with a topological group structure. Its admits a basis of neighbourhood of the identity consisting of open compact subgroups. A complex representation $V$ of $G$ is called smooth if for every $v \in V$, the fixator of $v$ in $G$ is open. To every compact open subgroup $K$ of $G$ is associated a Hecke algebra $\mathcal{H}_K$, which is the space of $K$-bi-invariant functions from $G$ to $\mathbb{C}$ which have compact support. Let $V$ be a smooth representation of $G$. Then the space of $K$-invariant vectors $V^K$ is naturally equipped with the structure of an $\mathcal{H}_K$-module, and we can prove that this assignment induces a bijection between the irreducible smooth representations of $G$ admitting a non zero $K$-invariant vector and the irreducible representations of $\mathcal{H}_K$.

Kac-Moody groups are infinite dimensional generalizations of reductive groups. For example, if $\mathring{\mathfrak{G}}$ is a split reductive group and $\mathcal{F}$ is a field, then the associated affine Kac-Moody group is a central extension of $\mathring{\mathfrak{G}}(\mathcal{F}[u, u^{-1}]) \rtimes \mathcal{F}^*$, where $u$ is an indeterminate. Let now $G = \mathfrak{G}(\mathcal{K})$ be a split Kac-Moody group over $\mathcal{K}$. Recently, Hecke algebras were associated to $G$. In [BK11] and [GR14], Braverman and Kazhdan (in the affine case) and Gaussent and Rousseau (in the general case) associated a spherical Hecke algebra $\mathcal{H}_s$ to $G$, i.e an algebra associated to the spherical subgroup $\mathfrak{G}(\mathcal{O})$ of $G$, where $\mathcal{O}$ is the ring of integers of $\mathcal{K}$. In [BKP16] and [BGR16], Braverman, Kazhdan and Patnaik and Bardy-Panse, Gaussent and Rousseau defined the Iwahori-Hecke algebra $\mathcal{H}_I$ of $G$ (associated to the Iwahori subgroup $K_I$ of $G$). In [AH19], together with Abdellatif, we associated Hecke algebras to certain parahoric subgroups of $G$, which generalizes the construction of

the Iwahori-Hecke algebra of $G$. In [Héb22b], [Héb21a] and [Héb21b], we associated and studied principal series representations of $\mathcal{H}_I$.

For the moment, there is no link between the representations of $G$ and the representations of its Hecke algebras. It seems natural to try to attach an irreducible representation of $G$ to each irreducible representation of $\mathcal{H}_I$. A more modest task would be to associate to each principal series representation $I_\tau$ of $\mathcal{H}_I$ a principal series representation $I(\tau)$ of $G$, which is irreducible when $I_\tau$ is.

Let $T$ be a maximal split torus of $G$ and $Y$ be the cocharacter lattice of $(G,T)$. Let $B$ be a Borel subgroup of $G$ containing $T$. Let $T_{\mathbb{C}} = \mathrm{Hom}_{\mathrm{Gr}}(Y, \mathbb{C}^*)$ and $\tau \in T_{\mathbb{C}}$. Then $\tau$ can be extended to a character $\tau : B \to \mathbb{C}^*$. Assume that $G$ is reductive. Then the principal series representation $I(\tau)$ of $G$ is the induction of $\tau\delta^{1/2}$ from $B$ to $G$, where $\delta : B \to \mathbb{R}_+^*$ is the modulus character of $B$. More explicitly, this is the space of locally constant functions $f : G \to \mathbb{C}$ such that $f(bg) = \tau\delta^{1/2}(b)f(g)$ for every $g \in G$ and $b \in B$. Then $G$ acts on $I(\tau)$ by right translation. Then $I_\tau := I(\tau)^{K_I}$ is a representation of $\mathcal{H}_I$. Assume now that $G$ is a Kac-Moody group. Then we do not know what "locally constant" mean, but we can define the representation $\widehat{I(\tau)}$ of $G$ as the set of functions $f : G \to \mathbb{C}$ such that $f(bg) = \tau\delta^{1/2}(b)f(g)$ for every $g \in G$ and $b \in B$. Let $\mathscr{T}_G$ be a topology of topological group on $G$ such that $K_I$ is open. Then

$$I(\tau)_{\mathscr{T}_G} := \{f \in \widehat{I(\tau)} \mid f \text{ is locally constant for } \mathscr{T}_G\} \tag{1.1}$$

is a subrepresentation of $G$ containing $\widehat{I(\tau)}^{K_I}$. Thus if we look for an irreducible representation containing $\widehat{I(\tau)}^{K_I}$ it is natural to search it inside $I(\tau)_{\mathscr{T}_G}$. Moreover, the more $\mathscr{T}_G$ is coarse, the smaller $I(\tau)_{\mathscr{T}_G}$ is. We thus look for the coarsest topology of topological group on $G$ for which $K_I$ is open.

## 1.2 Topology on $G$, masure and main results

We now assume that $\mathcal{K}$ is any field equipped with a valuation $\omega : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ such that $\omega(\mathcal{K}^*) \supset \mathbb{Z}$. We no longer assume $\mathcal{K}$ to be local, and $\omega(\mathcal{K}^*)$ can be dense in $\mathbb{R}$. Let $\mathcal{O}$ be its ring of integers. Let $\mathfrak{G}$ be a split Kac-Moody group (à la Tits, as defined in [Tit87]) and $G = \mathfrak{G}(\mathcal{K})$. In [GR08] and [Rou16], Gaussent and Rousseau associated to $G$ a kind of Bruhat-Tits building, called a masure, on which $G$ acts (when $G$ is reductive, $\mathcal{I}$ is the usual Bruhat-Tits building). They defined the spherical subgroup $K_s$ as the fixator of some vertex 0 in the masure (we prove in Proposition 3.1 that $K_s = \mathfrak{G}^{\min}(\mathcal{O})$, where $\mathfrak{G}^{\min}$ is the minimal Kac-Moody group defined by Marquis in [Mar18]). They also define the Iwahori subgroup $K_I$ as the fixator of some alcove $C_0^+$ of $\mathcal{I}$. Then we define the topology $\mathscr{T}_{\mathrm{Fix}}$ on $G$ as follows. A subset $V$ of $G$ is open if for every $g \in V$, there exists a finite subset $F$ of $\mathcal{I}$ such that $G_F.g \subset V$, where $G_F$ is the fixator of $F$ in $G$. Then we prove that $\mathscr{T}_{\mathrm{Fix}}$ is the coarsest topology of topological group on $G$ for which $K_I$ is open (see Proposition 4.14). However, it is not Hausdorff in general. Indeed, let $\mathcal{Z} \subset T$ be the center of $G$ and $\mathcal{Z}_{\mathcal{O}} = \mathcal{Z} \cap \mathfrak{T}(\mathcal{O})$. Then $\mathcal{Z}_{\mathcal{O}}$ is the fixator of $\mathcal{I}$ in $G$ and when $\mathcal{Z}_{\mathcal{O}}$ is nontrivial (which already happens for $\mathrm{SL}_2(\mathcal{K})$), $\mathscr{T}_{\mathrm{Fix}}$ is not Hausdorff.

To address this issue, we define an other topology, $\mathscr{T}$, finer than $\mathscr{T}_{\mathrm{Fix}}$ and Hausdorff. Let $\mathbb{A} = Y \otimes \mathbb{R}$ be the standard apartment of $\mathcal{I}$ and $\Phi \subset \mathbb{A}^*$ be the set of roots of $(G,T)$. Then $\mathcal{I} = \bigcup_{g\in G} g.\mathbb{A}$. Let us begin with the case where $G = \mathrm{SL}_m(\mathcal{K})$, for $m \in \mathbb{N}_{\geq 2}$. Let $\varpi \in \mathcal{O}$ be such that $\omega(\varpi) = 1$. For $n \in \mathbb{N}^*$, let $\pi_n : \mathrm{SL}_2(\mathcal{O}) \to \mathrm{SL}_2(\mathcal{O}/\varpi^n\mathcal{O})$ be the natural projection. Then a basis of the neighbourhood of the identity is given by the $(\ker \pi_n)_{n\in\mathbb{N}^*}$. Let $U^+ = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ and $U^- = \left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$. Then one can prove that $\ker \pi_n = (U^+ \cap \ker \pi_n).(U^- \cap \ker \pi_n).(T \cap \ker \pi_n)$. Let $\alpha, -\alpha$ be the two roots of $(G,T)$. Let $x_\alpha : a \mapsto \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ and $x_{-\alpha} : a \mapsto \left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right)$. Then $x_\alpha(\varpi^n\mathcal{O})$ fixes $\{a \in \mathbb{A} \mid \alpha(a) \geq -n\}$ and $x_{-\alpha}(\varpi^n\mathcal{O})$ fixes $\{a \in \mathbb{A} \mid \alpha(a) \leq n\}$. Therefore if $\lambda \in \mathbb{A}$ is such that $\alpha(\lambda) = 1$, we have

$$\ker \pi_n = \left(U^+ \cap \mathrm{Fix}_G([-n\lambda, n\lambda])\right) . \left(U^- \cap \mathrm{Fix}_G([-n\lambda, n\lambda])\right) .(T \cap \ker \pi_n).$$

We now return to the general case for $G$. We prove that the topology associated to $(\ker \pi_n)_{n\in\mathbb{N}^*}$ is not a topology of topological group if $G$ is not reductive (see Lemma 3.3). Let $(\alpha_i)_{i\in I}$ be the set of simple roots of $(G,T)$ and $C_f^v = \{x \in \mathbb{A} \mid \alpha_i(x) > 0, \forall i \in I\}$. Let $W^v$ be the Weyl group of $(G,T)$ and $\lambda \in Y \cap \bigsqcup_{w\in W^v} w.C_f^v$. We define the following subset $\mathcal{V}_{n\lambda}$ of $G$, for $n \in \mathbb{N}^*$:

$$\mathcal{V}_{n\lambda} = \left(U^+ \cap \mathrm{Fix}_G([-n\lambda, n\lambda])\right) . \left(U^- \cap \mathrm{Fix}_G([-n\lambda, n\lambda])\right) .(T \cap \ker \pi_{2N(\lambda)}),$$

where $N(\lambda) = \min\{|\alpha(\lambda)| \mid \alpha \in \Phi_+\}$. We prove the following theorem:

**Theorem 1.1.** *(see Theorem 4.8, Lemma 4.2, Proposition 4.21 and Proposition 5.11):*

1. *For $n \in \mathbb{N}^*$ and $\lambda \in Y \cap \bigsqcup_{w\in W^v} w.C_f^v$, $\mathcal{V}_{n\lambda}$ is a subgroup of $\mathfrak{G}^{\min}(\mathcal{O})$.*
2. *The topology $\mathscr{T}$ associated with $(\mathcal{V}_{n\lambda})_{n\in\mathbb{N}^*}$ is Hausdorff, independent of the choice of $\lambda$ and equips $G$ with the structure of a topological group.*
3. *The topology $\mathscr{T}$ is finer than $\mathscr{T}_{\mathrm{Fix}}$ and if $\mathcal{K}$ is Henselian, we have $\mathscr{T} = \mathscr{T}_{\mathrm{Fix}}$ if and only if $\mathcal{Z}_{\mathcal{O}} = \{1\}$.*
4. *Every compact subset of $G$ has empty interior (for $\mathscr{T}$).*

Note that $\mathscr{T}$ and $\mathscr{T}_{\mathrm{Fix}}$ induce the same topologies on $U^+$ and $U^-$. The main difference comes from what happens in $T$. As the elements of $I(\tau)_{\mathscr{T}}$ and $I(\tau)_{\mathscr{T}_{\mathrm{Fix}}}$ are left $\mathfrak{T}(\mathcal{O})$-invariant, these two spaces are actually equal (see Remark 4.22).

In [HKM13], based on works of Kac and Peterson on the topology of $\mathfrak{G}(\mathbb{C})$, Hartnick, Köhl and Mars defined a Kac-Peterson topology on $\mathfrak{G}(\mathcal{F})$, for any local field $\mathcal{F}$ (Archimedean or not). Assume that $\mathcal{K}$ is local and let $\mathscr{T}_{KP}$ be the Kac-Peterson topology on $G$. We prove that when $G$ is not reductive, then $\mathscr{T}$ is strictly coarser than $\mathscr{T}_{KP}$ (see Proposition 5.3) and thus $\mathscr{T}$ seems more adapted for our purpose.

Assume that $\mathfrak{G}$ is affine $\mathrm{SL}_2$ (with a nonfree set of simple coroots). Then $G = \mathrm{SL}_2(\mathcal{K}[u,u^{-1}]) \rtimes \mathcal{K}^*$. Up to the assumption that $\ker \pi_n \subset \left(\begin{smallmatrix} 1+\varpi^n\mathcal{O}[u,u^{-1}] & \varpi^n\mathcal{O}[u,u^{-1}] \\ \varpi^n\mathcal{O}[u,u^{-1}] & 1+\varpi^n\mathcal{O}[u,u^{-1}] \end{smallmatrix}\right) \rtimes$

$(1+\varpi^n\mathcal{O})$, for $n \in \mathbb{N}^*$, we prove that the topology on $G$ is associated to the filtration $(H_n)_{n\in\mathbb{N}^*}$, where $H_n = \ker(\pi_n) \cap \left( \begin{pmatrix} \mathcal{O}[(\varpi u)^n,(\varpi u^{-1})^n] & \mathcal{O}[(\varpi u)^n,(\varpi u^{-1})^n] \\ \mathcal{O}[(\varpi u)^n,(\varpi u^{-1})^n] & \mathcal{O}[(\varpi u)^n,(\varpi u^{-1})^n] \end{pmatrix}, \mathcal{K}^* \right)$.

The paper is organized as follows. In section 2, we define Kac-Moody groups (as defined by Tits, Mathieu and Marquis) and the masures.

In section 3, we define and study the subgroups $\ker \pi_n$ of $\mathfrak{G}^{\min}(\mathcal{O})$.

In section 4, we define the topologies $\mathscr{T}$ and $\mathscr{T}_{\text{Fix}}$, and compare them.

In section 5, we study the properties of $\mathscr{T}$ and $\mathscr{T}_{\text{Fix}}$: we prove that $\mathscr{T}_{KP}$ is strictly finer than $\mathscr{T}$, we describe the topology in the case of affine $\mathrm{SL}_2$, and we prove that usual subgroups of $G$ (i.e $T$, $N$, $B$, etc.) are closed for $\mathscr{T}$.

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# 2 Kac-Moody groups and masures

In this section, we define Kac-Moody groups and masures. Let $\mathcal{K}$ be a field. There are several possible definitions of Kac-Moody groups and we are interested in the minimal one $\mathfrak{G}(\mathcal{K})$, as defined by Tits in [Tit87]. However, because of the lack of commutation relations in $\mathfrak{G}(\mathcal{K})$, it is convenient to embed it in its Mathieu's positive and negative completions $\mathfrak{G}^{pma}(\mathcal{K})$ and $\mathfrak{G}^{nma}(\mathcal{K})$. Then one define certain subgroups of $\mathfrak{G}(\mathcal{K})$ as the intersection of a subgroup of $\mathfrak{G}^{pma}(\mathcal{K})$ and $\mathfrak{G}(\mathcal{K})$. For example if $\mathfrak{G}$ is affine $\mathrm{SL}_2$ (with a nonfree set of simple roots and coroots), then $\mathfrak{G}(\mathcal{K}) = \mathrm{SL}_2(\mathcal{K}[u,u^{-1}])$, $\mathfrak{G}^{pma}(\mathcal{K}) = \mathrm{SL}_2(\mathcal{K}(\!(u)\!))$ and $\mathfrak{G}^{nma}(\mathcal{K}) = \mathrm{SL}_2(\mathcal{K}(\!(u^{-1})\!))$.

As we want to define congruence subgroups in our framework, we also need to work with Kac-Moody groups over rings: if $\mathcal{K}$ is equipped with a valuation $\omega$ and $\varpi$ is such that $\omega(\varpi) = 1$, then we want to define $\ker \pi_n \subset \mathfrak{G}(\mathcal{O})$, where $\pi_n : \mathfrak{G}(\mathcal{O}) \to \mathfrak{G}(\mathcal{O}/\varpi^n\mathcal{O})$ is the natural projection. The functor defined by Tits in [Tit87] goes from the category of rings to the category of groups. However the fact that it satisfies the axioms defined by Tits is proved only for fields (see [Tit87, 3.9 Theorem 1]) and we do not know if it is "well-behaved" on rings, so we will consider it only as a functor from the category of fields to the category of groups. In [Mar18, 8.8], Marquis introduces a functor $\mathfrak{G}^{\min}$ which goes from the category of rings to the category of groups and he proves that it has nice properties (see [Mar18, Proposition 8.128]), especially on Bézout domains. We will use its functor $\mathfrak{G}^{\min}$. We have $\mathfrak{G}^{\min}(\mathcal{F}) \simeq \mathfrak{G}(\mathcal{F})$ for any field $\mathcal{F}$. This functor is defined as a subfunctor of $\mathfrak{G}^{pma}$, so we first define Tits's functor, then Mathieu's functors and then Marquis's functor.

## 2.1 Standard apartment of a masure

### 2.1.1 Root generating system

A **Kac-Moody matrix** (or generalized Cartan matrix) is a square matrix $A = (a_{i,j})_{i,j\in I}$ indexed by a finite set $I$, with integral coefficients, and such that :

$(i)$ $\forall\ i \in I,\ a_{i,i} = 2$;

$(ii)$ $\forall\ (i,j) \in I^2, (i \neq j) \Rightarrow (a_{i,j} \leq 0)$;

$(iii)$ $\forall\ (i,j) \in I^2,\ (a_{i,j} = 0) \Leftrightarrow (a_{j,i} = 0)$.

A **root generating system** is a 5-tuple $\mathcal{S} = (A, X, Y, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ made of a Kac-Moody matrix $A$ indexed by the finite set $I$, of two dual free $\mathbb{Z}$-modules $X$ and $Y$ of finite rank, and of a family $(\alpha_i)_{i\in I}$ (respectively $(\alpha_i^\vee)_{i\in I}$) of elements in $X$ (resp. $Y$) called **simple roots** (resp. **simple coroots**) that satisfy $a_{i,j} = \alpha_j(\alpha_i^\vee)$ for all $i, j$ in $I$. Elements of $X$ (respectively of $Y$) are called **characters** (resp. **cocharacters**).

Fix such a root generating system $\mathcal{S} = (A, X, Y, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ and set $\mathbb{A} := Y \otimes \mathbb{R}$. Each element of $X$ induces a linear form on $\mathbb{A}$, hence $X$ can be seen as a subset of the dual $\mathbb{A}^*$. In particular, the $\alpha_i$'s (with $i \in I$) will be seen as linear forms on $\mathbb{A}$. This allows us to define, for any $i \in I$, a **simple reflection** $r_i$ of $\mathbb{A}$ by setting $r_i.v := v - \alpha_i(v)\alpha_i^\vee$ for any $v \in \mathbb{A}$. One defines the **Weyl group of** $\mathcal{S}$ as the subgroup $W^v$ of $\mathrm{GL}(\mathbb{A})$ generated by $\{r_i \mid i \in I\}$. The pair $(W^v, \{r_i \mid i \in I\})$ is a Coxeter system, hence we can consider the length $\ell(w)$ with respect to $\{r_i \mid i \in I\}$ of any element $w$ of $W^v$.

The following formula defines an action of the Weyl group $W^v$ on $\mathbb{A}^*$:

$$\forall\ x \in \mathbb{A}, w \in W^v, \alpha \in \mathbb{A}^*,\ (w.\alpha)(x) := \alpha(w^{-1}.x).$$

Let $\Phi := \{w.\alpha_i \mid (w,i) \in W^v \times I\}$ (resp. $\Phi^\vee = \{w.\alpha_i^\vee \mid (w,i) \in W^v \times I\}$) be the set of **real roots** (resp. **real coroots**): then $\Phi$ (resp. $\Phi^\vee$) is a subset of the **root lattice** $Q := \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ (resp. **coroot lattice** $Q^\vee = \bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee$). By [Kum02, 1.2.2 (2)], one has $\mathbb{R}\alpha^\vee \cap \Phi^\vee = \{\pm\alpha^\vee\}$ and $\mathbb{R}\alpha \cap \Phi = \{\pm\alpha\}$ for all $\alpha^\vee \in \Phi^\vee$ and $\alpha \in \Phi$.

We define the **height** $\mathrm{ht} : Q \to \mathbb{Z}$ by $\mathrm{ht}(\sum_{i\in I} n_i\alpha_i) = \sum_{i\in I} n_i$, for $(n_i) \in \mathbb{Z}^I$.

### 2.1.2 Vectorial apartment

As in the reductive case, define the **fundamental chamber** as $C_f^v := \{v \in \mathbb{A} \mid \forall i \in I, \alpha_i(v) > 0\}$.

Let $\mathcal{T} := \bigcup_{w\in W^v} w.\overline{C_f^v}$ be the **Tits cone**. This is a convex cone (see [Kum02, 1.4]).

For $J \subset I$, set $F^v(J) = \{x \in \mathbb{A} \mid \alpha_j(x) = 0,\ \forall j \in J \text{ and } \alpha_j(x) > 0,\ \forall j \in I \setminus J\}$. A **positive vectorial face** (resp. **negative**) is a set of the form $w.F^v(J)$ $(-w.F^v(J))$ for some $w \in W^v$ and $J \subset I$. Then by [Rém02, 5.1 Théorème (ii)], the family of positive vectorial faces of $\mathbb{A}$ is a partition of $\mathcal{T}$ and the stabilizer of $F^v(J)$ is $W_J = \langle J \rangle$.

One sets $Y^{++} = Y \cap \overline{C_f^v}$ and $Y^+ = Y \cap \mathcal{T}$. An element of $Y^+$ is called **regular** if it does not belong to any wall, i.e if it belongs to $\bigsqcup_{w\in W^v} w.C_f^v$.

**Remark 2.1.** *By [Kac94, §4.9] and [Kac94, § 5.8] the following conditions are equivalent:*

1. *the Kac-Moody matrix $A$ is of finite type (i.e. is a Cartan matrix),*
2. $\mathbb{A} = \mathcal{T}$
3. $W^v$ *is finite.*

## 2.2 Split Kac-Moody groups over fields

### 2.2.1 Minimal Kac-Moody groups over fields

Let $\mathfrak{G} = \mathfrak{G}_{\mathcal{S}}$ be the group functor associated in [Tit87] with the root generating system $\mathcal{S}$, see also [Rém02, 8]. Let $\mathcal{K}$ be a field. Let $G = \mathfrak{G}(\mathcal{K})$ be the **split Kac-Moody group over $\mathcal{K}$ associated with $\mathcal{S}$**. The group $G$ is generated by the following subgroups:

- the fundamental torus $T = \mathfrak{T}(\mathcal{K})$, where $\mathfrak{T} = \mathrm{Spec}(\mathbb{Z}[X])$,
- the root subgroups $U_\alpha = \mathfrak{U}_\alpha(\mathcal{K})$, for $\alpha \in \Phi$, each isomorphic to $(\mathcal{K}, +)$ by an isomorphism $x_\alpha$.

The groups $X$ and $Y$ correspond to the character lattice $\mathrm{Hom}(\mathfrak{T}, \mathbb{G}_m)$ and cocharacter lattice $\mathrm{Hom}(\mathbb{G}_m, \mathfrak{T})$ of $\mathfrak{T}$ respectively. One writes $\mathfrak{U}^\pm$ the subgroup of $\mathfrak{G}$ generated by the $\mathfrak{U}_\alpha$, for $\alpha \in \Phi^\pm$ and $U^\pm = \mathfrak{U}^\pm(\mathcal{K})$.

By a simple computation in $\mathrm{SL}_2$, we have for $\alpha \in \Phi$ and $a, b \in \mathcal{K}$ such that $ab \neq -1$:

$$\begin{aligned} x_{-\alpha}(b)x_\alpha(a) &= x_\alpha(a(1+ab)^{-1})\alpha^\vee(1+ab)x_{-\alpha}(b(1+ab)^{-1}) \\ &= x_\alpha(a(1+ab)^{-1})x_{-\alpha}(b(1+ab))\alpha^\vee(1+ab), \end{aligned} \tag{2.1}$$

where $\alpha^\vee = w.\alpha_i^\vee$ if $\alpha = w.\alpha_i$, for $i \in I$ and $w \in W^v$.

Let $\mathfrak{N}$ be the group functor on rings such that if $\mathscr{R}'$ is a ring, $\mathfrak{N}(\mathscr{R}')$ is the subgroup of $\mathfrak{G}(\mathscr{R}')$ generated by $\mathfrak{T}(\mathscr{R}')$ and the $\widetilde{r}_i$, for $i \in I$, where

$$\widetilde{r}_i = x_{\alpha_i}(1)x_{-\alpha_i}(-1)x_{\alpha_i}(1). \tag{2.2}$$

Then if $\mathscr{R}'$ is a field with at least 4 elements, $\mathfrak{N}(\mathscr{R}')$ is the normalizer of $\mathfrak{T}(\mathscr{R}')$ in $\mathfrak{G}(\mathscr{R}')$.

Let $N = \mathfrak{N}(\mathcal{K})$ and $\mathrm{Aut}(\mathbb{A})$ be the group of affine automorphisms of $\mathbb{A}$. Then by [Rou06, 1.4 Lemme], there exists a group morphism $\nu^v : N \to \mathrm{GL}(\mathbb{A})$ such that:

1. for $i \in I$, $\nu^v(\widetilde{r}_i)$ is the simple reflection $r_i \in W^v$,
2. $\ker \nu^v = T$.

The aim of the next two subsubsections is to define Mathieu's Kac-Moody group. This group is defined by assembling three ingredients: the group $\mathfrak{U}^{pma}$, which corresponds to a maximal positive unipotent subgroup of $\mathfrak{G}^{pma}$, the torus $\mathfrak{T}$ and copies of $\mathrm{SL}_2$, one for each simple root $\alpha_i$, $i \in I$.

### 2.2.2 The affine group scheme $\mathfrak{U}^{pma}$

In this subsubsection, we define $\mathfrak{U}^{pma}$. Let $\mathfrak{g}$ be the Kac-Moody Lie algebra over $\mathbb{C}$ associated with $\mathcal{S}$ (see [Kum02, 1.2]) and $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$ be its enveloping algebra. The group $\mathfrak{U}^{pma}(\mathbb{C})$, will be defined as a subgroup of a completion of $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$. As we want to define $\mathfrak{U}^{pma}(\mathscr{R})$, for any ring $\mathscr{R}$, we will also consider $\mathbb{Z}$-forms of $\mathfrak{g}$ and $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$.

The Lie algebra $\mathfrak{g}$ decomposes as $\mathfrak{g} = \bigoplus_{\alpha\in\Delta}\mathfrak{g}_\alpha$, where $\Delta \subset Q$ is the **set of roots** and $\mathfrak{g}_\alpha$ is the proper space associated with $\alpha$, for $\alpha \in \Delta$ (see [Kum02, 1.2]). We have $\Delta = \Delta_+ \sqcup \Delta_-$, where $\Delta_+ = \Delta \cap Q_+$ and $\Delta_- = -\Delta_+$. We have $\Phi \subset \Delta$. The elements of $\Phi = \Delta_{re}$ are called **real roots** and the elements of $\Delta_{im} = \Delta \setminus \Phi$ are called **imaginary roots**.

Following [Tit87, 4] one defines $\mathcal{U}$ as the $\mathbb{Z}$-subalgebra of $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$ generated by $e_i^{(n)} := \frac{e_i^n}{n!}$, $f_i^{(n)} := \frac{f_i^n}{n!}$, $\binom{h}{n}$, for $i \in I$ and $h \in Y$ (where the $e_i$, $f_i$ are the generators of $\mathfrak{g}$, see [Kum02, 1.1]). This is a $\mathbb{Z}$-form of $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$. The algebra $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$ decomposes as $\mathcal{U}_{\mathbb{C}}(\mathfrak{g}) = \bigoplus_{\alpha\in Q}\mathcal{U}_{\mathbb{C}}(\mathfrak{g})_\alpha$ where we use the standard $Q$-graduation on $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$ induced by the $Q$-graduation of $\mathfrak{g}$ (for $i \in I$, $\deg(e_i) = \alpha_i$, $\deg(f_i) = -\alpha_i$, $\deg(h) = 0$, for $h \in Y$, $\deg(xy) = \deg(x) + \deg(y)$ for all $x, y \in \mathcal{U}_{\mathbb{C}}(\mathfrak{g})$ which can be written as a product of nonzero elements of $\mathfrak{g}$). For $\alpha \in Q$, one sets $\mathcal{U}_\alpha = \mathcal{U}_{\mathbb{C}}(\mathfrak{g})_\alpha \cap \mathcal{U}$ and $\mathcal{U}_{\alpha,\mathscr{R}} = \mathcal{U}_\alpha \otimes \mathscr{R}$.

For a ring $\mathscr{R}$, we set $\mathcal{U}_{\mathscr{R}} = \mathcal{U} \otimes_{\mathbb{Z}} \mathscr{R}$. One sets $\widehat{\mathcal{U}}^+ = \prod_{\alpha\in Q_+}\mathcal{U}_\alpha$ and $\widehat{\mathcal{U}}^+_{\mathscr{R}} = \prod_{\alpha\in Q_+}\mathcal{U}_{\alpha,\mathscr{R}}$. This is the completion of $\mathcal{U}^+$ with respect to the $Q_+$-gradation.

If $(u_\alpha) \in \prod_{\alpha\in Q_+}\mathcal{U}_{\alpha,\mathscr{R}}$, we write $\sum_{\alpha\in Q_+} u_\alpha$ the corresponding element of $\widehat{\mathcal{U}}^+_{\mathscr{R}}$. A sequence $(\sum_{\alpha\in Q_+} u_\alpha^{(n)})_{n\in\mathbb{N}}$ converges in $\widehat{\mathcal{U}}_{\mathscr{R}}$ if and only if for every $\alpha \in \Delta_+$, the sequence $(u_\alpha^{(n)})_{n\in\mathbb{N}}$ is stationary.

Let $(E, \leq)$ be a totally ordered set. Let $(u^{(e)}) \in (\widehat{\mathcal{U}}_{\mathscr{R}})^E$. For $e \in E$, write $u = \sum_{\alpha\in Q_+} u_\alpha^{(e)}$, with $u_\alpha^{(e)} \in \mathcal{U}_{\alpha,\mathscr{R}}$, for $\alpha \in Q_+$. We assume that for every $\alpha \in Q_+$, $\{e \in E \mid u_\alpha^{(e)} \neq 0\}$ is finite. Then one sets $\prod_{e\in E} u^{(e)} = \sum_{\alpha\in Q_+} u_\alpha$, where

$$u_\alpha = \sum_{\substack{(\beta_1,\dots,\beta_k)\in Q_+^{(\mathbb{N})},\\ \beta_1+\dots+\beta_k=\alpha}} \sum_{\substack{(e_1,\dots,e_k)\in E,\\ e_1<\dots<e_k}} u_{\beta_1}^{(e_1)}\dots u_{\beta_k}^{(e_k)} \in \mathcal{U}_{\alpha,\mathscr{R}},$$

for $\alpha \in \Delta_+$. This is well-defined since in the sum defining $u_\alpha$, only finitely many nonzero terms appear.

Let $\mathcal{A} = \bigoplus_{\alpha\in Q_+}\mathcal{U}_\alpha^*$, where $\mathcal{U}_\alpha^*$ denotes the dual of $\mathcal{U}_\alpha$ (as a $\mathbb{Z}$-module). We have a natural $\mathscr{R}$-modules isomorphism between $\widehat{\mathcal{U}}^+_{\mathscr{R}}$ and $\mathrm{Hom}_{\mathbb{Z}-\mathrm{lin}}(\mathcal{A}, \mathscr{R})$, for any ring $\mathscr{R}$ (see [Mar18, (8.26)]) and we now identify these two spaces. The algebra $\mathcal{A}$ is equipped with a Hopf algebra structure (see [Mar18, Definition 8.42]). This additional structure equips

$$\mathfrak{U}^{pma}(\mathscr{R}) := \mathrm{Hom}_{\mathbb{Z}-\mathrm{Alg}}(\mathcal{A}, \mathscr{R})$$

with the structure of a group (see [Mar18, Appendix A.2.2]). Otherwise said, $\mathcal{A}$ is the representing algebra of the (infinite dimensional in general) affine group scheme $\mathfrak{U}^{pma} : \mathbb{Z}\text{-Alg} \to \mathrm{Grp}$.

Let $\alpha \in \Delta \cup \{0\}$ and $x \in \mathfrak{g}_{\alpha,\mathbb{Z}}$. An **exponential sequence** for $x$ is a sequence $(x^{[n]})_{n\in\mathbb{N}}$ of elements of $\mathcal{U}$ such that $x^{[0]} = 1$, $x^{[1]} = x$ and $x^{[n]} \in \mathcal{U}_{n\alpha}$ for $n \in \mathbb{Z}_{\geq 1}$ and satisfying the conditions of [Mar18, Defiintion 8.45]. By [Rou16, Proposition 2.7] or [Mar18, Proposition 8.50], such a sequence exists. Note that it is not unique in general. However, if $\alpha \in \Phi_+$, the unique exponential sequence for $x$ is $(x^{(n)})_{n\in\mathbb{N}} = (\frac{1}{n!}x^n)$ by [Rou16, 2.9 2)]. For $r \in \mathscr{R}$, one then sets

$$[\exp](rx) =: \sum_{n\in\mathbb{N}} x^{[n]} \otimes r^n \in \widehat{\mathcal{U}}^+_{\mathscr{R}}.$$

This is the **twisted exponential** of $rx$ associated with the sequence $(x^{[n]})_{n\in\mathbb{N}}$.

We fix for every $\alpha \in \Delta_+$ a $\mathbb{Z}$-basis $\mathcal{B}_\alpha$ of $\mathfrak{g}_{\alpha,\mathbb{Z}} := \mathfrak{g}_\alpha \cap \mathcal{U}$. Set $\mathcal{B} = \bigcup_{\alpha\in\Delta_+} \mathcal{B}_\alpha$. We fix an order on each $\mathcal{B}_\alpha$ and on $\Delta_+$. Let $\alpha \in \Delta_+$. One defines $X_\alpha : \mathfrak{g}_{\alpha,\mathbb{Z}} \otimes \mathscr{R} \to \mathfrak{U}^{pma}(\mathscr{R})$ by $X_\alpha(\sum_{x\in\mathcal{B}_\alpha} \lambda_x.x) = \prod_{x\in\mathcal{B}_\alpha} [\exp]\lambda_x.x$, for $(\lambda_x) \in \mathscr{R}^{\mathcal{B}_\alpha}$. When $\alpha \in \Phi_+$, we have $\mathfrak{g}_{\alpha,\mathbb{Z}} = \mathbb{Z}e_\alpha$, where $e_\alpha$ is defined in [Mar18, Remark 7.6]. One sets $x_\alpha(r) = [\exp](re_\alpha)$, for $r \in \mathscr{R}$. One has $X_\alpha(\mathfrak{g}_{\alpha,\mathbb{Z}}) = x_\alpha(\mathscr{R}) := \mathfrak{U}_\alpha(\mathscr{R})$. By [Mar18, Theorem 8.5.1], every $g \in \mathfrak{U}^{pma}(\mathscr{R})$ can be written in a unique way as a product

$$g = \prod_{\alpha\in\Delta_+} X_\alpha(c_\alpha), \tag{2.3}$$

where $c_\alpha \in \mathfrak{g}_{\alpha,\mathbb{Z}} \otimes \mathscr{R}$, for $\alpha \in \Delta_+$, where the product is taken in the given order on $\Delta_+$.

Let $\Psi \subset \Delta_+$. We say that $\Psi$ is **closed** if for all $\alpha, \beta \in \Psi$, for all $p, q \in \mathbb{N}^*$, $p\alpha+q\beta \in \Delta_+$ implies $p\alpha + q\beta \in \Psi$. Let $\Psi \subset \Delta_+$ be a closed subset. One sets

$$\mathfrak{U}_\Psi(\mathscr{R}) = \prod_{\alpha\in\Psi} X_\alpha(\mathfrak{g}_{\alpha,\mathbb{Z}} \otimes \mathscr{R}) \subset \mathfrak{U}^{pma}(\mathscr{R}).$$

This is a subgroup of $\mathfrak{U}^{pma}$, which does not depend on the chosen order on $\Delta_+$ (for the product). This is not the definition given in [Rou16] or [Mar18, page 210], but it is equivalent by [Mar18, Theorem 8.51].

### 2.2.3 Mathieu's group $\mathfrak{G}^{pma}$

The **Borel subgroup** (it will be a subgroup of $\mathfrak{G}^{pma}$) is $\mathfrak{B} = \mathfrak{B}_\mathcal{S} = \mathfrak{T}_\mathcal{S} \ltimes \mathfrak{U}^{pma}$, where $\mathfrak{T}$ acts on $\mathfrak{U}^{pma}$ as follows. Let $\mathscr{R}$ be a ring, $\alpha \in \Delta_+$, $t \in \mathfrak{T}(\mathscr{R})$, $r \in \mathscr{R}$ and $x \in \mathfrak{g}_{\alpha,\mathscr{R}}$,

$$t[\exp](rx)t^{-1} = [\exp](\alpha(t)rx). \tag{2.4}$$

In particular, if $\alpha \in \Phi$, we have

$$tx_\alpha(r)t^{-1} = x_\alpha(\alpha(t)r).$$

For $i \in I$, let $\mathfrak{U}^Y_{\alpha_i}$ be the reductive group associated with the root generating system $((2), X, Y, \alpha_i, \alpha_i^\vee)$. For each $i \in I$, Mathieu defines an (infinite dimensional) affine group scheme $\mathfrak{P}_i = \mathfrak{U}^Y_{\alpha_i} \ltimes \mathfrak{U}^{ma}_{\Delta_+\setminus\{\alpha_i\}}$ (see [Mar18, Definition 8.65] for the definition of the action of $\mathfrak{U}_{-\alpha_i}$ on $\mathfrak{U}^{ma}_{\Delta_+\setminus\{\alpha_i\}}$).

We do not detail the definition of $\mathfrak{G}^{pma}$ and we refer to [Mat89], [Mar18, 8.7] or [Rou16, 3.6]. This is an ind-group scheme containing the $\mathfrak{P}_i$ for every $i \in I$. Let $w \in W^v$ and write $w = r_{i_1} \dots r_{i_k}$, with $k = \ell(w)$ and $i_1, \dots, i_k \in I$. Then the multiplication map $\mathfrak{P}_{i_1} \times \dots \times \mathfrak{P}_{i_k} \to \mathfrak{G}^{pma}$ is a scheme morphism, and we have $\mathfrak{G}^{pma}(\mathscr{R}) = \bigcup_{(i_1,\dots,i_n)\in \mathrm{Red}(W^v)} \mathfrak{P}_{i_1}(\mathscr{R}) \times \dots \times \mathfrak{P}_{i_k}(\mathscr{R})$, where $\mathrm{Red}(W^v)$ is the set of reduced words of $W^v$ (i.e $\mathrm{Red}(W^v) = \{(i_1, \dots, i_k) \in I^{(\mathbb{N})} \mid \ell(r_{i_1} \dots r_{i_k}) = k\}$).

Let $w \in W^v$, $i \in I$ and $\alpha = w.\alpha_i$. One sets $\mathfrak{U}_\alpha = \widetilde{w}.\mathfrak{U}_{\alpha_i}.\widetilde{w}^{-1}$, where

$$\widetilde{w} = \widetilde{r}_{i_1} \dots \widetilde{r}_{i_k}, \tag{2.5}$$

if $w = r_{i_1} \dots r_{i_k}$ is a reduced decomposition of $w$. There is an isomorphism of group schemes $x_\alpha : \mathbb{G}_a \to \mathfrak{U}_\alpha$ (see [Mar18, page 262]). The $\mathfrak{G}^{pma}$ is generated by the $\mathfrak{P}_i$, $i \in I$. Moreover, if $i \in I$, then $\mathfrak{P}_i$ is generated by $\mathfrak{T}$, $\mathfrak{U}_{\pm\alpha_i}$ and $\widetilde{r}_i = x_{\alpha_i}(1)x_{-\alpha_i}(1)x_{\alpha_i}(1)$. Thus $\mathfrak{G}^{pma}$ is generated by $\mathfrak{U}^{pma}$, $\mathfrak{T}$, $\mathfrak{U}_{-\alpha_i}$ and the $\widetilde{r}_i$, for $i \in I$ and thus we have:

$$\mathfrak{G}^{pma} = \langle \mathfrak{U}^{pma}, \mathfrak{T}, \mathfrak{U}_\alpha, \alpha \in \Phi_- \rangle. \tag{2.6}$$

There is a group functor morphism $\iota : \mathfrak{G} \to \mathfrak{G}^{pma}$ such that for any ring $\mathscr{R}$, $\iota_{\mathscr{R}}$ maps $x_\alpha(r)$ to $x_\alpha(r)$ and $t$ to $t$, for each $\alpha \in \Phi$, $r \in \mathscr{R}$, $t \in \mathfrak{T}(\mathscr{R})$. When $\mathscr{R}$ is a field, this morphism is injective (see [Rou16, 3.12] or [Mar18, Proposition 8.117]).

**Proposition 2.2.** *Let $\mathscr{R}$ and $\mathscr{R}'$ be two rings and $\varphi : \mathscr{R} \to \mathscr{R}'$ be a ring morphism.*

*Let $f_\varphi^{\widehat{\mathcal{U}}^+} : \widehat{\mathcal{U}}^+_{\mathscr{R}} \to \widehat{\mathcal{U}}^+_{\mathscr{R}'}$ and $f_\varphi : \mathfrak{G}^{pma}(\mathscr{R}) \to \mathfrak{G}^{pma}(\mathscr{R}')$ be the induced morphisms. Then $f_\varphi^{\widehat{\mathcal{U}}^+}(\mathfrak{U}^{pma}(\mathscr{R})) \subset \mathfrak{U}^{pma}(\mathscr{R}')$ and we have:*

1. *For every $(r_x) \in \mathscr{R}^{\mathcal{B}}$, $f_\varphi^{\widehat{\mathcal{U}}^+}\left(\prod_{x\in\mathcal{B}}[\exp](r_x x)\right) = \prod_{x\in\mathcal{B}}[\exp](\varphi(r_x)x)$.*
2. *For $\alpha \in \Delta_+$ and $(\lambda_x) \in \mathscr{R}^{\mathcal{B}_\alpha}$, we have $\varphi\left(X_\alpha(\sum_{x\in\mathcal{B}_\alpha} \lambda_x x)\right) = X_\alpha\left(\sum_{x\in\mathcal{B}_\alpha} \varphi(\lambda_x)x\right)$.*
3. *We have $f_\varphi(u) = f_\varphi^{\widehat{\mathcal{U}}^+}(u)$ for $u \in \mathfrak{U}^{pma}(\mathscr{R})$, $f_\varphi(x_\alpha(r)) = x_\alpha(\varphi(r))$, for $\alpha \in \Phi$ and $r \in \mathscr{R}$, and $f_\varphi(\chi(r)) = \chi(\varphi(r))$, for $\chi \in Y$ and $r \in \mathscr{R}^\times$.*
4. *If $\varphi$ is surjective, then $f_\varphi$ is surjective.*

*Proof.* (1), (2) By definition, we have

$$f_\varphi^{\widehat{\mathcal{U}}^+}\Big(\sum_{\alpha\in Q^+}\sum_{j\in J_\alpha} u_{\alpha,j}\otimes r_j\Big) = \sum_{\alpha\in Q^+}\sum_{j\in J_\alpha} u_{\alpha,j}\otimes\varphi(r_j)$$

if $J_\alpha$ is a finite set and $(r_j) \in \mathscr{R}^{J_\alpha}$ and $u_{\alpha,j} \in \mathcal{U}_{\alpha,\mathscr{R}}$, for every $\alpha \in Q_+$. Thus $\varphi$ commutes with infinite sums and product, which proves (1) and (2).

(3) Let $i \in I$. Then the morphism $\mathfrak{P}_i(\mathscr{R}) \to \mathfrak{P}_i(\mathscr{R}')$ induced by $\varphi$ satisfies the formula above. Using the fact that $x_\alpha = \widetilde{w}x_{-\alpha_i}\widetilde{w}^{-1}$, for $\alpha = -w.\alpha_i$, with $w \in W^v$, $i \in I$ and $\widetilde{w}$ defined as in (2.5), we have (3).

(4) Assume $\varphi$ is surjective. By (2.3) and (1), the restriction of $f_\varphi$ to $\mathfrak{U}^{pma}(\mathscr{R})$ is surjective. By (3), the restriction of $f_\varphi(\mathfrak{U}^-(\mathscr{R})) = \mathfrak{U}^-(\mathscr{R}')$ and $f_\varphi(\mathfrak{T}(\mathscr{R})) = \mathfrak{T}(\mathscr{R}')$. We conclude by using the fact that $\mathfrak{G}^{pma}$ is generated by $\mathfrak{U}^{pma}$, $\mathfrak{U}^-$ and $\mathfrak{T}$ (see (2.6)). □

### 2.2.4 Minimal Kac-Moody group over rings

For $i \in I$, there is a natural group morphism $\varphi_i : \mathrm{SL}_2 \to \mathfrak{U}^Y_{\alpha_i}$.

For a ring $\mathscr{R}$, one sets

$$\mathfrak{G}^{\min}(\mathscr{R}) = \langle \varphi_i \left(\mathrm{SL}_2(\mathscr{R})\right), \mathfrak{T}(\mathscr{R}) \rangle \subset \mathfrak{G}^{pma}(\mathscr{R}).$$

This group is introduced by Marquis in [Mar18, Definition 8.126]. By [Mar18, Proposition 8.129], it is a nondegenerate Tits functor in the sense of [Mar18, Definition 7.83] and we have $\mathfrak{G}^{\min}(\mathscr{R}) \simeq \mathfrak{G}(\mathscr{R})$, for any field $\mathscr{R}$.

Note that if $\varphi$ is a ring morphism between two rings $\mathscr{R}$ and $\mathscr{R}'$, the induced morphism $\mathfrak{G}^{pma}(\mathscr{R}) \to \mathfrak{G}^{pma}(\mathscr{R}')$ restricts to a morphism $\mathfrak{G}^{\min}(\mathscr{R}) \to \mathfrak{G}^{\min}(\mathscr{R}')$.

Let $\mathscr{R}$ be a semilocal ring, i.e a ring with finitely many maximal ideals, then by [HO89, 4.3.9 Theorem], $\mathrm{SL}_2(\mathscr{R})$ is generated by $\left(\begin{smallmatrix} 1 & \mathscr{R} \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ \mathscr{R} & 1 \end{smallmatrix}\right)$. Therefore,

$$\mathfrak{G}^{\min}(\mathscr{R}) = \langle \mathfrak{U}_{\pm\alpha_i}(\mathscr{R}), \mathfrak{T}(\mathscr{R}) \mid i \in I \rangle \subset \mathfrak{G}^{pma}(\mathscr{R}). \tag{2.7}$$

## 2.3 Split Kac-Moody groups over valued fields and masures

We now fix a field $\mathcal{K}$ equipped with a valuation $\omega : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ such that $\Lambda := \omega(\mathcal{K}^*)$ contains $\mathbb{Z}$. Let $\mathcal{O} = \{x \in \mathcal{K} \mid \omega(x) \geq 0\}$ be its ring of valuation. We defined Mathieu's positive completion $\mathfrak{G}^{pma}$. Replacing $\Delta_+$ by $w.\Delta_+$, for $w \in W^v$, one can also define a group $\mathfrak{G}^{pma,w}$. Replacing $\Delta_+$ by $\Delta_-$ or by $w.\Delta_-$, for $w \in W^v$, one can also define $\mathfrak{G}^{nma}$ or $\mathfrak{G}^{nma,w}$.

We set $G = \mathfrak{G}(\mathcal{K})$, $G^{pma} = \mathfrak{G}^{pma}(\mathcal{K})$ and $G^{nma} = \mathfrak{G}^{nma}(\mathcal{K})$.

### 2.3.1 Action of $N$ on $\mathbb{A}$

Let $N = \mathfrak{N}(\mathcal{K})$ and $\mathrm{Aut}(\mathbb{A})$ be the group of affine automorphism of $\mathbb{A}$. Then by [Rou16, 4.2], there exists a group morphism $\nu : N \to \mathrm{Aut}(\mathbb{A})$ such that:

1. for $i \in I$, $\nu(\widetilde{r}_i)$ is the simple reflection $r_i \in W^v$, it fixes 0,
2. for $t \in \mathfrak{T}(\mathcal{K})$, $\nu(t)$ is the translation on $\mathbb{A}$ by the vector $\nu(t)$ defined by $\chi(\nu(t)) = -\omega(\chi(t))$, for all $\chi \in X$.
3. we have $\nu(N) = W^v \ltimes (Y \otimes \Lambda) := W_\Lambda$.

### 2.3.2 Affine apartment

A **local face** in $\mathbb{A}$ is the germ $F(x, F^v) = germ_x(x + F^v)$ where $x \in \mathbb{A}$ and $F^v$ is a vectorial face (i.e, $F(x, F^v)$ is the filter of all neighbourhoods of $x$ in $x + F^v$). It is an a **local panel**, positive, or negative if $F^v$ is. If $F^v$ is a chamber, we call $F(x, F^v)$ an **alcove** (or a local chamber). We denote by $C_0^+$ the **fundamental alcove**, i.e, $C_0^+ = germ_0(C_f^v)$.

A **sector** in $\mathbb{A}$ is a subset $\mathfrak{q} = x + C^v$, for $x$ a point in $\mathbb{A}$ and $C^v$ a vectorial chamber. Its **sector germ** (at infinity) is the filter $\mathfrak{Q} = germ_\infty(\mathfrak{q})$ of subsets of $\mathbb{A}$ containing another

sector $x+y+C^v$, with $y \in C^v$. It is entirely determined by its direction $C^v$. This sector or sector germ is said positive (resp. negative) if $C^v$ has this property. We denote by $\pm\infty$ the germ at infinity of $\pm C_f^v$.

For $\alpha \in \Delta$ and $k \in \Lambda \cup \{+\infty\}$, we set $D(\alpha, k) = \{x \in \mathbb{A} \mid \alpha(x) + k \geq 0\}$. A set of the form $D(\alpha, k)$, for $\alpha \in \Delta$ and $k \in \Lambda$ is called a **half-apartment**.

### 2.3.3 Parahoric subgroups

In [Rou16] and [GR08], the masure $\mathcal{I}$ of $G$ is constructed as follows. To each $x \in \mathbb{A}$ is associated a group $\hat{P}_x = G_x$. Then $\mathcal{I}$ is defined in such a way that $G_x$ is the fixator of $x$ in $G$ for the action on $\mathcal{I}$. We actually associate to each filter $\Omega$ on $\mathbb{A}$ a subgroup $G_\Omega \subset G$ (with $G_{\{x\}} = G_x$ for $x \in \mathbb{A}$). Even though the masure is not yet defined, we use the terminology "fixator" to speak of $G_\Omega$, as this will be the fixator of $\Omega$ in $G$. The definition of $G_\Omega$ involves the completed groups $G^{pma}$ and $G^{nma}$.

If $\Omega$ is a non empty subset of $\mathbb{A}$ we sometimes regard it as a filter on $\mathbb{A}$ by identifying it with the filter consisting of the subsets of $\mathbb{A}$ containing $\Omega$. Let $\Omega \subset \mathbb{A}$ be a non empty set or filter. One defines a function $f_\Omega : \Delta \to \mathbb{R}$ by

$$f_\Omega(\alpha) = \inf\{r \in \mathbb{R} \mid \Omega \subset D(\alpha + r)\} = \inf\{r \in \mathbb{R} \mid \alpha(\Omega) + r \subset [0, +\infty[\},$$

for $\alpha \in \Delta$. For $r \in \mathbb{R}$, one sets $\mathcal{K}_{\omega \geq r} = \{x \in \mathcal{K} \mid \omega(x) \geq r\}$, $\mathcal{K}_{\Omega = r} = \{x \in \mathcal{K} \mid \omega(x) = r\}$.

If $\Omega$ is a set, we define the subgroup $U_\Omega^{pma} = \prod_{\alpha \in \Delta_+} X_\alpha(\mathfrak{g}_{\alpha,\mathbb{Z}} \otimes \mathcal{K}_{\omega \geq f_\Omega(\alpha)}) \subset G^{pma}$. Actually, for $\alpha \in \Phi^+ = \Delta_{re}^+$, $X_\alpha(\mathfrak{g}_{\alpha,\mathbb{Z}} \otimes \mathcal{K}_{\omega \geq f_\Omega(\alpha)}) = x_\alpha(\mathcal{K}_{\omega \geq f_\Omega(\alpha)}) =: U_{\alpha,\Omega}$. We then define

$$U_\Omega^{pm+} = U_\Omega^{pma} \cap G = U_\Omega^{pma} \cap U^+,$$

see [Rou16, 4.5.2, 4.5.3 and 4.5.7]. When $\Omega$ is a filter, we set $U_\Omega^{pma} := \cup_{S \in \Omega} U_S^{pma}$ and $U_\Omega^{pm+} := U_\Omega^{pma} \cap G$

We may also consider the negative completion $G^{nma} = \mathfrak{G}^{nma}(\mathcal{K})$ of $G$, and define the subgroup $U_\Omega^{ma-} = \prod_{\alpha \in \Delta_-} X_\alpha(\mathfrak{g}_{\alpha,\mathbb{Z}} \otimes \mathcal{K}_{\omega \geq f_\Omega(\alpha)})$. For $\alpha \in \Phi^- = \Delta_{re}^-$, $X_\alpha(\mathfrak{g}_{\alpha,\mathbb{Z}} \otimes \mathcal{K}_{\omega \geq f_\Omega(\alpha)}) = x_\alpha(\mathcal{K}_{\omega \geq f_\Omega(\alpha)}) =: U_{\alpha,\Omega}$. We then define $U_\Omega^{nm-} = U_\Omega^{ma-} \cap G = U_\Omega^{ma-} \cap U^-$.

Let $\Psi$ be a closed subset of $\Delta_+$. One sets $U_\Omega(\Psi) = \mathfrak{U}_\Psi \cap U_\Omega^{pm+}$. By the uniqueness in the decomposition of the elements of $U^{pma}$ as a product, we have $U_\Omega(\Psi) = \prod_{\alpha \in \Psi} X_\alpha(\mathcal{K}_{\omega \geq f_\Omega(\alpha)})$. We define $U_\Omega(-\Psi)$ similarly.

Let $\Omega$ be a filter on $\mathbb{A}$. We denote by $N_\Omega$ the fixator of $\Omega$ in $N$ (for the action of $N$ on $\mathbb{A}$). If $\Omega$ is not a set, we have $N_\Omega = \bigcup_{S \in \Omega} N_S$. Note that we drop the hats used in [Rou16]. When $\Omega$ is open one has $N_\Omega = N_\mathbb{A} = \mathfrak{T}(\mathcal{O}) := \mathfrak{T}(\mathcal{K}_{\omega \geq 0}) = \mathfrak{T}(\mathcal{K}_{\Omega = 0})$.

If $x \in \mathbb{A}$, we set $G_x = U_x^{pm+}.U_x^{nm-}.N_x$. This is a subgroup of $G$. If $\Omega \subset \mathbb{A}$ is a set, we set $G_\Omega = \bigcap_{x \in \Omega} G_x$ and if $\Omega$ is a filter, we set $G_\Omega = \bigcup_{S \in \Omega} G_S$. Note that in [Rou16], the definition of $G_x$ is much more complicated (see [Rou16, Définition 4.13]). However it is equivalent to this one by [Rou16, Proposition 4.14].

A filter is said to have a "good fixator" if it satisfies [Rou16, Définition 5.3]. There are many examples of filters with good fixators (see [Rou16, 5.7]): points, local faces, sectors, sector germs, $\mathbb{A}$, walls, half apartments, ... For such a filter $\Omega$, we have:

$$G_\Omega = U_\Omega^{pm+}.U_\Omega^{nm-}.N_\Omega = U_\Omega^{nm-}.U_\Omega^{pm+}.N_\Omega. \tag{2.8}$$

We then have:

$$U_\Omega^{pm+} = G_\Omega \cap U^+ \text{ and } U_\Omega^{nm-} = G_\Omega \cap U^-, \tag{2.9}$$

as $U^- \cap U^+.N = U^+ \cap N = \{1\}$, by [Rou16, Remarque 3.17] and [Rém02, 1.2.1 (RT3)].

When $\Omega = C_0^+ = germ_0(C_f^v)$ is the (fundamental) positive local chamber in $\mathbb{A}$, $K_I := G_\Omega$ is called the (fundamental) **Iwahori subgroup**. When $\Omega$ is a face of $C_0^+$, $G_\Omega$ is called a **parahoric subgroup**.

For $\Omega$ a set or a filter, one defines:

$U_\Omega = \langle U_{\alpha,\Omega} \mid \alpha \in \Phi \rangle$ , $U_\Omega^\pm = U_\Omega \cap U^\pm$ and $U_\Omega^{\pm\pm} = \langle U_{\alpha,\Omega} \mid \alpha \in \Phi^\pm \rangle$. Then one has $U_\Omega^{++} \subset U_\Omega^+ \subset U_\Omega^{pm+}$, but these inclusions are not equalities in general, contrary to the reductive case (see [Rou16, 4.12.3 a and 5.7 3)]).

**Lemma 2.3.** *Let $(u_+, u_-, t), (u'_+, u'_-, t') \in U^+ \times U^- \times T$. Assume that $u_+tu_- = u'_+t'u'_-$ or $u_+u_-t = u'_+u'_-t'$ or $tu_+u_- = t'u'_+u'_-$. Then $u_- = u'_-$, $u_+ = u'_+$ and $t = t'$.*

*Proof.* Assume $u_+tu_- = u'_+t'u'_-$. We have $(u'_+)^{-1}u_+t = t'u_-(u'_-)^{-1}$. As $t$ normalizes $U^-$, we deduce the existence of $u''_-$ such that $(u'_+)^{-1}u_+tt'^{-1} = u''_-$. By [Rou06, Proposition 1.5 (DR5)] (there is a misprint in this proposition, $Z$ is in fact $T$), we deduce $(u'_+)^{-1}u_+tt'^{-1} = 1$ and hence $u'_+ = u_+$ and $t = t'$. Therefore $u_- = u'_-$. The other cases are similar. □

By [Rou16, 4.10] and (2.4), we have the following lemma.

**Lemma 2.4.** *Let $\Omega$ be a filter on $\mathbb{A}$, $t \in T$ and $\Psi$ be a closed subset of $\Delta_+$. Then $tU_\Omega^{pm+}t^{-1} = U_{t.\Omega}^{pm+}$, $tU_\Omega^{pm}(\Psi)t^{-1} = U_{t.\Omega}^{pm}(\Psi)$, $tU_\Omega^{nm-}(\Psi)t^{-1} = U_{t.\Omega}^{nm-}(\Psi)$.*

### 2.3.4 Masure

We now define the masure $\mathcal{I} = \mathcal{I}(\mathfrak{G}, \mathcal{K}, \omega)$. As a set, $\mathcal{I} = G \times \mathbb{A}/\sim$, where $\sim$ is defined as follows:

$$\forall (g,x),(h,y) \in G \times \mathbb{A}, (g,x) \sim (h,y) \Leftrightarrow \exists n \in N \mid y = \nu(n).x \text{ and } g^{-1}hn \in G_x.$$

We regard $\mathbb{A}$ as a subset of $\mathcal{I}$ by identifying $x$ and $(1,x)$, for $x \in \mathbb{A}$. The group $G$ acts on $\mathcal{I}$ by $g.(h,x) = (gh,x)$, for $g,h \in G$ and $x \in \mathbb{A}$. An **apartment** is a set of the form $g.\mathbb{A}$, for $g \in G$. The stabilizer of $\mathbb{A}$ in $G$ is $N$ and if $x \in \mathbb{A}$, then the fixator of $x$ in $G$ is $G_x$. More generally, when $\Omega \subset \mathbb{A}$, then $G_\Omega$ is the fixator of $\Omega$ in $G$ and $G_\Omega$ permutes transitively the apartments containing $\Omega$. If $A$ is an apartment, we transport all the notions that are preserved by $W_\Lambda$ (for example segments, walls, faces, chimneys, etc.) to $A$. Then by [Héb22a, Corollary 3.7], if $(\alpha_i)_{i\in I}$ and $(\alpha_i^\vee)_{i\in I}$ are free, then $\mathcal{I}$ satisfies the following axioms:

(MA II) : Let $A, A'$ be two apartments. Then $A \cap A'$ is a finite intersection of half-apartments and there exists $g \in G$ such that $g.A = A'$ and $g$ fixes $A \cap A'$.

(MA III): if $\mathfrak{R}$ is the germ of a splayed chimney and if $F$ is a local face or a germ of a chimney, then there exists an apartment containing $\mathfrak{R}$ and $F$.

We did not recall the definition of a chimney and we refer to [Rou11] for such a definition. We will only use the fact that a sector-germ is a particular case of a chimney-germ.

We also have:

- The stabilizer of $\mathbb{A}$ in $G$ is $N$ and $N$ acts on $\mathbb{A} \subset \mathcal{I}$ via $\nu$.
- The group $U_{\alpha,r} := \{x_\alpha(u) \mid u \in \mathcal{K}, \omega(u) \geq r\}$, for $\alpha \in \Phi, r \in \Lambda$, fixes the half-apartment $D(\alpha, r)$. It acts simply transitively on the set of apartments in $\mathcal{I}$ containing $D(\alpha, r)$.

**Remark 2.5.** *In 2.1.1, we did not assume the freeness of the families $(\alpha_i)_{i\in I}$ and $(\alpha_i^\vee)_{i\in I}$, since there are interesting Kac-Moody groups, which do not satisfy this assumption. For example, $G := \mathrm{SL}_n(\mathcal{K}[u, u^{-1}]) \rtimes \mathcal{K}^*$ is naturally equipped with the structure of a Kac-Moody group associated with a root generating system $\mathcal{S}$ having nonfree coroots. This group is particularly interesting for examples, since it is one of the only Kac-Moody groups in which we can make explicit computations. In [Héb22a], we proved that if $(\alpha_i)_{i\in I}$ and $(\alpha_i^\vee)_{i\in I}$ are free families, then the masure associated with $G$ satisfies (MA II) and (MA III). Without this assumption we do not know. In [Rou16, Théorème 5.16], Rousseau proves that $\mathcal{I}$ satisfies the axioms (MA2) to (MA5) of [Rou11]. We did not introduce these axioms since they are more complicated and a bit less convenient. However it is easy to adapt the proofs of this paper to use the axioms of [Rou11] instead of those of [Héb22a], for example, retractions are already available in [Rou11].*

### 2.3.5 Retraction centred at a sector-germ

Let $\mathfrak{Q}$ be a sector-germ of $\mathbb{A}$. If $x \in \mathcal{I}$, then by (MA III), there exists an apartment $A$ of $\mathcal{I}$ containing $\mathfrak{Q}$ and $x$. By (MA II), there exists $g \in G$ such that $g.A = \mathbb{A}$ and $g$ fixes $A \cap \mathbb{A}$. One sets $\rho_{\mathfrak{Q}}(x) = g.x \in \mathbb{A}$. This is well-defined, independently of the choices of $A$ and $g$, by (MA II). This defines the **retraction $\rho_{\mathfrak{Q}} : \mathcal{I} \to \mathbb{A}$ onto $\mathbb{A}$ and centred at $\mathfrak{Q}$**. When $\mathfrak{Q} = +\infty$, we denote it $\rho_{+\infty}$. If $x \in \mathcal{I}$, then $\rho_{+\infty}(x)$ is the unique element of $U^+.x \cap \mathbb{A}$.

### 2.3.6 Topology defined by a filtration

A **filtration of $G$ by subgroups** is a sequence $(V_n)_{n\in\mathbb{N}^*}$ of subgroups of $G$ such that $V_{n+1} \subset V_n$ for all $n \in \mathbb{N}^*$. Let $(V_n)$ be a filtration of $G$ by subgroups. The associated topology $\mathscr{T}((V_n))$ is the topology on $G$ for which a set $V$ is open if for all $g \in V$, there exists $n \in \mathbb{N}^*$ such that $g.V_n \subset V$.

Let $(V_n), (\widetilde{V}_n)$ be two filtrations of $G$ by subgroups. We say that $(V_n)$ and $(\widetilde{V}_n)$ are equivalent if for all $n \in \mathbb{N}$, there exist $m, \widetilde{m} \in \mathbb{N}$ such that $V_m \subset \widetilde{V}_n$ and $\widetilde{V}_{\widetilde{m}} \subset V_n$. This defines an equivalence relation on the set of filtrations of $G$ by subgroups. Then $(V_n)$ and $(\widetilde{V}_n)$ are equivalent filtrations, if and only if $\mathscr{T}((V_n)) = \mathscr{T}\left((\widetilde{V}_n)\right)$.

We say that $(V_n)$ is **conjugation-invariant** if for all $g \in G$, $(gV_ng^{-1})$ is equivalent to $(V_n)$. Then $\mathscr{T}((V_n))$ equips $G$ with the structure of a topological group if and only if $(V_n)$ is conjugation invariant, by [Mar18, Exercise 8.5].

# 3 Congruence subgroups

In this section, we define and study the congruence subgroups. They will be a key tool in order to define the topology $\mathscr{T}$ in section 4. We prove however that the filtration $(\ker \pi_n)_{n\in\mathbb{N}^*}$ is not conjugation-invariant. We also study how they decompose.

## 3.1 Definition of the congruence subgroup

**Proposition 3.1.** *The fixator $G_0$ of 0 in $G$ is the group $\mathfrak{G}^{\min}(\mathcal{O})$.*

*Proof.* For $i \in I$, $x_{\alpha_i}(\mathcal{O})$, $x_{-\alpha_i}(\mathcal{O})$ and $\mathfrak{T}(\mathcal{O})$ fix 0. Therefore by (2.7), $\mathfrak{G}^{\min}(\mathcal{O}) \subset G_0$.

By [Rou16, Proposition 4.14]

$$G_0 = U_0^{pm+} U_0^{nm-} N_0, \tag{3.1}$$

where $N_0 = \{n \in N \mid n.0 = 0\}$. By [BHR22, 2.4.1 2)], we have

$$U_0^{pm+} = U_0^+ := \langle x_\alpha(u) \mid \alpha \in \Phi, u \in \mathcal{O}\rangle \cap U^+ \subset \mathfrak{G}^{\min}(\mathcal{O})$$

and

$$U_0^{nm-} = U_0^- := \langle x_\alpha(u) \mid \alpha \in \Phi, u \in \mathcal{O}\rangle \cap U^- \subset \mathfrak{G}^{\min}(\mathcal{O}).$$

For $i \in I$, set $\widetilde{r}_i = x_{\alpha_i}(1)x_{-\alpha_i}(-1)x_{\alpha_i}(1) \in \mathfrak{G}^{\min}(\mathcal{O})$. We have $N = \langle \mathfrak{T}(\mathcal{K}), \widetilde{r}_i \mid i \in I\rangle$. Let $n \in N_0$. Write $\nu^v(n) = w \in W^v$, where $\nu^v$ was defined in 2.2.1. Write $w = r_{i_1} \dots r_{i_k}$, with $k = \ell(w)$ and $i_1, \dots, i_k \in I$. Let $n' = \widetilde{r}_{i_1} \dots \widetilde{r}_{i_k} \in N_0$. By [Rou16, 1.6 4)] $\nu^v(n') = w$ and $t := n'^{-1}n \in T \cap \ker(\nu)$. By [Rou16, 4.2 3)], $t \in \mathfrak{T}(\mathcal{O})$. Therefore

$$N_0 = \langle \widetilde{r}_i \mid i \in I\rangle.\mathfrak{T}(\mathcal{O}). \tag{3.2}$$

and in particular, $N_0 \subset \mathfrak{G}^{\min}(\mathcal{O})$. Proposition follows. □

Recall that we assumed that $\Lambda = \omega(\mathcal{K}^*) \supset \mathbb{Z}$. If $\omega(\mathcal{K}^*)$ is discrete, we can normalize $\omega$ so that $\Lambda = \mathbb{Z}$. We fix $\varpi \in \mathcal{O}$ such that $\omega(\varpi) = 1$.

For $n \in \mathbb{N}^*$, we denote by $\pi_n^{pma} : \mathfrak{G}^{pma}(\mathcal{O}) \to \mathfrak{G}^{pma}(\mathcal{O}/\varpi^n\mathcal{O})$ and $\pi_n^{nma} : \mathfrak{G}^{nma}(\mathcal{O}) \to \mathfrak{G}^{nma}(\mathcal{O}/\varpi^n\mathcal{O})$ the morphisms associated with the canonical projection $\mathcal{O} \twoheadrightarrow \mathcal{O}/\varpi^n\mathcal{O}$. We denote by $\pi_n$ the restriction of $\pi_n^{pma}$ to $\mathfrak{G}^{\min}(\mathcal{O})$. By Proposition 2.2, $\pi_n$ is also the restriction of $\pi_n^{nma}$ to $\mathfrak{G}^{\min}(\mathcal{O})$ (it is also the restrictions of $\pi_n^{pma,w} : \mathfrak{G}^{pma,w}(\mathcal{O}) \to \mathfrak{G}^{pma,w}(\mathcal{O}/\varpi^n\mathcal{O})$ and $\pi_n^{nma,w} : \mathfrak{G}^{nma,w}(\mathcal{O}) \to \mathfrak{G}^{nma,w}(\mathcal{O}/\varpi^n\mathcal{O})$, for $w \in W^v$). By Proposition 2.2 and (2.7), $\pi_n$, $\pi_n^{pma}$ and $\pi_n^{nma}$ are surjective. Their kernels are respectively called the **$n$-th congruence subgroups** of $\mathfrak{G}^{\min}(\mathcal{O})$, $\mathfrak{G}^{pma}(\mathcal{O})$ and $\mathfrak{G}^{nma}(\mathcal{O})$.

The family $(\ker \pi_n)_{n\in\mathbb{N}^*}$ is a filtration of $G$. We prove below that it is not conjugation-invariant when $W^v$ is infinite, which motivates the introduction of other filtrations $(\mathcal{V}_{n\lambda})_{n\in\mathbb{N}^*}$, for $\lambda \in Y^+$ regular, in section 4.

**Lemma 3.2.** *Let $x \in \mathbb{A}$ be such that $\alpha_i(x) > 0$ for all $i \in I$. Suppose that $W^v$ is infinite. Then for all $n \in \mathbb{N}^*$, there exists $g \in \ker(\pi_n)$ such that $g.x \neq x$.*

*Proof.* Let $n \in \mathbb{N}^*$. As $\Phi^+$ is infinite, there exists $\beta \in \Phi^+$ such that $\mathrm{ht}(\beta) > \frac{n}{\min_{i\in I} \alpha_i(x)}$. Then $\beta(x) > n$. Let $g = x_{-\beta}(\varpi^n) \in \ker \pi_n$. Then the subset of $\mathbb{A}$ fixed by $g$ is $\{y \in \mathbb{A} \mid -\beta(y) + n \geq 0\}$, which does not contain $x$. □

**Lemma 3.3.** *Assume that $W^v$ is infinite. Then $(\ker(\pi_n))_{n\in\mathbb{N}^*}$ is not conjugation-invariant.*

*Proof.* Suppose that $(\ker(\pi_n))$ is conjugation-invariant. Then the topology $\mathscr{T}((\ker(\pi_n))$ equips $G$ with the structure of a topological group. We have $\ker(\pi_1) \subset \mathfrak{G}^{\min}(\mathcal{O}) = G_0$ and in particular $G_0 = \bigcup_{g\in G_0} g.\ker(\pi_1)$ is open. Let $\lambda \in Y^+$ be such that $\alpha_i(\lambda) = 1$ for all $i \in I$ and $t \in T$ be such that $t.0 = \lambda$. Then $H := tG_0t^{-1}$ is open (since $G$ is a topological group). As $1 \in H$, we deduce the existence of $n \in \mathbb{N}^*$ such that $\ker(\pi_n) \subset H$. As $H$ fixes $\lambda$, this implies that $W^v$ is finite, by Lemma 3.2. □

## 3.2 On the decompositions of the congruence subgroups

Let $\mathfrak{m} = \{x \in \mathcal{O} \mid \omega(x) > 0\}$ be the maximal ideal of $\mathcal{O}$ and $\Bbbk = \mathcal{O}/\mathfrak{m}$. Let $\pi_\Bbbk : \mathfrak{G}^{\min}(\mathcal{O}) \to \mathfrak{G}^{\min}(\Bbbk)$ be the morphism induced by the natural projection $\mathcal{O} \twoheadrightarrow \Bbbk$. When $\omega(\mathcal{K}^*) = \mathbb{Z}$, $\pi_\Bbbk = \pi_1$.

In this subsection we study $\ker \pi_\Bbbk$: we prove that it decomposes as the product of its intersections with $U^-$, $U^+$ and $T$ (see Proposition 3.5), using the masure $\mathcal{I}$ of $G$. We also describe $U^- \cap \ker \pi_\Bbbk$ and $U^+ \cap \ker \pi_\Bbbk$ through their actions on $\mathcal{I}$ and we deduce that $\ker \pi_\Bbbk$ fixes $C_0^+ \cup C_0^-$. It would be interesting to prove similar properties for $\ker \pi_n$ instead of $\ker \pi_\Bbbk$, for $n \in \mathbb{N}^*$. The difficulty is that when $\omega$ is not discrete or $n \geq 2$, $\mathcal{O}/\varpi^n\mathcal{O}$ is no longer a field and very few is known for Kac-Moody groups over rings.

Let $C, C'$ be two alcoves of the same sign based at 0. By [Héb20, Proposition 5.17], there exists an apartment $A$ containing $C$ and $C'$. Let $g \in G$ be such that $g.A = \mathbb{A}$ and $g.C = C_0^+$. Then $g.C'$ is an alcove of $\mathbb{A}$ based at 0 and thus there exists $w \in W^v$ such that $g.C' = w.C_0^+$. We set $d^{W^+}(C, C') = w$, which is well-defined, independently of the choices we made (note that in [BGR16, 1.11] the "$W$-distance" $d^{W^+}$ is defined for more general pairs of alcoves). Then $d^{W^+}$ is $G$-invariant.

**Lemma 3.4.** *Let $C$ be a positive alcove of $\mathcal{I}$ based at 0 and $w \in W^v$. Write $w = r_{i_1} \dots r_{i_k}$, with $k = \ell(w)$ and $i_1, \dots, i_k \in I$. Let $\beta_1 = \alpha_{i_1}$, $\beta_2 = r_{i_1}.\alpha_{i_2}$, …, $\beta_k = r_{i_1} \dots r_{i_{k-1}}.\alpha_{i_k}$. Then $\beta_1, \dots, \beta_k \in \Phi_+$ and we have $\rho_{+\infty}(C) = w.C_0^+$ if and only if there exists $a_1, \dots, a_k \in \mathcal{O}$ such that $C = x_{\beta_1}(a_1) \dots x_{\beta_k}(a_k).\widetilde{r}_{i_1} \dots \widetilde{r}_{i_k}.C_0^+$.*

*Proof.* As $x_{\beta_1}(\mathcal{O}).\dots.x_{\beta_k}(\mathcal{O})$ fixes 0, an element of $x_{\beta_1}(\mathcal{O}) \dots x_{\beta_k}(\mathcal{O}).\widetilde{r}_{i_1} \dots \widetilde{r}_{i_k}.C_0^+$ is a positive alcove based at 0. The fact that $\beta_1, \dots, \beta_k \in \Phi_+$ follows from [Kum02, 1.3.14 Lemma]. Thus $x_{\beta_1}(\mathcal{O}).\dots.x_{\beta_k}(\mathcal{O}).\widetilde{r}_{i_1} \dots \widetilde{r}_{i_k}.C_0^+ \subset U^+.w.C_0^+$ and we have one implication.

We prove the reciprocal by induction on $\ell(w)$. Assume $w = 1$. Then $\rho_{+\infty}(C) = C_0^+$. Let $A$ be an apartment containing $+\infty$ and $C$. Let $g \in G$ be such that $g.A = \mathbb{A}$ and $g$ fixes $A \cap \mathbb{A}$. Then $C = g^{-1}.C_0^+$, by definition of $\rho_{+\infty}$. Moreover, $A$ contains 0 and $+\infty$ and thus it contains $\mathrm{conv}(0, +\infty) \supset C_0^+$. Therefore $C = C_0^+$ and the lemma is clear in this case. Assume now that $\ell(w) \geq 1$.

Let $C'_0 = C_0^+$, $C'_1 = r_{i_1}.C_0^+$, ..., $C'_k = r_{i_1}\dots r_{i_k}.C_0^+ = C'$. Let $C$ be a positive alcove based at 0 and such that $\rho_{+\infty}(C) = C'_k$. Let $A$ be an apartment containing $C$ and $+\infty$. Let $g \in G$ be such that $g.A = \mathbb{A}$ and $g$ fixes $A \cap \mathbb{A}$. Set $C_i = g^{-1}.C'_i$, for $i \in [\![0,k]\!]$. Then $g$ fixes $+\infty$ and hence $\rho_{+\infty}(x) = g.x$ for every $x \in \mathbb{A}$. Therefore $\rho_{+\infty}(C_i) = C'_i$, for $i \in [\![0,k]\!]$. In particular, $\rho_{+\infty}(C_{k-1}) = C'_{k-1}$. By induction, we may assume that there exist $a_1, \dots, a_{k-1} \in \mathcal{O}$ such that $C_{k-1} = u\widetilde{v}.C_0^+$, where $u = x_{\beta_1}(a_1)\dots x_{\beta_k}(a_{k-1})$ and $\widetilde{v} = \widetilde{r}_{i_1}\dots\widetilde{r}_{i_{k-1}}$. Moreover we have

$$d^{W^+}(C_{k-1}, C_k) = d^{W^+}(C'_{k-1}, C'_k) = r_{i_k} = d^{W^+}\left(u^{-1}.C_{k-1}, u^{-1}.C_k\right) = d^{W^+}\left(\widetilde{v}.C_0^+, u^{-1}.C_k\right).$$

Let $P$ be the panel common to $\widetilde{v}.C_0^+$ and $u^{-1}.C_k$. Then $P \subset \beta_k^{-1}(\{0\})$. Let $D$ be the half-apartment delimited by $\beta_k^{-1}(\{0\})$ and containing $C_{k-1}$. Then as $\beta_k(C_{k-1}) = r_{i_1}\dots r_{i_{k-1}}.\alpha_{i_k}(r_{i_1}\dots r_{i_{k-1}}.C_0^+) > 0$, $D$ contains $+\infty$. By [Rou11, Proposition 2.9], there exists an apartment $B$ containing $D$ and $u^{-1}.C_k$. Let $g' \in G$ be such that $g'.B = \mathbb{A}$ and $g'$ fixes $\mathbb{A} \cap B$. We have $g'.u^{-1}.C_k = \rho_{+\infty}(C_k) = C'_k$. By [Rou16, 5.7 7)], $g' \in \mathfrak{T}(\mathcal{O})U_{\beta_k,0}$ and as $\mathfrak{T}(\mathcal{O})$ fixes $\mathbb{A}$, we can assume $g' \in U_{\beta_k,0} = x_{\beta_k}(\mathcal{O})$. Write $g' = x_{\beta_k}(-a_k)$, with $a_k \in \mathcal{O}$. Then $C_k = u.x_{\beta_k}(a_k).C_k = x_{\beta_1}(a_1)\dots x_{\beta_k}(a_k).\widetilde{r}_{i_1}\dots\widetilde{r}_{i_k}.C_0^+$, which proves the lemma. □

**Proposition 3.5.** *1. We have* $U^{nm-}_{C_0^+} = U^- \cap \ker(\pi_{\Bbbk})$ *and* $U^{pm+}_{-C_0^+} = U^+ \cap \ker(\pi_{\Bbbk})$.

*2. We have* $\ker(\pi_{\Bbbk}) = (\ker(\pi_{\Bbbk}) \cap U^+)\,.\,(\ker(\pi_{\Bbbk}) \cap U^-)\,.\,(\ker(\pi_{\Bbbk}) \cap \mathfrak{T}(\mathcal{O}))\,.$

*3. We have* $\ker(\pi_{\Bbbk}) \subset G_{C_0^+ \cup C_0^-}$.

*Proof.* 2) Let $u \in U^{nm-}_{C_0^+}$. By definition, there exists $\Omega \in C_0^+$ such that $u \in U^{nm-}_{\Omega}$. Let $x \in C_f^v \cap \Omega$. Then $u \in \prod_{\alpha\in\Delta_-} X_\alpha(\mathfrak{g}_{\alpha,\mathbb{Z}} \otimes \mathcal{K}_{\omega\geq -\alpha(x)}) \cap G_0$. As $-\alpha(x) > 0$ for every $\alpha \in \Delta_-$, Proposition 2.2 implies:

$$U^{nm-}_{C_0^+} \subset \ker(\pi_{\Bbbk}). \tag{3.3}$$

Let $g \in \ker(\pi_{\Bbbk}) \subset \mathfrak{G}^{\min}(\mathcal{O})$. Then $g$ fixes 0 and $g.C_0^+$ is a positive alcove based at 0. Write $\rho_{+\infty}(g.C_0^+) = w.C_0^+$, with $w \in W^v$. Write $w = r_{i_1}\dots r_{i_m}$, with $m = \ell(w)$ and $i_1, \dots, i_m \in I$. Let $\widetilde{n} = \widetilde{r}_{i_1}\dots\widetilde{r}_{i_m} \in \mathfrak{N}(\mathcal{K})$. By Lemma 3.4, there exists $u \in \langle x_\beta(\mathcal{O}) \mid \beta \in \Phi_+\rangle$ such that $g.C_0^+ = u\widetilde{n}.C_0^+$. Then $g = u\widetilde{n}i$, with $i \in G_{C_0^+}$. As $C_0^+$ has a good fixator ([Rou16, 5.7 2)]), we have (by (2.8):

$$G_{C_0^+} = U^{pm+}_{C_0^+} U^{nm-}_{C_0^+} N_{C_0^+}.$$

As every element of $C_0^+$ has non empty interior, $N_{C_0^+} = \mathfrak{T}(\mathcal{O})$. Moreover, $U^{pm+}_{C_0^+} = U_0^{pm+}$ and $\mathcal{T}(\mathcal{O})$ normalizes $U_0^{pm+}$ and $U^{nm-}_{C_0^+}$. Therefore,

$$G_{C_0^+} = \mathfrak{T}(\mathcal{O}).U_0^{pm+}.U^{nm-}_{C_0^+}.$$

Write $i = tu_+u_-$, with $t \in \mathfrak{T}(\mathcal{O})$, $u_+ \in U_0^{pm+}$ and $u_- \in U^{nm-}_{C_0^+}$.

Therefore by (3.3), we have

$$\pi_{\Bbbk}(g) = 1 = \pi_{\Bbbk}(u\widetilde{n}tu_+u_-) = \pi_{\Bbbk}(u)\pi_{\Bbbk}(\widetilde{n}t)\pi_{\Bbbk}(u_+)\pi_{\Bbbk}(u_-) = \pi_{\Bbbk}(u)\pi_{\Bbbk}(\widetilde{n}t)\pi_{\Bbbk}(u_+).$$

By [Rou16, 3.16 Proposition] or [Mar18, Theorem 8.118],
$(\mathfrak{G}^{pma}(\Bbbk), \mathfrak{N}(\Bbbk), \mathfrak{U}^{pma}(\Bbbk), \mathfrak{U}^{-}(\Bbbk), \mathfrak{T}(\Bbbk), \{r_i \mid i \in I\})$ is a refined Tits system. By [Rou16, 3.16 Remarque], we have the Birkhoff decomposition

$$\mathfrak{G}^{pma}(\Bbbk) = \bigsqcup_{n\in\mathfrak{N}(\Bbbk)} \mathfrak{U}^{+}(\Bbbk)n\mathfrak{U}^{pma}(\Bbbk).$$

As $\pi_{\Bbbk}(u) \in \mathfrak{U}^{+}(\Bbbk)$, $\pi_{\Bbbk}(\widetilde{n}t) \in \mathfrak{N}(\Bbbk)$ and $\pi_{\Bbbk}(u_+) \in \mathfrak{U}^{pma}(\Bbbk)$, we deduce $\pi_{\Bbbk}(\widetilde{n}t) = 1$.

By [Rou06, 1.4 Lemme and 1.6], there exists a group morphism $\nu^v_{\Bbbk} : \mathfrak{N}(\Bbbk) \to W^v$ such that $\nu^v_{\Bbbk}(\widetilde{r}_i) = r_i$ for $i \in I$ and $\nu^v_{\Bbbk}(\mathfrak{T}(\Bbbk)) = 1$. Then $\nu^v_{\Bbbk}(\widetilde{n}t) = w = 1$. Therefore $w = 1$ and $g = utu_+u_- = u'tu_-$, for some $u' \in U_0^{pm+}$, since $t$ normalizes $U_0^{pm+}$. By Lemma 2.3 and by symmetry of the roles of $U^{nm-}$ and $U^{pm+}$, we have $u' \in U^{pm+}_{-C_0^+}$. By (3.3) applied to $U^{pm+}_{-C_0^+}$, we have $\pi_{\Bbbk}(g) = 1 = \pi_{\Bbbk}(u')\pi_{\Bbbk}(t)\pi_{\Bbbk}(u-) = \pi_{\Bbbk}(t)$ and thus

$$g \in U^{pm+}_{-C_0^-}.(T \cap \ker \pi_{\Bbbk}).U^{nm-}_{C_0^+} = U^{pm+}_{-C_0^-}.U^{nm-}_{C_0^+}.(T \cap \ker \pi_{\Bbbk}).$$

By (3.3), we deduce 2).

3) We have $U^{pm+}_{-C_0^+} = U^{pm+}_{-C_0^+\cup C_f^v} \subset G_{C_0^+\cup -C_0^+}$, $U^{nm-}_{C_0^+} = U^{nm-}_{C_0^+\cup -C_f^v} \subset G_{C_0^+\cup -C_0^+}$ and $T \cap \ker \pi_{\Bbbk} \subset T \cap G_0 \subset G_{\mathbb{A}}$, which proves 3).

1) We already proved one inclusion. Let $u \in \ker \pi_{\Bbbk} \cap U^-$. Then by what we proved above, $u \in U^{pm+}_{-C_0^-}.(T \cap \ker \pi_{\Bbbk}).U^{nm-}_{C_0^+}$. By Lemma 2.3, $u \in U^{nm-}_{C_0^+}$, and the proposition follows. □

**Corollary 3.6.** *Let $n \in \mathbb{N}^*$. Then $\ker \pi_n \subset U^{pm+}_{-C_0^+}.\mathfrak{T}(\mathcal{O}).U^{nm-}_{C_0^+}$.*

**Remark 3.7.** *Let $u \in U^- \cap \ker \pi_{\Bbbk} = U^{nm-}_{C_0}$. Write $u = \prod_{\alpha\in\Delta_-} X_\alpha(v_\alpha)$, where $v_\alpha \in \mathfrak{g}_{\alpha,\mathbb{Z}}\otimes\mathcal{O}$, for every $\alpha \in \Delta_-$. Define $\omega : \mathfrak{g}_{\alpha,\mathbb{Z}}\otimes\mathcal{K} \to \mathbb{R}\cup\{+\infty\}$ by $\omega(v) = \inf\{x \in \mathbb{R} \mid v \in \mathfrak{g}_{\alpha,\mathbb{Z}}\otimes\mathcal{K}_{\omega\geq x}\}$, for $v \in \mathfrak{g}_{\alpha,\mathbb{Z}}\otimes\mathcal{K}$. Let $\lambda \in Y$ be such that $\alpha_i(\lambda) = 1$ for every $i \in I$. Let $\Omega \in C_0^+$ be such that $u \in U^{nm-}_{\Omega}$ and $\eta \in \mathbb{R}^*_+$ be such $\eta\lambda \in \Omega$. Then $u \in U^{nm-}_{\Omega}$ implies $\omega(v_\alpha) \geq |\alpha(\eta\lambda)| = \eta\mathrm{ht}(\alpha)$ for every $\alpha \in \Delta_-$. In particular, $\omega(v_\alpha)$ goes to $+\infty$ when $-\mathrm{ht}(\alpha)$ goes to $+\infty$.*

# 4 Definition of topologies on $G$

In this section, we define two topologies $\mathscr{T}$ and $\mathscr{T}_{\mathrm{Fix}}$ on $\mathcal{I}$ and compare them. For the first one, we proceed as follows. We define a set $\mathcal{V}_\lambda$ for every regular $\lambda \in Y^+$. We prove that it is actually a subgroup of $\mathfrak{G}^{\min}(\mathcal{O})$ (Lemma 4.2) and we define $\mathscr{T}$ as the topology associated with $(\mathcal{V}_{n\lambda})_{n\in\mathbb{N}^*}$. We then prove that $\mathscr{T}$ does not depend on the choice of $\lambda$ and that it is conjugation-invariant (Theorem 4.8) and thus that $(G, \mathscr{T})$ is a topological group. We then introduce the topology $\mathscr{T}_{\mathrm{Fix}}$ associated with the fixators of finite subsets of $\mathcal{I}$ and we end up by a comparison of $\mathscr{T}$ and $\mathscr{T}_{\mathrm{Fix}}$.

## 4.1 Subgroup $\mathcal{V}_\lambda$

For $n \in \mathbb{N}^*$, we set $T_n = \ker \pi_n \cap T \subset \mathfrak{T}(\mathcal{O})$. For $\lambda \in Y^+$ regular, we set

$$N(\lambda) = \min\{|\alpha(\lambda)| \mid \alpha \in \Phi\} \in \mathbb{N}^* \text{ and } \mathcal{V}_\lambda = U^{pm+}_{[-\lambda,\lambda]}.U^{nm-}_{[-\lambda,\lambda]}.T_{2N(\lambda)}.$$

By (2.9), we have

$$\mathcal{V}_\lambda = (U^+ \cap G_{[-\lambda,\lambda]}).(U^- \cap G_{[-\lambda,\lambda]}).(T \cap \ker \pi_{2N(\lambda)}) \subset G_{[-\lambda,\lambda]}. \tag{4.1}$$

The $2N(\lambda)$ appearing comes from $x_{-\alpha}(\varpi^n\mathcal{O}).x_\alpha(\varpi^n\mathcal{O}) \subset x_\alpha(\varpi^n\mathcal{O})x_{-\alpha}(\varpi^n\mathcal{O})\alpha^\vee(1+\varpi^{2n}\mathcal{O})$ for $\alpha \in \Phi$ and $n \in \mathbb{N}^*$, which follows from (2.1). To prove that $\mathcal{V}_\lambda$ is a group, the main difficulty is to prove that it is stable by left multiplication by $U^{nm-}_{[-\lambda,\lambda]}$. If $G$ is reductive, we have $U^{nm-}_{[-\lambda,\lambda]} = U^{--}_{[-\lambda,\lambda]} := \langle x_\alpha(a) \mid \alpha \in \Phi_-, a \in \mathcal{K}, \omega(a) \geq |\alpha(\lambda)|\rangle$. By induction, it then suffices to prove that $x_{-\alpha}(\varpi^{|\alpha(\lambda)|}\mathcal{O})\mathcal{V}_\lambda \subset \mathcal{V}_\lambda$, for $\alpha \in \Phi_+$. When $G$ is no longer reductive, we have $U^{--}_{[-\lambda,\lambda]} \subsetneq U^{nm-}_{[-\lambda,\lambda]}$ in general (see [Rou16, 4.12 3)]). The group $U^{nm-}_{[-\lambda,\lambda]}$ is defined as a set of infinite products and so its seems difficult to reason by induction in our case. We could try to use the group $U^-_{[-\lambda,\lambda]} := \langle x_\alpha(a) \mid \alpha \in \Phi, a \in \mathcal{K}, \omega \geq |\alpha(a)|\rangle \cap U^-$ since it is sometimes equal to $U^{nm-}_{[-\lambda,\lambda]}$ (for example when $\lambda \in C^v_f$, $U^{nm-}_{[-\lambda,\lambda]} = U^{nm-}_{-\lambda} = U^-_\lambda = U^-_{[-\lambda,\lambda]}$ by [BHR22, 2.4.1 2)]). However it seems difficult since if $\alpha \in \Phi_+$, the condition $\omega(a)+\alpha(\lambda) \geq 0$ allows elements with a negative valuation. In order to overcome these difficulties, we use the morphisms $\pi_n$, for $n \in \mathbb{N}$.

By definition,

$$U^{pm+}_{[-\lambda,\lambda]} = G \cap \prod_{\alpha\in\Delta_+} X_\alpha\left(\mathfrak{g}_{\alpha,\mathbb{Z}} \otimes \mathcal{K}_{\omega\geq f_\Omega([-\lambda,\lambda])}\right) \subset G \cap \prod_{\alpha\in\Delta_+} X_\alpha\left(\mathfrak{g}_{\alpha,\mathbb{Z}} \otimes \varpi^{N(\lambda)}\mathcal{O}\right).$$

By Proposition 2.2 we deduce $U^{pm+}_{[-\lambda,\lambda]} \subset \ker\left(\pi_{N(\lambda)}\right)$. Using a similar reasoning for $U^{nm-}_{[-\lambda,\lambda]}$ we deduce

$$\mathcal{V}_\lambda \subset \ker(\pi_{N(\lambda)}). \tag{4.2}$$

For $n \in \mathbb{N}$ and $\alpha^\vee \in \Phi^\vee$, one sets $T_{\alpha^\vee,n} = \alpha^\vee(1+\varpi^n\mathcal{O}) \subset \mathfrak{T}(\mathcal{O})$.

**Lemma 4.1.** *Let $\lambda \in Y^+$ be regular and $\alpha \in \Phi$. Then $U_{\alpha,[-\lambda,\lambda]}.U_{-\alpha,[-\lambda,\lambda]}.T_{\alpha^\vee,2N(\lambda)}$ is a subgroup of $G$.*

*Proof.* Set $\Omega = [-\lambda,\lambda]$. Set $H = U_{\alpha,\Omega}.U_{-\alpha,\Omega}.T_{\alpha^\vee,2N(\lambda)}$. It suffices to prove that $H$ is stable under left multiplication by $U_{\alpha,\Omega}, U_{-\alpha,\Omega}$ and $T_{\alpha^\vee,2N(\lambda)}$. The first stability is clear and the third follows from Lemma 2.4 and the fact that $T_{\alpha^\vee,2N(\lambda)} \subset \mathfrak{T}(\mathcal{O})$ fixes $\mathbb{A}$. Let $u_-, \widetilde{u_-} \in U_{-\alpha,\Omega}$, $u_+ \in U_{\alpha,\Omega}$ and $t \in T_{\alpha^\vee,2N(\lambda)}$. Write $u_- = x_{-\alpha}(a_-)$, $\widetilde{u}_- = x_{-\alpha}(\widetilde{a}_-)$ and $u_+ = x_\alpha(a_+)$, for $a_-, \widetilde{a}_-, a_+ \in \mathcal{K}$. We have $\omega(\widetilde{a}_-), \omega(a_-), \omega(a_+) \geq |\alpha(\lambda)| \geq N(\lambda)$. Then by (2.1), we have

$$\widetilde{u}_-u_+u_-t = x_\alpha\left(a_+(1+\widetilde{a}_-a_+)^{-1}\right) x_{-\alpha}\left(\widetilde{a}_-(1+\widetilde{a}_-a_+)^{-1} + a_-\right) \alpha^\vee(1+\widetilde{a}_-a_+)t.$$

As $\omega(1+\widetilde{a}_-a_+) = 1$, we deduce that $\widetilde{u}_-u_+u_-t \in H$, which proves the lemma. ∎

Let $w \in W^v$ and $\Omega$ be a filter on $\mathbb{A}$. Recall that $\mathfrak{G}^{pma,w}$ and $\mathfrak{G}^{nma,w}$ are the completions of $\mathfrak{G}$ with respect to $w.\Delta_+$ and $w.\Delta_-$ respectively. One defines $U_\Omega^{pm}(w.\Delta_+)$ and $U_\Omega^{nm}(w.\Delta_-)$ similarly as $U_\Omega^{pm+}$ and $U_\Omega^{nm-}$ in these groups.

**Lemma 4.2.** *Let $\lambda \in Y^+$ be regular. Set $\mathcal{V}_\lambda = U^{pm+}_{[-\lambda,\lambda]}.U^{nm-}_{[-\lambda,\lambda]}.T_{2N(\lambda)}$. Then*

1. *$\mathcal{V}_\lambda = U^{pm+}_{[-\lambda,\lambda]}(w.\Delta_+).U^{nm-}_{[\lambda,\lambda]}(w.\Delta_-).T_{2N(\lambda)}$ for every $w \in W^v$,*
2. *$\mathcal{V}_\lambda$ is a subgroup of $G$.*

*Proof.* (1) This follows the proof of [GR08, Proposition 3.4]. Set $\Omega = [-\lambda, \lambda]$. Let $i \in I$ and $\alpha = \alpha_i$. By [GR08, 3.3.4)] and Lemma 4.1,

$$\begin{aligned}
\mathcal{V}_\lambda &= U_\Omega^{pm}(\Delta_+ \setminus \{\alpha\}).U_\Omega^{nm}(\Delta_- \setminus \{\alpha\}).U_{\alpha,\Omega}.U_{-\alpha,\Omega}.T_{2N(\lambda)} \\
&= U_\Omega^{pm}(\Delta_+ \setminus \{\alpha\}).U_\Omega^{nm}(\Delta_- \setminus \{\alpha\}).U_{-\alpha,\Omega}.U_{\alpha,\Omega}.T_{2N(\lambda)} \\
&= U_\Omega^{pm}(\Delta_+ \setminus \{\alpha\}).U_{-\alpha,\Omega}.U_\Omega^{nm}(\Delta_- \setminus \{\alpha\}).U_{\alpha,\Omega}.T_{2N(\lambda)} \\
&= U_\Omega^{pm}(r_i.\Delta_+).U_\Omega^{nm}(r_i.\Delta_-).T_{2N(\lambda)}
\end{aligned}$$

Therefore $\mathcal{V}_\lambda$ does not change when $\Delta_+$ is replaced by $w.\Delta_+$, for $w \in W^v$, which proves (1).

Let $w \in W^v$ be such that $\lambda \in w.C_f^v$. By (1) we have

$$\mathcal{V}_\lambda = U_\Omega^{pm}(w.\Delta_+).U_\Omega^{nm}(w.\Delta_-).T_{2N(\lambda)}.$$

Let $t \in T$ be such that $t.0 = \lambda$. By Lemma 2.4, we have

$$tU_\Omega^{pm}(w.\Delta_+)t^{-1} = U_{t.\Omega}^{pm}(w.\Delta_+) = U_{[0,2\lambda]}^{pm}(w.\Delta_+) = U_0^{pm}(w.\Delta_+).$$

Similarly, $tU_\Omega^{nm}(w.\Delta_-)t^{-1} = U_{2\lambda}^{nm}(w.\Delta_-)$. As $T$ is commutative, we also have $tT_{2N(\lambda)}t^{-1} = T_{2N(\lambda)}$. In order to prove (2), it suffices to prove that

$$H := tU_\Omega^{pm}(w.\Delta_+)U_\Omega^{nm}(w.\Delta_-)T_{2N(\lambda)}t^{-1} = U_0^{pm}(w.\Delta_+)U_{2\lambda}^{nm}(w.\Delta_-)T_{2N(\lambda)}$$

is a subgroup of $G$. It suffices to prove that $H$ is stable under left multiplication by $U_0^{pm}(w.\Delta_+)$, $U_{2\lambda}^{nm}(w.\Delta_-)$ and $T_{2N(\lambda)}$. The first stability is clear and the third follows from Lemma 2.4.

First note that by [Rou16, 5.7 1)],

$$G_{[0,2\lambda]} = U_0^{pm}(w.\Delta_+)U_{2\lambda}^{nm}(w.\Delta_-)\mathfrak{T}(\mathcal{O}) \tag{4.3}$$

is a subgroup of $G$.

Let $u_-, \widetilde{u_-} \in U_{2\lambda}^{nm}(w.\Delta_-)$, $u_+ \in U_0^{pm}(w.\Delta_+)$, $t \in T_{2N(\lambda)}$. Let us prove that $\widetilde{u}_-u_+u_-t \in H$. We have $\widetilde{u}_-u_+ = u_+(u_+^{-1}\widetilde{u}_-u_+)$. By Proposition 2.2, $U_{2\lambda}^{nm}(w.\Delta_-) \subset \ker \pi_{2N(\lambda)}$. By Proposition 3.1, $u_+ \in \mathfrak{G}^{\min}(\mathcal{O})$ and we have $u_+^{-1}\widetilde{u}_-u_+ \in G_{[0,2\lambda]} \cap \ker \pi_{2N(\lambda)}$. Therefore (4.3)

implies that we can write $u_+^{-1}\widetilde{u}_-u_+ = u_1^+u_1^-t_1$, where $u_1^+ \in U_0^{pm}(w.\Delta_+)$, $u_1^- \in U_{2\lambda}^{nm}(w.\Delta_-)$ and $t_1 \in \mathfrak{T}(\mathcal{O})$. We have

$$\pi_{2N(\lambda)}(u_1^+u_1^-t_1) = 1 = \pi_{2N(\lambda)}(u_1^+t_1) = \pi_{2N(\lambda)}(u_1^+)\pi_{2N(\lambda)}(t_1).$$

As $\mathfrak{B}(\mathcal{O}/\varpi^{2N(\lambda)}\mathcal{O}) = \mathfrak{T}(\mathcal{O}/\varpi^{2N(\lambda)}\mathcal{O})\ltimes\mathfrak{U}^{pma,w}(\mathcal{O}/\varpi^{2N(\lambda)}\mathcal{O})$ and as $\pi_{2N(\lambda)}(\mathfrak{T}(\mathcal{O})) \subset \mathfrak{T}(\mathcal{O}/\varpi^{2N(\lambda)}\mathcal{O})$ and $\pi_{2N(\lambda)}(\mathfrak{U}^{pma,w}(\mathcal{O})) \subset \mathfrak{U}^{pma,w}(\mathcal{O}/\varpi^{2N(\lambda)}\mathcal{O})$, we deduce $\pi_{2N(\lambda)}(u_1^+) = \pi_{2N(\lambda)}(t_1) = 1$ and $t_1 \in T_{2N(\lambda)}$. We have

$$\widetilde{u}_-u_+u_-t = u_+u_1^+u_1^-t_1u_-t$$

and as $T_{2N(\lambda)}$ normalizes $U_{2\lambda}^{nm}(w.\Delta_-)$, $\widetilde{u}_-u_+u_-t \in H$, which proves the lemma.

□

**Remark 4.3.** *1. Note that if $\lambda \in Y^+$ is regular, then $U_{[-\lambda,\lambda]}^{pm+}U_{[-\lambda,\lambda]}^{nm-}T_{2N(\lambda)+1}$ is not a subgroup of $G$. Indeed, take $\alpha \in \Phi^+$ such that $|\alpha(\lambda)| = N(\lambda)$. Then $x_{-\alpha}(\varpi^{N(\lambda)})x_\alpha(\varpi^{N(\lambda)}) = x_\alpha(\varpi^{N(\lambda)}(1+\varpi^{2N(\lambda)})^{-1})x_{-\alpha}(\varpi^{N(\lambda)}(1+\varpi^{2N(\lambda)})^{-1})\alpha^\vee(1+\varpi^{2N(\lambda)})$ and by Lemma 2.3, this does not belong to $U_{[-\lambda,\lambda]}^{pm+}U_{[-\lambda,\lambda]}^{nm-}T_{2N(\lambda)+1}$.*

*2. For every $k \in [\![0, 2N(\lambda)]\!]$, $U_{[-\lambda,\lambda]}^{pm+}.U_{[-\lambda,\lambda]}^{nm-}.T_k = \mathcal{V}_\lambda.T_k$ is a subgroup of $G$, since $T_k$ normalizes $U_{[-\lambda,\lambda]}^{pm+}$ and $U_{[-\lambda,\lambda]}^{nm-}$ by Lemma 2.4. Note that for the definition of a topology, we could also have taken the filtration $(U_{[-n\lambda,n\lambda]}^{pm+}.U_{[-n\lambda,n\lambda]}^{nm-}.T_{k(n)})_{n\in\mathbb{N}^*}$, for any $k(n) \in [\![0, 2nN(\lambda)]\!]$ such that $k(n) \underset{n\to+\infty}{\to} +\infty$.*

*3. As $\mathcal{V}_\lambda = \mathcal{V}_\lambda^{-1}$ and by Lemma 2.4, we have $\mathcal{V}_\lambda = U_{[-\lambda,\lambda]}^{nm-}U_{[-\lambda,\lambda]}^{pm+}T_{2N(\lambda)}$.*

## 4.2 Filtration $(\mathcal{V}_{n\lambda})_{n\in\mathbb{N}^*}$

Let $\Omega$ be a filter on $\mathbb{A}$. One defines $\mathrm{cl}^\Delta(\Omega)$ as the filter on $\mathbb{A}$ consisting of the subsets $\Omega'$ of $\mathbb{A}$ for which there exists $(k_\alpha) \in \prod_{\alpha\in\Delta}\Lambda_\alpha \cup \{+\infty\}$ such that $\Omega' \supset \bigcap_{\alpha\in\Delta} D(\alpha, k_\alpha) \supset \Omega$, where $\Lambda_\alpha = \Lambda$ if $\alpha \in \Phi$ and $\Lambda_\alpha = \mathbb{R}$ otherwise. Note that $\mathrm{cl}^\Delta$ is denoted cl in [Rou11] and [Rou16]. By definition of $U_\Omega^{pm+}$ and $U_\Omega^{nm-}$, we have

$$U_\Omega^{pm+} = U_{\Omega'}^{pm+} = U_{\mathrm{cl}^\Delta(\Omega)}^{pm+} \text{ and } U_\Omega^{nm-} = U_{\Omega'}^{nm-} = U_{\mathrm{cl}^\Delta(\Omega)}^{nm-}, \tag{4.4}$$

for any filter $\Omega'$ such that $\Omega \subset \Omega' \subset \mathrm{cl}^\Delta(\Omega)$.

**Lemma 4.4.** *Let $\lambda \in C_f^v$ and $w \in W^v$. Then $\mathrm{cl}^\Delta([-w.\lambda, w.\lambda]) \supset (-w.\lambda + w.\overline{C_f^v}) \cap (w.\lambda - w.\overline{C_f^v})$.*

*Proof.* As $\Delta$ and $\Phi$ are $W^v$-invariant, we have $w.\mathrm{cl}^\Delta(\Omega) = \mathrm{cl}^\Delta(w.\Omega)$ for every $w \in W^v$. Thus it suffices to determine $\mathrm{cl}^\Delta([-\lambda, \lambda])$. Let $(k_\alpha) \in \prod_{\alpha\in\Delta}\Lambda_\alpha \cup \{+\infty\}$ be such that $\bigcap_{\alpha\in\Delta} D(\alpha, k_\alpha) \supset [-\lambda, \lambda]$. Let $\alpha \in \Delta_+$. Write $\alpha = \sum_{i\in I} n_i\alpha_i$, with $n_i \in \mathbb{N}$ for $i \in I$. Then $k_\alpha \geq \alpha(\lambda) = \sum_{i\in I} n_i\alpha_i(\lambda)$. Let $\alpha \in \Delta_-$. We also have $k_{-\alpha} \geq \sum_{i\in I} n_i\alpha_i(\lambda)$.

Let $x \in (-\lambda + C_f^v) \cap (\lambda - C_f^v)$. Then $-\alpha_i(\lambda) \leq \alpha_i(x) \leq \alpha_i(\lambda)$ for every $i \in I$. Then $k_{-\alpha} \leq \sum_{i\in I} n_i\alpha_i(x) \leq k_\alpha$ and thus $k_\alpha + \alpha(x) \geq 0$ and $k_{-\alpha} - \alpha(x) \geq 0$. Consequently, $x \in \bigcap_{\alpha\in\Delta} D(\alpha, k_\alpha)$ and thus

$$(-\lambda + C_f^v) \cap (\lambda - C_f^v) \subset \bigcap_{\alpha\in\Delta} D(\alpha, k_\alpha),$$

which proves the lemma. □

The following lemma will be crucial throughout the paper. This is a rewriting of [BHR22, Lemma 3.2 and Lemma 3.5]. Although $\omega$ is assumed to be discrete in [BHR22], the proofs of these lemma do not use this assumption.

**Lemma 4.5.** *1. Let $a \in \mathbb{A}$ and $g \in U^+$. Then there exists $b \in a - C_f^v$ such that $g^{-1}U_b^{pm+}g \subset U_a^{pm+}$.*

*2. Let $y \in \mathcal{I}$. Then there exists $a \in \mathbb{A}$ such that $U_a^{pm+}$ fixes $y$.*

*3. Let $\lambda \in Y^+$ be regular and $y \in \mathcal{I}$. Then for $n \in \mathbb{N}$ large enough, $U_{[-n\lambda,n\lambda]}^{pm+}$ fixes $y$.*

*Proof.* By [BHR22, Lemma 3.2 and 3.5], we have 1) and 2). Let $\lambda \in Y^+$ be regular. Write $\lambda = w.\lambda^{++}$, with $\lambda^{++} \in C_f^v$ and $w \in W^v$. Let $a \in \mathbb{A}$ be such that $U_a^{pm+}$ fixes $y$. For $n \in \mathbb{N}^*$, we have $\mathrm{cl}^\Delta([-n\lambda, n\lambda]) \supset n\left((-w.\lambda^{++} + w.\overline{C_f^v}) \cap (w.\lambda^{++} - w.\overline{C_f^v})\right)$, which contains $a$, for $n \gg 0$. So for $n \gg 0$, we have $U_{[-n\lambda,n\lambda]}^{pm+} \subset U_a^{pm+}$, which proves 2). □

**Lemma 4.6.** *Let $\lambda, \mu \in Y^+$ be regular. Then $(\mathcal{V}_{n\lambda})_{n\in\mathbb{N}^*}$ and $(\mathcal{V}_{n\mu})_{n\in\mathbb{N}^*}$ are equivalent.*

*Proof.* Write $\mu = v.\mu^{++}$, where $v \in W^v$ and $\mu^{++} \in C_f^v$. For $m \in \mathbb{N}$, set

$$\Omega_{m\mu} = (-m\mu + v.\overline{C_f^v}) \cap (m\mu - w.\overline{C_f^v}).$$

By Lemma 4.4, $\Omega_{m\mu} \subset \mathrm{cl}^\Delta([-m\mu, m\mu])$. Let $n \in \mathbb{N}^*$. As $\Omega_{m\mu} = m\Omega_\mu$ and $\Omega_\mu$ contains 0 in its interior, there exists $m \in \mathbb{Z}_{\geq n}$ such that $\Omega_{m\mu} \supset [-n\lambda, n\lambda]$. Moreover, by (4.4), $U_{[-m\mu,m\mu]}^{pm+} = U_{\Omega_{m\mu}}^{pm+} \subset U_{[-n\lambda,n\lambda]}^{pm+}$, since $\Omega' \mapsto U_{\Omega'}^{pm+}$ is decreasing for $\subset$. With the same reasoning for $U^{nm-}$, we deduce $\mathcal{V}_{m\mu} \subset \mathcal{V}_{n\lambda}$. By symmetry of the roles of $\lambda$ and $\mu$, we deduce the lemma. □

The end of this subsection is devoted to the proof of the fact that for every $\lambda \in Y^+$ regular, $(\mathcal{V}_{n\lambda})_{n\in\mathbb{N}^*}$ is conjugation-invariant.

**Lemma 4.7.** *Let $\alpha \in \Phi$ and $a \in \mathcal{K}$. Let $\lambda \in Y^+$ be regular. Then $(x_\alpha(a).\mathcal{V}_{n\lambda}.x_\alpha(-a))_{n\in\mathbb{N}^*}$ is equivalent to $(\mathcal{V}_{n\lambda})_{n\in\mathbb{N}^*}$.*

*Proof.* Let $\epsilon \in \{-,+\}$ be such that $\alpha \in \Phi_\epsilon$. Let $w \in W^v$ be such that $\epsilon w^{-1}.\alpha$ is simple. By symmetry, we may assume that $\epsilon = +$. By Lemma 4.6 we may assume that $\lambda \in w.C_f^v$. Let $n \in \mathbb{N}^*$. By Lemma 4.2, we have

$$\begin{aligned}\mathcal{V}_{n\lambda} &= U^{pm}_{[-n\lambda,n\lambda]}(w.\Delta_+)U^{nm}_{[-n\lambda,n\lambda]}(w.\Delta_-)T_{2nN(\lambda)}.\\ &= U^{pm}_{-n\lambda}(w.\Delta_+)U^{nm}_{n\lambda}(w.\Delta_-)T_{2nN(\lambda)}.\end{aligned}$$

Write $\alpha = w.\alpha_i$, for $i \in I$. By Lemma 4.5,

$$x_{w.\alpha_i}(a)U^{pm}_{-m\lambda}(w.\Delta_+)x_{w.\alpha_i}(-a) \subset U^{pm}_{-n.\lambda}(w.\Delta_+), \tag{4.5}$$

for $m \gg 0$.

By [Rou16, Lemma 3.3], $U^{nm}_{[-n\lambda,n\lambda]}(w.\Delta_-) = U^{nm}_{[-n\lambda,n\lambda]}(w.(\Delta_- \setminus \{-\alpha_i\})).U_{-w.\alpha_i,[-n\lambda,n\lambda]}$ and $U_{-w.\alpha_i,[-n\lambda,n\lambda]}$ normalizes $U^{nm}_{[-n\lambda,n\lambda]}(w.(\Delta_- \setminus \{-\alpha_i\}))$.

By [Kum02, 1.3.11 Theorem (b4)], $r_i.(\Delta_- \setminus \{-\alpha_i\}) = \Delta_- \setminus \{-\alpha_i\}$ and thus for $m \in \mathbb{N}$, we have

$$\begin{aligned}U^{nm}_{[-m\lambda,m\lambda]}\left(w.(\Delta_- \setminus \{-\alpha_i\})\right) &= U^{nm}_{[-m\lambda,m\lambda]}\left(wr_i(\Delta_- \setminus \{-\alpha_i\})\right)\\ &= U^{nm}(wr_i(\Delta_- \setminus \{-\alpha_i\})) \cap U^{nm}_{[-m\lambda,m\lambda]}(wr_i.\Delta_-).\end{aligned}$$

Moreover,

$$x_{w.\alpha_i}(a)U^{nm}(wr_i(\Delta_- \setminus \{-\alpha_i\}))x_{w.\alpha_i}(-a) = U^{nm}(wr_i(\Delta_- \setminus \{-\alpha_i\})), \tag{4.6}$$

by [Rou16, Lemma 3.3] (applied to $wr_i.(\Delta_- \setminus \{-\alpha_i\}) \subset wr_i.\Delta_-$). By Lemma 4.5, for $m \gg 0$,

$$x_{w.\alpha_i}(a)U^{nm}_{[-m\lambda,m\lambda]}(wr_i.\Delta_-)x_{w.\alpha_i}(-a) \subset U^{nm}_{[-n\lambda,n\lambda]}(wr_i.\Delta_-). \tag{4.7}$$

Combining (4.6) and (4.7), we get

$$x_{w.\alpha_i}(a)U^{nm}_{[-m\lambda,m\lambda]}(w.\Delta_- \setminus \{-\alpha_i\})x_{w.\alpha_i}(-a) \subset U^{nm}_{[-n\lambda,n\lambda]}\left(w.(\Delta_- \setminus \{-\alpha_i\})\right), \tag{4.8}$$

for $m \gg 0$.

For $b \in \mathcal{K}$ such that $1 + ab \neq 0$, we have

$$x_{w.\alpha_i}(a)x_{-w.\alpha_i}(b)x_{w.\alpha_i}(-a) = x_{-w.\alpha_i}\left(b(1+ab)^{-1}\right)\alpha^\vee(1+ab)x_{w.\alpha_i}(-a^2b(1+ab)^{-1}).$$

Therefore if $m \gg 0$, $x_{w.\alpha_i}(a)U_{-w.\alpha_i,[-m\lambda,m\lambda]}x_{w.\alpha_i}(-a) \subset \mathcal{V}_{n\lambda}$. Combined with (4.8) we get

$$x_{w.\alpha_i}(a)U^{nm}_{-m\lambda}(w.\Delta_-)x_{w.\alpha_i}(-a) \subset \mathcal{V}_{n\lambda}, \tag{4.9}$$

for $m \gg 0$, since $\mathcal{V}_{n\lambda}$ is a group.

Let $m \in \mathbb{N}^*$ and $t \in T_{2m}$. Then:

$$x_{w.\alpha_i}(a)tx_{w.\alpha_i}(-a) = tx_{w.\alpha_i}\left(a\left(w.\alpha_i(t^{-1}) - 1\right)\right)$$

Therefore if $m \geq nN(\lambda)$ and $\omega\left(a\left(w.\alpha_i(t^{-1}) - 1\right)\right) \geq \omega(a) + 2mN(\lambda)$ is greater than $|w.\alpha_i(n\lambda)|$, then $x_{w.\alpha_i}\left(a\left(w.\alpha_i(t^{-1}) - 1\right)\right) \in \mathcal{V}_{n\lambda}$ and $x_{w.\alpha_i}(a)tx_{w.\alpha_i}(-a) \in \mathcal{V}_{n\lambda}$. Combined with (4.9) and (4.5), we get $x_\alpha(a).\mathcal{V}_{m\lambda}.x_\alpha(-a) \subset \mathcal{V}_{n\lambda}$, for $m \gg 0$. Applying this to $(x_\alpha(-a).\mathcal{V}_{k\lambda}.x_\alpha(a))_{k\in\mathbb{N}^*}$, we get the other inclusion needed to prove that $(\mathcal{V}_{n\lambda})$ and $(x_\alpha(a).\mathcal{V}_{n\lambda}.x_\alpha(-a))$ are equivalent. □

**Theorem 4.8.** *Let $\lambda \in Y^+$ be regular. For $n \in \mathbb{N}^*$, set $\mathcal{V}_{n\lambda} = U^{pm+}_{[-n\lambda,n\lambda]}.U^{nm-}_{[-n\lambda,n\lambda]}.T_{2N(\lambda)}$. Then $(\mathcal{V}_{n\lambda})$ is conjugation-invariant. Therefore, the associated topology $\mathscr{T}((\mathcal{V}_{n\lambda}))$ equips $G$ with the structure of a Hausdorff topological group.*

*Proof.* We need to prove that for every $g \in G$, $g(\mathcal{V}_{n\lambda})g^{-1}$ is equivalent to $(\mathcal{V}_{n\lambda})$. Using Lemma 4.6, we may assume $\lambda \in C_f^v$. By [Rou06, Proposition 1.5], $G$ is generated by $T$ and the $x_\alpha(a)$, for $\alpha \in \Phi$ and $a \in \mathcal{K}$. By Lemma 4.7, it remains only to prove that if $t \in T$, then $(t\mathcal{V}_{n\lambda}t^{-1})_{n\in\mathbb{N}^*}$ is equivalent to $(\mathcal{V}_{n\lambda})$. Let $t \in T$ and $m \in \mathbb{N}^*$. Then by Lemma 2.4,

$$\begin{aligned} t\mathcal{V}_{m\lambda}t^{-1} &= tU^{pm+}_{[-m\lambda,m\lambda]}t^{-1}.tU^{nm-}_{[-m\lambda,m\lambda]}t^{-1}.T_{2nN(\lambda)} \\ &= U^{pm+}_{[t.(-m\lambda),t.m\lambda]}U^{nm-}_{[t.(-m\lambda),t.m\lambda]}T_{2nN(\lambda)} \end{aligned}$$

Set $\Omega = (-\lambda + \overline{C_f^v}) \cap (\lambda - \overline{C_f^v})$. Then by Lemma 4.4, $\mathrm{cl}^\Delta([-m\lambda, m\lambda]) \supset m\Omega$. Moreover, $\mathrm{cl}^\Delta([t.(-m\lambda), t.m\lambda]) = t.\mathrm{cl}^\Delta([-m\lambda, m\lambda]) \supset t.m\Omega$. Let $n \in \mathbb{N}^*$. Then as $\Omega$ contains 0 in its interior, $t.m\Omega \supset n\Omega$ for $m \gg 0$. Therefore (by (4.4)) $U^{pm+}_{[t.(-m\lambda),t.m\lambda]} \subset U^{pm+}_{[-n\lambda,n\lambda]}$ and $U^{nm-}_{[t.(-m\lambda),t.m\lambda]} \subset U^{nm-}_{[-n\lambda,n\lambda]}$ for $m \gg 0$. Consequently, $t\mathcal{V}_{m\lambda}t^{-1} \subset \mathcal{V}_{n\lambda}$ for $m \gg 0$, which proves that $(\mathcal{V}_{n\lambda})$ is conjugation-invariant.

It remains to prove that $\mathscr{T}((\mathcal{V}_{n\lambda}))$ is Hausdorff. For that it suffices to prove that $\bigcap_{n\in\mathbb{N}^*} \mathcal{V}_{n\lambda} = \{1\}$. Let $g \in \bigcap_{n\in\mathbb{N}^*} \mathcal{V}_{n\lambda}$. Let $n \in \mathbb{N}^*$. Then as $[-n\lambda, n\lambda]$ has a good fixator ([Rou16, 5.7 1)] and (2.8)) $g \in G_{[-n\lambda,n\lambda]} = U^{pm+}_{[-n\lambda,n\lambda]}.U^{nm-}_{[-n\lambda,n\lambda]}.\mathfrak{T}(\mathcal{O})$, so we can write $g = u_n^+u_n^-t_n$, with $(u_n^+, u_n^-, t_n) \in U^{pm+}_{[-n\lambda,n\lambda]} \times U^{nm-}_{[-n\lambda,n\lambda]} \times \mathfrak{T}(\mathcal{O})$. By Lemma 2.3, $u_+ := u_n^+$ does not depend on $n$ and thus $u^+ \in \bigcap_{n\in\mathbb{N}^*} U^{pm+}_{[-n\lambda,n\lambda]} = U^{pm+}_{\mathbb{A}} = \{1\}$. Similarly, $u^- := u_n^- = 1$. Therefore $t \in T \cap \bigcap_{n\in\mathbb{N}^*} \ker \pi_n$. Let $(\chi_i)_{i\in[\![1,m]\!]}$ be a $\mathbb{Z}$-basis of $Y$. Write $t = \prod_{i=1}^m \chi_i(a_i)$, with $a_i \in \mathcal{O}^*$, for $i \in [\![1, m]\!]$. Let $n \in \mathbb{N}$. Then $\pi_n(t) = \prod_{i=1}^m \chi_i(\pi_n(a_i)) = 1$ and thus $a_i \in \bigcap_{n\in\mathbb{N}^*} \varpi^n\mathcal{O} = \{0\}$. Consequently, $t = 1$ and $g = 1$. Therefore $\bigcap_{n\in\mathbb{N}^*} \mathcal{V}_{n\lambda} = \{1\}$ and $\mathscr{T}((\mathcal{V}_{n\lambda}))$ is Hausdorff. □

We denote by $\mathscr{T}$ the topology $\mathscr{T}((\mathcal{V}_{n\lambda}))$, for any $\lambda \in Y^+$ regular.

**Corollary 4.9.** *Assume that $W^v$ is infinite. Then the filtrations $(\ker \pi_n)_n$ and $(\mathcal{V}_{n\lambda})_{n\in\mathbb{N}^*}$ are not equivalent.*

*Proof.* This follows from Lemma 3.3 and Theorem 4.8. □

**Remark 4.10.** 1. *If $n \in \mathbb{N}^*$, then $\ker \pi_n$ is open for $\mathscr{T}$, by (4.2).*

2. *The Iwahori subgroup $G_{C_0^+} = K_I$ is open. Indeed, if $\lambda \in Y \cap C_f^v$, then $\mathcal{V}_\lambda \subset K_I$ by (4.1). In particular, $\mathfrak{G}^{\min}(\mathcal{O}) \supset K_I$ is open.*

## 4.3 Topology of the fixators

### 4.3.1 Definition of the topology

Recall that if $F$ is a subset of $\mathcal{I}$, we denote by $G_F$ its fixator in $G$. In this subsection we study the topology $\mathscr{T}_{\mathrm{Fix}}$ which is defined as follows. A subset $V$ of $G$ is open if for every $v$

in $V$, there exists a finite subset $F$ of $\mathcal{I}$ such that $v.G_F \subset V$. Note that $G_0 = \mathfrak{G}^{\min}(\mathcal{O})$ is open for this topology.

We begin by constructing increasing sequences of finite sets of vertices $(F_n) = (F_n(\lambda))_{n\in\mathbb{N}^*}$ such that $\mathscr{T}_{\mathrm{Fix}}$ is the topology associated with $(G_{F_n})_{n\in\mathbb{N}}$.

We fix $\lambda \in Y^+$ regular. We set $F_0 = \varnothing$. For $n \in \mathbb{N}^*$, we set

$$F_n = F_n(\lambda) = \{n\lambda, -n\lambda\} \cup \{x_{\alpha_i}(\varpi^{-n}).0 \mid i \in I\} \cup F_{n-1}. \tag{4.10}$$

Let $n \in \mathbb{N}^*$. By [Rou16, 5.7 1)], $[-n\lambda, n\lambda]$ has a good fixator. Therefore

$$G_{F_n} \subset G_{[-n\lambda,n\lambda]} = U^{pm+}_{[-n\lambda,n\lambda]}.U^{nm-}_{[-n\lambda,n\lambda]}.\mathfrak{T}(\mathcal{O}). \tag{4.11}$$

We chose $F_n$ as above for the following reasons. We want that when $x \in \mathcal{I}$ and $n \gg 0$, an element of $G_{F_n}$ fixes $x$. By Lemma 4.5, if $u \in U^+$ (resp. $U^-$), and if $u$ fixes $-n\lambda$ (resp. $n\lambda$), for some $n$ large, then $u$ fixes $x$. However, if $t \in T$, as $\mathfrak{T}(\mathcal{O})$ fixes $\mathbb{A}$, we need to require that $t$ fixes elements outside of $\mathbb{A}$, and the choice of $x_{\alpha_i}(\varpi^n).0$ is justified by the lemma below.

**Lemma 4.11.** *Let $i \in I$, $n \in \mathbb{N}$ and $t \in T$. Then $t$ fixes $x_{\alpha_i}(\varpi^{-n}).0$ if and only if $\omega(\alpha_i(t) - 1) \geq n$.*

*Proof.* We have $t.x_{\alpha_i}(\varpi^{-n}).0 = x_{\alpha_i}(\varpi^{-n}).0$ if and only if $x_{\alpha_i}(-\varpi^{-n})x_{\alpha_i}(\alpha_i(t)\varpi^{-n}).t.0 = 0$. We have $\rho_{+\infty}(x_{\alpha_i}(-\varpi^{-n})x_{\alpha_i}(\alpha_i(t)\varpi^{-n}).t.0) = t.0$ and thus if $t.x_{\alpha_i}(\varpi^{-n}).0 = x_{\alpha_i}(\varpi^{-n}).0$, we have $t.0 = 0$. Thus $t.x_{\alpha_i}(\varpi^{-n}).0 = x_{\alpha_i}(\varpi^{-n}).0$ if and only if $x_{\alpha_i}((\alpha_i(t) - 1)\varpi^{-n}).0 = 0$ if and only if $\omega(\alpha_i(t) - 1) \geq n$. □

For $n \in \mathbb{N}^*$, we set $T_{n,\Phi} = \{t \in T \mid \omega(\alpha_i(t) - 1) \geq n, \forall i \in I\}$.

**Lemma 4.12.** *Let $y \in \mathcal{I}$. Then there exists $M \in \mathbb{N}$ such that $T_{M,\Phi}$ fixes $y$.*

*Proof.* By the Iwasawa decomposition ((MA III)), $y \in U^+.z$, where $z = \rho_{+\infty}(y)$. Write $y = x_{\beta_1}(a_1)\dots x_{\beta_k}(a_k).z$, with $k \in \mathbb{N}$ and $\beta_1, \dots, \beta_k \in \Phi_+$. Let $t \in T$. We have $t.y$ if and only if

$$\begin{aligned} z &= x_{\beta_k}(-a_k)\dots x_{\beta_1}(-a_1)tx_{\beta_1}(a_1)\dots x_{\beta_k}(a_k).z \\ &= x_{\beta_k}(-a_k)\dots x_{\beta_2}(-a_2)tx_{\beta_1}\left((1-\beta_1(t^{-1}))a_1\right)x_{\beta_2}(a_2)\dots x_{\beta_k}(a_k).z \end{aligned}$$

Let $M \in \mathbb{N}^*$. Assume that $\alpha_i(t) - 1 \in \varpi^M\mathcal{O}$ for every $i \in I$.

Write $\beta_1 = \sum_{i\in I} m_i\alpha_i$, with $m_i \in \mathbb{N}$ for every $i \in I$. Then $\beta_1(t) = \prod_{i\in I} \alpha_i^{m_i}(t)$. Therefore $\beta_1(t) \in 1 + \varpi^M\mathcal{O}$. For $M \gg 0$, $x_{\beta_1}\left((1-\beta_1(t^{-1}))a_1\right)$ fixes $x_{\beta_2}(a_2)\dots x_{\beta_k}(a_k).z$, by Lemma 4.5. By induction on $k$ we deduce that $t$ fixes $z$ for $M \gg 0$. □

**Lemma 4.13.** *1. Let $F$ be a finite subset of $\mathcal{I}$. Then there exists $n \in \mathbb{N}^*$ such that $G_{F_n}$ fixes $F$.*

*2. Let $\lambda, \mu \in Y^+$ be regular. Then the filtrations $(G_{F_n(\lambda)})_{n\in\mathbb{N}^*}$ and $(G_{F_n(\mu)})_{n\in\mathbb{N}^*}$ are equivalent.*

*3. The topology $\mathscr{T}_{\mathrm{Fix}}$ is the topology associated with $\mathscr{T}((G_{F_n})_{n\in\mathbb{N}})$.*

*Proof.* 1) It suffices to prove that if $y \in \mathcal{I}$, then $G_{F_n}$ fixes $y$ for $n \gg 0$. Let $n \in \mathbb{N}^*$. Then $F_n \supset [-n\lambda, n\lambda]$.

By Lemma 4.5, there exists $n_1 \in \mathbb{N}$ such that $U^{pm+}_{[-n\lambda,n\lambda]}$ and $U^{nm-}_{[-n\lambda,n\lambda]}$ fix $y$, for $n \geq n_1$. Let $n \geq n_1$. Let $n_2 = n_2(n) \geq n$ be such that for every $i \in I$, $\langle U^{pm+}_{[-n_2.\lambda,n_2.\lambda]}, U^{nm-}_{[-n_2.\lambda,n_2.\lambda]}\rangle$ fixes $x_{\alpha_i}(\varpi^{-n}).0$ for every $i \in I$. Let $g \in G_{F_{n_2}}$. Using (4.11), we write $g = u_+u_-t$, with $(u_+, u_-, t) \in U^{pm+}_{[-n_2\lambda,n_2\lambda]} \times U^{nm-}_{[-n_2\lambda,n_2\lambda]} \times \mathfrak{T}(\mathcal{O})$. Then $g$ fixes $y$ if and only if $t$ fixes $y$. Moreover, $u_+u_-t$ fixes $F_n$ and thus $t$ fixes $F_n$. By Lemma 4.11, we deduce $\omega((\alpha_i(t) - 1) \geq n$ for every $i \in I$. By Lemma 4.12 we deduce that $t$ fixes $y$, for $n \gg 0$. Thus $G_{F_{n_2(n)}}$ fixes $y$ for $n \gg 0$.

2) Follows from 1) by applying to $F = F_m(\mu)$ and $F_n = F_n(\lambda)$, for $m, n \in \mathbb{N}^*$ and by symmetry of the roles of $\lambda$ and $\mu$.

3) As $F_n$ is finite for every $n \in \mathbb{N}^*$, $\mathscr{T}((G_{F_n}))$ is coarser than $\mathscr{T}_{\mathrm{Fix}}$. But by 1), $\mathscr{T}_{\mathrm{Fix}}$ is coarser that $(\mathscr{T}_{G_{F_n}})$. □

**Proposition 4.14.** *The topology $\mathscr{T}_{\mathrm{Fix}}$ is the coarsest topology of topological group on $G$ such that $\mathfrak{G}^{\min}(\mathcal{O})$ is open.*

*Proof.* For $n \in \mathbb{N}^*$, $G_{F_n} \subset \mathfrak{G}^{\min}(\mathcal{O})$ and thus $\mathfrak{G}^{\min}(\mathcal{O})$ is open for $\mathscr{T}_{\mathrm{Fix}}$.

Let now $\mathscr{T}'$ be a topology of topological group on $G$ such that $\mathfrak{G}^{\min}(\mathcal{O})$ is open. Let $n \in \mathbb{N}^*$. Then for every element $a$ of $F_n$, there exists $g_a \in G$ such that $g_a.0 = a$. Then $G_{F_n} = \bigcap_{a\in F_n} g_a.\mathfrak{G}^{\min}(\mathcal{O}).g_a^{-1}$ is open in $G$. Proposition follows. □

### 4.3.2 Relation between $\mathscr{T}_{\mathrm{Fix}}$ and $\mathscr{T}$

In this subsection, we compare $\mathscr{T}_{\mathrm{Fix}}$ and $\mathscr{T}$. We prove that $\mathscr{T}$ is finer than $\mathscr{T}_{\mathrm{Fix}}$. When $\mathcal{K}$ is Henselian , we prove that $\mathscr{T} = \mathscr{T}_{\mathrm{Fix}}$ if and only if the fixator of $\mathcal{I}$ in $G$ is $\{1\}$ (see Proposition 4.21).

Let $\lambda \in Y^+$ be regular. For $n \in \mathbb{N}$, we define $F_n = F_n(\lambda)$ as in (4.10).

For $n \in \mathbb{N}$, one sets $T_{n,\Phi} = \{t \in T \mid \alpha_i(t) \in 1 + \varpi^n\mathcal{O}, \forall i \in I\}$. Let $\mathcal{Z} = \bigcap_{n\in\mathbb{N}} T_{n,\Phi}$ and $\mathcal{Z}_{\mathcal{O}} = \mathcal{Z} \cap \mathfrak{T}(\mathcal{O})$. Then $\mathcal{Z} = \{t \in T \mid \alpha_i(t) = 1, \forall i \in I\}$ is the center of $G$ by [Rém02, 8.4.3 Lemme].

**Lemma 4.15.** *The fixator $G_{\mathcal{I}}$ of $\mathcal{I}$ in $G$ is $\mathcal{Z}_{\mathcal{O}}$ and $\bigcap_{n\in\mathbb{N}^*} G_{F_n} = \mathcal{Z}_{\mathcal{O}}$.*

*Proof.* We have $G_{\mathcal{I}} \subset G_{\mathbb{A}} = \mathfrak{T}(\mathcal{O})$, by [Rou16, 5.7 5)]. By Lemma 4.11, $G_{\mathcal{I}} \subset T_{n,\Phi} \cap \mathfrak{T}(\mathcal{O})$ for every $n \in \mathbb{N}$ and thus $G_{\mathcal{I}} \subset \mathcal{Z}_{\mathcal{O}}$. Let $z \in \mathcal{Z}_{\mathcal{O}}$ and $x \in \mathcal{I}$. Write $x = g.a$, with $a \in \mathbb{A}$. Then $z.x = gz.a = g.a = x$ and $z \in G_{\mathcal{I}}$.

Now let $g \in \bigcap_{n\in\mathbb{N}^*} G_{F_n}$. Then by Lemma 4.13, $g$ fixes $\mathcal{I}$, which proves the lemma. □

**Lemma 4.16.** *There exists an increasing map $M : \mathbb{N} \to \mathbb{N}$ whose limit is $+\infty$ and such that for every $m \in \mathbb{N}$,*

$$G_{F_n} \subset \mathcal{V}_{m\lambda}.(T_{M(n),\Phi} \cap \mathfrak{T}(\mathcal{O})),$$

*for $m, n \in \mathbb{N}^*$ such that $M(n) \geq m$ and $n \geq m$.*

*Proof.* Let $n \in \mathbb{N}^*$. Let $M(n) \in \mathbb{N}$ be maximum such that $U^{nm-}_{[-n\lambda,n\lambda]}$, $U^{pm+}_{[-n\lambda,n\lambda]}$, fix $F_{M(n)}$. By Lemma 4.5, $M(n) \to +\infty$. Let $g \in G_{F_n}$. Using (4.11) we write $g = u_+u_-t$, with $u_+ \in U^{pm+}_{[-n\lambda,n\lambda]}$, $u_- \in U^{nm-}_{[-n\lambda,n\lambda]}$ and $t \in \mathfrak{T}(\mathcal{O})$. Let $m' = M(n)$. Then $m' \leq n$ and $u_+, u_-$ fix $F_{m'}$. As $g$ fixes $F_{m'}$ we deduce $t$ fixes $F_{m'}$. By Lemma 4.12, $t \in T_{m',\Phi} \cap \mathfrak{T}(\mathcal{O})$. Therefore $g \in \mathcal{V}_{m\lambda}.(T_{M(n),\Phi} \cap \mathfrak{T}(\mathcal{O}))$, which proves the lemma. □

**Lemma 4.17.** *Let $\mathbb{A}^* = X \otimes \mathbb{R}$, $Q' = (\bigoplus_{i\in I} \mathbb{Q}\alpha_i) \cap X \subset \mathbb{A}^*$ and $d$ be the dimension of $Q'$ as a $\mathbb{Q}$-vector space. Then there exists a $\mathbb{Z}$-basis $(\chi_1, \dots, \chi_\ell)$ of $X$ such that $(\chi_1, \dots, \chi_d)$ is a $\mathbb{Z}$-basis of $Q'$.*

*Proof.* Let $x \in X$ and $n \in \mathbb{Z} \setminus \{0\}$. Assume that $nx \in Q'$. Then $x \in Q'$. Therefore $X/Q'$ is torsion-free. Let $(e_{d+1}, \dots, e_\ell) \in (X/Q')^{\ell-d}$ be a $\mathbb{Z}$-basis of $X/Q'$. For $j \in [\![d+1, \ell]\!]$, take $\chi_j \in X$ whose reduction modulo $Q'$ is $e_j$. Choose a $\mathbb{Z}$-basis $(\chi_1, \dots, \chi_d)$ of $X'$. Then $(\chi_1, \dots, \chi_\ell)$ satisfies the condition of the lemma. □

**Lemma 4.18.** *Assume $\mathcal{K}$ to be Henselian. Let $a \in \mathcal{O}$ and $m \in \mathbb{N}^*$. Assume $\omega(a^m - 1) > 0$. Then we can write $a = b + c$, with $b \in \mathcal{O}$ such that $b^m = 1$ and $\omega(c) > 0$.*

*Proof.* Let $\Bbbk = \mathcal{O}/\mathfrak{m}$ be the residual field and $\pi_\Bbbk : \mathcal{O} \twoheadrightarrow \mathcal{O}/\mathfrak{m}$ be the natural projection. Let $p$ be the characteristic of $\Bbbk$. If $p = 0$, we set $m' = m$ and $k = 0$. If $p > 0$, we write $m = p^k m'$, with $k \in \mathbb{N}$ and $m' \in \mathbb{N}$ prime to $p$. We have $\pi_\Bbbk(a^m) = \pi_\Bbbk(a)^m = \pi_\Bbbk(1)$. We have $(\pi_\Bbbk(a)^{m'} - 1)^{p^k} = 0$ and thus $\pi_\Bbbk(a)^{m'} = \pi_\Bbbk(1)$. Let $Z$ be an indeterminate. We have $\overline{Z^{m'} - 1} = \overline{(Z - a)}Q_\Bbbk$, where the bar denotes the reduction modulo $\mathfrak{m}[Z]$ and $Q_\Bbbk \in \Bbbk[Z]$ is prime to $\overline{Z - a}$. As $\mathcal{O}$ is Henselian, we can write $Z^{m'} - 1 = (Z - b)Q$, where $b \in \mathcal{O}$ is such that $\pi_\Bbbk(b) = \pi_\Bbbk(a)$ and $Q \in \mathcal{O}[Z]$ is such that $\overline{Q} = Q_\Bbbk$. Then $\pi_\Bbbk(b - a) = 0$ and we get the lemma, with $c = b - a$. □

The following lemma was suggested to me by Guy Rousseau.

**Lemma 4.19.** *Assume $\mathcal{K}$ to be Henselian. Let $m \in \mathbb{N}^*$. Then there exists $K \in \mathbb{N}^*, K' \in \mathbb{N}$ such that for every $n \in \mathbb{N}^*$ and $a \in \mathcal{O}$ such that $\omega(a^m - 1) \geq n$, we can write $a = b + c$, with $b, c \in \mathcal{O}$ such that $b^m = 1$ and $\omega(c) \geq n/K - K'$.*

*Proof.* We first assume that $\mathcal{K}$ has characteristic $p > 0$. Let $n \in \mathbb{N}^*$ and $a \in \mathcal{O}$ be such that $\omega(a^m - 1) \geq n$. Write $m = m'p^k$, with $k \in \mathbb{N}$ and $m' \in \mathbb{N}$ prime to $p$. We have $a^m - 1 = (a^{m'} - 1)^{p^k}$ and thus $\omega(a^{m'} - 1) = \omega(a^m - 1)/p^k \geq n/p^k > 0$. By Lemma 4.18, we can write $a = b + c$, with $b, c \in \mathcal{O}$, $b^{m'} = 1$ and $\omega(c) > 0$. We have $a^{m'} - 1 = \overline{m'}b^{m'-1}c + \sum_{i=2}^{m'} \overline{\binom{m'}{i}} c^i b^{m-i}$, where $\overline{x}$ is the image of $x$ in $\mathcal{K}$, if $x \in \mathbb{Z}$. As $m'$ is prime to $p$, $\overline{m'}$ is a root of 1 and thus we have $\omega(\overline{m'}) = 0$. As $b^{m'} = 1$, $\omega(b) = 0$. Therefore $\omega(c) = \omega(\overline{m'}b^{m-1}c) < \omega(\overline{\binom{m'}{i}}b^{m'-i}c^i)$ for $i \in [\![2, m']\!]$. Consequently $\omega(a^{m'} - 1) = \omega(c)$ and $\omega(a^m - 1) = p^k\omega(a^{m'} - 1) = p^k\omega(c) \geq n$. This proves the lemma in this case, with $K' = 0$ and $K = p^k$.

We now assume that $\mathcal{K}$ has characteristic 0. Then by [Gre66, Theorem 1] and [Rou77, Annexe A4] (for the case where $\omega(\mathcal{K}^*)$ is not discrete) applied with $F = \{Z^m - 1\}$ (where $Z$ is an indeterminate), there exist $K \in \mathbb{N}^*$, $K' \in \mathbb{N}$ such that for every $n \in \mathbb{N}^*$, for every

$a \in \mathcal{O}$ such that $\omega(a^m - 1) \geq n$, we can write $a = b + c$, with $b, c \in \mathcal{O}$, $b^m = 1$ and $\omega(c) \geq n/K - K'$, which proves the lemma in this case. □

**Lemma 4.20.** *Assume $\mathcal{K}$ to be Henselian. There exist $K_1 \in \mathbb{R}_+^*$, $L \in \mathbb{N}$ such that for every $n \in \mathbb{Z}_{\geq L}$, $T_{n,\Phi} \cap \mathfrak{T}(\mathcal{O}) \subset \mathcal{Z}_{\mathcal{O}}.T_{n/K_1}$.*

*Proof.* We keep the same notation as in Lemma 4.17. Let $(\chi_1^\vee, \dots, \chi_\ell^\vee) \in Y^\ell$ be the dual basis of $(\chi_1, \dots, \chi_\ell)$. For $i \in I$, we write $\alpha_i = \sum_{j=1}^{\ell} n_{j,i}\chi_j$, with $n_{j,i} \in \mathbb{Z}$ for all $i, j$. We have $n_{j,i} = 0$ for $j \in [\![d+1, \ell]\!]$. Set $\widetilde{t} = \prod_{j=d+1}^{\ell} \chi_j^\vee(\chi_j(t)) \in \mathfrak{T}(\mathcal{O})$. Then $\alpha_i(\widetilde{t}) = 1$ for every $i \in I$ and thus $\widetilde{t} \in \mathcal{Z}_{\mathcal{O}}$.

For $j \in [\![1, d]\!]$, write $\chi_j = \sum_{i \in I} m_{i,j}\alpha_i$, with $m_{i,j} \in \mathbb{Q}$ for every $i \in I$. Take $m \in \mathbb{N}^*$ such that $mm_{i,j} \in \mathbb{Z}$ for every $(i, j) \in I \times [\![1, d]\!]$. Let $j \in [\![1, d]\!]$. We have

$$\chi_j(t)^m = \prod_{j=1}^{d} \alpha_j(t)^{mm_{i,j}} \in 1 + \varpi^n\mathcal{O}.$$

Using Lemma 4.19 we can write $\chi_j(t) = b_j + c_j$, with $b_j, c_j \in \mathcal{O}$ such that $b_j^m = 1$ and $\omega(c_j) \geq n/K - K'$, with the same notation as in Lemma 4.19. Set $c_j' = c_j b_j^{-1} \in \mathcal{O}$. As $b_j$ is a root of 1, we have $\omega(b_j) = 0$ and thus $\omega(c_j) = \omega(c_j') \geq n/K - K'$. We have $b_j + c_j = b_j(1 + c_j')$.

Set $t' = \prod_{j=1}^{d} \chi_j^\vee(b_j)$ and $t'' = \prod_{j=1}^{d} \chi_j^\vee(1 + c_j')$. Then $\chi_j(t') = b_j$ and $\chi_j(t'') = 1 + c_j'$, for $j \in [\![1, d]\!]$. For $i \in I$, we have $\alpha_i(t) = \alpha_i(t')\alpha_i(t'')\alpha_i(\widetilde{t}) = \alpha_i(t')\alpha_i(t'')$ and $\alpha_i(t'') \in 1 + \varpi^{n/K-K'}\mathcal{O}$ (when $n/K - K' \notin \mathbb{N}$, $\varpi^{n/K-K'}\mathcal{O}$ is just a notation for $\mathcal{K}_{\omega \geq n/K-K'}$). As $\alpha_i(t) \in 1 + \varpi^{n/K-K'}\mathcal{O}$ we deduce $\alpha_i(t') \in 1 + \varpi^{n/K-K'}\mathcal{O}$ (replacing $K$ by $K + 1$ if $K \leq 1$).

Let $F = \{\xi \in \mathcal{O} \mid \xi^m = 1\}$. Then $F$ is finite. Let $L' = \max\{\omega(\xi - 1) \mid \xi \in F \setminus \{1\}\}$. Let $L \in \mathbb{N}$ be such that $L/K - K' \geq L'$. For $n \in \mathbb{Z}_{\geq L}$, we have $n/K - K' \geq L'$, we have $\alpha_i(t') = 1$ for $i \in I$ and $t' \in \mathcal{Z}_{\mathcal{O}}$. Maybe increasing $L$, we can assume that $K_1 := 1/K - K'/L > 0$. Then for $n \geq L$, we have $n/K - K' \geq n(1/K - K'/L) \geq nK_1$.

Consequently, for $n \in \mathbb{Z}_{\geq L}$ and $t \in T_{n,\Phi}$, we have $t = t'\widetilde{t}t''$, with $t'\widetilde{t} \in \mathcal{Z}_{\mathcal{O}}$ and $t'' \in T_{n/K_1}$, which proves the lemma. □

**Proposition 4.21.** *The topology $\mathscr{T}$ is finer than $\mathscr{T}_{\mathrm{Fix}}$. If $\mathcal{K}$ is Henselian, then we have $\mathscr{T} = \mathscr{T}_{\mathrm{Fix}}$ if and only if $\mathcal{Z}_{\mathcal{O}} = \{1\}$ if and only if $\mathscr{T}_{\mathrm{Fix}}$ is Hausdorff.*

*Proof.* Let $x \in \mathcal{I}$ and $m \in \mathbb{N}^*$. Then by Lemma 4.5, $U^{pm+}_{[-m\lambda,m\lambda]}$ and $U^{nm-}_{[-m\lambda,m\lambda]}$ fix $x$, for $m \gg 0$. By Lemma 4.12, $T_{2mN(\lambda)}$ fix $x$ and thus $\mathcal{V}_{m\lambda}$ fixes $x$ for $m \gg 0$. Thus if $n \in \mathbb{N}^*$, $\mathcal{V}_{m\lambda} \subset G_{F_n}$ for $m \gg 0$ and $\mathscr{T}$ is finer than $\mathscr{T}_{\mathrm{Fix}}$.

If $\mathscr{T} = \mathscr{T}_{\mathrm{Fix}}$, then by Theorem 4.8, $\mathscr{T}_{\mathrm{Fix}}$ is Hausdorff. Therefore $\bigcap_{n \in \mathbb{N}^*} G_{F_n} = \mathcal{Z}_{\mathcal{O}} = \{1\}$ by Lemma 4.15.

Assume $\mathcal{K}$ is Henselian. Let $m \in \mathbb{N}^*$. Then by Lemma 4.16 and Lemma 4.20, there exist $K_1 \in \mathbb{R}_+^*$ and $L \in \mathbb{N}$ such that

$$G_{F_n} \subset \mathcal{V}_{m\lambda}.\mathcal{Z}_{\mathcal{O}}.T_{M(n)/K_1},$$

for $n \geq \min(m, L)$, with $M(n) \underset{n\to+\infty}{\to} +\infty$. Therefore if $\mathcal{Z}_{\mathcal{O}} = 1$ we have

$$G_{F_n} \subset \mathcal{V}_{m\lambda}.T_{M(n)/K_1} \subset \mathcal{V}_{m\lambda},$$

for $n$ such that $M(n)/K \geq 2mN(\lambda)$, and thus $(\mathcal{V}_{n\lambda})$ and $(G_{F_n})$ are equivalent, which proves the proposition. □

**Remark 4.22.** *1. If $(\alpha_i)_{i\in I}$ is a $\mathbb{Z}$-basis of $X$, then $\mathcal{T} = \mathcal{T}_{\mathrm{Fix}}$. Indeed, assume that $(\alpha_i)_{i\in I}$ is a $\mathbb{Z}$-basis of $X$. Let $(\chi_i^\vee)_{i\in I}$ be the dual basis. Let $n \in \mathbb{N}^*$ and $t \in T_{n,\Phi} \cap \mathfrak{T}(\mathcal{O})$. Write $t = \prod_{i\in I} \chi_i^\vee(a_i)$, with $a_i \in \mathcal{O}^*$ for $i \in I$. Then $\pi_n(t) = \prod_{i\in I} \chi_i^\vee(\pi_n(a_i))$ and $\pi_n(t) = 1$ if and only if $\pi_n(a_i) = 1$ for all $i \in I$. Now $\alpha_i(t) = a_i$ and thus $t \in T_{n,\Phi}$ if and only if $t \in T_n$. Therefore $\mathcal{Z}_{\mathcal{O}} = \bigcap_{n\in\mathbb{N}} T_n = \{1\}$.*

*2. Note that by Lemma 4.16, the set of left $\mathfrak{T}(\mathcal{O})$-invariant open subsets of $G$ are the same for $\mathscr{T}_{\mathrm{Fix}}$ and $\mathscr{T}$. Indeed, let $V \subset G$ be a non empty left $\mathfrak{T}(\mathcal{O})$-invariant open subset of $G$ for $\mathscr{T}_{\mathrm{Fix}}$. Then for every $v \in V$, there exists $n \in \mathbb{N}^*$ such that $G_{F_n}.v \subset V$. By Lemma 4.16, $\mathfrak{T}(\mathcal{O}).G_{F_n} \subset \mathcal{V}_{n\lambda}$ and thus $V$ is open for $\mathscr{T}$.*

*3. Assume that $\mathcal{K}$ is local. By 2), if $\tau \in \mathrm{Hom}_{\mathrm{Gr}}(Y, \mathbb{C}^*)$, then $I(\tau)_{\mathscr{T}} = I(\tau)_{\mathscr{T}_{\mathrm{Fix}}}$ (see (1.1) for the definition). Indeed, $\delta^{1/2}$ and $\tau$ are maps from $Y = T/\mathfrak{T}(\mathcal{O})$ to $\mathbb{C}^*$ and thus their extensions to $B$ are left $\mathfrak{T}(\mathcal{O})$-invariant. Therefore any element of $\widehat{I(\tau)}$ is left $\mathfrak{T}(\mathcal{O})$-invariant.*

# 5 Properties of the topologies

In this section, we study the properties of the topologies $\mathscr{T}$ and $\mathscr{T}_{\mathrm{Fix}}$. In 5.1, we prove that when $G$ is not reductive, $\mathscr{T}$ is strictly coarser than the Kac-Peterson topology on $G$ (Proposition 5.3). In 5.2, we prove that certain subgroups of $G$ are closed for $\mathscr{T}$. In 5.3, we prove that the compact subsets of $G$ have empty interior. In 5.4, we describe the topology in the case of affine $\mathrm{SL}_2$, under some assumption.

## 5.1 Comparison with the Kac-Peterson topology on $G$

In [KP83], Kac and Peterson defined a topology of topological group on $\mathfrak{G}(\mathbb{C})$. This topology was then studied in [HKM13] and generalized in [HKM13, 7]: Hartnick, Köhl and Mars define a topology of topological group on $\mathfrak{G}(\mathcal{F})$ for $\mathcal{F}$ a local field (Archimedean or not), taking into account the topology of $\mathcal{F}$. The aim of this section is to prove that the topologies we defined on $G = \mathfrak{G}(\mathcal{K})$ are strictly coarsest than the Kac-Peterson topology on $G$, unless $G$ is reductive. As $\mathscr{T}_{\mathrm{Fix}}$ is coarser than $\mathscr{T}$ it suffices to prove that $\mathscr{T}$ is strictly coarser than $\mathscr{T}_{KP}$. To that end, we prove that the topologies induced on $G$ by $\mathscr{T}$ and $\mathscr{T}_{KP}$ on $B := TU^+$ are different using the description of $\mathscr{T}_{KP}|_B$ as a colimit.

**Lemma 5.1.** *Let $w \in W^v$. Assume that $wr_i < w$ for every $i \in I$ (for the Bruhat order on $W^v$). Then $W^v$ is finite.*

*Proof.* By [Kum02, 1.3.13 Lemma], we have $w.\alpha_i \in \Phi_-$ for every $i \in I$. Let $\lambda \in C_f^v$. Then $\alpha_i(w^{-1}.\lambda) < 0$ for every $i \in I$ and thus $w^{-1}.\lambda \in -C_f^v$. Thus $\lambda \in \mathring{\mathcal{T}} \cap -\mathring{\mathcal{T}}$. By [Kum02, 1.4.2 Proposition] we deduce that $\Phi$ is finite and thus $W^v$ is finite. □

We equip $W^v$ with the **right weak Bruhat order** $\preceq$: for every $v, w \in W^v$, $v \preceq w$ if $\ell(v) + \ell(v^{-1}w) = \ell(w)$. We assume that $W^v$ is infinite. By Lemma 5.1, there exists a sequence $(w_i)_{i\in\mathbb{N}} \in (W^v)^{\mathbb{N}}$ such that $w_0 = 1$, $\ell(w_{i+1}) = \ell(w_i) + 1$ and $w_i \preceq w_{i+1}$ for every $i \in \mathbb{N}$.

For $w \in W^v$, one sets $\mathrm{Inv}(w) = \{\alpha \in \Phi_+ \mid w^{-1}.\alpha \in \Phi_-\}$. Let $U_w = \langle U_\alpha \mid \alpha \in \mathrm{Inv}(w)\}$. By [CR09, Lemma 5.8], if $w = r_{i_1} \dots r_{i_k}$, with $k = \ell(w)$ and $i_1, \dots, i_k \in I$, then $U_w = U_{\alpha_{i_1}}.U_{r_{i_1}.\alpha_{i_2}} \dots U_{r_{i_1}\dots r_{i_{k-1}}.\alpha_{i_k}}$ and every element of $U_w$ admits a unique decomposition in this product. By [HKM13, Proposition 7.27], as a topological space, $B$ is the colimit $\lim_{\to} TU_w$ (note that $(W^v, \preceq)$ is not directed). Let $U' = \bigcup_{n\in\mathbb{N}} U_{w_n}$. Then the topology induced on $TU'$ by $\mathscr{T}_{KP}$ is the topology of the direct limit $\lim_{\to} TU_{w_n}$: a subset $V$ of $TU'$ is open if and only if $V \cap TU_{w_n}$ is open for every $n \in \mathbb{N}$.

For $n \in \mathbb{N}$, write $w_{n+1} = w_n r_i$, where $i \in I$. Set $\beta[n] = w_n.\alpha_i$. Then $\mathrm{Inv}(w_n) = \{\beta[i] \mid i \in [\![1, n]\!]\}$, by [Kum02, 1.3.14 Lemma].

By [HKM13, Lemma 7.26], the map $m = m_n : T\times(\mathcal{K})^n \to TU_{w_n}$ defined by $m(t, a_1, \dots, a_n) = t x_{\beta[1]}(a_1) \dots x_{\beta[n]}(a_n)$ is a homeomorphism, when $TU_{w_n}$ for $\mathscr{T}_{KP}|_{TU_{w_n}}$.

Recall that $\mathscr{T} = \mathscr{T}((\mathcal{V}_{n\lambda}))$ for any $\lambda \in Y^+ \cap C_f^v$.

Define $\mathrm{ht} : Q_+ = \bigoplus_{i\in I} \mathbb{Z}\alpha_i \to \mathbb{Z}$ by $\mathrm{ht}(\sum_{i\in I} n_i\alpha_i) = \sum_{i\in I} n_i$, for $(n_i) \in \mathbb{Z}^I$.

**Lemma 5.2.** *Assume that $W^v$ is infinite. For $n \in \mathbb{N}^*$, set $V_n = T\prod_{i=1}^n x_{\beta[i]}(\varpi^{(\mathrm{ht}(\beta[i])!}\mathcal{O})$ and set $V = \bigcup_{n\in\mathbb{N}} V_n$. Then $V$ is open in $(TU', \mathscr{T}_{KP})$ but not in $(TU', \mathscr{T})$. In particular, $\mathscr{T}$ and $\mathscr{T}_{KP}$ are different.*

*Proof.* Let $n \in \mathbb{N}^*$. Let $v \in V \cap TU_{w_n}$ and choose $k \in \mathbb{N}^*$ such that $v \in V_k$. If $k \le n$, then $v \in V_k \subset V_n$. Suppose now $k \ge n$. Write $v = t\prod_{i=1}^k x_{\beta[i]}\left(\varpi^{(\mathrm{ht}(\beta_i)!}a_i\right)$, with $a_1, \dots, a_k \in \mathcal{O}$ and $t \in T$. By [HKM13, Lemma 7.26], we have $a_i = 0$ for every $i \in [\![n+1, k]\!]$ and thus $v \in V_n$. Therefore $V \cap TU_{w_n} = V_n$. By [HKM13, Lemma 7.26], $V_n$ is open in $TU_{w_n}$ and thus $V$ is open in $(TU', \mathscr{T}_{KP})$.

Let $\lambda \in C_f^v \cap Y$ be such that $\alpha_i(\lambda) = 1$ for every $i \in I$. Let us prove that for every $n \in \mathbb{N}^*$, $U' \cap U_{-n\lambda}^{pm+}$ is not contained in $V$. For $k, n \in \mathbb{N}^*$, set $x_{k,n} = \prod_{i=1}^n x_{\beta[i]}(\varpi^{n\mathrm{ht}(\beta[i])}) \in U' \cap U_{-n\lambda}^{pm+}$. By [HKM13, Lemma 7.26], if $x_{k,n} \in V$, then $n\mathrm{ht}(\beta[i]) \ge (\mathrm{ht}(\beta[i]))!$, for every $i \in [\![0, k]\!]$. As $\mathrm{ht}(\beta[i]) \underset{i\to+\infty}{\longrightarrow} +\infty$, there exists $k \in \mathbb{N}$ such that $x_{k,n} \in U' \cap U_{-n\lambda}^{pm+} \setminus V$ and thus, $U' \cap U_{-n\lambda}^{pm+} \not\subset V$. Using Lemma 2.3 we deduce that there exists no $n \in \mathbb{N}^*$ such that $\mathcal{V}_{n\lambda} \cap TU' \subset V$ and thus $V$ is not open for $\mathscr{T}$. □

We equip $\mathrm{SL}_2(\mathcal{K})$ with the topology associated to $(\ker \pi_n^{\mathrm{SL}_2})_{n\in\mathbb{N}^*}$, where $\pi_n^{\mathrm{SL}_2} : \mathrm{SL}_2(\mathcal{O}) \to \mathrm{SL}_2(\mathcal{O}/\varpi^n\mathcal{O})$ is the natural projection. We denote by $x_+$ (resp. $x_-$) the morphism of algebraic groups $a \mapsto \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ (resp. $a \mapsto \left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right)$) for $a$ in a ring $\mathscr{R}$. Using Corollary 3.6, it is easy to check that

$$\ker \pi_n^{\mathrm{SL}_2} = x_+(\varpi^n\mathcal{O}).x_-(\varpi^n\mathcal{O}).\left(\left(\begin{smallmatrix} 1+\varpi^n\mathcal{O} & 0 \\ 0 & 1+\varpi^n\mathcal{O} \end{smallmatrix}\right) \cap \mathrm{SL}_2(\mathcal{K})\right),$$

and thus $(\mathcal{V}_{n\lambda}^{\mathrm{SL}_2})$ is equivalent to $(\ker \pi_n^{\mathrm{SL}_2})$ for any regular $\lambda \in Y_{\mathrm{SL}_2}$.

We equip $T$ with its usual topology, via the isomorphism $T \simeq (\mathcal{K}^*)^m$, for $m$ the rank of $X$. This is the topology $\mathscr{T}((\ker \pi_n|_T)_{n\in\mathbb{N}^*})$. Note that by Lemma 2.3, this is the topology induced by $\mathscr{T}$ on $T$.

**Proposition 5.3.** *1. Let $\alpha \in \Phi_+$ and $\varphi_\alpha : \mathrm{SL}_2(\mathcal{K}) \to G$ be the group morphism defined by $\varphi_\alpha \circ x_\pm = x_{\pm\alpha}$. Fix a basis $(\chi_1^\vee, \dots, \chi_\ell^\vee)$ of $Y$ and define $\iota : (\mathcal{K}^*)^\ell \xrightarrow{\sim} T \subset G$ by $\iota((a_1, \dots, a_\ell)) = \chi_1^\vee(a_1) \dots \chi_\ell^\vee(a_\ell)$, for $a_1, \dots, a_\ell \in \mathcal{K}^*$. Then the $\varphi_\alpha$, $\alpha \in \Phi$ and $\iota$ are continuous when $G$ is equipped with $\mathscr{T}$.*

*2. If $\Phi$ is infinite, the topology $\mathscr{T}$ is strictly coarser than $\mathscr{T}_{KP}$.*

*Proof.* (1) Let $\alpha \in \Phi$ and $\lambda \in Y^+$ be regular. Let $g \in \varphi_\alpha^{-1}(\mathcal{V}_\lambda)$. By [Rou16, 3.16] and [Rém02, 1.2.4 Proposition], we have the Birkhoff decomposition in $\mathrm{SL}_2(\mathcal{K})$ (where $N_{\mathrm{SL}_2}$ is the set of monomial matrices with coefficient in $\mathcal{K}^*$) and $G$:

$$\mathrm{SL}_2(\mathcal{K}) = \bigsqcup_{n\in N_{\mathrm{SL}_2}} x_+(\mathcal{K}) n x_-(\mathcal{K}) \text{ and } G = \bigsqcup_{n\in N} U^+ n U^-.$$

Let $n \in N_{\mathrm{SL}_2}$ be such that $g \in x_+(\mathcal{K})nx_-(\mathcal{K})$. If $n \notin T_{\mathrm{SL}_2}$, then $\varphi_\alpha(g) \in U^+\varphi_\alpha(n)U^-$ and $\nu^v(\varphi_\alpha(n))$ acts as the reflection with respect to $\alpha$ on $\mathbb{A}$. Then $\varphi_\alpha(g) \notin U^+TU^-$ which contradicts Corollary 3.6. Therefore $n \in T$. Write $g = x_+(a_+)nx_-(a_-)$, with $a_+, a_- \in \mathcal{K}$. Then by Lemma 2.3, we have $x_\alpha(a_+) \in U^{pm+}_{[-\lambda,\lambda]}$, $x_{-\alpha}(a_-) \in U^{nm-}_{[-\lambda,\lambda]}$ and $\varphi_\alpha(n) \in T_{2N(\lambda)}$. Consequently, $\omega(a_+), \omega(a_-) \geq |\alpha(\lambda)|$ and

$$t \in T_{\mathrm{SL}_2,2N(\lambda)} := \begin{pmatrix} 1+\varpi^{2N(\lambda)}\mathcal{O} & 0 \\ 0 & 1+\varpi^{2N(\lambda)}\mathcal{O} \end{pmatrix} \cap \mathrm{SL}_2(\mathcal{K}).$$

Therefore $\varphi_\alpha^{-1}(\mathcal{V}_\lambda) \subset x_+(\mathcal{K}_{\omega\geq|\alpha(\lambda)|})T_{\mathrm{SL}_2,2N(\lambda)}x_-(\mathcal{K}_{\omega\geq|\alpha(\lambda)|})$. Conversely, $\varphi_\alpha(x_-(\mathcal{K}_{\omega\geq|\alpha(\lambda)|}))$, $\varphi_\alpha(x_+(\mathcal{K}_{\omega\geq|\alpha(\lambda)|})$, $\varphi_\alpha(T_{\mathrm{SL}_2,2N(\lambda)}) \subset \mathcal{V}_\lambda$ and thus $\varphi_\alpha^{-1}(\mathcal{V}_\lambda) = x_+(\mathcal{K}_{\omega\geq|\alpha(\lambda)|})T_{\mathrm{SL}_2,2N(\lambda)}x_-(\mathcal{K}_{\omega\geq|\alpha(\lambda)|})$ is open in $\mathrm{SL}_2(\mathcal{K})$. Therefore $\varphi_\alpha$ is continuous.

Let $n \in \mathbb{N}^*$. Then $\iota^{-1}(T_n) = (1+\varpi^n\mathcal{O})^\ell$ and thus $\iota$ is continuous.

(2) By [HKM13, Proposition 7.21] and (1), $\mathscr{T}$ is coarser than $\mathscr{T}_{KP}$. By 5.2, $\mathscr{T}$ is different from $\mathscr{T}_{KP}$, which concludes the proof. □

## 5.2 Properties of usual subgroups of $G$ for $\mathscr{T}$ and $\mathscr{T}_{\mathrm{Fix}}$

In this subsection, we prove that many subgroups important in this theory (such as $B$, $T$, $U_\alpha$, $\alpha \in \Phi$, etc.) are open or closed. We have $\mathcal{T}_{\mathrm{Fix}} \subset \mathcal{T}$ and thus every subset of $G$ open or closed for $\mathcal{T}_{\mathrm{Fix}}$ is open or closed for $\mathcal{T}$. As the Kac-Peterson topology $\mathscr{T}_{KP}$ is finer than $\mathscr{T}$, this improves the corresponding results of [HKM13]. Note that we consider $B = B^+$ and $U^+$, but the same results hold for $B^-$ and $U^-$, by symmetry.

If $g \in G$, we say that $g$ **stabilizes** (resp. **pointwise fixes**) $+\infty$ if $g.+\infty = +\infty$ (resp. if there exists $Q \in +\infty$ such that $g$ pointwise fixes $Q$). We denote by $\mathrm{Stab}_G(+\infty)$ the stabilizer of $+\infty$ in $G$.

By [Héb18, 3.4.1], $\mathrm{Stab}_G(+\infty) = B := TU^+$. We denote by $\mathrm{Ch}(\partial\mathcal{I}^+)$ the set of positive sector-germs at infinity of $\mathcal{I}$. For $c \in \mathrm{Ch}(\partial\mathcal{I}^+)$ and $x \in \mathcal{I}$, there exists an apartment $A$ containing $x$ and $c$. We denote by

$$x + c \tag{5.1}$$

the convex hull of $x$ and $c$ in this apartment. This does not depend on the choice of $A$, by (MA II). Fix $\lambda_0 \in C_f^v$. For $r \in \mathbb{R}_+$, we set $\mathcal{C}_r = \{c \in \mathrm{Ch}(\partial\mathcal{I}^+) \mid [0, r.\lambda_0] \subset 0 + c\}$. This set is introduced in [CMR20, Definition 3.1] where it is denoted $U_{0,r,c}$ or $U_{r,c}$.

**Proposition 5.4.** *1. The subgroup $B$ is closed in $G$ for $\mathscr{T}$ and $\mathscr{T}_{\mathrm{Fix}}$.*

*2. The subgroup $U^+$ of $G$ is closed for $\mathscr{T}$. It is closed for $\mathscr{T}_{\mathrm{Fix}}$ if and only if $\mathscr{T} = \mathscr{T}_{\mathrm{Fix}}$.*

*Proof.* 1) Let $g \in G \setminus B$. Then $g.(+\infty) \neq +\infty$. By [CMR20, Lemma 7.6], $\bigcap_{r\in\mathbb{R}_+} \mathcal{C}_r = \{+\infty\}$. Thus there exists $n \in \mathbb{N}^*$ such that $g^{-1}.+\infty \notin \mathcal{C}_n$. Then $\mathcal{V}_{n\lambda_0}$ fix $[0, n\lambda_0]$. Let $v \in \mathcal{V}_{n\lambda_0}$. Then

$$v.(0 + \infty) = v.0 + (v. + \infty) = 0 + (v. + \infty) \supset v.[0, n\lambda_0] = [0, n\lambda_0].$$

Therefore $\mathcal{V}_{n\lambda_0}.(+\infty) \subset \mathcal{C}_n$. Consequently, $g^{-1}.(+\infty) \notin \mathcal{V}_{n\lambda_0}. + \infty$ and thus $g\mathcal{V}_{n\lambda_0}. + \infty \not\ni +\infty$. Thus $g.\mathcal{V}_{n\lambda_0} \subset G \setminus B$, which proves that $G \setminus B$ is open for $\mathscr{T}_{\mathrm{Fix}}$.

2) Let $g \in G \setminus U^+$. If $g \in G \setminus U^+T$, then by 1), there exists $V \in \mathcal{T}$ such that $gV \subset G \setminus U^+T \subset G \setminus U^+$. We now assume $g \in U^+T \setminus U$. Write $g = u_+t$, with $u_+ \in U^+$ and $t \in T \setminus \{1\}$. Let $\lambda \in Y \cap C_f^v$ and assume that $g\mathcal{V}_\lambda \cap U^+ \neq \varnothing$. Then there exists $(u'_+, u'_-, t') \in U^{pm+}_{[-\lambda,\lambda]} \times U^{nm-}_{[-\lambda,\lambda]} \times T_{2N(\lambda)}$ such that $u_+tu'_+u'_-t' = u''_+$, where $u''_+ \in U^+$. As $t$ normalizes $U^+$ and $U^-$, we can write $tu'_+u'_- = u^{(3)}_+u^{(3)}_-t$, for some $u^{(3)}_+ \in U^+, u^{(3)}_- \in U^-$. Then we have

$$u''^{-1}_+ u_+ u^{(3)}_+ u^{(3)}_- tt' = 1.$$

By Lemma 2.3 we deduce $tt' = 1$. Therefore $t \in T_{2N(\lambda)}$. Thus if $\lambda$ is sufficiently dominant, $g\mathcal{V}_\lambda \cap U^+ = \varnothing$ and $g\mathcal{V}_\lambda \subset G \setminus U^+$. We deduce that $G \setminus U^+$ is closed for $\mathcal{T}$.

Suppose now $\mathscr{T} \neq \mathscr{T}_{\mathrm{Fix}}$. Then by Proposition 4.21, $\mathcal{Z}_\mathcal{O} \neq \{1\}$. Then every non empty open subset of $G$ for $\mathcal{T}_{\mathrm{Fix}}$ contains $\mathcal{Z}_\mathcal{O}$. Take $z \in \mathcal{Z}_\mathcal{O} \setminus \{1\}$. As $z \in \mathfrak{T}(\mathcal{O})$, $z \in G \setminus U^+$. Moreover, for any non empty open subset $V$ of $G$, $zV \ni 1 \in U^+$. Therefore $G \setminus U^+$ is not open, which completes the proof of the proposition.

□

**Proposition 5.5.** *1. Let $x \in \mathcal{I}$. Then the fixator $G_x$ of $x$ in $G$ is open (for $\mathcal{T}_{\mathrm{Fix}}$ and $\mathcal{T}$). In particular, $\mathfrak{G}^{\min}(\mathcal{O})$ is open in $G$.*

*2. Let $E \subset \mathcal{I}$. Then the fixator and the stabilizer of $E$ in $G$ are closed for $\mathcal{T}_{\mathrm{Fix}}$ and $\mathcal{T}$.*

*Proof.* 1) By Lemma 4.13, $G_{F_n} \subset G_x$ for $n \gg 0$, which proves 1).

2) Let $g \in G \setminus G_E$. Let $x \in E$ be such that $g.x \neq x$. Then $g.G_x \subset G \setminus G_E$ and hence $G_E$ is open. Let $g \in G \setminus \mathrm{Stab}_G(E)$. Let $x \in E$ be such that $g.x \notin E$. Then $g.G_x \subset G \setminus \mathrm{Stab}_G(E)$ and hence $\mathrm{Stab}_G(E)$ is closed.

□

**Corollary 5.6.** *The subgroups $N$ and $T$ are closed in $G$ for $\mathcal{T}_{\mathrm{Fix}}$ and $\mathcal{T}$.*

*Proof.* By Proposition 5.5, $N = \mathrm{Stab}_G(\mathbb{A})$ is closed. We have $T = \mathrm{Stab}_G(+\infty) \cap N$. Indeed, it is clear that $T \subset \mathrm{Stab}_G(+\infty) \cap N$. Conversely, let $g \in \mathrm{Stab}_G(+\infty) \cap N$. Let $w \in W^v$ and $\lambda \in \Lambda$ be such that $g.x = \lambda + w.x$ for every $x \in \mathbb{A}$. Then $w.C_f^v = C_f^v$ and thus $w = 1$. Therefore $g$ acts by translation on $\mathbb{A}$ and hence $g \in T$. This proves that $T = \mathrm{Stab}_G(+\infty) \cap N$ and we conclude with Proposition 5.4. □

**Remark 5.7.**

1. *The fixator $K_I$ of $C_0^+$ is open for $\mathscr{T}_{\mathrm{Fix}}$. Indeed, let $\lambda \in C_f^v \cap Y$. Then $G_{[0,\lambda]} = G_0 \cap G_\lambda$ is open for $\mathscr{T}_{\mathrm{Fix}}$ and $G_{[0,\lambda]} \subset K_I$.*

2. *For $x, y \in \mathcal{I}$, one writes $x \leq y$ if there exists $g \in G$ such that $g.x, g.y \in \mathbb{A}$ and $g.y - g.x \in \mathcal{T} = \bigcup_{w \in W^v} w.\overline{C_f^v}$. By $W^v$-invariance of $\mathcal{T}$, this does not depend on the choice of $g$ and by [Rou11, Théorème 5.9], $\leq$ is a preorder on $\mathcal{I}$. One sets*

$$G^+ = \{g \in G \mid g.0 \geq 0\}.$$

*This is a subsemigroup of $G$ which is crucial for the definition of the Hecke algebras associated with $G$ (when $\mathcal{K}$ is local), see [BK11], [BKP16], [GR14] or [BGR16]. Then $G^+ \supset G_0 = \mathfrak{G}^{\min}(\mathcal{O})$ and thus $G^+$ is open in $G$.*

**Lemma 5.8.** *Let $g \in G$. Then there exists $n \in N$ such that $g.a = n.a$ for every $a \in \mathbb{A} \cap g^{-1}.\mathbb{A}$.*

*Proof.* Let $h \in G$ be such that $hg.\mathbb{A} = \mathbb{A}$ and $h$ fixes $\mathbb{A} \cap g.\mathbb{A}$, which exists by (MA II). Then $n := hg$ stabilizes $\mathbb{A}$ and thus it belongs to $N$. Moreover, $hg.a = n.a = g.a$ for every $a \in \mathbb{A} \cap g^{-1}.\mathbb{A}$, which proves the lemma. □

**Lemma 5.9.** *Let $\alpha \in \Phi$. Write $\alpha = \epsilon w.\alpha_i$, for $w \in W^v$, $\epsilon \in \{-, +\}$ and $i \in I$. Let $\mathfrak{Q}$ be the sector-germ at infinity of $-\epsilon w r_i(C_f^v)$. Then $U_\alpha T = \mathrm{Stab}_G(w.\epsilon\infty) \cap \mathrm{Stab}_G(\mathfrak{Q})$.*

*Proof.* There is no loss of generality in assuming that $w = 1$ and $\epsilon = +$. Let $u \in U_{\alpha_i}$. Then $u$ fixes a translate of $\alpha_i^{-1}(\mathbb{R}_+)$. Therefore $TU_{\alpha_i}$ stabilizes $\mathfrak{Q}$ and $+\infty$. Conversely, let $g \in \mathrm{Stab}_G(+\infty) \cap \mathrm{Stab}_G(\mathfrak{Q})$. Then there exist $x, x' \in \mathbb{A}$ such that $g.(x + C_f^v) = x' + C_f^v$. Then by Lemma 5.8, there exists $n \in N$ such that $g.x'' = n.x''$ for every $x'' \in \mathbb{A} \cap g^{-1}.\mathbb{A}$. Then $n$ fixes $+\infty$ and thus $n \in T$ (by the proof of Corollary 5.6). Then $n^{-1}g.x = x$. Considering $n^{-1}g$ instead of $g$, we may assume that $g$ pointwise fixes $+\infty$. Therefore $g$ pointwise fixes $\mathfrak{Q}$. There exists $a, a' \in \mathbb{A}$ such that $g$ fixes $a + C_f^v$ and $g$ fixes $a' - r_i(C_f^v)$. Let $A = g.\mathbb{A}$. Then $A \cap \mathbb{A}$ is a finite intersection of half-apartments by (MA II) and thus either $A = \mathbb{A}$ or $A \cap \mathbb{A}$ is a translate of $\alpha_i^{-1}(\mathbb{R}_+)$. Moreover, $g$ fixes $A \cap \mathbb{A}$ since it fixes an open subset of $A \cap \mathbb{A}$. By [Rou16, 5.7 3)], $g \in U_{\alpha_i} \mathfrak{T}(\mathcal{O})$. Consequently $\mathrm{Stab}_G(+\infty) \cap \mathrm{Stab}_G(\mathfrak{Q}) \subset U_{\alpha_i} T$, and the lemma follows. □

**Proposition 5.10.** *Let $\alpha \in \Phi$.*

1. *The group $U_\alpha T$ is closed for $\mathcal{T}$ and $\mathcal{T}_{\mathrm{Fix}}$.*

2. *The group $U_\alpha$ is closed for $\mathcal{T}$.*

3. *If $\mathscr{T} \neq \mathscr{T}_{\mathrm{Fix}}$, then $U_\alpha$ is not closed for $\mathscr{T}_{\mathrm{Fix}}$.*

*Proof.* 1) Let $\mathfrak{Q}$ be a sector-germ of $\mathcal{I}$ (positive or negative). Then by Proposition 5.4 (or the similar proposition for $B^-$ if $\mathfrak{Q}$ is negative), $\mathrm{Stab}_G(\mathfrak{Q})$ is closed in $G$ for $\mathscr{T}_{\mathrm{Fix}}$. By Lemma 5.9 and Proposition 5.5, we have 1).

2) We have $U_\alpha = U_\alpha T \cap U^+$, by Lemma 2.3 and thus 2) follows from 1) and Proposition 5.4. The proof of 3) is similar to the proof of the corresponding result of Proposition 5.4. □

## 5.3 Compact subsets have empty interior

By [AH19, Theorem 3.1], for any topology of topological group on $G$, $G_{C_0^+}$ or $G_0$ are not compact and open. In particular, $G_0$ and $G_{C_0^+}$ are not compact for $\mathscr{T}$. With a similar reasoning, we have the following.

**Proposition 5.11.** *Assume that $W^v$ is infinite.*

1. *Let $n \in \mathbb{N}^*$ and $\lambda \in Y^+$ regular. Then $\mathcal{V}_{n\lambda}/\mathcal{V}_{(n+1)\lambda}$ is infinite.*

2. *Every compact subset of $(G, \mathscr{T})$ has empty interior.*

*Proof.* 1) Set $H = G_{[-n\lambda,(n+1)\lambda]} \subset G$. Then $H \supset \mathcal{V}_{(n+1)\lambda}$, by (4.1). Thus $|\mathcal{V}_{n\lambda}/\mathcal{V}_{(n+1)\lambda}| \geq |\mathcal{V}_{n\lambda}/(H \cap \mathcal{V}_{n\lambda})|$ and it suffices to prove that $\mathcal{V}_{n\lambda}/(H \cap \mathcal{V}_{n\lambda})$ is infinite. We have $\mathcal{V}_{n\lambda} = \bigsqcup_{v \in \mathcal{V}_{n\lambda}/(H\cap\mathcal{V}_{n\lambda})} v.(H \cap \mathcal{V}_{n\lambda})$. Moreover if $v, v' \in \mathcal{V}_{n\lambda}$, then $v.((n+1)\lambda) = v'.((n+1)\lambda)$ if and only if $v'.(G_{(n+1)\lambda} \cap \mathcal{V}_{n\lambda}) = v.(G_{(n+1)\lambda} \cap \mathcal{V}_{n\lambda})$.

Let us prove that $G_{(n+1)\lambda} \cap \mathcal{V}_{n\lambda} = H \cap \mathcal{V}_{n\lambda}$. Let $g \in G_{(n+1)\lambda} \cap \mathcal{V}_{n\lambda}$. Then by (4.1), $g$ fixes $[-n\lambda, n\lambda]$ and $(n+1)\lambda$. Then $g.\mathbb{A}$ is an apartment containing $[-n\lambda, n\lambda] \cup \{(n+1)\lambda\}$. As $g.\mathbb{A} \cap \mathbb{A}$ is convex, $g.\mathbb{A}$ contains $[-n\lambda, (n+1)\lambda]$. By (MA II), there exists $h \in G$ such that $g.\mathbb{A} = h.\mathbb{A}$ and $h$ fixes $\mathbb{A} \cap g.\mathbb{A}$. Then $h^{-1}g.\mathbb{A} = \mathbb{A}$ and $h^{-1}g$ acts on $\mathbb{A}$ by an affine map. As $h^{-1}g$ fixes $[-n\lambda, n\lambda]$, it fixes $[-n\lambda, (n+1)\lambda]$. Therefore $g$ fixes $[-n\lambda, (n+1)\lambda]$ and thus $g \in H$. Therefore $G_{(n+1)\lambda} \cap \mathcal{V}_{n\lambda} = H \cap \mathcal{V}_{n\lambda}$. Consequently,

$$\mathcal{V}_{n\lambda}.((n+1)\lambda) = \bigsqcup_{v \in \mathcal{V}_{n\lambda}/(H\cap\mathcal{V}_{n\lambda})} \{v.(n+1)\lambda\} \text{ and } |\mathcal{V}_{n\lambda}/(H \cap \mathcal{V}_{n\lambda})| = |\mathcal{V}_{n\lambda}.((n+1)\lambda)|.$$

Let $(\beta_\ell) \in (\Phi_+)^{\mathbb{N}}$ be an injective sequence. Write $\beta_\ell = \sum_{i\in I} m_i^{(\ell)}\alpha_i$, with $m_i^{(\ell)} \in \mathbb{N}$ for $\ell \in \mathbb{N}$. Then $\beta_\ell(\lambda) \geq (\sum_{i\in I} m_i^{(\ell)})(\min_{i\in I} \alpha_i(\lambda)) \underset{\ell\to\infty}{\to} +\infty$. Let $\ell \in \mathbb{N}$. For $k \in [\![\beta_\ell(n\lambda), \beta_\ell((n+1)\lambda) - 1]\!]$, $x_{-\beta_\ell}(\varpi^k) \in U^{nm-}_{[-n\lambda,n\lambda]} \subset \mathcal{V}_{n\lambda}$. Set

$$x_k = x_{-\beta_\ell}(\varpi^k).((n+1)\lambda) \in \mathcal{V}_{n\lambda}.((n+1)\lambda).$$

Let $k' \in [\![\beta_\ell(n\lambda), \beta_\ell((n+1)\lambda) - 1]\!]$. Then $x_k = x_{k'}$ if and only if $x_{-\beta_\ell}(\varpi^k).((n+1)\lambda) = x_{-\beta_\ell}(\varpi^{k'}).((n+1)\lambda)$ if and only if $x_{-\beta_\ell}(\varpi^k - \varpi^{k'}).((n+1)\lambda) = (n+1)\lambda$ if and only if

$\omega(\varpi^k - \varpi^{k'}) \geq (n+1)\beta_\ell(\lambda)$ if and only $k = k'$. Therefore $|\mathcal{V}_{n\lambda}.((n+1)\lambda)| \geq \beta_\ell(\lambda)$. As this is true for every $\ell \in \mathbb{N}$, $|\mathcal{V}_{n\lambda}.((n+1)\lambda)|$ if infinite, which proves 1).

2) Let $V$ be a compact subset of $G$ and assume that $V$ has non empty interior. Considering $v^{-1}.V$ instead of $V$, we may assume $1 \in V$. Then there exists $\lambda \in Y^+ \cap C_f^v$ such that $\mathcal{V}_\lambda \subset V$, and we have $\mathcal{V}_{2\lambda} \subset \mathcal{V}_\lambda$. As $\mathcal{V}_\lambda$ is closed, it is compact. By 1), $\mathcal{V}_\lambda/\mathcal{V}_{2\lambda}$ is infinite. Therefore $\mathcal{V}_\lambda = \bigsqcup_{v \in \mathcal{V}_\lambda/\mathcal{V}_{2\lambda}} v.\mathcal{V}_{2\lambda}$ is a cover of $\mathcal{V}_\lambda$ by open subsets from which we can not extract a finite subcover: we reach a contradiction. Thus every compact subset of $G$ has empty interior.

□

## 5.4 Example of affine $\mathrm{SL}_2$

In this subsection, we determine an explicit filtration equivalent to $(\mathcal{V}_{n\lambda})$ in the case of affine $\mathrm{SL}_2$ (quotiented by the central extension).

Let $Y = \mathbb{Z}\mathring{\alpha}^\vee \oplus \mathbb{Z}d$, where $\mathring{\alpha}^\vee, d$ are some symbols, corresponding to the positive root of $\mathrm{SL}_2(\mathcal{K})$ and to the semi-direct extension by $\mathcal{K}^*$ respectively. Let $X = \mathbb{Z}\mathring{\alpha} \oplus \mathbb{Z}\delta$, where $\mathring{\alpha}$, $\delta : Y \to \mathbb{Z}$ are the $\mathbb{Z}$-module morphisms defined by $\mathring{\alpha}(\mathring{\alpha}^\vee) = 2$, $\mathring{\alpha}(d) = 0$, $\delta(\mathring{\alpha}^\vee) = 0$ and $\delta(d) = 1$. Let $\alpha_0 = \delta - \mathring{\alpha}$, $\alpha_1 = \mathring{\alpha}$, $\alpha_0^\vee = -\mathring{\alpha}^\vee$ and $\alpha_1^\vee = \mathring{\alpha}^\vee$. Then $\mathcal{S} = (\left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right), X, Y, \{\alpha_0, \alpha_1\}, \{\alpha_0^\vee, \alpha_1^\vee\})$ is a root generating system. Let $\mathfrak{G}$ be the Kac-Moody group associated with $\mathcal{S}$ and $G = \mathfrak{G}(\mathcal{K})$. Then by [Kum02, 13] and [Mar18, 7.6], $G = \mathrm{SL}_2(\mathcal{K}[u, u^{-1}]) \rtimes \mathcal{K}^*$, where $u$ is an indeterminate and if $(M, z), (M_1, z_1) \in G$, with $M = \left(\begin{smallmatrix} a(\varpi,u) & b(\varpi,u) \\ c(\varpi,u) & d(\varpi,u) \end{smallmatrix}\right)$, $M_1 = \left(\begin{smallmatrix} a_1(\varpi,u) & b_1(\varpi,u) \\ c_1(\varpi,u) & d_1(\varpi,u) \end{smallmatrix}\right)$, we have

$$(M, z).(M_1, z_1) = (M \left(\begin{smallmatrix} a_1(\varpi,zu) & b_1(\varpi,zu) \\ c_1(\varpi,zu) & d_1(\varpi,zu) \end{smallmatrix}\right), zz_1). \tag{5.2}$$

Note that the family $(\alpha_0^\vee, \alpha_1^\vee)$ is not free. We have $\Phi = \{\alpha + k\delta \mid \alpha \in \{\pm\mathring{\alpha}\}, k \in \mathbb{Z}\}$ and $(\alpha_0, \alpha_1)$ is a basis of this root system. We denote by $\Phi^+$ (resp. $\Phi^-$) the set $\Phi \cap (\mathbb{N}\alpha_0 + \mathbb{N}\alpha_1)$ (resp $-\Phi_+$). For $k \in \mathbb{Z}$ and $y \in \mathcal{K}$, we set $x_{\mathring{\alpha}+k\delta}(y) = (\left(\begin{smallmatrix} 1 & u^k y \\ 0 & 1 \end{smallmatrix}\right), 1) \in G$ and $x_{-\mathring{\alpha}+k\delta}(y) = (\left(\begin{smallmatrix} 1 & 0 \\ u^k y & 1 \end{smallmatrix}\right), 1) \in G$.

Let $f, g \in \mathcal{K}$ be such that $\omega(f) = \omega(g) = 0$. Let $\ell, n \in \mathbb{Z}$. Then $\left(\left(\begin{smallmatrix} f\varpi^\ell & 0 \\ 0 & f^{-1}\varpi^{-\ell} \end{smallmatrix}\right), g\varpi^n\right)$ acts on $\mathbb{A}$ by the translation of vector $-\ell\mathring{\alpha}^\vee - nd$. For $\mu = \ell\mathring{\alpha}^\vee + nd \in Y$, we set $t_\mu = (\left(\begin{smallmatrix} \varpi^{-\ell} & 0 \\ 0 & \varpi^\ell \end{smallmatrix}\right), \varpi^{-n})$, which acts by the translation of vector $\mu$ on $\mathbb{A}$. We set $\lambda = \mathring{\alpha}^\vee + 3d$. We have $\alpha_0(\lambda) = 1$, $\alpha_1(\lambda) = 2$ and thus $\lambda \in C_f^v$.

By [Rou16, 4.12 3 b], $U_0^{pm+} = \left(\left(\begin{smallmatrix} 1+u\mathcal{O}[u] & \mathcal{O}[u] \\ u\mathcal{O}[u] & 1+u\mathcal{O}[u] \end{smallmatrix}\right), 1\right) \cap G$ and similarly

$$U_0^{nm-} = \left(\left(\begin{smallmatrix} 1+u^{-1}\mathcal{O}[u^{-1}] & u^{-1}\mathcal{O}[u^{-1}] \\ \mathcal{O}[u^{-1}] & 1+u^{-1}\mathcal{O}[u^{-1}] \end{smallmatrix}\right), 1\right) \cap G.$$

We make the following assumption:

$$\forall n \in \mathbb{N}^*, \ker \pi_n \subset \left(\begin{smallmatrix} 1+\varpi^n\mathcal{O}[u,u^{-1}] & 1+\varpi^n\mathcal{O}[u,u^{-1}] \\ 1+\varpi^n\mathcal{O}[u,u^{-1}] & 1+\varpi^n\mathcal{O}[u,u^{-1}] \end{smallmatrix}\right) \rtimes (1 + \varpi^n\mathcal{O}). \tag{5.3}$$

If for any $n \in \mathbb{N}^*$, we have $\mathfrak{G}(\mathcal{O}/\varpi^n\mathcal{O}) \simeq \begin{pmatrix} (\mathcal{O}/\varpi^n\mathcal{O})[u,u^{-1}] & (\mathcal{O}/\varpi^n\mathcal{O})[u,u^{-1}] \\ (\mathcal{O}/\varpi^n\mathcal{O})[u,u^{-1}] & (\mathcal{O}/\varpi^n\mathcal{O})[u,u^{-1}] \end{pmatrix} \rtimes (\mathcal{O}/\varpi^n\mathcal{O})^\times$ and $\pi_n$ is the canonical projection, then the assumption is satisfied. However we do not know if it is true. In [Mar18, 7.6], $\mathfrak{G}$ is described only on fields and in [Kum02, 13], only on $\mathbb{C}$.

For $n \in \mathbb{N}^*$, we set $H_n = \ker(\pi_n) \cap \left( \begin{pmatrix} \mathcal{O}[(\varpi u)^n,(\varpi u^{-1})^n] & \mathcal{O}[(\varpi u)^n,(\varpi u^{-1})^n] \\ \mathcal{O}[(\varpi u)^n,(\varpi u^{-1})^n] & \mathcal{O}[(\varpi u)^n,(\varpi u^{-1})^n] \end{pmatrix}, \mathcal{K}^* \right)$.

**Proposition 5.12.** *If* (5.3) *is true, then the filtrations* $(H_n)_{n\in\mathbb{N}^*}$ *and* $(\mathcal{V}_{n\lambda})_{n\in\mathbb{N}^*}$ *are equivalent.*

*Proof.* Let $n \in \mathbb{N}^*$. By Lemma 2.4, we have $U^{pm+}_{[-n\lambda,n\lambda]} = U^{pm+}_{-n\lambda} = t_{-n\lambda} U^{pm+}_0 t_{n\lambda}$. We have $t_{n\lambda} = t_{n\mathring{\alpha}^\vee} t_{3nd}$. We have

$$t_{-3nd} U^{pm+}_0 t_{3nd} \subset \begin{pmatrix} 1+(\varpi^{3n}u)\mathcal{O}[\varpi^{3n}u] & \mathcal{O}[\varpi^{3n}u] \\ (\varpi^{3n}u)\mathcal{O}[\varpi^{3n}u] & 1+(\varpi^{3n}u)\mathcal{O}[\varpi^{3n}u] \end{pmatrix} \ltimes \{1\}.$$

We have

$$\begin{aligned} & t_{-n\mathring{\alpha}^\vee} \left( \begin{pmatrix} 1+\varpi^{3n}u\mathcal{O}[\varpi^{3n}u] & \mathcal{O}[\varpi^{3n}u] \\ (\varpi^{3n}u)\mathcal{O}[\varpi^{3n}u] & 1+\varpi^{3n}u\mathcal{O}[\varpi^{3n}u] \end{pmatrix} \ltimes \{1\} \right) t_{n\mathring{\alpha}^\vee} \\ & \subset \left( \begin{pmatrix} 1+\varpi^{3n}u\mathcal{O}[\varpi^{3n}u] & \varpi^{2n}\mathcal{O}[\varpi^{3n}u] \\ \varpi^{-2n}(\varpi^{3n}u)\mathcal{O}[\varpi^{3n}u] & 1+\varpi^{3n}u\mathcal{O}[\varpi^{3n}u] \end{pmatrix} \ltimes \{1\} \right) \\ & \subset \left( \begin{pmatrix} 1+(\varpi^{n}u)\mathcal{O}[\varpi^{n}u] & \mathcal{O}[\varpi^{n}u] \\ \varpi^{n}u\mathcal{O}[\varpi^{n}u] & 1+\varpi^{n}u\mathcal{O}[\varpi^{n}u] \end{pmatrix} \ltimes \{1\} \right) \\ & \subset \mathrm{SL}_2(\mathcal{O}[\varpi^n u]) \ltimes \{1\} \subset \mathrm{SL}_2(\mathcal{O}[\varpi^n u, \varpi^n u^{-1}]) \ltimes \{1\}. \end{aligned}$$

Similarly, $U^{nm-}_{[-n\lambda,n\lambda]} \subset \mathrm{SL}_2(\mathcal{O}[\varpi^n u, \varpi^n u^{-1}]) \ltimes \{1\}$.

As $T_{2n} \subset \mathrm{SL}_2(\mathcal{O}[\varpi^n u, \varpi^n u]) \ltimes \{1 + \varpi^n\mathcal{O}\}$, we deduce $\mathcal{V}_{n\lambda} \subset H_n$, since $\mathcal{V}_{n\lambda} \subset \ker(\pi_n)$.

Now let $M \in \mathrm{SL}_2(\mathcal{O}[\varpi^{2n}u, \varpi^{2n}u^{-1}]) \cap \ker(\pi_{2n})$ and $a \in \mathcal{K}^*$. Using (5.3), we write

$$M = \begin{pmatrix} 1 + a_0\varpi^{2n} + \sum_{|i|\geq 1} a_i \varpi^{2n|i|} u^i & b_0\varpi^{2n} + \sum_{|i|\geq 1} a_i \varpi^{2n|i|} u^i \\ c_0\varpi^{2n} + \sum_{|i|\geq 1} c_i \varpi^{2n|i|} u^i & 1 + d_0\varpi^{2n} + \sum_{|i|\geq 1} d_i \varpi^{2n|i|} u^i \end{pmatrix},$$

with $a_i, b_i, c_i, d_i \in \mathcal{O}$, for all $i$. Then

$$t_{-nd}(M,a)t_{nd} = \left( \begin{pmatrix} 1 + a_0\varpi^{2n} + \sum_{|i|\geq 1} a_i \varpi^{n(2|i|-i)} u^i & b_0\varpi^{2n} + \sum_{|i|\geq 1} a_i \varpi^{n(2|i|-i)} u^i \\ c_0\varpi^{2n} + \sum_{|i|\geq 1} c_i \varpi^{n(2|i|-i)} u^i & 1 + d_0\varpi^{2n} + \sum_{|i|\geq 1} d_i \varpi^{n(2|i|-i)} u^i \end{pmatrix}, a \right).$$

Therefore

$$\begin{aligned} & t_{-n\mathring{\alpha}^\vee - nd}(M,a)t_{n(\mathring{\alpha}^\vee + d)} \\ & = \left( \begin{pmatrix} 1 + a_0\varpi^{2n} + \sum_{|i|\geq 1} a_i \varpi^{n(2|i|-i)} u^i & \varpi^{2n}(b_0\varpi^{2n} + \sum_{|i|\geq 1} a_i \varpi^{n(2|i|-i)} u^i) \\ \varpi^{-2n}\left(c_0\varpi^{2n} + \sum_{|i|\geq 1} c_i \varpi^{n(2|i|-i)} u^i\right) & 1 + d_0\varpi^{2n} + \sum_{|i|\geq 1} d_i \varpi^{n(2|i|-i)} u^i \end{pmatrix}, a \right) \\ & \in \mathrm{SL}_2(\mathcal{O}[u,u^{-1}]) \ltimes \mathcal{O}^*. \end{aligned}$$

By [BHR22, Lemma 6.10], $H_{2n}$ fixes $n\lambda'$, where $\lambda' = \mathring{\alpha}^\vee + d$. Similarly, it fixes $-n\lambda'$. Therefore $H_{2n} \subset G_{[-n\lambda',n\lambda']} \cap \ker \pi_{2n}$. We have $G_{[-n\lambda',n\lambda']} = U^{pm+}_{[-n\lambda',n\lambda']}.U^{nm-}_{[-n\lambda',n\lambda']}.\mathfrak{T}(\mathcal{O})$, by (2.8). Using the inclusion $\langle U^{pm+}_{[-n\lambda',n\lambda']}, U^{nm-}_{[-n\lambda',n\lambda']} \rangle \subset \ker \pi_n$, we deduce that $G_{[-n\lambda',n\lambda']} \cap \ker \pi_n \subset \mathcal{V}_{n\lambda'}$. As $(\mathcal{V}_{m\lambda})$ and $(\mathcal{V}_{m\lambda'})$ are equivalent, we deduce that $(H_m)$ and $(\mathcal{V}_{n\lambda})$ are equivalent. □

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