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AN ANALYTIC VIEWPOINT ON THE HASSE PRINCIPLE

VLERË MEHMETI

Abstract. Working on non-Archimedean analytic curves, we propose a geometric approach to the study of the Hasse principle over function fields of curves defined over a complete discretely valued field. Using it, we show the Hasse principle to be verified for certain families of projective homogeneous spaces. As a consequence, we prove that said principle holds for quadratic forms and homogeneous varieties over unitary groups, results originally shown in [9], [36] and [33].

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Introduction

History. Let K be a number field, and G/K a semisimple simply connected linear algebraic group. By results of Kneser ([21], [22]), Harder ([18], [19]) and Chernousov ([6]), we know that the following *Hasse principle* holds: a torsor over G has a K-rational point if and only if it has rational points over all the completions of K. It is worth mentioning here that to this day there is no uniform proof of the Hasse principle in this setting: Kneser proves it for classical groups, Harder for exceptional groups with the exception of E_8, and Chernousov for E_8.

Later on, Harder ([20]) proved that a Hasse principle continues to hold if K is the function field of a curve defined over a finite field, meaning it holds for all global fields. The methods used in [20] involve yet again case-by-case considerations.

During the last couple of decades, similar questions are being studied over function fields of curves which are defined over more general fields. In [16], Harbater, Hartmann and Krashen introduced a new technique to the study of such questions: *algebraic patching*. Let F be the function field of a curve defined over a complete discretely valued field. Let G/F be a linear algebraic group that is a rational variety. Through algebraic patching,

the aforementioned authors showed that a *local-global principle* holds for projective homogeneous varieties X over G. This means there exist larger fields $F_i, i \in I$, such that X has an F-rational point if and only if it has F_i-rational points for all i.

In terms of the Hasse principle in this setting, in [9], Colliot-Thélène, Parimala and Suresh propose the following conjecture:

Conjecture 0.1 ([9, Conjecture 1])**.** *Let k be a p-adic field. Let F be the function field of a smooth projective geometrically integral curve defined over k. Let Ω denote the set of discrete (rank 1) valuations on F which either extend the norm of k or are trivial on k. Let X/F be a projective homogeneous space over a connected linear algebraic group G/F. Then, $X(F) \neq \emptyset \iff X(F_v) \neq \emptyset \ \forall v \in \Omega$, where F_v denotes the completion of F with respect to v.*

In [9], it is shown that the conjecture is true in the case of quadratic forms. Considerable progress has been made toward Conjecture 0.1 for classical linear algebraic groups in [33], [36], [34] and [15]. In [9] and [33], algebraic patching is a crucial ingredient. In [8], the same technique is used to prove that a Hasse principle is verified if G is defined over k°-the valuation ring of k. The authors also provide a counterexample to the conjecture when k is *not* a p-adic field.

In [29], we extend algebraic patching to Berkovich analytic curves, where it acquires a very geometric form. Through it, we generalize the results of [16], and show that Conjecture 0.1 is true if G is a rational linear algebraic group and provided we replace Ω by the set of *all* (rank 1) valuations on F which either extend the norm on k or are trivial there. The result in question does not depend on k being discretely valued. More precisely, we show:

Theorem 0.2 ([29, Corollary 3.18])**.** *Let k be a complete ultrametric non-trivially valued field. Let F denote the function field of a normal irreducible projective algebraic curve C defined over k. Let us denote by $\overline{\Omega}$ the set of all (rank 1) non-trivial valuations v on F such that $v_{|k}$ is either trivial or induces the norm on k.*

Let X/F be a variety on which a rational linear algebraic group G/F acts strongly transitively*. Then*

$$X(F) \neq \emptyset \iff X(F_v) \neq \emptyset \ \forall v \in \overline{\Omega},$$

where F_v denotes the completion of F with respect to v.

We recall:

Definition 0.3. The linear algebraic group G/F is said to act *strongly transitively* on the variety X/F if for any field extension L/F, either $X(L) = \emptyset$ or $G(L)$ acts transitively on $X(L)$.

If G is reductive, then by [16, Remark 3.9], for any projective homogeneous F-variety X over G, the action of G on X is strongly transitive.

A geometric approach. We see from Theorem 0.2 that in order to prove Conjecture 0.1 in the case of rational linear algebraic groups it suffices to show that

$$X(F_v) \neq \emptyset \ \forall v \in \Omega \Longrightarrow X(F_v) \neq \emptyset \ \forall v \in \overline{\Omega}. \tag{A}$$

The points of a Berkovich analytic curve are in bijective correspondence to the valuations in $\overline{\Omega}$ (see [29, Proposition 3.15]). The set of points on the analytic curve corresponding

to the discrete valuations (*i.e.* to the points of Ω) can be well described and has been extensively studied in the theory of these analytic spaces.

From now on, we suppose that k is discretely valued. Following the notation of Theorem 0.2, let C^{an} denote the Berkovich analytification of the curve C. We denote by $\mathscr{M}$ the sheaf of meromorphic functions on C^{an}. By [1, Proposition 3.6.2], we know that $F = \mathscr{M}(C^{\mathrm{an}})$. Let $val : C^{\mathrm{an}} \to \overline{\Omega}$ denote the bijection constructed in [29, Proposition 3.15] between the points of C^{an} and the valuations in $\overline{\Omega}$. In [29, Corollary 3.17], we show that for any $x \in C^{\mathrm{an}}$, $X(\mathscr{M}_x) \neq \emptyset$ if and only if $X(F_{val(x)}) \neq \emptyset$. Theorem 0.2 is proved as a corollary of the following equivalence:

$$X(F) \neq \emptyset \iff X(\mathscr{M}_x) \neq \emptyset \ \forall x \in C^{\mathrm{an}}.$$

Set $S_{\mathrm{disc}} := \{x \in C^{\mathrm{an}} : val(x) \text{ is discrete}\}$. Relation (A) is then equivalent to the following:

$$X(\mathscr{M}_x) \neq \emptyset \ \forall x \in S_{\mathrm{disc}} \Longrightarrow X(\mathscr{M}_x) \neq \emptyset \ \forall x \in C^{\mathrm{an}}. \tag{B}$$

The study of relation (B) for any variety X/F provides a geometric approach to Conjecture 0.1 when G is a rational variety, and is the topic of study of this manuscript.

We remark here that S_{disc} is a *dense* subset of C^{an}. Its points can be described topologically: we recall that a Berkovich analytic curve has the structure of a graph; S_{disc} contains all its *branching points* (i.e. the *type 2* points) and some of its *extreme points* (i.e. some of the *type 1* points).

Main statements. In this manuscript we show that several groups of varieties satisfy relation (B), and hence (A), above. Here is one of the main results we obtain:

Theorem 0.4 (Corollary 2.14, Proposition 2.19)**.** *Let k be a complete discretely valued field. Let C/k be a smooth irreducible projective algebraic curve and C^{an} its Berkovich analytification. Set $F := k(C)$. Let X/F be a smooth proper variety such that $X(F_v) \neq \emptyset$ for all $v \in \Omega$.*

(1) *Let Q be a finite subset of $S_{\mathrm{disc}} \subset C^{\mathrm{an}}$. For any $x \in Q$, there exists a neighborhood V_x of x in C^{an} such that $X(\mathscr{M}(V_x)) \neq \emptyset$. Set $U := C^{\mathrm{an}} \setminus \bigcup_{x \in Q} V_x$. If X has a* smooth proper *model over the ring*

$$\mathcal{O}^\circ(U) := \{f \in \mathcal{O}(U) : |f(x)| \leqslant 1 \ \forall x \in U\},$$

then $X(F_v) \neq \emptyset$ for all $v \in \overline{\Omega}$.

(2) *We can construct open virtual discs and open virtual annuli $B_1, B_2, \ldots, B_n$ in C^{an} depending on X such that if X has proper smooth models over $\mathcal{O}^\circ(B_i), i = 1, 2, \ldots, n$, then $X(F_v) \neq \emptyset$ for all $v \in \overline{\Omega}$.*

If, in addition, there exists a rational linear algebraic group G/F acting strongly transitively on X, then $X(F) \neq \emptyset$.

The proof of part (1) (Corollary 2.14) of this statement is based on topological considerations of the analytic curve C^{an}, as well as the nature of the rings $\mathcal{O}^\circ(V)$ for certain open subsets V. The proof of part (2) (Proposition 2.19) is obtained as a consequence of part (1) and the analytic structure of C^{an} (using *triangulations*).

Let $\mathcal{C}$ be a proper model of the algebraic curve C/k over the valuation ring k° of k. In [2], Berkovich constructed a *specialization morphism* $\pi : C^{\mathrm{an}} \to \mathcal{C}_s$, where $\mathcal{C}_s$ is the special fiber of $\mathcal{C}$. Using this, and a result of Bosch (see Theorem 1.2), we can interpret Theorem 0.4 over models of C.

Theorem 0.5 (Remark 3.18)**.** *Assume $X(F_v) \neq \emptyset$ for all $v \in \Omega$. A regular proper model $\mathscr{C}$ of C and closed points $Q_1, Q_2, \ldots, Q_n \in \mathscr{C}_s$ depending on X can be constructed such that: if X has* proper smooth *models over the rings $\mathcal{O}_{\mathscr{C},Q_i}$, $i = 1, 2, \ldots, n$, then $X(F_v) \neq \emptyset$ for all $v \in \overline{\Omega}$.*

If in addition there exists a rational linear algebraic group G/F acting strongly transitively on X, then $X(F) \neq \emptyset$.

We start by showing there exists a non-empty subset $U \subsetneq \mathscr{C}_s$ such that if X has a proper model over $\mathscr{C}$ which is smooth over U, then (A) and (B) are satisfied. The "finer" the model, the smaller U has to be.

These smoothness results can also be interpreted over the residue fields of the completions of F (Theorems 4.10, 4.12).

By combining Theorem 0.4 above with a theorem of Springer, we prove that Conjecture 0.1 is true for quadratic forms, a result originally shown in [9]. Recall that a quadratic form defined over a field K is said to be *K-isotropic* if it has a non-trivial zero over K. We continue using the same notations as in Theorem 0.4. Let us remark here that the case of residue characteristic two remains unknown.

Theorem 0.6 (Theorem 5.7)**.** *Suppose* $\operatorname{char} \widetilde{k} \neq 2$*, where $\widetilde{k}$ denotes the residue field of k. Let q be a quadratic form defined over F. Then q is F_v-isotropic for all $v \in \Omega$ if and only if it is F_v-isotropic for all $v \in \overline{\Omega}$. Consequently, if* $\dim q > 2$*, q is isotropic over F if and only if it is isotropic over F_v for all $v \in \Omega$.*

More generally, if the variety satisfies such a Springer-type theorem, then by combining it with Theorem 0.4, it should amount to a proof of Conjecture 0.1. In [25], such a "Springer-type result" is shown for Hermitian forms. One can then show the following (see Theorem 7.10 for the precise statement which includes some additional technical conditions):

Theorem 0.7. *Assume k is a local field with* $\operatorname{char} \widetilde{k} \neq 2$*. Let A be a central simple algebra over F with an involution of the second (resp. first) kind σ. Let h be a Hermitian form on (A, σ). Let $G := U(A, \sigma, h)$ be the unitary (resp. $G = SU(A, \sigma, h)$-the special unitary) group associated to (A, σ, h). Let X/F be a projective homogeneous variety over G. Then $X(F) \neq \emptyset$ if and only if $X(F_v) \neq \emptyset$ for all $v \in \Omega$.*

We remark that this result was already shown in [36] and [33]. Here we merely translate the proof to our setting by using the tools developed in *loc.cit.* and Theorem 0.4.

By relying heavily on the structure of Berkovich analytic curves, something can also be said about the case of *constant varieties*.

Theorem 0.8 (Theorem 6.12)**.** *Suppose k is a complete non-trivially valued ultrametric field. Let F be the function field of a smooth connected projective algebraic curve C/k. Let X be a variety defined over k. Suppose there exists a rational linear algebraic group G/F acting strongly transitively on the F-variety $X \times_k F$. Under certain conditions on the curve C, if $X(F_v) \neq \emptyset$ for all $v \in \Omega$, then*

1. *if the value group $|k^\times|$ is dense in $\mathbb{R}_{>0}$, then $X(F) \neq \emptyset$;*
2. *if k is discretely valued, then X has a zero cycle of degree one over F.*

The conditions on the curve C mentioned in the above statement are satisfied for example by curves with semi-stable reduction over k (in the discretely valued case) and

Mumford curves (in general). As for the zero cycles, there has been an extensive study of varieties for which having a zero cycle of degree one is equivalent to having a rational point. This is in particular true for quadratic forms.

Structure of the manuscript. For an easier reading we start with a section of preliminairies. Among other things, in it we include the construction and some of the properties of the *specialization morphism*. We also include a result on the connection between models of an algebraic curve and certain finite subsets (called *vertex sets*) of its Berkovich analytification.

In Section 2, we use the structure of the rings $\mathcal{O}^\circ(U)$ for certain opens U to prove a result of local nature. Through it, we prove the main results of this part: Corollary 2.14 and Proposition 2.19.

In Section 3, we translate the results of Section 2 to the language of models of algebraic curves. More precisely, we construct several models on which, under certain smoothness hypotheses, the implications (A) and (B) are satisfied. We prove Theorems 3.6, 3.11 and 3.16.

In Section 4, we interpret the smoothness assumptions we encounter in the previous sections over the residue fields of the completions of the function field F. The main results we show in this section are Corollary 4.4 and Theorems 4.10, 4.12.

In Section 5, we use the techniques of Section 2 and a theorem of Springer to prove that Conjecture 0.1 is true for quadratic forms (Theorem 5.7). We work here with sncd models of curves.

In Section 6, we study Conjecture 0.1 for constant varieties. More precisely, we construct isomorphisms of the analytic curve in order to prove that relation (B) is satisfied. The techniques that are used are of different nature from those of the previous parts. The main statement of this section is Theorem 6.12.

Finally, in Appendix 7, we add another example to which the techniques of Section 2 apply: homogeneous varieties over (special) unitary groups. This comes down to translating to our setting the tools and proof of [36] and [33], where the result was first shown.

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1. Preliminaires

Throughout this section, k will denote a complete ultrametric field (possibly trivially valued). We will denote by k° the valuation ring of k and by $\widetilde{k}$ its residue field.

1.1. On Berkovich analytic curves.

Let C/k denote a k-analytic curve.

1.1.1. *The completed residue field.* For any $x \in C$, the local ring $\mathcal{O}_{C,x}$ is endowed with a semi-norm with kernel m_x–the maximal ideal of $\mathcal{O}_{C,x}$ (see [28, Lemma 1.4.21]). The *completed residue field* of x, denoted $\mathcal{H}(x)$ is the completion of the residue field $\kappa(x) := \mathcal{O}_x/m_x$ of x with respect to the norm induced on $\kappa(x)$ from said semi-norm.

We remark that $\mathcal{H}(x)$ is a complete ultrametric field. We denote by $\widetilde{\mathcal{H}}(x)$ its residue field.

The completed residue field is constructed similarly for the points of a general k-analytic space, not only a curve (see [1, Remark 1.2.2]).

1.1.2. *Classification of points (by* [1]*).* For any $x \in C$, let

$$s_x := \deg \operatorname{tr}_{\widetilde{k}} \widetilde{\mathcal{H}(x)},\ t_x := \dim_{\mathbb{Q}} |\mathcal{H}(x)^\times|/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}.$$

By Abhyankar's inequality, $s_x + t_x \leqslant 1$ for all $x \in C$. The point x is said to be of:

(1) *type 1* if $\mathcal{H}(x) \subseteq \widehat{\overline{k}}$ (remark that $s_x = t_x = 0$),
(2) *type 2* if $s_x = 1$,
(3) *type 3* if $t_x = 1$,
(4) *type 4* if $s_x = t_x = 0$ and x is not of type 1.

In addition, if x is of type 1 and $\mathcal{H}(x)/k$ is a finite field extension, then x is said to be a *rigid point*.

1.1.3. *Branch issued from a point (by* [10, 1.7]*).* By [10, Théorème 3.5.1], C has the structure of a "real graph" (see [10, 1.3.1] for a precise definition). This graph is said to be a *tree* if it is uniquely arc-wise connected, meaning for any two points of C there exists a unique injective path in C connecting them.

Let $x \in C$. A *branch issued from* x is an element of $\varprojlim \pi_0(V \backslash \{x\})$, where the projective limit is taken with respect to open neighborhoods V of x in C which are trees. By (1.7.1) of *loc.cit.*, the set of branches issued from x is in bijection with $\pi_0(V \backslash \{x\})$ for any such V.

1.2. **The specialization morphism.** We will be using the notion of *specialization morphism* in the sense of Berkovich (see [2, Section 1] and [3, Section 1]).

1.2.1. *The affine case.* Let $\mathscr{X} = \operatorname{Spec} A$ be a flat finite type scheme over k°. The formal completion $\widehat{\mathscr{X}}$ of $\mathscr{X}$ along its special fiber is $\operatorname{Spf}(\widehat{A})$, where $\widehat{A}$ is a topologically finitely presented ring over k° (*i.e.* isomorphic to some $k^\circ\{T_1, \ldots, T_n\}/I$, where I is a finitely generated ideal; see [2, pg. 541] for the definition of $k^\circ\{T_1, \ldots, T_n\}$). The *analytic generic fiber* of $\widehat{\mathscr{X}}$, denoted by $\widehat{\mathscr{X}_\eta}$, is defined to be $\mathcal{M}(\widehat{A} \otimes_{k^\circ} k)$, where $\mathcal{M}(\cdot)$ denotes the *Berkovich spectrum* (see [1, 1.2]).

There exists a *specialization morphism* $\pi : \widehat{\mathscr{X}_\eta} \to \widehat{\mathscr{X}_s}$, where $\widehat{\mathscr{X}_s}$ is the special fiber of $\widehat{\mathscr{X}}$, which is *anti-continuous*, meaning the pre-image of a closed subset is open. We remark that $\widehat{\mathscr{X}_s} = \mathscr{X}_s$, where $\mathscr{X}_s := \operatorname{Spec}(A \otimes_{k^\circ} \widetilde{k})$ is the special fiber of $\mathscr{X}$. Let us describe π explicitly.

There are embeddings $A \hookrightarrow \widehat{A} \hookrightarrow (\widehat{A} \otimes_{k^\circ} k)^\circ$, where $(\widehat{A} \otimes_{k^\circ} k)^\circ$ is the set of all elements f of $\widehat{A} \otimes_{k^\circ} k$ for which $|f(x)| \leqslant 1$ for all $x \in \mathcal{M}(\widehat{A} \otimes_{k^\circ} k)$. Let $x \in \mathcal{M}(\widehat{A} \otimes_{k^\circ} k)$. This point then determines a bounded morphism $A \to \mathcal{H}(x)^\circ$, which induces an application $\varphi_x : A \otimes_{k^\circ} \widetilde{k} \to \widetilde{\mathcal{H}(x)}$. The specialization morphism π sends x to $\ker \varphi_x$.

Remark 1.1. In [1, 2.4], Berkovich constructs a *reduction map* $r : \mathcal{M}(\widehat{A} \otimes_{k^\circ} k) \to \operatorname{Spec}(\widetilde{\widehat{A} \otimes_{k^\circ} k})$; here $\widetilde{\widehat{A} \otimes_{k^\circ} k} := (\widehat{A} \otimes_{k^\circ} k)^\circ/(\widehat{A} \otimes_{k^\circ} k)^{\circ\circ}$, where $(\widehat{A} \otimes_{k^\circ} k)^{\circ\circ}$ is the set of all elements $f \in \widehat{A} \otimes_{k^\circ} k$ such that for any $x \in \mathcal{M}(\widehat{A} \otimes_{k^\circ} k)$, $|f(x)| < 1$. In [29, Proposition 4.1], we show that if A is a normal domain, then the canonical morphism $\phi : \operatorname{Spec}(\widetilde{\widehat{A} \otimes_{k^\circ} k}) \to \operatorname{Spec}(A \otimes_{k^\circ} \widetilde{k})$ is a bijection and $\pi = \phi \circ r$ (see [29, 4.1] for more details). In particular, this means that some of the properties shown for r in [1, 2.4] will remain true for π.

1.2.2. *The proper case.* The construction in the previous section has nice glueing properties. Let $\mathscr{X}$ be a finite type scheme over k°, and $\widehat{\mathscr{X}}$ its formal completion along the special fiber. Then, the *analytic generic fiber* $\widehat{\mathscr{X}_\eta}$ of $\widehat{\mathscr{X}}$ is the k-analytic space we obtain by glueing the analytic generic fibers of an open affine cover of the formal scheme $\widehat{\mathscr{X}}$. In general, $\widehat{\mathscr{X}_\eta}$ is a compact analytic domain of the Berkovich analytification $\mathscr{X}^{\mathrm{an}}$ of $\mathscr{X}$. If $\mathscr{X}$ is proper, then $\mathscr{X}^{\mathrm{an}} = \widehat{\mathscr{X}_\eta}$ (see [30, 2.2.2]).

Similarly, there exists an anti-continuous *specialization morphism* $\pi : \widehat{\mathscr{X}_\eta} \to \mathscr{X}_s$, where $\mathscr{X}_s$ is the special fiber of $\mathscr{X}$. In particular, if $\mathscr{X}$ is proper, then we have the specialization morphism $\pi : \mathscr{X}^{\mathrm{an}} \to \mathscr{X}_s$.

1.3. **The Theorem of Bosch.**

1.3.1. *The sheaf of bounded functions.* Let X be a k-analytic space. We define the subsheaf $\mathcal{O}_X^\circ$ of $\mathcal{O}_X$ as follows: for any open U of X, let

$$\mathcal{O}_X^\circ(U) = \{f \in \mathcal{O}_X(U) : |f|_x \leqslant 1 \ \forall x \in U\}.$$

When there is no risk of ambiguity, we will simply write $\mathcal{O}^\circ$.

1.3.2. *The statement.* Let $\mathscr{C}$ be a flat normal irreducible proper k°-analytic curve. Let us denote by C its generic fiber, and by $\mathscr{C}_s$ its special fiber. Also, let $k(C)$ denote the function field of C. The specialization morphism constructed in Section 1.2 gives us an anti-continuous morphism $\pi : C^{\mathrm{an}} \to \mathscr{C}_s$, where C^{an} denotes the Berkovich analytification of C.

Theorem 1.2 ([4, Theorem 5.8], [27, Theorem 3.1])**.** *Let $P \in \mathscr{C}_s$ be a closed point. Then $\widehat{\mathcal{O}_{\mathscr{C},P}} = \mathcal{O}^\circ(\pi^{-1}(P))$, where $\widehat{\mathcal{O}_{\mathscr{C},P}}$ is the completion of the local ring $\mathcal{O}_{\mathscr{C},P}$ with respect to its maximal ideal.*

We remark here that as P is a closed point and π is anti-continuous, $\pi^{-1}(P)$ is an open subset of C^{an}. For a proof of Theorem 1.2 in this setting, see [29, Proposition 4.5].

Remark 1.3. By [29, Lemma 4.3], if $x \in \mathscr{C}_s$ is the generic point of an irreducible component of $\mathscr{C}_s$, then $\pi^{-1}(x)$ is a single type 2 point of C^{an}. Moreover, by the proof of *loc.cit.*, the valuation on $k(C)$ determined by x is the same as that determined by $\pi^{-1}(x)$.

1.4. **Vertex sets and models of curves.** Let C/k be a proper normal irreducible algebraic curve. We denote by C^{an} its Berkovich analytification.

Definition 1.4.

(1) A non-empty finite set of type 2 points of C^{an} is said to be a *vertex set* of C^{an}. (See [10, 6.3.17].)
(2) A connected k-analytic curve X/k is said to be an *open virtual disc* (resp. *open virtual annulus*) if $X \times_k \widehat{\overline{k}}$ is a finite disjoint union of open discs (resp. annuli) over $\widehat{\overline{k}}$-the completion of an algebraic closure of k. One can similarly define *closed virtual discs* and *closed virtual annuli* over k. (See [10, pg. 210], [11, Section 3].)
(3) A *triangulation* of C^{an} is a vertex set S of C^{an} such that each connected component of $C^{\mathrm{an}} \backslash S$ is either an open virtual disc or an open virtual annulus. (See [10, 5.1.13].)

By [10, Théorème 5.1.14], if C generically smooth, then there exists a triangulation of C^{an}.

Remark 1.5. Let $\mathscr{C}$ be a proper flat normal model of C over k°. Let $\mathrm{Gen}(\mathscr{C}_s)$ denote the set of generic points of the irreducible components of the special fiber $\mathscr{C}_s$ of $\mathscr{C}$. By Remark 1.3, $S_{\mathscr{C}} := \pi^{-1}(\mathrm{Gen}(\mathscr{C}_s))$ is a vertex set of C^{an}, where π is the specialization morphism $C^{\mathrm{an}} \to \mathscr{C}_s$.

Theorem 1.6 ([11, Theorem 4.3], [10, 6.3.14])**.** *The map $\mathscr{C} \mapsto S_{\mathscr{C}}$ induces a bijection between the following two partially ordered sets:*

(1) *the isomorphism classes of flat normal proper models of C over k°, ordered by morphisms of models,*
(2) *the vertex sets of C^{an}, ordered by inclusion.*

Furthermore, for a flat normal proper model $\mathscr{C}$ of C with special fiber $\mathscr{C}_s$, the corresponding specialization morphism π induces a bijection:

$$\{\text{closed points on } \mathscr{C}_s\} \cong \{\text{connected components of } C^{\mathrm{an}} \backslash S_{\mathscr{C}}\}$$
$$P \mapsto \pi^{-1}(P)$$

If $P \in \mathscr{C}_s$ is a closed point, then the boundary of $\pi^{-1}(P)$ consists precisely of the preimages by π of the generic points of the irreducible components of $\mathscr{C}_s$ containing P.

As a consequence, thanks to Hironaka's resolution of singularities:

Corollary 1.7. *Let T be a finite set of type 2 points of C^{an}. There exists a proper regular model $\mathscr{C}$ of C over k° such that $T \subseteq S_{\mathscr{C}}$. The same remains true when replacing "regular" with "sncd" (see Remark 5.2(3)).*

1.5. **Strongly transitive action.** We recall the following definition, originally introduced in [16].

Definition 1.8. Let K be a field. Let X be a K-variety and G a linear algebraic group over K. We say that G acts *strongly transitively* on X if G acts on X, and for any field extension L/K, either $X(L) = \emptyset$ or $G(L)$ acts transitively on $X(L)$.

Remark 1.9. By [16, Remark 3.9], if X/K is a projective homogeneous variety over a lineau algebraic group G/K, then G acts strongly transitively on X. We also remark that if X is a G-torsor (*i.e.* a principal homogeneous space), then G acts strongly transitively on X.

In [29], we show that the statement of Theorem 0.2 in the Introduction is true if G acts strongly transitively on X.

2. A smoothness criterion over analytic curves

We will be using the following notation throughout this section.

Notation 2.1. (1) Let k be a complete discretely valued field. We will denote by k° its valuation ring, and $\widetilde{k}$ its residue field. Let us also fix a uniformizer t of k.

(2) Let C/k be a proper normal irreducible k-analytic curve. Set $F = \mathscr{M}(C)$, where $\mathscr{M}$ denotes the sheaf of meromorphic functions on C (see [28, 1.7] for the definition of $\mathscr{M}$ and some of its properties).

(3) We will denote by C^{al} the unique projective k-algebraic curve whose Berkovich analytification is C (see [10, Théorème 3.7.2]). By [1, Proposition 3.4.3, Theorem 3.4.8(iii)], C^{al} is normal and irreducible. Moreover, $k(C) = F$ ([1, Proposition 3.6.2]).

(4) We denote by $V(F)$ the set of rank 1 non-trivial valuations on F such that for any $v \in V(F)$, $v_{|k}$ either induces the norm on k or is trivial. For any $v \in V(F)$, we denote by F_v the completion of F with respect to v.

Moreover:

Hypothesis 1. Let X/F be a smooth proper variety such that $X(F_v) \neq \emptyset$ for all discrete valuations $v \in V(F)$.

Remark 2.2. (1) By [29, Remark 3.14], for any $x \in C$, the field $\mathscr{M}_x$ of germs of meromorphic functions on x is endowed with a norm. We will denote by $\widehat{\mathscr{M}_x}$ its completion.

(2) We recall that by [29, Proposition 3.15], there is a bijective correspondence $val : C \longleftrightarrow V(F)$ between the points of C and the valuations in $V(F)$. Moreover, if $x \mapsto v_x$, then $\widehat{\mathscr{M}_x} = F_{v_x}$. By the proof of *loc.cit.*, the map *val* induces a bijection between the rigid points of C and the valuations v in $V(F)$ such that $v_{|k}$ is trivial.

(3) The canonical morphism $C \to C^{\mathrm{al}}$ maps (bijectively) rigid points to Zariski closed points (see [1, Theorem 3.4.1]). If $x \in C$ is a rigid point, the discrete valuation $val(x)$ on F is the same as the one induced by the (unique) corresponding Zariski closed point on C^{al}.

If $x \in C$ is a type 2 point, then $|\mathcal{H}(x)^\times|/|k^\times|$ is a finite group, so $\mathcal{H}(x)$ is also discretely valued. Hence, $val(x)$ is a discrete valuation on F which extends the norm on k.

(4) We have established that both rigid and type 2 points determine (uniquely) discrete valuations on F. Consequently, Hypothesis 1 implies that $X(\widehat{\mathscr{M}_x}) \neq \emptyset$ for all $x \in C$ either a rigid or type 2 point. By [29, Corollary 3.17], this in turn is equivalent to the hypothesis that $X(\mathscr{M}_x) \neq \emptyset$ for all rigid or type 2 points $x \in C$.

Remark 2.3. Let Y/F be a smooth variety. As a consequence of Remark 2.2(2) and [29, Corollary 3.17], the following are equivalent:

(1) $Y(\mathscr{M}_x) \neq \emptyset$ for all $x \in C$,
(2) $Y(F_v) \neq \emptyset$ for all $v \in V(F)$.

Remark 2.4. We recall that as C is proper, strict affinoid domains (meaning affinoid domains with only type 2 points in their boundaries) form a basis of neighborhoods of C. This implies that for any variety Y/F, if $z \in C$ is such that $Y(\mathscr{M}_z) \neq \emptyset$, then there exists a strict affinoid neighborhood V_z of z such that $Y(\mathscr{M}(V_z)) \neq \emptyset$.

Lemma 2.5. *Assume V is a strict affinoid domain of C. If $X(\mathscr{M}(V)) \neq \emptyset$, then $X(\mathscr{M}_x) \neq \emptyset$ for all $x \in V$.*

Proof. The statement is clearly true for any $x \in \operatorname{Int} V$. As $V = \partial V \cup \operatorname{Int} V$ (see [28, Corollary 1.8.11]), it only remains to prove it for points $x \in \partial V$. This follows immediately from Remark 2.2(4), seeing as ∂V contains only type 2 points. □

In what follows, we will make use of the subsheaf $\mathcal{O}^\circ$ of $\mathcal{O}$ (see Subsection 1.3.1).

Theorem 2.6. *Let $x \in C$. Suppose there exists a connected open neighborhood T_x of x in C such that ∂T_x contains only type 2 and 3 points, and X has a proper smooth model $\mathcal{X} \to \operatorname{Spec} \mathcal{O}^\circ(T_x)$. Then there exists a neighborhood $U_x \subseteq T_x$ of x such that $X(\mathscr{M}(U_x)) \neq \emptyset$. In particular, $X(\mathscr{M}_x) \neq \emptyset$.*

Remark 2.7. In the statement of Theorem 2.6, we assume that ∂T_x contains only type 2 and 3 points because otherwise it might not make sense to speak of *a model of X over $\mathcal{O}^\circ(T_x)$*. More precisely, it could be possible for F to *not* be contained in $\operatorname{Frac} \mathcal{O}^\circ(T_x)$.

(1) For example, if we take $C = \mathbb{P}^{1,\mathrm{an}}_k, T_x = \mathbb{A}^{1,\mathrm{an}}_k$, then the meromorphic function T on $\mathbb{P}^{1,\mathrm{an}}_k$ is not contained in Frac $\mathcal{O}^\circ(\mathbb{A}^{1,\mathrm{an}}_k)$.
(2) If ∂T_x contains only type 2 and 3 points, then $F \subseteq$ Frac $\mathcal{O}^\circ(T_x)$. To see this, let $\overline{T_x}$ denote the closure of T_x in C. By assumption, $\partial\overline{T_x}$ contains (only) type 2 and 3 points, and thus $\overline{T_x} \neq C$. Let $y \in C\backslash\overline{T_x}$. For any $z \in \overline{T_x}$, let U_z be an affinoid neighborhood of z such that $y \notin U_z$ (recall C is separated). By compacity, there exists a finite number of points $z \in \overline{T_x}$ such that $\overline{T_x} \subseteq \bigcup_z U_z$; moreover $T_x \subseteq V := \bigcup_z U_z \neq C$. By [28, Theorem 1.8.15(2)], V is an affinoid domain of C. For any $a \in F \subseteq \mathscr{M}(V)$, there exist $b, c \in \mathcal{O}(V)$ such that $a = \frac{b}{c}$. The functions b, c are bounded on V. Hence, for a large enough n, $t^n b$ and $t^n c$ are bounded by 1 in V. Consequently, they are bounded by 1 in T_x, implying $a \in$ Frac $\mathcal{O}^\circ(T_x)$.

Proof of Theorem 2.6. Let $V_x \subseteq T_x$ be a strict affinoid neighborhood of x, so that ∂V_x is a finite set of type 2 points. We remark that X has a proper smooth model over $\mathcal{O}^\circ(V_x)$. Let $\mathscr{C}$ be a proper regular model of C^{al} over k° corresponding to a vertex set S of C containing ∂V_x (see Corollary 1.7). We denote by $\mathscr{C}_s$ its special fiber and by π the corresponding specialization morphism $C \to \mathscr{C}_s$. If $x \in S$, then clearly $X(\mathscr{M}_x) \neq \emptyset$. Let us assume that $x \notin S$.

Set $P_x := \pi(x)$. This is a closed point of $\mathscr{C}_s$ (see Remark 1.5). By Theorem 1.6, $U_x := \pi^{-1}(P_x)$ is a connected component of $C\backslash S$. In particular, $U_x \cap S = \emptyset$.

Lemma 2.8. *The following is satisfied:* $U_x \subseteq V_x$.

Proof. Assume that there exists $y \in U_x\backslash V_x$. As U_x is connected, there exists an injective path $[x, y]$ connecting x and y which is entirely contained in U_x. But as $x \in V_x$ and $y \notin V_x$, $[x, y] \cap \partial V_x \neq \emptyset$. This implies that $U_x \cap S \neq \emptyset$, contradiction. □

As a consequence of Lemma 2.8, X has a proper smooth model over $\mathcal{O}^\circ(U_x)$, which we will continue to denote by $\mathcal{X}$. We recall that by Theorem 1.2, $\widehat{\mathcal{O}_{\mathscr{C},P_x}} = \mathcal{O}^\circ(\pi^{-1}(P_x))$. In particular, this means that $\mathcal{O}^\circ(U_x)$ is a complete regular local ring of dimension 2.

Lemma 2.9. *The following is satisfied:* $\mathcal{X}(\mathcal{O}^\circ(U_x)) \neq \emptyset$.

Proof. Let (α, β) be the maximal ideal of $\mathcal{O}^\circ(U_x)$ such that the generators α, β form a regular set of parameters (see Remark 5.2(1)). Then the localization $\mathcal{O}^\circ(U_x)_{(\alpha)}$ is a discrete valuation ring with uniformizer α ([35, Tag 0AFS]). As $F \subseteq$ Frac $\mathcal{O}^\circ(T_x) \subseteq$ Frac $\mathcal{O}^\circ(U_x)$ (see Remark 2.7(2)), we obtain that the completion Frac $\widehat{\mathcal{O}^\circ(U_x)_{(\alpha)}}$ of Frac $\mathcal{O}^\circ(U_x)_{(\alpha)}$ is a complete discretely valued field containing F. Moreover, by Remark 2.11, it either extends the norm on k or is trivial there. Consequently, by assumption, $X(\text{Frac } \widehat{\mathcal{O}^\circ(U_x)_{(\alpha)}}) \neq \emptyset$, and so $\mathcal{X}(\text{Frac } \widehat{\mathcal{O}^\circ(U_x)_{(\alpha)}}) \neq \emptyset$.

By the valuative criterion of properness, $\mathcal{X}(\widehat{\mathcal{O}^\circ(U_x)_{(\alpha)}}) \neq \emptyset$. By taking the residue field, we obtain $\mathcal{X}(\mathcal{O}^\circ(U_x)_{(\alpha)}/(\alpha)) = \mathcal{X}(\text{Frac } \mathcal{O}^\circ(U_x)/(\alpha)) \neq \emptyset$. Again, $\mathcal{O}^\circ(U_x)/(\alpha)$ is a discrete valuation ring (with uniformizer β, see [35, Tag 00NQ]), so by applying the valuative criterion of properness, $\mathcal{X}(\mathcal{O}^\circ(U_x)/(\alpha)) \neq \emptyset$.

As the local ring $(\mathcal{O}^\circ(U_x), (\alpha, \beta))$ is complete, it is also complete with respect to the topology induced by the ideal (α). In particular, $(\mathcal{O}^\circ(U_x), (\alpha))$ is a Henselian couple. Seeing as $\mathcal{X}$ is smooth over $\mathcal{O}^\circ(U_x)$, by applying the Hensel lifting property (see [14, Théorème I.8]), we obtain that $\mathcal{X}(\mathcal{O}^\circ(U_x)) \neq \emptyset$. □

Consequently, $X(\text{Frac } \mathcal{O}^\circ(U_x)) \neq \emptyset$, so $X(\mathscr{M}(U_x)) \neq \emptyset$ and $X(\mathscr{M}_x) \neq \emptyset$. □

Remark 2.10. If the point x in the statement of Theorem 2.6 is rigid or of type 2, then $X(\mathscr{M}_x) \neq \emptyset$ without any additional hypotheses (see Remark 2.2(4)).

Remark 2.11. In the setting of Lemma 2.9, let α, β be generators of the maximal ideal of $\mathcal{O}^\circ(U_x)$. The localizations $\mathcal{O}^\circ(U_x)_{(\alpha)}, \mathcal{O}^\circ(U_x)_{(\beta)}$ are discretely valued rings. As $k^\circ \subseteq \mathcal{O}^\circ(U_x)$, depending on whether the uniformizer t of k° is in (α) (resp. (β)) or not, the restriction of the discrete valuation on $\mathcal{O}^\circ(U_x)_{(\alpha)}$ (resp. $\mathcal{O}^\circ(U_x)_{(\beta)}$) to k° either induces the norm on k or is trivial, respectively.

Remark 2.12.
(1) We recall that the variety X has good reduction on a Zariski open subset of the algebraic curve C^{al}. Hence, there are only finitely many points in C^{al} on which X has bad reduction, and they are Zariski closed.
(2) We also recall that the canonical analytification morphism $C \to C^{\text{al}}$ induces a bijection between the rigid points of C and the Zariski closed points of C^{al}. Moreover, if $s \mapsto s'$ via this map, then the residue fields of s and s' are isomorphic. By an abuse of terminology and notation, we will identify the rigid points of C with the Zariski closed ones of C^{al} (see [1, Theorem 3.4.1]).
(3) If the hypotheses of Theorem 2.6 are satisfied, then $C \backslash T_x$ will contain the rigid points on which X has bad reduction. To see this, assume $s \in T_x$ is a rigid point on which X has bad reduction. The map $\mathcal{O}^\circ(T_x) \to \mathcal{O}_{C,s}$, induces $\mathcal{O}^\circ(T_x) \to \kappa(s)$, where $\kappa(s)$ denotes the residue field of s. But then X has a smooth model over $\kappa(s)$, meaning it has good reduction over the point s, contradiction.

Before giving a similar global criterion for checking that $X(\mathscr{M}_x) \neq \emptyset$, let us start by proving an auxiliary result.

Lemma 2.13. *Let $V_i, i = 1, 2, \ldots, n$, be strict affinoid domains of C. Let A be a finite set of type 2 points in C. Set $D := C \backslash (A \cup \bigcup_{i=1}^n V_i)$. Then D is an open subset of C and ∂D is a finite set of type 2 points contained in $A \cup \bigcup_{i=1}^n \partial V_i$.*

Proof. As affinoid domains are closed, $V_i, i = 1, 2, \ldots, n$, are closed. As C is separated, its points are closed, so A is a closed subset. Consequently, D is an open subset of C. As it is open, $D \cap \partial D = \emptyset$, so $\partial D \subseteq \bigcup_{i=1}^n V_i \cup A$. Suppose there exists $i' \in \{1, 2, \ldots, n\}$ such that $\partial D \cap \text{Int } V_{i'} \neq \emptyset$. Let $\eta \in \partial D \cap \text{Int } V_{i'}$. There exists a neighborhood U_η of η in C such that $U_\eta \subseteq V_{i'}$. But as $\eta \in \partial D$, $U_\eta \cap D \neq \emptyset$, so $V_{i'} \cap D \neq \emptyset$, contradiction. Thus, as $V_i = \partial V_i \cup \text{Int} V_i$ for any $i \in \{1, 2, \ldots, n\}$ (see [28, Corollary 1.8.11]), we obtain that $\partial D \subseteq \bigcup_{i=1}^n \partial V_i \cup A$, meaning ∂D contains only type 2 points. Moreover, as $\bigcup_{i=1}^n \partial V_i \cup A$ is finite (see [29, Proposition 2.5]), so is D. □

The following is a consequence of Theorem 2.6. We recall the notion of *strongly transitive action* in Definition 1.8 (see also Remark 1.9).

Corollary 2.14. *Let Q be a finite set of rigid and type 2 points of C. For any $z \in Q$, let V_z be a strict affinoid neighborhood of z in C such that $X(\mathscr{M}(V_z)) \neq \emptyset$. Set $U := C \backslash \bigcup_{z \in Q} V_z$ (see Figure 1 below for an illustration). If there exists a smooth proper model $\mathcal{X} \to \text{Spec } \mathcal{O}^\circ(U)$ of X, then $X(\mathscr{M}_x) \neq \emptyset$ for all $x \in C$. Equivalently, $X(F_v) \neq \emptyset$ for all $v \in V(F)$.*

Moreover, if there exists a rational linear algebraic group G/F acting strongly transitively on X, then $X(F) \neq \emptyset$.

Proof. By Lemma 2.13, U is an open subset of C and ∂U is a finite set of type 2 points. By Theorem 2.6, for any $x \in U$, we have that $X(\mathscr{M}_x) \neq \emptyset$. On the other hand, by construction, for all $z \in Q$, $X(\mathscr{M}(V_z)) \neq \emptyset$, so for any $x \in \bigcup_{z\in Q} V_z$, $X(\mathscr{M}_x) \neq \emptyset$ (see Lemma 2.5). Thus, $X(\mathscr{M}_x) \neq \emptyset$ for all $x \in C$. By Remark 2.3, $X(F_v) \neq \emptyset$ for all $v \in V(F)$. We can now conclude by [29, Corollary 3.18]. □

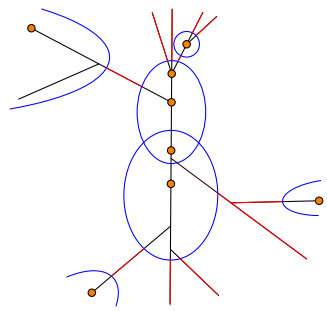

Figure 1. $Q, V_z, z \in Q, U$

Remark 2.15. (1) In the statement of Corollary 2.14, if the hypotheses are satisfied, we can show, using the same arguments as in Remark 2.12(3), that $\bigcup_{z\in Q} V_z$ contains the rigid points on which X has bad reduction. Consequently, without loss of generality, we may assume that Q contains the bad-reduction points of X.

(2) We remark that since U does not contain the bad reduction points of X, the latter has a proper smooth model over the ring $\mathcal{O}(U)$. However, as the ring $\mathcal{O}(U)$ is in general substiantially larger than its subring $\mathcal{O}^\circ(U)$, the assumption of smoothness over $\mathcal{O}^\circ(U)$ is stronger. In the next section, we will interpret it as a smoothness assumption over parts of the special fiber of a model of C^{al}.

We can show a result similar to Corollary 2.14 by also removing any finite set of type 2 points (such as any vertex set) of C from consideration.

Corollary 2.16. *With the notations of Corollary 2.14, let A be a finite set of type 2 points on C. Set $V = C\backslash(\bigcup_{z\in Q} V_z \cup A)$. If there exists a smooth proper model $\mathcal{X} \to \mathrm{Spec}\ \mathcal{O}^\circ(V)$ of X, then $X(\mathscr{M}_x) \neq \emptyset$ for all $x \in C$. Equivalently, $X(F_v) \neq \emptyset$ for all $v \in V(F)$.*

Moreover, if there exists a rational linear algebraic group G/F acting strongly transitively on X, then $X(F) \neq \emptyset$.

Remark 2.17. By Lemma 2.13, ∂V contains only type 2 points in its boundary. By Remark 2.7(2), this means that $F \subseteq \mathrm{Frac}\ \mathcal{O}^\circ(V)$.

Proof of Corollary 2.16. By Remark 2.2(4), for any $z \in A$, there exists a strict affinoid neighborhood V_z of z in C such that $X(\mathscr{M}(V_z)) \neq \emptyset$. Then $U' := C\backslash \bigcup_{z\in A\cup Q} V_z$ is an open subset of C. As $U' \subseteq V$, we obtain that X has a proper smooth model over $\mathcal{O}^\circ(U')$. By Remark 2.3, $X(F_v) \neq \emptyset$ for all $v \in V(F)$. We can now conclude by [29, Corollary 3.18]. □

As proven by the following remark and proposition, it suffices to show that X has a smooth model over $\mathcal{O}^\circ(U)$ for a *finite* number of open subsets $U \subseteq C$, which are considerably smaller than V from the statement of Corollary 2.16.

Remark 2.18. With the same notation as in Corollary 2.14, let S be any vertex set of C such that $\bigcup_{z\in Q} \partial V_z \subseteq S$. As it contains only type 2 points, for any $s \in S$, $X(\mathscr{M}_s) \neq \emptyset$.

Hence, there exists a neighborhood U_s of s in C such that $X(\mathscr{M}(U_s)) \neq \emptyset$. Without loss of generality, we may assume that U_s has finite boundary in C.

There exist only a finite number of connected components of $C \backslash S$ not entirely contained in $\bigcup_{s \in S} U_s$: if that were not the case, as S is finite, there exists $s_0 \in S$ such that there are infinitely many connected components of $C \backslash S$ not contained in $\bigcup_{s \in S} U_s$ and intersecting U_{s_0}. Then, they all intersect the boundary of U_{s_0}, implying the latter is infinite, contradiction.

Let $A_1, A_2, \ldots, A_n$ be the connected components of $C \backslash S$ not contained in $\bigcup_{s \in S} U_s$. If S is a triangulation, then A_i is an open virtual disc or open virtual annulus. We remark that $\partial A_i \subseteq S$. For any $i \in \{1, 2, \ldots, n\}$, set $B_i := A_i \backslash \bigcup_{z \in Q} V_z$. This implies that $\partial B_i \subseteq \partial A_i \cup \bigcup_{z \in Q} \partial V_z$. Hence, B_i is an open subset of C and $\partial B_i \subseteq S \cup \bigcup_{s \in S} \partial U_s$ (meaning it is a finite set of type 2 points). See Figure 2 for illustrations of such data.

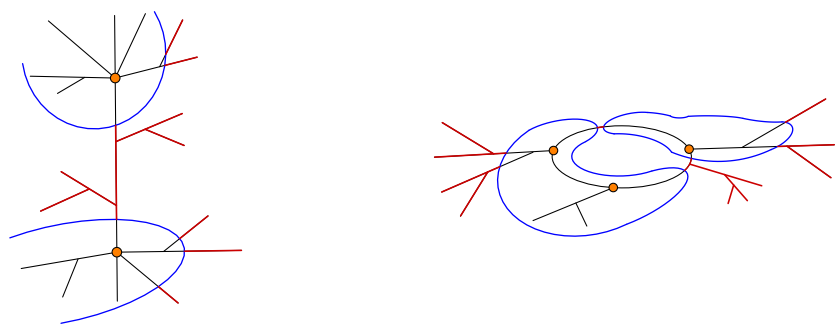

Figure 2. $S, U_s, s \in S, A_i$

Proposition 2.19. *If X has proper smooth models over the rings $\mathcal{O}^\circ(B_i)$, $i = 1, 2, \ldots, n$, then $X(\mathscr{M}_x) \neq \emptyset$ for all $x \in C$. Equivalently, $X(F_v) \neq \emptyset$ for all $v \in V(F)$.*

If, moreover, there exists a rational linear algebraic group G/F acting strongly transitively on X, then $X(F) \neq \emptyset$.

Proof. If $x \in \bigcup_{s \in S} U_s \cup \bigcup_{z \in Q} V_z$, then clearly $X(\mathscr{M}_x) \neq \emptyset$. Otherwise, suppose $x \in \bigcup_{i=1}^n B_i$. Let $\mathscr{C}$ be a proper regular model of C^{al} over k° corresponding to a vertex set S' such that $S \cup \bigcup_{s \in S} \partial U_s \subseteq S'$ (see Corollary 1.7). In particular, $\bigcup_{i=1}^n \partial B_i \subseteq S'$. We denote by $\mathscr{C}_s$ its special fiber and by π the corresponding specialization morphism $C \to \mathscr{C}_s$. Set $P_x := \pi(x)$ and $U_x := \pi^{-1}(P_x)$. The proof of Lemma 2.8 can be applied *mutatis mutandis* to show that $U_x \subseteq B_i$, meaning X has a proper smooth model over $\mathcal{O}^\circ(U_x)$. By Theorem 2.6, $X(\mathscr{M}_x) \neq \emptyset$. By Remark 2.3, $X(F_v) \neq \emptyset$ for all $v \in V(F)$. We can now conclude by [29, Corollary 3.18]. □

Let us illustrate how strong the hypotheses of Corollary 2.16 or Proposition 2.19 are with an example where the variety X is determined by a quadratic form.

Example 1. Let q be a quadratic form defined over the field F. Let us assume that q is diagonal with coefficients $a_1, a_2, \ldots, a_m \in F$. Let Q be a finite closed subset of C which contains all the zeroes and poles of $a_1, a_2, \ldots, a_m$ in C. We use the same notation as in Remark 2.18. Let $i \in \{1, 2, \ldots, n\}$.

By multiplying with a high enough power of t (this does not change the Weil divisors of $a_1, a_2, \ldots, a_m$ in C), we may assume that $a_1, a_2, \ldots, a_m \in \mathcal{O}^\circ(B_i)$. Moreover, we may assume (by changing to an equivalent quadratic form if necessary) that $t^{-2} a_j \notin \mathcal{O}^\circ(B_i)$ for all $j \in \{1, 2, \ldots, m\}$.

The quadratic form q is smooth over $\mathcal{O}^\circ(B_i)$ if and only if for any maximal ideal M of $\mathcal{O}^\circ(B_i)$ and any $j \in \{1, 2, \ldots, m\}$, $a_j \notin M$. Consequently, $a_j \in \mathcal{O}^\circ(B_i)^\times$ for all such j. This implies that for any $x \in B_i$ and any j, $|a_j|_x = 1$.

In Section 5, by using the techniques of this section and a theorem of Springer, we prove a general result for quadratic forms (provided char $\widetilde{k} \neq 2$).

Remark 2.20. An argument similar to that of Example 1 can be applied to a general variety X. As the smoothness of X is checked by the non-vanishing of minors ϵ of a certain matrix, it suffices to show that for any $i \in \{1, 2, \ldots, n\}$, there exists a model of X over $\mathcal{O}^\circ(B_i)$ with corresponding minors ϵ_i satisfying $|\epsilon_i|_x = 1$ for all $x \in B_i$.

Example 2. Let $k = \mathbb{Q}_p$, $C = \mathbb{P}^{1,\mathrm{an}}_{\mathbb{Q}_p}$, and $F = \mathbb{Q}_p(T)$. Let q be the quadratic form $X_1^2 - (1 + pT)X_2^2 + TX_3^2 - (T + p)X_4^2$. The bad reduction points of the corresponding quadric are those for which $|T| = 0$, $|T| = \infty$, $|T + p| = 0$ or $|1 + pT| = 0$.

By taking $X_1 = \sqrt{1 + pT}$, $X_2 = 1$, and $X_3 = X_4 = 0$, we obtain a zero of q defined on a neighborhood N of the point $|T| = 0$ (referred to as 0). We remark that N is a disk centered at 0 and of radius the radius of convergence of the series expansion of $\sqrt{1 + pT}$. The latter is > 1, so said zero is defined over the open unit disk D (*i.e.* $D \subsetneq N$). Remark that the point $|T + p| = 0$ is also in D.

On the other hand, by taking $X_1 = X_2 = 0$, $X_3 = \sqrt{1 + \frac{p}{T}}$ and $X_4 = 1$, we obtain a solution of $q = 0$ on a neighborhood of the point $|T| = \infty$ (referred to as ∞). Similarly, this solution is defined on the open disk D_∞ centered at ∞ and of radius 1. We remark that D_∞ contains the point $|1 + pT| = 0$.

Let U be a connected component of $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{Q}_p} \backslash (D \cup D_\infty)$. Then, by Theorem 1.2, the ring $\mathcal{O}^\circ(U)$ is local with maximal ideal $(p, P(T))$, which is a maximal ideal of $\mathbb{Z}_p[T]$ (seeing as $U \neq D_\infty$). Since $U \neq D$, $P(T) \neq T$. Consequently, the rank of q does not change in $\mathcal{O}^\circ(U)/(p, P(T))$, meaning the quadric defined by q is proper and smooth over $\mathcal{O}^\circ(U)$.

A crucial point here is that the Weil divisors in $\mathbb{P}^1_{\mathbb{Z}_p}$ of the coefficients of q are *not* vertical. See also Section 3.

We remark that, in this particular case, as $N \cup D_\infty = \mathbb{P}^{1,\mathrm{an}}_{\mathbb{Q}_p}$, $q = 0$ has non-trivial solutions over $\mathscr{M}_x$ for all $x \in \mathbb{P}^{1,\mathrm{an}}_{\mathbb{Q}_p}$, so it has a non-trivial solution over $\mathbb{Q}_p(T)$.

3. A smoothness criterion over fine models of algebraic curves

We will use notations similar to those of Section 2.

Notation 3.1. (1) Let k be a complete discretely valued field. We will denote by k° its valuation ring, and $\widetilde{k}$ its residue field. Let us also fix a uniformizer t of k.

(2) Let C/k be a proper normal irreducible k-algebraic curve. Set $F = k(C)$.

(3) We will denote by C^{an} the proper analytic curve that is the Berkovich analytification of C. By [1, Proposition 3.4.3, Theorem 3.4.8(iii)], C^{an} is normal and irreducible. Moreover, $\mathscr{M}(C^{\mathrm{an}}) = F$, where $\mathscr{M}$ denotes the sheaf of meromorphic functions on C^{an} ([1, Proposition 3.6.2]).

(4) We denote by $V(F)$ the set of rank 1 non-trivial valuations on F such that for any $v \in V(F)$, $v_{|k}$ either induces the norm on k or is trivial. For any $v \in V(F)$, we denote by F_v the completion of F with respect to v.

Moreover:

Hypothesis 2. Let X/F be a smooth proper variety such that $X(F_v) \neq \emptyset$ for all discrete valuations $v \in V(F)$.

Let $Z \subseteq C$ be a Zariski closed subset such that X has good reduction over $C \backslash Z$. We remark that Z need not be the smallest such subset of C.

We will now use Bosch's theorem to interpret over a model of the algebraic curve C the hypotheses we encoutered in the main statements of Section 2. They translate to asking for good reduction of the variety X over certain parts of a nicely chosen model of C. In what follows, we show that the "finer" the model, the "smaller" said parts have to be.

Let us start with a local statement.

Lemma 3.2. *Let $\mathcal{C}$ be a proper regular model of C over k°. Let $\mathcal{C}_s$ denote its special fiber and π the corresponding specialization morphism $C^{\mathrm{an}} \to \mathcal{C}_s$. Let $P \in \mathcal{C}_s$ be a closed point. If X has a smooth proper model over $\mathcal{O}_{\mathcal{C},P}$, then $X(\mathscr{M}(\pi^{-1}(P))) \neq \emptyset$. In particular, for any $x \in \pi^{-1}(P)$, $X(\mathscr{M}_x) \neq \emptyset$.*

Proof. By Theorem 1.6, $U := \pi^{-1}(P)$ is a connected open subset of C^{an}, and its boundary contains only type 2 points. By assumption, X has a proper smooth model $\mathcal{X}$ over $\mathcal{O}^\circ(U) = \widehat{\mathcal{O}_{\mathcal{C},P}}$ (Theorem 1.2). By the proof of Lemma 2.9, $\mathcal{X}(\mathcal{O}^\circ(U)) \neq \emptyset$, implying $X(\mathscr{M}(U)) \neq \emptyset$, and so for all $x \in \pi^{-1}(P)$, $X(\mathscr{M}_x) \neq \emptyset$. □

Let us show that if $z \in C$ is such that X has bad reduction over z, then the hypothesis of Lemma 3.2 is never satisfied for $P := \pi(z)$. We start with an auxiliary result. Recall that for any closed point $z \in C$, the closure $\overline{\{z\}}$ of $\{z\}$ in $\mathcal{C}$ is some set $\{z, P_z\}$, where P_z is a closed point of $\mathcal{C}_s$ (see [26, Definition 10.1.31]).

Lemma 3.3. *We use the same notation as in Lemma 3.2. Let $z \in C$ be a Zariski closed point and let us also denote by z its corresponding rigid point in C^{an}. Then $\pi(z) = P_z$, where $\{z, P_z\}$ is the Zariski closure of $\{z\}$ in $\mathcal{C}$.*

Proof. Let $V = \operatorname{Spec} A$ be an open neighborhood of P_z and $\pi(z)$ in $\mathcal{C}$ (such a neighborhood exists by [12, 2.2]). As V is a curve over k°, A is a two-dimensional ring. Since $\overline{\{z\}} = \{z, P_z\}$, this means that $z \in V$. Then, as z is a point of V, but not a closed one, it corresponds to a principal prime ideal (a) of A. Also, the point P_z corresponds to the unique maximal ideal m of A containing (a). We recall that the radical ideal $\sqrt{(t,a)}$ of (t,a) is $\bigcap_{(t,a)\subseteq I} I$, where the intersection is taken over prime ideals I of A. As $(a) \subsetneq I$, we obtain that I is a maximal ideal. Hence, $I = m$, and $\sqrt{(t,a)} = m$ is the ideal corresponding to the closed point P_z.

On the other hand, let us denote $P'_z := \pi(z)$. By Remark 1.5, P'_z is a closed point of $\mathcal{C}_s$. By [2, pg. 541], $\pi^{-1}(P'_z) \subseteq \widehat{V_\eta}$, where $\widehat{V_\eta}$ denotes the analytic generic fiber of V. The restriction of π to $\widehat{V_\eta}$ induces the specialization map $\widehat{V_\eta} \to V_s$ corresponding to V. The point $z \in C^{\mathrm{an}}$ induces a morphism $\widehat{A} \otimes_{k^\circ} k \to \mathcal{H}(z)$. As $A \subseteq (\widehat{A} \otimes_{k^\circ} k)^\circ$, this in turn induces a morphism $A \to \mathcal{H}(z)^\circ$, and thus one $f_z : A/(t) \to \widetilde{\mathcal{H}(z)}$. Then $\pi(z) = \ker f_z = P'_z$ and this is a maximal ideal of A. As z is the point corresponding to $|a| = 0$ in V^{an}, we have that $(t,a) \subseteq \ker f_z$. Consequently, we obtain that $\ker f_z = \sqrt{(t,a)}$, implying $P_z = P'_z$. □

Remark 3.4. If X has a proper smooth model over $\mathcal{O}_{\mathcal{C},P}$ for some closed point $P \in \mathcal{C}_s$, then it has a proper smooth model over some open neighborhood N of P in $\mathcal{C}$. If $z \in C$ is any Zariski closed point for which $\overline{\{z\}} = \{z, P\}$, meaning $P = \pi(z)$ by Lemma 3.3, then

clearly $z \in N$, so X has a proper smooth model over $\kappa(z)$. This means that X has good reduction over z. This is why the bad reduction points of X in C will automatically be excluded in Lemma 3.2.

We will now give similar, but global, versions of Lemma 3.2. In order to be able to also deal with the points on C^{an} where the variety X has bad reduction, we will need to construct and work on "fine enough" models of the curve C.

3.1. **Removing a finite set of the special fiber from consideration.** We use Notation 3.1 and Hypothesis 2. We also recall the notion of *strongly transitive action* in Definition 1.8 (see also Remark 1.9), which we will use several times throughout this section.

Construction 1 (The model $\mathscr{C}_1$)**.** By Remark 2.4, for any $z \in Z$, there exists a strict affinoid neighborhood V_z of z in C^{an} such that $X(\mathscr{M}(V_z)) \neq \emptyset$. We remark that as V_z is a strict affinoid domain, ∂V_z is a finite set of type 2 points. Let $\mathscr{C}_1$ denote a *proper regular model* of C over k° corresponding to a vertex set S of C^{an} such that $\bigcup_{z\in Z} \partial V_z \subseteq S$. We will denote by $\mathscr{C}_{1,s}$ its special fiber and by π_1 the corresponding specialization morphism.

See Figure 3 below for a couple of illustrations of $\mathscr{C}_{1,s}$ and π_1, where the bijection of Theorem 1.6 associates to $\eta_i \in S$ the irreducible component I_i of $\mathscr{C}_{1,s}$.

Remark 3.5. Let $Z' \subseteq C$ be any Zariski closed subset. If, for any $z \in Z'$, there exists an affinoid neighborhood V'_z of z in C^{an} such that $X(\mathscr{M}(V'_z)) \neq \emptyset$, and $\bigcup_{z\in Z'} V'_z = C^{\mathrm{an}}$, then for any $x \in C^{\mathrm{an}}$, $X(\mathscr{M}_x) \neq \emptyset$ (see Lemma 2.5), and equivalently $X(F_v) \neq \emptyset$ for all $v \in V(F)$. Hence, without loss of generality, we may assume that $\bigcup_{z\in Z} V_z \neq C^{\mathrm{an}}$.

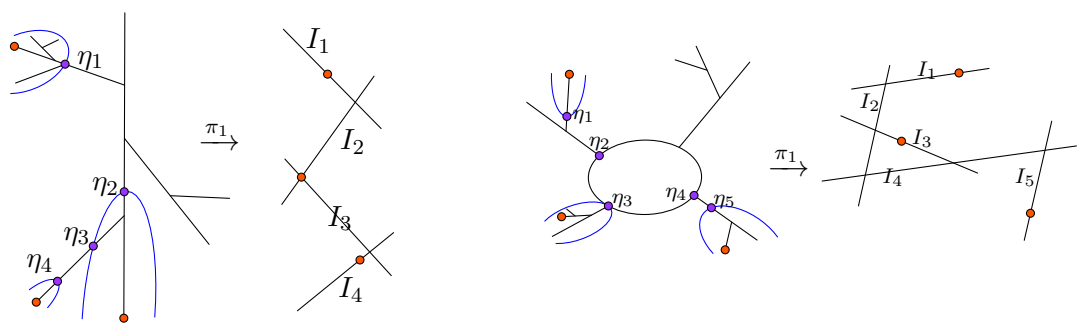

Figure 3. $Z, \pi_1(Z), V_z, z \in Z, \eta_i \in S$

Theorem 3.6. *If there exists a proper model $\mathcal{X} \to \mathscr{C}_1$ of X/F which is smooth over $\mathscr{C}_{1,s} \backslash \pi_1(Z)$, then $X(\mathscr{M}_x) \neq \emptyset$ for all $x \in C^{\mathrm{an}}$, and, equivalently, $X(F_v) \neq \emptyset$ for all $v \in V(F)$.*

Furthermore, if there exists a rational linear algebraic group G/F acting strongly transitively on X, then $X(F) \neq \emptyset$.

Remark 3.7. Let P be any closed point of $\mathscr{C}_1$. As $\mathcal{O}_{\mathscr{C}_1,P}$ and $\mathcal{O}_{\mathscr{C}_{1,s},P}$ are local rings with the same residue field, the respective base change of $\mathcal{X}$ is smooth over $\mathcal{O}_{\mathscr{C}_1,P}$ if and only if it is smooth over $\mathcal{O}_{\mathscr{C}_{1,s},P}$.

Proof of Theorem 3.6. If $x \in \bigcup_{z\in Z} V_z \cup S$, then clearly $X(\mathscr{M}_x) \neq \emptyset$ (see Remark 2.2(4) and Lemma 2.5). Otherwise, let $x \in C^{\mathrm{an}} \backslash (\bigcup_{z\in Z} V_z \cup S)$. We will show that the closed point $P_x := \pi_1(x) \in \mathscr{C}_{1,s}$ satisfies $P_x \notin \pi_1(Z)$. Let us assume that there exists $z_0 \in Z$ such that $P_x = \pi_1(z_0)$. Set $U_x := \pi_1^{-1}(P_x)$. Then $z_0 \in U_x$. By Theorem 1.6, U_x is a connected component of $C^{\mathrm{an}} \backslash S$. By its connectedness, there exists an injective path $[x, z_0]$ connecting x and z_0 that is entirely contained in U_x. As $x \notin V_{z_0}$, the path $[x, z_0]$ must intersect ∂V_{z_0}, meaning $[x, z_0] \cap S \neq \emptyset$. As a consequence, $U_x \cap S \neq \emptyset$, which is impossible. Thus, $P_x \notin \pi_1(Z)$. We can now conclude via Lemma 3.2 that $X(\mathscr{M}_x) \neq \emptyset$. We have shown that $X(\mathscr{M}_x) \neq \emptyset$ for all $x \in C^{\mathrm{an}}$. By Remark 2.3, $X(F_v) \neq \emptyset$ for all $v \in V(F)$. We can now conclude by [29, Corollary 3.18]. □

Remark 3.8. In light of Remark 3.4, we can take Z to be any Zariski closed subset of C, and then the hypothesis of Theorem 3.6 will imply that $\pi_1^{-1}(\pi_1(Z))$ contains the rigid points of C^{an} over which X has bad reduction. We thus assumed, without loss of generality, that Z itself contains them.

3.2. **Almost removing irreducible components from consideration.** Let us now explore how by further refining the model $\mathscr{C}_1$ from Section 3.1, we can forget more points from the special fiber in the statement of Theorem 3.6. We will do this in two steps, the first of which consists of being more restrictive when constructing the neighborhoods V_z of $z \in Z$.

We recall Notation 3.1 and Hypothesis 2, which we will be using here.

Hypothesis 1. From now on, throughout this section, we will assume that the curve C is smooth. (This is stronger than normal when k is not a perfect field.)

Remark 3.9. In practice, we only need C to be smooth at the points of the subset Z. This is needed to insure the existence of certain special neighborhoods of these points in the analytic curve C^{an}.

See Subsection 1.1.3 for some elements from the branch language of [10, 1.7] that we use here.

Lemma 3.10. *Let N be a closed virtual disc in C^{an}. Then*

(1) *∂N is a single type 2 point $\{\tau\}$,*
(2) *there exists a unique branch b issued from η in C^{an} not contained in N.*

If $z \in C^{\mathrm{an}}$ is a rigid point, then there exists a neighborhood N of z which is a virtual closed disc, and

(3) *$X(\mathscr{M}(N)) \neq \emptyset$.*

Proof. By [10, Théorème 4.5.4], open (and hence closed) virtual discs form a basis of neighborhoods of z in C^{an}. Hence, as type 2 points are dense in C^{an}, there exists a neighborhood N–a closed virtual disc–of z satisfying conditions (1) and (3) of the statement. It only remains to show that a closed virtual disc in C^{an} satisfies condition (2). Let $\widehat{\overline{k}}$ denote the completion of an algebraic closure of k. Then $N_{\widehat{\overline{k}}} := N \times_k \widehat{\overline{k}}$ is a finite disjoint union of closed discs. Let D be one of those closed discs, which is, by construction embedded in $C^{\mathrm{an}}_{\widehat{\overline{k}}} := C^{\mathrm{an}} \times_k \widehat{\overline{k}}$. As it is a disc, it can also be embedded in $\mathbb{P}^{1,\mathrm{an}}_{\widehat{\overline{k}}}$. Let us denote by ω its unique boundary point.

By [10, 1.7.2], it suffices to show that there exists a unique branch issued from ω in $C^{\mathrm{an}}_{\widehat{\bar{k}}}$ that is not contained in D. By [10, 4.2.11.1], the number of such branches is the same as the number of branches issued from ω in $\mathbb{P}^{1,\mathrm{an}}_{\widehat{\bar{k}}}$ and not contained in D. Thus, there is a unique such branch, meaning condition (2) is also satisfied. □

Construction 2 (The model $\mathscr{C}_2$)**.** For any point $z \in Z$, let N_z denote an affinoid neighborhood of z in C^{an} satisfying the conditions of Lemma 3.10. As in Remark 3.5, without loss of generality, we may assume that $\bigcup_{z\in Z} N_z \neq C^{\mathrm{an}}$. Let $s_0 \in C^{\mathrm{an}}$ be a type 2 point such that $s_0 \notin \bigcup_{z\in Z} N_z$. Set $T := \bigcup_{z\in Z} \partial N_z$.

Let $\mathscr{C}_2$ be a *proper sncd model* of the curve C over k° such that the corresponding vertex set S of C^{an} satisfies $T \cup \{s_0\} \subseteq S$ (see Corollary 1.7). We denote by $\mathscr{C}_{2,s}$ the special fiber of $\mathscr{C}_2$, and by $\pi_2 : C^{\mathrm{an}} \to \mathscr{C}_{2,s}$ the corresponding specialization morphism.

We recall that the correspondence given in Theorem 1.6 is such that there is a bijection between the points y of S and the irreducible components I_y of $\mathscr{C}_{2,s}$. See Figure 4 below for an illustration of $\mathscr{C}_{2,s}$ and π_2. We now proceed with a statement analogous to that of Section 3.1.

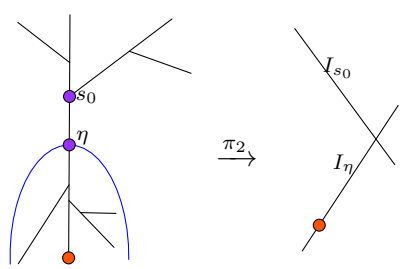

Figure 4. $Z, \pi_2(Z), N_z, z \in Z, S$

Theorem 3.11. *If there exists a proper model $\mathcal{X} \to \mathscr{C}_2$ of X/F which is smooth over $\left(\bigcup_{s\in S\setminus T} I_s\right) \cup \left(\bigcup_{i\neq j\in T} I_i \cap I_j\right)$, then $X(\mathscr{M}_x) \neq \emptyset$ for all $x \in C^{\mathrm{an}}$, and, equivalently, $X(F_v) \neq \emptyset$ for all $v \in V(F)$.*

Furthermore, if there exists a rational linear algebraic group G/F acting strongly transitively on X, then $X(F) \neq \emptyset$.

Recall Remark 3.7.

Proof of Theorem 3.11. Let $x \in C^{\mathrm{an}}$.

(a) If $x \in \bigcup_{z\in Z} N_z \cup S$, then $X(\mathscr{M}_x) \neq \emptyset$ (see Remark 2.2(4) and Lemma 2.5).

(b) Suppose $x \notin \bigcup_{z\in Z} N_z \cup S$. Set $U := C^{\mathrm{an}} \setminus (\bigcup_{z\in Z} N_z \cup S)$. Let us denote by U_x the connected component of $C\setminus S$ containing x. By Theorem 1.6, $\pi_2(U_x) =: \{P_x\}$, where P_x is a closed point of $\mathscr{C}_{2,s}$, and $\pi_2^{-1}(P_x) = U_x$. By *loc.cit.*, $\partial U_x \subseteq S$, and so depending on whether P_x is a double point or not, ∂U_x consists either of two points or one. Let us show that $P_x \in \left(\bigcup_{s\in S\setminus T} I_s\right) \cup \left(\bigcup_{i\neq j\in T} I_i \cap I_j\right)$.

(1) Assume P_x is a double point, meaning ∂U_x consists of two points. In this case, there is nothing to check as the set $\left(\bigcup_{s\in S\setminus T} I_s\right) \cup \left(\bigcup_{i\neq j\in T} I_i \cap I_j\right)$ contains all the double points of $\mathscr{C}_{2,s}$.

(2) If ∂U_x is a singleton, meaning P_x is a non-double point of $\mathscr{C}_{2,s}$, then $P_x \in \bigcup_{i\in T} I_i$ if and only if $\partial U_x \in T$ (by Theorem 1.6). Assume this is the case. Let $z_0 \in Z$ be such that $\partial U_x = \partial N_{z_0} =: \{\eta\}$. There are three possibilities:
   (a) $N_{z_0} \cup U_x = C^{\mathrm{an}}$, in which case $S\backslash\{\eta\} \subseteq N_{z_0}$, meaning $s_0 \in N_{z_0}$, which is in contradiction with the choice of s_0;
   (b) $U_x \subseteq N_{z_0}$, in which case $x \in N_{z_0}$, contradiction;
   (c) $N_{z_0} \cup U_x \neq C^{\mathrm{an}}$ and $U_x \not\subseteq N_{z_0}$; let $a \in C^{\mathrm{an}}\backslash(N_{z_0} \cup U_x)$; let us denote by $[a, \eta)$ an injective path in C^{an} connecting a and η (but without containing η); the path $[a, \eta)$ intersects a branch issued from η, and as such, by Lemma 3.12 below, intersects either U_x or N_{z_0}; both of these possibilities amount to a contradiction, seeing as then $[a, \eta)$ would intersect either ∂U_x or ∂N_{z_0}, respectively, meaning $\eta \in [a, \eta)$.

   Thus, we must have $\partial U_x \notin T$. Consequently, $P_x \in \bigcup_{s\in S\backslash T} I_s$.

Lemma 3.12. *Suppose $N_{z_0} \cup U_x \neq C^{\mathrm{an}}$ and $U_x \not\subseteq N_{z_0}$. The unique branch issued from η in C^{an} not contained in N_{z_0} is contained in U_x.*

Proof. Let b be the branch issued from η that is not contained in N_{z_0}. Suppose $b \subsetneq U_x$, meaning $b \cap U_x = \emptyset$. Let $[c, \eta)$ be an injective path in U_x connecting c and η, without containing η. This path intersects a branch issued from η and is not contained in N_{z_0} (otherwise $\partial N_{z_0} \cap [c, \eta) \neq \emptyset$, impossible). Consequently, $[c, \eta) \subseteq b$, contradiction. Thus, $b \subseteq U_x$. □

To resume, we have shown that $\pi_2(x) = \pi_2(U_x) = P_x \in \left(\bigcup_{s\in S\backslash T} I_s\right) \cup \left(\bigcup_{i\neq j\in T} I_i \cap I_j\right)$. This means that for any $x \in U$, X has a proper smooth model over $\mathcal{O}_{\mathscr{C}_2,P_x}$, hence by Lemma 3.2, $X(\mathscr{M}_x) \neq \emptyset$.

By combining points (a) and (b) above, we obtain that $X(\mathscr{M}_x) \neq \emptyset$ for all $x \in C^{\mathrm{an}}$. By Remark 2.3, $X(F_v) \neq \emptyset$ for all $v \in V(F)$. We can now conclude by [29, Corollary 3.18]. □

Remark 3.13. Let us briefly explain why, in order to apply the techniques of this manuscript, the point s_0 in Construction 2 is necessary. Let us look at the example illustrated in Figure 4. If $S := \{\eta\} = T$ instead of $\{s_0, \eta\}$, then $\mathscr{C}_{1,s}$ would contain a unique irreducible component I_η, making the hypothesis of Theorem 3.11 empty, and the technique of proof non-applicable. If we took $s_0 \in N_z$, then the hypothesis insures good reduction of X over I_{s_0}, but the points in $C^{\mathrm{an}}\backslash N_z$ all map to $I_\eta\backslash I_{s_0}$ via π_2, meaning we can't apply the technique to them, hence they remain unaccounted for.

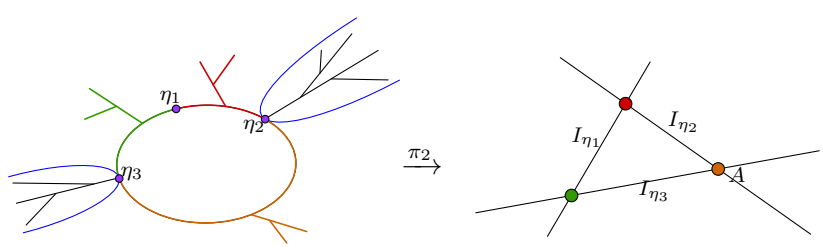

Figure 5. $T = \{\eta_2, \eta_3\}, S = T \cup \{\eta_1\}$, $N_z, z \in Z$

Remark 3.14. Considering the double points in the statement of Theorem 3.11 is necessary. To see this, let us consider the illustration of $\mathscr{C}_{2,s}$ and π_2 in Figure 5 above. If we

assume that X has good reduction only over I_{η_1}, then all the points of C^{an} that map to the point A via π_2 remain unaccounted for.

3.3. **Removing irreducible components from consideration.** We now show that by blowing up at some of the double points of the special fiber $\mathscr{C}_{2,s}$ in the statement of Theorem 3.11, we *can* get rid of them.

Recall Notation 3.1 and Hypothesis 2, which we will be using. We also adopt Hypothesis 1.

Construction 3 (The model $\mathscr{C}_3$)**.** For any point $z \in Z$, let N_z denote an affinoid neighborhood of z in C^{an} as in Lemma 3.10. Without loss of generality, we may assume $\bigcup_{z\in Z} N_z \neq C^{\mathrm{an}}$ (see Remark 3.5). Set $T := \bigcup_{z\in Z} \partial N_z$. This is a finite set of type 2 points. For any $a, b \in T$, and any injective path $[a, b]$ in C^{an} connecting a and b, let c denote a type 2 point in $[a, b]\backslash\{a, b\}$. By [28, Proposition 1.8.14], there are only finitely many such paths $[a, b]$ in C^{an}, so the set T_1 of all such points c, for any $a, b \in T$, is a finite set of type 2 points.

Let $\mathscr{C}_3$ be a *proper sncd model* of C over k° corresponding to a vertex set S on C^{an} such that $T \cup T_1 \subseteq S$ (see Corollary 1.7). We denote by $\mathscr{C}_{3,s}$ the special fiber of $\mathscr{C}_3$, and by π_3 the corresponding specialization morphism.

For any $s \in S$, we denote by I_s the unique irreducible component of $\mathscr{C}_{3,s}$ which corresponds to it (Theorem 1.6). We remark that $\mathscr{C}_3$ is finer than the model $\mathscr{C}_2$, which is in turn finer than the model $\mathscr{C}_1$ of the curve C. See Figure 6 for an illustration of an example of $\mathscr{C}_{3,s}$ and π_3.

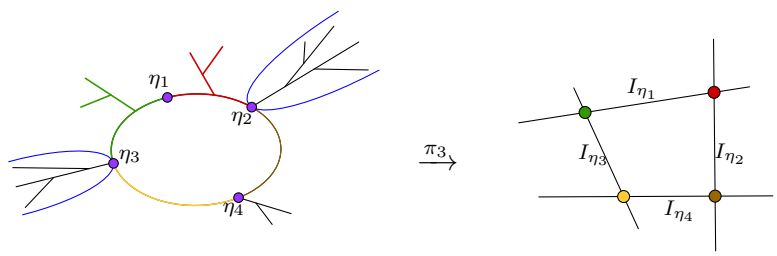

Figure 6. $T = \{\eta_2, \eta_3\}, S = T \cup \{\eta_1, \eta_4\}, N_z, z \in Z$

Remark 3.15. To see why S was constructed this way, let us go back to the illustration in Figure 5. By Remark 3.14, one needs good reduction over the double point A as well as the irreducible component I_{η_1}.

Theorem 3.16. *If there exists a proper model $\mathcal{X} \to \mathscr{C}_3$ of X/F which is smooth over $\bigcup_{s\in S\backslash T} I_s$, then $X(\mathscr{M}_x) \neq \emptyset$ for all $x \in C^{\mathrm{an}}$. Equivalently, $X(F_v) \neq \emptyset$ for all $v \in V(F)$.*

Furthermore, if there exists a rational linear algebraic group G/F acting strongly transitively on X, then $X(F) \neq \emptyset$.

Recall Remark 3.7.

Proof of Theorem 3.16. Let $x \in C^{\mathrm{an}}$.

(a) If $x \in \bigcup_{z\in Z} N_z \cup S$, then clearly $X(\mathscr{M}_x) \neq \emptyset$ (see Remark 2.2(4) and Lemma 2.5).

(b) Let $x \in C^{\mathrm{an}}\backslash(\bigcup_{z\in Z} N_z \cup S)$. Set $P_x := \pi_3(x)$ and $U_x := \pi_3^{-1}(x)$. We will show that $P_x \in \bigcup_{s\in S\backslash T} I_s$. By Theorem 1.6, ∂U_x consists of two points if P_x is a double point of $\mathscr{C}_{3,s}$ and is a singleton otherwise. Let us consider these cases separately.

(1) Let $\partial U_x =: \{\alpha, \beta\}$. Then $P_x \notin \bigcup_{s\in S\setminus T} I_s$ if and only if $\alpha, \beta \in T$. Suppose this is the case. There exists an injective path (α, β) in U_x that connects its border points α and β (without containing them). But then, by the construction of the vertex set S, we have $S \cap (\alpha, \beta) \neq \emptyset$, so $S \cap U_x \neq \emptyset$, contradiction. Hence, $P_x \in \bigcup_{s\in T\setminus S} I_s$.

(2) Let ∂U_x be a singleton. Then $P_x \in \bigcup_{s\in T} I_s$ is not possible by part (2) of the proof of Theorem 3.11. Consequently, $P_x \in \bigcup_{s\in S\setminus T} I_s$.

Finally, we have shown that X has a smooth model over $\mathcal{O}_{\mathscr{C}_3,P_x}$, so by Lemma 3.2, $X(\mathscr{M}_x) \neq \emptyset$.

By combining points (a) and (b) above, we obtain that $X(\mathscr{M}_x) \neq \emptyset$ for all $x \in C^{\mathrm{an}}$. By Remark 2.3, $X(F_v) \neq \emptyset$ for all $v \in V(F)$. We can now conclude by [29, Corollary 3.18]. □

Remark 3.17. If $T = \emptyset$, meaning $Z = \emptyset$, then the variety X/F has good reduction over all points of the curve C. In that case, in the statements above (Theorems 3.6, 3.11 and 3.16), we have to check that X has a proper smooth model over the entire special fiber $\mathscr{C}_{3,s}$ of the model $\mathscr{C}_3$ of C. Hence, this condition is directly related to the uniformizer t of k°. More precisely, we check smoothness of X via the nonvanishing of certain minors ϵ of a matrix defined over F. As ϵ doesn't vanish anywhere on the proper curve C, it is a constant, meaning defined over k. Then checking whether X has a model that is smooth over $\mathscr{C}_{3,s}$ comes down to checking whether ϵ is invertible in k°, or equivalently, whether it is non-zero on the residue field $\widetilde{k}$ of k.

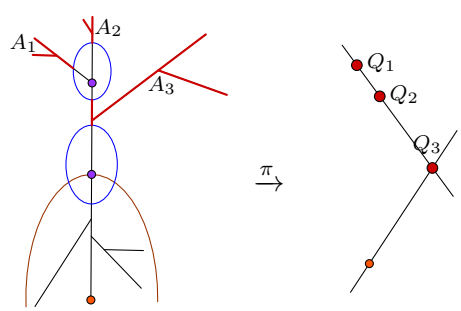

Figure 7. $Z, N_z, z \in Z, S, U_s, s \in S, Q_i = \pi(A_i)$

Remark 3.18. Let us go back and use the notations of Remark 2.18. By Theorem 1.6, we may assume that the vertex set S of C^{an} corresponds to a *regular* proper model $\mathscr{C}$ over k° of the algebraic curve C. Let us denote by $\mathscr{C}_s$ its special fiber and by π the corresponding specialization morphism $C^{\mathrm{an}} \to \mathscr{C}_s$. By *loc.cit.*, for any $i \in \{1, 2, \ldots, n\}$, $\pi(B_i) = \pi(A_i) =: \{Q_i\}$, where Q_i is a closed point of $\mathscr{C}$. By Theorem 1.2, $\widehat{\mathcal{O}_{\mathscr{C},Q_i}} = \mathcal{O}^\circ(A_i) \subseteq \mathcal{O}^\circ(B_i)$. Hence, by Proposition 2.19, if X has proper smooth models over the rings $\mathcal{O}_{\mathscr{C},Q_i}$, then $X(\mathscr{M}_x) \neq \emptyset$ for all $x \in C^{\mathrm{an}}$, and, equivalently, $X(F_v) \neq \emptyset$ for all $v \in V(F)$. See Figure 7 above for an illustration.

4. A smoothness criterion over residue fields of completions

We recalled the notion of the completed residue field $\mathcal{H}(\cdot)$ in Subsection 1.1.1. As mentioned there, it is a complete ultrametric field. We will denote by $\widetilde{\mathcal{H}(\cdot)}$ its residue field.

4.1. **Over the analytic curve.** We will use Notation 2.1 and Hypothesis 1 in this subsection.

Remark 4.1. Let l be a complete ultrametric field (not necessarily discretely valued), and C'/l an analytic curve. Then, for any open U of C' and any point $x \in U$, we have maps $\mathcal{O}^\circ(U) \to \widetilde{\mathcal{H}(x)}$. This is a direct consequence of the definition of the sheaf $\mathcal{O}^\circ$, which gives us a map $\mathcal{O}^\circ(U) \to \mathcal{H}(x)^\circ$, where $\mathcal{H}(x)^\circ$ denotes the valuation ring of $\mathcal{H}(x)$.

As in Section 2, we start by proving a local statement.

Proposition 4.2. *Let $x \in C$. Suppose there exists a strict affinoid neighborhood N of x in C such that X has a proper model $\mathcal{X} \to \operatorname{Spec} \mathcal{O}^\circ(N)$ which is smooth over $\widetilde{\mathcal{H}(y)}$ for all but a finite number of*

(1) *rigid points, or*
(2) *type 2 points*

y of N. Then there exists a neighborhood $U_x \subseteq N$ of x such that $X(\mathscr{M}(U_x)) \neq \emptyset$, and hence $X(\mathscr{M}_x) \neq \emptyset$.

Proof. By Remark 4.1, it makes sense to consider the model $\mathcal{X}$ of X (by applying the appropriate base change) over $\widetilde{\mathcal{H}(y)}, y \in N$. By restricting to a smaller neighborhood of x if necessary, as C is separated, we may assume without loss of generality that $\mathcal{X}$ is smooth over $\widetilde{\mathcal{H}(y)}$ for *all* rigid (resp. type 2) points y of N.

Let $\mathscr{C}$ be a proper regular model of C^{al} over k° corresponding to a vertex set S of C such that $\partial N \subseteq S$ (see Corollary 1.7). We denote by $\mathscr{C}_s$ its special fiber and by π the corresponding specialization morphism $C \to \mathscr{C}_s$. Set $P_x := \pi(x)$ and $U_x := \pi^{-1}(P_x)$. The point P_x is closed in $\mathscr{C}_s$ and the open U_x is connected in C. Moreover, $\partial U_x \subseteq S$, so ∂U_x contains only type 2 points. By the proof of Lemma 2.8, $U_x \subseteq N$, and so for any rigid (resp. type 2) point $\eta \in U_x$, the model $\mathcal{X}$ of X (meaning its respective base change) is smooth over $\widetilde{\mathcal{H}(\eta)}$.

By [1, 2.4, pg. 35] (see also Remark 1.1), the specialization map π induces an embedding $\kappa(P_x) \hookrightarrow \widetilde{\mathcal{H}(\eta)}$. Consequently, $\mathcal{X}$ is smooth over $\kappa(P_x)$, and thus over $\widehat{\mathcal{O}_{\mathscr{C},P_x}} = \mathcal{O}^\circ(U_x)$. We can now conclude by Lemma 2.9 that $X(\mathscr{M}(U_x)) \neq \emptyset$, and hence that $X(\mathscr{M}_x) \neq \emptyset$. □

Remark 4.3. As we can see from the proof, the hypotheses of Proposition 4.2 can be relaxed to: *for any neighborhood M of x there exists a rigid (resp. type 2) point $y_M \in M$ such that $\mathcal{X}$ is smooth over $\widetilde{\mathcal{H}(y_M)}$.*

Let us now give a similar global version of Proposition 4.2 akin to Corollary 2.14. We recall the notion of *strongly transitive action* in Definition 1.8.

Corollary 4.4. *Let Q be a finite set of rigid and type 2 points of C. For any $z \in Q$, let V_z be a strict affinoid neighborhood of z in C such that $X(\mathscr{M}(V_z)) \neq \emptyset$. Set $U := C \setminus \bigcup_{z \in Q} V_z$. If X has a proper model $\mathcal{X} \to \operatorname{Spec} \mathcal{O}^\circ(U)$ such that at least one of the following conditions is satisfied:*

(1) *$\mathcal{X}$ is smooth over $\widetilde{\mathcal{H}(x)}$ for all but a finite number of* rigid *points $x \in U$,*
(2) *$\mathcal{X}$ is smooth over $\widetilde{\mathcal{H}(x)}$ for all but a finite number of* type 2 *points $x \in U$,*

then $X(\mathscr{M}_x) \neq \emptyset$ for all $x \in C$, and, equivalently, $X(F_v) \neq \emptyset$ for all $v \in V(F)$.

If, moreover, there exists a rational linear algebraic group G/F acting strongly transitively on X, then $X(F) \neq \emptyset$.

Proof. By Lemma 2.13, ∂U is a finite set of type 2 points, so by Remark 2.7(2), $F \subseteq \operatorname{Frac} \mathcal{O}^\circ(U)$. By Remark 4.1, we can take a base change of $\mathcal{X} \to \operatorname{Spec} \mathcal{O}^\circ(U)$ to the fields $\widetilde{\mathcal{H}(x)}$, $x \in U$, thus obtaining a model of X over these fields. By Proposition 4.2, for all $x \in U$, $X(\mathscr{M}_x) \neq \emptyset$. By construction, for any $x \in \bigcup_{z \in Q} V_z$, $X(\mathscr{M}_x) \neq \emptyset$. Thus, $X(\mathscr{M}_x) \neq \emptyset$ for all $x \in C$. Equivalently, $X(F_v) \neq \emptyset$ for all $v \in V(F)$ by Remark 2.3. We can now conclude by [29, Corollary 3.18]. □

Remark 4.5. One can show the same result (akin to Corollary 2.16) for $U := C \backslash (\bigcup_{z \in Q} V_z \cup A)$, where $A \subseteq C$ is a finite set of type 2 points.

As proven by the following remark and proposition, it suffices to show that for a *finite* number of fixed open subsets $U \subseteq C$, there exists $x_U \in U$ a rigid or type 2 point, such that X has a proper model over $\mathcal{O}^\circ(U)$ which is smooth over $\widetilde{\mathcal{H}(x_U)}$. Moreover, U depends on the variety X.

Remark 4.6. With the same notation as in Corollary 4.4, let S be any vertex set of C corresponding to a proper regular model $\mathscr{C}$ of C^{al} and such that $\bigcup_{z \in Q} \partial V_z \subseteq S$ (see Corollary 1.7). As it contains only type 2 points, for any $s \in S$, $X(\mathscr{M}_s) \neq \emptyset$. Hence, there exists an open neighborhood U_s of s in C such that $X(\mathscr{M}(U_s)) \neq \emptyset$. By Remark 2.18, there exist only finitely many connected components $A_1, A_2, \ldots, A_n$ of $C \backslash S$ which are not entirely contained in $\bigcup_{s \in S} U_s$. Recall also an illustration given in Figure 2.

Proposition 4.7. *If for any $i \in \{1, 2, \ldots, n\}$ there exists a rigid or type 2 point $x_i \in A_i$ such that X has a proper model $\mathcal{X} \to \operatorname{Spec} \mathcal{O}^\circ(A_i)$ which is smooth over $\widetilde{\mathcal{H}(x_i)}$, then $X(\mathscr{M}_x) \neq \emptyset$ for all $x \in C$. Equivalently, $X(F_v) \neq \emptyset$ for all $v \in V(F)$.*

If, moreover, there exists a rational linear algebraic group G/F acting strongly transitively on X, then $X(F) \neq \emptyset$.

Proof. If $x \in \bigcup_{s \in S} U_s \cup \bigcup_{z \in Q} V_z$, then $X(\mathscr{M}_x) \neq \emptyset$. Otherwise, suppose there exists $i_0 \in \{1, 2, \ldots, n\}$ such that $x \in A_{i_0}$. By Theorem 1.6, $\pi(A_{i_0}) = \pi(x) = \pi(x_{i_0}) =: P_{i_0} \in \mathscr{C}_s$, where $\mathscr{C}_s$ is the special fiber of $\mathscr{C}$ and π the corresponding specialization morphism.

By [1, 2.4, pg. 35] (see also Remark 1.1), π induces an embedding $\kappa(P_{i_0}) \subseteq \widetilde{\mathcal{H}(x_{i_0})}$, where $\kappa(P_{i_0})$ is the residue field of the point P_{i_0}, hence the residue field of the local ring $\mathcal{O}^\circ(A_{i_0}) = \widehat{\mathcal{O}_{\mathscr{C}, P_{i_0}}}$. Consequently, $\mathcal{X}$ (or rather, its respective base change) is smooth over $\kappa(P_{i_0})$, meaning $\mathcal{X}$ is smooth over $\mathcal{O}^\circ(A_{i_0})$. By Lemma 2.9, this implies that $\mathcal{X}(\mathcal{O}^\circ(A_{i_0})) \neq \emptyset$, hence that $X(\mathscr{M}_x) \neq \emptyset$. By Remark 2.3, $X(F_v) \neq \emptyset$ for all $v \in V(F)$. We can now conclude by [29, Corollary 3.18]. □

4.2. **Over a model of the algebraic curve.** Given the connection between points of a Berkovich curve and the valuations on its function field, the results of Section 4.1 can also be stated over models and using valuations.

We will use the same notation as in Section 3, see Notation 3.1 and Hypothesis 2.

Remark 4.8. Let v be a discrete valuation on F such that $v_{|k}$ is trivial. The residue field $\kappa(v) = \widetilde{F_v}$ is a finite field extension of k. In particular, it is uniquely endowed with a discrete valuation extending that of k. As a consequence, it makes sense to look at its residue field, which we will call the *double residue field* of F_v and denote by $\widetilde{\widetilde{F_v}}$.

Remark 4.9. Let v be a discrete valuation on F. Let $x_v \in C^{\mathrm{an}}$ be the unique point corresponding to the valuation v. By the proof of [29, Proposition 3.15], if v extends the

norm on k, then $F_v = \mathcal{H}(x_v)$, so $\widetilde{F_v} = \widetilde{\mathcal{H}(x_v)}$. If, on the other hand, v is trivial on k, then by *loc.cit.*, $\widetilde{F_v} = \mathcal{H}(x_v)$, so $\widetilde{\widetilde{F_v}} = \widetilde{\mathcal{H}(x_v)}$.

For the next statement only, we will use the setting and notation of Section 3.1. In particular, we work over the model $\mathscr{C}_1$ of the curve C (see Construction 1).

Theorem 4.10. *Let $\mathcal{X} \to \mathscr{C}_1$ be a proper model of X. If either of the following hypotheses is satisfied:*

(1) *$\mathcal{X} \to \widetilde{\widetilde{F_v}}$ is smooth for all but a finite number of discrete valuations v on F such that $v_{|k}$ is trivial and with center c_v satisying $\pi_1(c_v) \in \mathscr{C}_{1,s} \backslash \pi_1(Z)$,*

(2) *$\mathcal{X} \to \widetilde{F_v}$ is smooth for all but a finite number of discrete valuations v on F such that $v_{|k}$ induces the norm on k and with center c_v satisfying $c_v \in \mathscr{C}_{1,s} \backslash \pi_1(Z)$,*

then $X(F_v) \neq \emptyset$ for all $v \in V(F)$.

Furthermore, if there exists a rational linear algebraic group G/F acting strongly transitively on X, then $X(F) \neq \emptyset$.

Remark 4.11. We recall that if v is a discrete valuation on F which is trivial on k, then it corresponds to a unique rigid point x_v of C^{an} (*i.e.* to a Zariski closed point of C), and by [26, Remark 8.3.19], $c_v = x_v$. On the other hand, if v is a discrete valuation on F which extends the norm of k, then by *loc.cit.*, $c_v = \pi_1(x_v)$ (see also Lemma 3.3).

Proof of Theorem 4.10. Let us start by showing that a base change of $\mathcal{X} \to \mathscr{C}_1$ over $\widetilde{\widetilde{F_v}}$ (resp. $\widetilde{F_v}$) makes sense. For any point $P \in \mathscr{C}_{1,s}$, the morphism $\mathcal{X} \to \mathscr{C}_1$ gives rise (by a base change) to a morphism $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_{\mathscr{C}_1,P}$, hence to a morphism $\mathcal{X} \to \operatorname{Spec} \kappa(P)$. For any $x \in C^{\mathrm{an}}$, by [1, 2.4, pg. 35] (see also Remark 1.1), the specialization morphism π_1 induces an embedding $\kappa(P_x) \subseteq \widetilde{\mathcal{H}(x)}$, where $P_x := \pi_1(x)$. As a consequence, $\mathcal{X}$ gives rise (by a base change) to a model over $\widetilde{\mathcal{H}(x)}$ for all $x \in C^{\mathrm{an}}$. By Remark 4.9, this implies the existence of such a model over all the fields $\widetilde{\widetilde{F_v}}$ (resp. $\widetilde{F_v}$) in the statement.

By Remark 2.3, it suffices to show that $X(\mathscr{M}_x) \neq \emptyset$ for all $x \in C^{\mathrm{an}}$. This is true for any $x \in \bigcup_{z \in Z} V_z \cup S$ by construction (see the notation in Construction 1). Let $x \in C^{\mathrm{an}} \backslash (\bigcup_{z \in Z} V_z \cup S)$. Set $P_x := \pi_1(x) \in \mathscr{C}_{1,s}$, and $U_x := \pi_1^{-1}(P_x)$. We recall that by Theorem 1.6, U_x is a connected open subset of C^{an} and $\partial U_x \subseteq S$. By the proof of Theorem 3.6, $P_x \notin \pi_1(Z)$, so for all but a finite number of rigid (resp. type 2) points $y \in U_x$, the model $\mathcal{X}$ is smooth (after a base change) over the field $\widetilde{\mathcal{H}(y)}$. Hence, by Proposition 4.2, $X(\mathscr{M}_x) \neq \emptyset$. By Remark 2.3, $X(F_v) \neq \emptyset$ for all $v \in V(F)$. We can conclude by [29, Corollary 3.18]. □

For the next statement only, we use the setting and notation of Section 3.2 (resp. Section 3.3). In particular, we work over the model $\mathscr{C}_2$ (resp. $\mathscr{C}_3$) of the curve C (see Construction 2, resp. Construction 3).

Theorem 4.12. *Let $\mathcal{X} \to \mathscr{C}_2$ (resp. $\mathcal{X} \to \mathscr{C}_3$) be a proper model of X. If either of the following hypotheses is satisfied:*

(1) *$\mathcal{X} \to \widetilde{\widetilde{F_v}}$ is smooth for all but a finite number of discrete valuations v on F such that $v_{|k}$ is trivial and with center c_v satisying $\pi_2(c_v) \in \left(\bigcup_{s \in S \backslash T} I_s\right) \cup \left(\bigcup_{i \neq j \in T} I_i \cap I_j\right)$ (resp. $\pi_3(c_v) \in \bigcup_{s \in S \backslash T} I_s$),*

(2) $\mathcal{X} \to \widetilde{F_v}$ *is smooth for all but a finite number of discrete valuations* v *on* F *such that* $v_{|k}$ *induces the norm on* k *and with center* c_v *satisfying* $c_v \in \left(\bigcup_{s\in S\setminus T} I_s\right) \cup \left(\bigcup_{i\neq j\in T} I_i \cap I_j\right)$ *(resp.* $c_v \in \bigcup_{s\in S\setminus T} I_s$*),*

then $X(F_v) \neq \emptyset$ *for all* $v \in V(F)$*.*

Furthermore, if there exists a rational linear algebraic group G/F *acting strongly transitively on* X*, then* $X(F) \neq \emptyset$*.*

Proof. The proof of Theorem 4.10 can be applied *mutatis mutandis* by replacing the reference of Theorem 3.6 with Theorem 3.11 (resp. Theorem 3.16). □

5. The case of quadratic forms

We will now use the techniques from Section 2 to show a Hasse principle for quadratic forms in non-dyadic residue characteristic. This is a result originally shown in [9, Theorem 3.1].

We use the following notation.

Notation 5.1. (1) Let k be a complete discretely valued field. We will denote by k° its valuation ring, and $\widetilde{k}$ its residue field. Let us also fix a uniformizer t of k.

(2) Let C/k be a proper normal irreducible k-algebraic curve. Set $F = k(C)$.

(3) We will denote by C^{an} the Berkovich analytification of C. It is a proper, normal and irreducible k-analytic curve. Moreover, $\mathscr{M}(C^{\mathrm{an}}) = F$, where $\mathscr{M}$ denotes the sheaf of meromorphic functions on C^{an} (see Notation 3.1(3)).

(4) For a valuation v on F, we denote by F_v the completion of F with respect to v.

Remark 5.2. Let us recall a couple of notions whose properties we will explicitely use for the proof of Theorem 5.7 below.

(1) Let (R, m) be a regular local ring. A minimal set of generators $a_1, a_2, \dots, a_n$ for the maximal ideal m will be called a *regular system of parameters*. In that case, $\dim R = n$ (see [35, Tag 00NN] for a more detailed account).

(2) Let D be a *strict normal crossings divisor* (from now on abreviated to *sncd*) on a locally Noetherian scheme Y defined through the invertible ideal sheaf $\mathcal{I}_D$. We recall that this means that for any $x \in D$: *(a)* $\mathcal{O}_{Y,x}$ is a regular local ring with maximal ideal m_x; *(b)* there exists a regular system of parameters $f_1, f_2, \dots, f_m \in m_x$ such that if x is contained in r irreducible components of D, then $r \leqslant m$ and $\mathcal{I}_{D,x} = f_1 f_2 \dots f_r \mathcal{O}_{Y,x}$ (see [35, Tag 0BI9]).

(3) A model $\mathcal{C}$ over k° of the curve C is said to be *sncd* if its special fiber is a sncd divisor of $\mathcal{C}$. In particular, we remark that an sncd model of C is always regular.

We will now prove a few (standard) results which deal purely with sncd models of curves. To that end, let us introduce some complementary notations.

Notation 5.3. (1) Let $\mathcal{C}$ denote a proper sncd model of the curve C over k°. We will denote by $\mathcal{C}_s$ its special fiber, and by $\mathcal{I}_{\mathcal{C}_s}$ the invertible ideal sheaf of $\mathcal{O}_{\mathcal{C}}$ defining $\mathcal{C}_s$.

(2) For $a \in F^\times$, we will denote by $[a]$ the Weil divisor associated to a in C, and by $\mathrm{div}(a)$ the Weil divisor associated to a in $\mathcal{C}$. We remark that $\mathrm{div}(a) \cap C = [a]$.

(3) For $a \in F^\times$, we will denote by $\overline{[a]}$ the Zariski closure of $[a]$ in $\mathcal{C}$.

In the following lemma, the role of D from Remark 5.2 is played by the special fiber $\mathcal{C}_s$ of the model $\mathcal{C}$, and that of $\mathcal{I}_D$ by $\mathcal{I}_{\mathcal{C}_s}$.

Lemma 5.4. *Let $P \in \mathcal{C}_s$ be a closed point. Let $a \in F^\times$ such that $P \notin \overline{[a]}$. If $a \in \mathcal{O}_{\mathcal{C},P}$, then either $a \in \mathcal{I}_{\mathcal{C}_s,P}$ or $a \in \mathcal{O}_{\mathcal{C},P}^\times$.*

Proof. Suppose $a \notin \mathcal{I}_{\mathcal{C}_s,P}$. Then it suffices to show that $P \notin \operatorname{div}(a)$. Suppose $P \in \operatorname{div}(a)$ and let I be an irreducible component of $\operatorname{div}(a)$ containing P. If $I \cap C = \emptyset$, then $I \subseteq \mathcal{C}_s$, which is impossible seeing as $a \notin \mathcal{I}_{\mathcal{C}_s,P}$. Hence, $I \cap C \neq \emptyset$, so there exists a Zariski closed point $z \in C$ such that $z \in \operatorname{div}(a)$. We remark that then $z \in [a]$. If the Zariski closure of $\{z\}$ in $\mathcal{C}$ is $\{z, Q\}$ (see [26, Definition 10.1.31]), then $\{Q\} \subsetneq \{z, Q\} \subseteq I$, so by an argument of dimension: $\{z, Q\} = I$ and $P = Q$. As then $I \subseteq \overline{[a]}$, this is in contradiction with the assumption that $P \notin \overline{[a]}$. □

Lemma 5.5. *Let $P \in \mathcal{C}_s$ be a closed point. There exists a regular system of parameters $\alpha, \beta \in \mathcal{O}_{\mathcal{C},P}$ such that for any $a \in F^\times$ for which $a \in \mathcal{O}_{\mathcal{C},P}$ and $P \notin \overline{[a]}$:*

1. *if P is not a double point of $\mathcal{C}_s$, then either $\mathcal{I}_{\mathcal{C}_s,P} = \alpha\mathcal{O}_{\mathcal{C},P}$ or $\mathcal{I}_{\mathcal{C}_s,P} = \beta\mathcal{O}_{\mathcal{C},P}$, and there exist $n \in \mathbb{N} \cup \{0\}$, $u \in \mathcal{O}_{\mathcal{C},P}^\times$ such that $a = u\alpha^n$, resp. $a = u\beta^n$;*
2. *if P is a double point of $\mathcal{C}_s$, then $\mathcal{I}_{\mathcal{C}_s,P} = \alpha\beta\mathcal{O}_{\mathcal{C},P}$, and there exist $m \in \mathbb{N} \cup \{0\}$, $v \in \mathcal{O}_{\mathcal{C},P}^\times$ such that $a = v\alpha^m\beta^m$.*

Proof. That either $\mathcal{I}_{\mathcal{C}_s,P} = \alpha\mathcal{O}_{\mathcal{C},P}$ or $\mathcal{I}_{\mathcal{C}_s,P} = \beta\mathcal{O}_{\mathcal{C},P}$ (resp. $\mathcal{I}_{\mathcal{C}_s,P} = \alpha\beta\mathcal{O}_{\mathcal{C},P}$) is immediate from Remark 5.2. For (1), let us assume, without loss of generality, that $\mathcal{I}_{\mathcal{C}_s,P} = \alpha\mathcal{O}_{\mathcal{C},P}$. Then there exist $n \in \mathbb{N} \cup \{0\}$ and $b \in \mathcal{O}_{\mathcal{C},P}$ (resp. $m \in \mathbb{N} \cup \{0\}$ and $c \in \mathcal{O}_{\mathcal{C},P}$) such that $a = b\alpha^n$ (resp. $a = c\alpha^m\beta^m$) and $b \notin \mathcal{I}_{\mathcal{C}_s,P}$ (resp. $c \notin \mathcal{I}_{\mathcal{C}_s,P}$).

As $\operatorname{Frac} \mathcal{O}_{\mathcal{C},P} = F$, one obtains $b \in F^\times$ (resp. $c \in F^\times$). Assume $P \in \overline{[b]}$. Then there exists $z \in [b]$ such that the closure of $\{z\}$ in $\mathcal{C}$ is $\{z, P\}$. As $b, \alpha \in \mathcal{O}_{\mathcal{C},P} \subseteq \mathcal{O}_{C,z}$, neither b or α have poles on z, implying b has a zero on z which is not a pole of α. This means that $z \in [a]$, contradiction because then $P \in \overline{[a]}$. Hence, $P \notin \overline{[b]}$. (One can show using the same arguments that, when applicable, $P \notin \overline{[c]}$). By Lemma 5.4, $b \in \mathcal{O}_{\mathcal{C},P}^\times$ (resp. $c \in \mathcal{O}_{\mathcal{C},P}^\times$). □

Corollary 5.6. *Let $P \in \mathcal{C}_s$ be a closed point. There exists a regular system of parameters $\alpha, \beta \in \mathcal{O}_{\mathcal{C},P}$ such that for any $a \in F^\times$ satisfying $P \notin \overline{[a]}$:*

1. *if P is not a double point of $\mathcal{C}_s$, then there exist $n \in \mathbb{Z}, u \in \mathcal{O}_{\mathcal{C},P}^\times$ such that either $a = u\alpha^n$ or $a = u\beta^n$;*
2. *if P is a double point of $\mathcal{C}_s$, then there exist $m \in \mathbb{Z}, v \in \mathcal{O}_{\mathcal{C},P}^\times$ such that $a = v\alpha^m\beta^m$.*

Proof. As $a \in F = \operatorname{Frac} \mathcal{O}_{\mathcal{C},P}$, there exist $b, c \in \mathcal{O}_{\mathcal{C},P}$ such that $a = b/c$. Moreover, as $\mathcal{O}_{\mathcal{C},P}$ is a regular local ring, it is a unique factorization domain, so without loss of generality, we may assume that b, c have no common prime divisors. Let us show that $P \notin \overline{[b]} \cup \overline{[c]}$.

Assume on the contrary that there exists $z \in [b] \cup [c]$ such that its closure in $\mathcal{C}$ is $\{z, P\}$. As $[a] = [b] - [c]$, and by assumption $z \notin [a]$, we obtain that $z \in [b] \cap [c]$. This implies that $b, c \in m_{C,z}$–the maximal ideal of $\mathcal{O}_{C,z}$. Let $m_{\mathcal{C},P}$ denote the maximal ideal of $\mathcal{O}_{\mathcal{C},P}$.

As z is the generic point of $\{z, P\}$, there exists a canonical embedding $\mathcal{O}_{\mathcal{C},P} \hookrightarrow \mathcal{O}_{C,z}$. As $t \notin m_{C,z}$, $\mathcal{O}_{\mathcal{C},P} \cap m_{C,z}$ is a prime, non-maximal, ideal of the two-dimensional ring $\mathcal{O}_{\mathcal{C},P}$. As such, it is principal; let us denote $\mathcal{O}_{\mathcal{C},P} \cap m_{C,z} =: q\mathcal{O}_{\mathcal{C},P}$. Consequently, $b, c \in q\mathcal{O}_{\mathcal{C},P}$, which is impossible seeing as b and c were assumed to be coprime.

We have shown that $P \notin \overline{[b]} \cup \overline{[c]}$. We can now conclude by Lemma 5.5, seeing as $a = b/c$. □

Let R be a commutative unitary ring, and q a quadratic form defined over R. We recall that q is said to be *R-isotropic* if it has a non-trivial zero over R.

Theorem 5.7. *Suppose* char $\widetilde{k} \neq 2$. *Let q/F be a quadratic form which is isotropic over F_v for all discrete valuations v on F such that $v_{|k}$ is either trivial or induces the norm on k. Then q is isotropic over F_v for all $v \in V(F)$. Furthermore, if* $\dim q \neq 2$, *then q is F-isotropic.*

Proof. By Witt decomposition ([24, I.4.1]), $q = q_t \perp q_r$, where q_r is a regular quadratic form over F and q_t a totally isotropic one. If $q_t \neq 0$, then clearly q is isotropic over F, so we may assume that $q_t = 0$, meaning q is a regular quadratic form. Since char $F \neq 2$, we may also assume that q is a diagonal quadratic form with (non-zero) coefficients $a_1, a_2, \ldots, a_n \in F$. By Remark 2.2(4), q is $\mathscr{M}_x$–isotropic for all rigid and type 2 points $x \in C^{\mathrm{an}}$. By Remark 2.3, it suffices to show that q is $\mathscr{M}_x$–isotropic for all $x \in C^{\mathrm{an}}$.

Let $Z \subseteq C$ be a Zariski closed subset such that $\bigcup_{i=1}^n [a_i] \subseteq Z$. We identify Z with a finite set of rigid points in C^{an} (see [1, Theorem 3.4.1]). For any $z \in Z$, let V_z be a strict affinoid neighborhood of z in C^{an} such that q is $\mathscr{M}(V_z)$–isotropic. Let $\mathscr{C}$ be a proper sncd model of C over k° corresponding to a vertex set S of C^{an} such that $\bigcup_{z \in Z} \partial V_z \subseteq S$ (see Corollary 1.7). We denote by $\mathscr{C}_s$ its special fiber and by $\pi : C^{\mathrm{an}} \to \mathscr{C}_s$ the corresponding specialization morphism. (This is the model $\mathscr{C}_1$ constructed for the smooth projective variety X determined by q; see Construction 1.)

Let $x \in C^{\mathrm{an}}$. If $x \in \bigcup_{z \in Z} V_z \cup S$, then q is isotropic over $\mathscr{M}_x$ by construction (see Remark 2.2(4) and Lemma 2.5). Suppose $x \notin \bigcup_{z \in Z} V_z \cup S$. Set $\pi(x) =: P_x$ and $\pi^{-1}(x) =: U_x$. By the proof of Theorem 3.6, $P_x \notin \pi(Z)$. Let $\alpha, \beta \in \mathcal{O}_{\mathscr{C}, P_x}$ be a regular system of parameters. Then $m_{\mathscr{C}, P_x} = (\alpha, \beta)$. By Lemma 3.3, $Z \cup \pi(Z)$ is the Zariski closure of Z in $\mathscr{C}$. As $P_x \notin Z \cup \pi(Z)$, we obtain that $P_x \notin \bigcup_{i=1}^n \overline{[a_i]}$. Let $a \in \{a_1, a_2, \ldots, a_n\}$. By Corollary 5.6, there exist $n \in \mathbb{Z}$ and $u_a \in \mathcal{O}^\times_{\mathscr{C}, P_x}$ such that $a = u_a \alpha^n$ or $a = u_a \beta^n$ if P_x is *not* a double point of $\mathscr{C}_s$, resp. $a = u_a \alpha^n \beta^n$ if P_x *is* a double point of $\mathscr{C}_s$.

By Theorem 1.2, $\mathcal{O}^\circ(U_x) = \widehat{\mathcal{O}_{\mathscr{C}, P_x}}$. Consequently, the quadratic form q is isomorphic over F to a quadratic form $q' := q_1 \perp \alpha q_2 \perp \beta q_3 \perp \alpha\beta q_4$, where q_1, q_2, q_3, q_4 are diagonal quadratic forms defined over $\mathcal{O}^\times_{\mathscr{C}, P_x} \subseteq \mathcal{O}^\circ(U_x)^\times$. As Frac $\widehat{\mathcal{O}^\circ(U_x)_{(\alpha)}}$ is a complete discretely valued field ([35, Tag 0AFS]) and contains F, the quadratic form q, and hence q', is Frac $\widehat{\mathcal{O}^\circ(U_x)_{(\alpha)}}$–isotropic. Without loss of generality, we may assume that q' is isotropic over the complete discrete valuation ring $\widehat{\mathcal{O}^\circ(U_x)_{(\alpha)}}$. We note here that α is one of its uniformizers.

By a theorem of Springer (see [24, VI, Proposition 1.9]), q' is isotropic over $\widehat{\mathcal{O}^\circ(U_x)_{(\alpha)}}$ if an only if $q_1 \perp \beta q_3$ or $q_2 \perp \beta q_4$ is isotropic over $\mathcal{O}^\circ(U_x)_{(\alpha)}/(\alpha) =$ Frac $\mathcal{O}^\circ(U_x)/(\alpha)$. Without loss of generality, we may assume that $q_1 \perp \beta q_3$ is isotropic over $\mathcal{O}^\circ(U_x)/(\alpha)$. By the same theorem of Springer, as $\mathcal{O}^\circ(U_x)/(\alpha)$ is a discrete valuation ring with uniformizer β ([35, Tag 00NQ]), q_1 or q_3 is isotropic over $(\mathcal{O}^\circ(U_x)/(\alpha))/(\beta) = \mathcal{O}^\circ(U_x)/(\alpha, \beta)$. Without loss of generality, we may assume that q_1 is isotropic over $\mathcal{O}^\circ(U_x)/(\alpha, \beta)$. As q_1 is defined over $\mathcal{O}^\circ(U_x)^\times$, the projective variety determined by it is smooth over $\mathcal{O}^\circ(U_x)$. Hence, by Hensel's Lemma, q_1 is isotropic over $\mathcal{O}^\circ(U_x)$, implying q' is isotropic over $\mathcal{O}^\circ(U_x)$. Consequently, q is isotropic over $\mathscr{M}(U_x)$, and hence over $\mathscr{M}_x$.

Thus, q is $\mathscr{M}_x$–isotropic for all $x \in C^{\mathrm{an}}$. If, in addition, $\dim q \neq 2$, then by [29, Theorem 3.12], q is F-isotropic. □

Remark 5.8. In the proof of Theorem 5.7, if P_x is not a double point, then q' is of the form $q_1 \perp \alpha q_2 \perp \beta q_3$ and, in general, both applications of the Theorem of Springer are necessary.

However, if P_x *is* a double point, then the quadratic form q' is of the kind $q_1 \perp \alpha\beta q_2$, so the first application of the Theorem of Springer is enough to conclude.

6. The case of constant varieties

The techniques and approach presented in this section are different from those of the previous sections. Let C/k be an analytic curve. Set $F = \mathscr{M}(C)$, and let X/F be a variety such that $X(\mathscr{M}_y) \neq \emptyset$ for some fixed point y. For a suitably chosen $x \in C$, we construct an isomorphism φ of the curve C such that $x \mapsto y$, with the purpose of showing that $X(\mathscr{M}_x) \neq \emptyset$. To insure such an implication, as φ does not fix F, we have to assume that X is defined over the smaller field k. See Remark 6.4 for more precise details.

We recall that if K is a complete ultrametric field, for $a \in K$ and $r \in \mathbb{R}_{\geqslant 0}$, the map

$$\eta_{a,r} : K[T] \to \mathbb{R}_{\geqslant 0},$$
$$\sum_n b_n (T-a)^n \mapsto \max_n \{|b_n| r^n\},$$

defines a multiplicative semi-norm on $K[T]$, meaning $\eta_{a,r}$ is a point of $\mathbb{P}^{1,\mathrm{an}}_K$. See Definition 2.2 and Proposition 2.3 of [29] for more details.

The following is a well-known auxiliary result.

Lemma 6.1. *Let K be a complete ultrametric field. For $\alpha \in K$ and $s > 0$, let*

$$D := \{x \in \mathbb{P}^{1,\mathrm{an}}_K : |T - \alpha| < s\}$$

be the open disc centered in α and of radius s. Let $\beta, \gamma \in K$.

(1) If, for some $r > 0$, $\eta_{\beta,r} \in D$, then $\eta_{\beta,0} \in D$.

(2) For any two rigid points $\eta_{\beta,0}, \eta_{\gamma,0} \in D$, $|\beta - \gamma| < s$.

(3) If $\eta_{\beta,0} \in D$ or equivalently, $|\alpha - \beta| < s$, then $D = \{x \in \mathbb{P}^{1,\mathrm{an}}_K : |T - \beta|_x < s\}$.

Proof. (1) If $\eta_{\beta,r} \in D$, then $|T - \alpha|_{\eta_{\beta,r}} = \max(|\alpha - \beta|, r) < s$, so $|T - \alpha|_{\eta_{\beta,0}} = |\beta - \alpha| < s$, implying $\eta_{\beta,0} \in D$.

(2) As $\eta_{\beta,0} \in D$, we have that $|T - \alpha|_{\eta_{\beta,0}} = |\alpha - \beta| < s$. Similarly, for $\eta_{\gamma,0}$ we obtain $|\alpha - \gamma| < s$. Consequently, $|\beta - \gamma| \leqslant \max(|\alpha - \beta|, |\alpha - \gamma|) < s$.

(3) Let $x \in D$, meaning $|T - \alpha|_x < s$. Then $|T - \beta|_x \leqslant \max(|T - \alpha|_x, |\alpha - \beta|) < s$. It is shown similarly that if $y \in \mathbb{P}^{1,\mathrm{an}}_K$ satisfies $|T - \beta|_y < s$, then $|T - \alpha|_y < s$. □

Throughout this section we will use the following:

Notation 6.2. (1) Let k be a complete non-trivially valued ultrametric field.

(2) We denote by $\overline{k}$ an algebraic closure of k, and by $\widehat{\overline{k}}$ its completion with respect to the unique norm extending that of k.

(3) For a k-analytic space Y and any *valued field extension* l/k (meaning l is a complete valued field with a norm extending that of k), we denote by Y_l the l-analytic space $Y \times_k l$.

(4) We denote by $\mathscr{M}_Y$ (or simply $\mathscr{M}$ when there is no risk of ambiguity) the sheaf of meromorphic functions on the k-analytic space Y (see [28, 1.7] for the definition and some details on $\mathscr{M}$).

Remark 6.3. By [10, 3.4.24], for any $b \in \widehat{\overline{k}}$, there exists a canonical retraction $d_b : \mathbb{P}^{1,\mathrm{an}}_{\widehat{\overline{k}}} \to \Gamma_b$ such that $|T - b|_u = |T - b|_{d_b(u)}$ for any $u \in \mathbb{P}^{1,\mathrm{an}}_{\widehat{\overline{k}}}$. Moreover, for any $u, u' \in \Gamma_b$, $u \neq u'$,

by *loc.cit.*, $|T-b|_u \neq |T-b|_{u'}$. Here $\Gamma_b := [\eta_{b,0}, \infty]$ is the unique injective path in $\mathbb{P}^{1,\mathrm{an}}_{\widehat{\overline{k}}}$ connecting $\eta_{b,0}$ and ∞.

In other words, $T-b$ is strictly increasing on Γ_b and locally constant on $\mathbb{P}^{1,\mathrm{an}}_{\widehat{\overline{k}}} \backslash \Gamma_b$.

In Definition 1.4, we briefly recall the notions of *virtual discs* and *virtual annuli*.

Remark 6.4. Let X/k be a variety. Let L/k be an open virtual disc or open virtual annulus that can be embedded in $\mathbb{P}^{1,\mathrm{an}}_k$. Let $y \in L$ be such that $X(\mathscr{M}_y) \neq \emptyset$. In what follows, we construct isomorphisms of L which send a random point x to a point like y, with the purpose of then obtaining that $X(\mathscr{M}_x) \neq \emptyset$.

Proposition 6.5. *Let L/k be an open virtual disc that is embedded in $\mathbb{P}^{1,\mathrm{an}}_k$. Let X/k be a variety. Assume that there exists an open neighborhood U in L of the end ω of L such that $X(\mathscr{M}(U)) \neq \emptyset$. Then for any $m \in \mathbb{N}$, there exists a finite field extension l/k such that $([l:k], m) = 1$ and $X(\mathscr{M}_{L_l,x}) \neq \emptyset$ for all $x \in L_l$.*

If $|k^\times|$ is dense in $\mathbb{R}_{>0}$, then one can take $l := k$.

Proof. Let us fix the positive integer m. Let $p : \mathbb{P}^{1,\mathrm{an}}_{\widehat{\overline{k}}} \to \mathbb{P}^{1,\mathrm{an}}_k$ denote the projection morphism. By [1, Corollary 1.3.6], the Galois group $G := \mathrm{Gal}(\overline{k}/k)$ acts on $\mathbb{P}^{1,\mathrm{an}}_{\widehat{\overline{k}}}$ in such a way that p induces an isomorphism $\mathbb{P}^{1,\mathrm{an}}_{\widehat{\overline{k}}}/G \cong \mathbb{P}^{1,\mathrm{an}}_k$. As L is an open virtual disc embedded in $\mathbb{P}^{1,\mathrm{an}}_k$, the preimage $p^{-1}(L)$ is a finite disjoint union $\bigsqcup_{i\in I} D_i$ of open discs D_i. By *loc.cit.*, the restriction of p to $L_{\widehat{\overline{k}}} = p^{-1}(L)$, which we will continue to denote by p, induces an isomorphism $L_{\widehat{\overline{k}}}/G \cong L$.

For any $i \in I$, let $\alpha_i \in \widehat{\overline{k}}$ and $s_i \in \mathbb{R}_{>0}$ be such that $D_i = \{x \in \mathbb{P}^{1,\mathrm{an}}_{\widehat{\overline{k}}} : |T-\alpha_i|_x < s_i\}$. We remark that the action of G on $\bigsqcup_{i\in I} D_i$ (which permutes the D_i) implies that for any $i', i'' \in I$, $s_{i'} = s_{i''} =: s$. We remark also that $p^{-1}(\omega)$ consists of the unique boundary points ω_i of D_i, and that $\omega_i = \eta_{\alpha_i,s}$ for all $i \in I$.

As U is an open neighborhood of ω, $p^{-1}(U)$ is a disjoint union of open neighborhoods U_i of ω_i in D_i, $i \in I$. As such, $\partial_{D_i} U_i$ is a finite set of points of type 2 or 3 of D_i. By [29, Proposition 2.3], for any $z \in \partial_{D_i} U_i$, there exists $\alpha_z \in \widehat{\overline{k}}$ and $r_z > 0$ such that $z = \eta_{\alpha_z, r_z}$.

For $z \in \partial_{U_i} D_i$, let γ_z denote the unique path in D_i connecting the point $\eta_{\alpha_z,0}$ to ω_i, meaning $\gamma_z = \{\eta_{\alpha_z,c} \in D_i : c > 0\}$ (see [28, Remark 1.8.26]). We recall that it is homeomorphic to the open interval $(0, s)$. As U_i is connected, $\gamma_z \cap U_i$ is a connected subset of γ_z. As U_i is an open neighborhood of ω_i, we obtain that $\gamma_z \cap U_i = \{\eta_{\alpha_z,c} \in D_i : c > r_z\} \subseteq U_i$, and for any $t \leqslant r_z, \eta_{\alpha_z,t} \notin U_i$.

Lemma 6.6. *Let $i \in I$. For any two points $z, z' \in \partial_{D_i} U_i$, $r_z < |\alpha_z - \alpha_{z'}|$.*

Proof. Suppose on the contrary that $r_z \geqslant |\alpha_z - \alpha_{z'}|$. Then $\eta_{\alpha_z,r_z} = \eta_{\alpha_{z'},r_z}$. As $z \neq z'$, $r_z \neq r_{z'}$. If $r_z < r_{z'}$, then $\eta_{\alpha_{z'},r_{z'}} = \eta_{\alpha_z,r_{z'}}$, which implies that $\eta_{\alpha_{z'},r_{z'}} \in \gamma_z \cap U_i \subseteq U_i$, contradiction. Similarly, if $r_{z'} < r_z$, then $\eta_{\alpha_z,r_z} = \eta_{\alpha_{z'},r_z} \in \gamma_{z'} \cap U_i \subseteq U_i$, contradiction. Hence, $r_z < |\alpha_z - \alpha_{z'}|$. □

By Lemma 6.1, parts (1) and (2), for any $z, z' \in \partial_{D_i} U_i$, $|\alpha_z - \alpha_{z'}| < s$. Let $a, b \in \mathbb{R}_{>0}$ be such that $\max_{i\in I} \max_{z,z'\in\partial_{D_i}U_i}(|\alpha_z - \alpha_{z'}|) < a < b < s$.

Lemma 6.7. *There exists a finite field extension l/k such that $([l:k], m) = 1$ and $\exists w \in l$ with $|w| := r \in (a, b)$ (with respect to the unique norm on l extending that of k).*

If $|k^\times|$ is dense in $\mathbb{R}_{>0}$, then one can take $l = k$.

Proof. The statement is immediate if $|k^\times|$ is dense in $\mathbb{R}_{>0}$. Let us assume this is not the case. Then k is a discretely valued field, and let us denote by π a uniformizer.

Seeing as the divisible closure $\sqrt{|k^\times|}$ of the value group $|k^\times|$ is dense in $\mathbb{R}_{>0}$, there exists a large enough integer h such that $(h, m) = 1$ and for which $(a, b) \cap \sqrt[h]{|k^\times|}$. Here $\sqrt[h]{|k^\times|} := \{r \in \mathbb{R}_{>0} : r^h \in |k^\times|\}$. Let $r \in (a, b) \cap \sqrt[h]{|k^\times|}$, meaning there exists $n \in \mathbb{Z}$ such that $r^h = |\pi|^n$.

Set $P(X) := X^h - \pi \in k[X]$. By Eisenstein's criterion of irreducibility, $P(X)$ is an irreducible polynomial over k. Set $l := k[X]/(P(X))$. Then $[l : k] = h$. Clearly, l contains a root α of $P(X)$, implying that $|\alpha| = |\pi|^{1/h}$. As a consequence, $r \in |l^\times|$, meaning there exists $w \in l$ such that $|w| = r \in (a, b)$. □

From now on, let l/k, $w \in l$ and r be as in Lemma 6.7.

Lemma 6.8. *Let $i \in I$. For any $z \in \partial_{D_i} U_i$, $\{y \in D_i : |T - \alpha_z + w|_y < r\} \subseteq U_i$.*

Proof. Suppose there exists $y \in D_i$ such that $|T - \alpha_z + w|_y < r$, but $y \notin U_i$. We remark that $|T - \alpha_z|_y = \max(|T - \alpha_z + w|_y, |w|) = r$, implying $d_{\alpha_z}(y) = \eta_{\alpha_z, r}$ (see Remark 6.3). If $y \notin U_i$, then the unique injective path $[y, \eta_{\alpha_z,r})$ in D_i connecting y and $\eta_{\alpha_z,r}$ (without containing $\eta_{\alpha_z,r}$) intersects $\partial_{D_i} U_i$ at a single point $\eta_{\alpha_{z'}, r_{z'}}$. As $d_{\alpha_z}(y) = d_{\alpha_z}(\eta_{\alpha_z,r}) = \eta_{\alpha_z,r}$ and $\eta_{\alpha_{z'},r_{z'}} \in [y, \eta_{\alpha_z,r})$, by Remark 6.3, $d_{\alpha_z}(\eta_{\alpha_{z'},r_{z'}}) = \eta_{\alpha_z,r}$. Consequently, $|T - \alpha_z|_{\eta_{\alpha_{z'},r_{z'}}} = r$. At the same time, $|T - \alpha_z|_{\eta_{\alpha_{z'},r_{z'}}} = \max(|\alpha_z - \alpha_{z'}|, r_{z'})$, so by Lemma 6.6, $|T - \alpha_z|_{\eta_{\alpha_{z'},r_{z'}}} = |\alpha_z - \alpha_{z'}|$. Thus, $r = |\alpha_z - \alpha_{z'}|$, which is in contradiction with the choice of r. □

A base change of the isomorphism $l[T] \to l[T], T \mapsto T + w$, induces an isomorphism $\psi : \mathbb{P}^{1,\mathrm{an}}_{\widehat{\overline{k}}} \to \mathbb{P}^{1,\mathrm{an}}_{\widehat{\overline{k}}}$ which sends $\eta_{\gamma,c}$ to $\eta_{\gamma+w,c}$ for any $\gamma \in \widehat{\overline{k}}$ and any $c \geqslant 0$. Since for any $i \in I$, $|T - \alpha_i|_x < s$ if and only if $|T - \alpha_i + w|_{\psi(x)} < s$, seeing as $|\alpha_i - (\alpha_i - w)| = |w| < s$, by Lemma 6.1(3), $\psi(D_i) = D_i$. This means that ψ (and hence the map $T \mapsto T + w$) induces an isomorphism $D_i \to D_i$ for $i \in I$, meaning an isomorphism $L_{\widehat{\overline{k}}} \to L_{\widehat{\overline{k}}}$, which we will continue to denote by ψ.

Let $x \in L \backslash U$. Let $x' \in p^{-1}(x)$. There exists $j \in I$ such that $x' \in D_j$. Moreover, seeing as $x \notin U$, we have $x' \notin U_j$.

Lemma 6.9. *The point $\psi(x')$ is in U_j.*

Proof. Let $[x', \omega_j) \subseteq D_j$ be the unique injective path connecting x' to ω_j. As $x' \notin U_j$, $[x', \omega_j)$ intersects $\partial_{D_j} U_j$ at a single point η_{α_z,r_z}. We remark that $\eta_{\alpha_z,r_z} \in [x', \omega_j) = [x', \eta_{\alpha_z,r_z}] \cup (\eta_{\alpha_z,r_z}, \omega_j)$.

Set $v := |T - \alpha_z|_{x'}$. By Lemma 6.1(3), $D_j = \{x \in \mathbb{P}^{1,\mathrm{an}}_{\widehat{\overline{k}}} : |T - \alpha_z|_x < s\}$. Its boundary point ω_j coincides with $\eta_{\alpha_z,s}$. Assume $v > r_z$. Then $\eta_{\alpha_z,v} \in (\eta_{\alpha_z,r_z}, \omega_j) \subseteq [x', \omega_j)$. This implies that $\eta_{\alpha_z,r_z} \in [x', \eta_{\alpha_z,v}) \subseteq [x'\omega_j)$.

By Remark 6.3, $d_{\alpha_z}(x') = \eta_{\alpha_z,v}$, so $[x', \eta_{\alpha_z,v}) \cap \Gamma_{\alpha_z} = \emptyset$, where $[x', \eta_{\alpha_z,v}) \subseteq [x', \omega_j)$. This is a contradiction, seeing as $\eta_{\alpha_z,r_z} \in [x', \eta_{\alpha_z,v}) \cap \Gamma_{\alpha_z}$.

Consequently, $v \leqslant r_z < r$. Since $|T - \alpha_z + w|_{\psi(x')} = |T - \alpha_z|_{x'} = v < r$, by Lemma 6.8, $\psi(x') \in U_j$. □

Let us denote by π_l the projection $\pi_l : L_l \to L$. Let us denote by $G' = \mathrm{Gal}(\overline{k}/l)$. We remark that G' is a subgroup of G. Seeing as the isomorphism $\psi : \mathbb{P}^{1,\mathrm{an}}_{\widehat{\overline{k}}} \to \mathbb{P}^{1,\mathrm{an}}_{\widehat{\overline{k}}}$

is defined over l, it is G'-equivariant, meaning it induces a G'-equivariant isomorphism $L_{\widehat{\overline{k}}} \to L_{\widehat{\overline{k}}}$. Consequently, we obtain an isomorphism $\varphi : L_{\widehat{\overline{k}}}/G' \to L_{\widehat{\overline{k}}}/G'$. By [1, Corollary 1.3.6], $L_{\widehat{\overline{k}}}/G' \cong L_l$, meaning we have constructed an isomorphism $\varphi : L_l \to L_l$, whose base change $l \subseteq \widehat{\overline{k}}$ induces ψ. Hence, if we denote by q the projection $L_{\widehat{\overline{k}}} \to L_l$, then the diagram (C) is commutative.

$$\text{(C)} \qquad \begin{tikzcd} L_{\widehat{\overline{k}}} \arrow[r, "\psi"] \arrow[d, "q"'] & L_{\widehat{\overline{k}}} \arrow[d, "q"] \\ L_l \arrow[r, "\varphi"'] & L_l \end{tikzcd}$$

Let $x \in L_l \backslash U_l$. Then for any $y \in q^{-1}(x)$, $y \notin q^{-1}(U_l) = \bigcup_{i \in I} U_i$. Let $j \in I$ and $y \in q^{-1}(x)$ such that $y \in D_j$. By Lemma 6.9, $\psi(y) \in U_j$, implying $q(\psi(y)) \in U_l$. As $q(\psi(y)) = \varphi(q(y)) = \varphi(x)$, we obtain that $\varphi(x) \in U_l$. The isomorphism φ induces an isomorphism of fields $\mathscr{M}_{L_l,x} \cong \mathscr{M}_{L_l,\varphi(x)}$ (which fixes l but not F, which is why we ask of X to be defined over the base field k rather than the function field F). As $\varphi(x) \in U_l$, $\pi_l(\varphi(x))) \in U$, so $\mathscr{M}_L(U) \subseteq \mathscr{M}_{L,\pi_l(\varphi(x))} \subseteq \mathscr{M}_{L_l,\varphi(x)}$. By assumption, $X(\mathscr{M}_L(U)) \neq \emptyset$, so $X(\mathscr{M}_{L_l,\varphi(x)}) \neq \emptyset$. As $\mathscr{M}_{L_l,\varphi(x)} \cong \mathscr{M}_{L_l,x}$, we obtain $X(\mathscr{M}_{L_l,x}) \neq \emptyset$. We have thus shown that for any $x \notin U_l, X(\mathscr{M}_{L_l,x}) \neq \emptyset$.

If $x \in U_l$, then $\mathscr{M}_L(U) \subseteq \mathscr{M}_{L,\pi_l(x)} \subseteq \mathscr{M}_{L_l,x}$, implying $X(\mathscr{M}_{L_l,x}) \neq \emptyset$, thus concluding the proof. □

Proposition 6.10. *Let L/k be an open virtual annulus that is embedded in $\mathbb{P}^{1,\mathrm{an}}_k$. Let us denote its ends by ω_1 and ω_2. Let X/k be a variety. Assume that there exists an open neighborhood U in L of the end ω_1 of L such that $X(\mathscr{M}(U)) \neq \emptyset$. Then for any $m \in \mathbb{N}$, there exists a finite field extension l/k such that $([l:k], m) = 1$ and $X(\mathscr{M}_{L_l,x}) \neq \emptyset$ for all $x \in L_l$.*

If $|k^\times|$ is dense in $\mathbb{R}_{>0}$, then one can take $l := k$.

Proof. Let $p : \mathbb{P}^{1,\mathrm{an}}_{\widehat{\overline{k}}} \to \mathbb{P}^{1,\mathrm{an}}_k$ denote the projection corresponding to the base change $\widehat{\overline{k}}/k$. Then $L_{\widehat{\overline{k}}} = p^{-1}(L)$ is a finite disjoint union $\bigsqcup_{i \in I} L_i$ of open annuli L_i. For any $i \in I$, there exist $\alpha_i \in \widehat{\overline{k}}$ and $r_i, s_i \in \mathbb{R}_{>0}$ such that $L_i = \{x \in \mathbb{P}^{1,\mathrm{an}}_{\widehat{\overline{k}}} : r_i < |T - \alpha_i|_x < s_i\}$. As the action of $G := \mathrm{Gal}(\overline{k}/k)$ on $\bigsqcup_{i \in I} L_i$ permutes the L_i, for any $i', i'' \in I$, $r_{i'} = r_{i''} =: r$ and $s_{i'} = s_{i''} =: s$. We remark that the ends of L_i are the points $\omega_{1,i} := \eta_{\alpha_i,s}$ and $\omega_{2,i} := \eta_{\alpha_i,r}$. The action of G on $L_{\widehat{\overline{k}}}$ permutes the sets $\{\omega_{1,i}\}_{i \in I}$ and $\{\omega_{2,i}\}_{i \in I}$. Consequently, $p(\{\omega_{1,i}\}_{i \in I})$ is a single point which is also an end of L; let us assume it is the point ω_1 (we can always reduce to this case by a change of coordinate on $\mathbb{P}^{1,\mathrm{an}}_{\widehat{\overline{k}}}$ if necessary). Similarly, $p(\{\omega_{2,i}\}_{i \in I}) = \omega_2$.

For any $i \in I$, let $D_i := \{x \in \mathbb{P}^{1,\mathrm{an}}_{\widehat{\overline{k}}} : |T - \alpha_i|_x < s\}$. This is an open disc satisfying $L_i \subseteq D_i$. We will now show that for any $i', i'' \in I$, if $i' \neq i''$, then $D_{i'} \cap D_{i''} = \emptyset$. Otherwise, as $D_{i'} \cap D_{i''}$ is open, it contains a rigid point $\eta_{\gamma,0}$ for some $\gamma \in \overline{k}$. Consequently, $|\alpha_{i'} - \gamma| < s$ and $|\alpha_{i''} - \gamma| < s$, implying $|\alpha_{i'} - \alpha_{i''}| < s$. But then, for any $x \in D_{i'}$, $|T - \alpha_{i''}|_x \leqslant \max(|\alpha_{i'} - \alpha_{i''}|, |T - \alpha_{i'}|_x) < s$, implying $x \in D_{i''}$, hence $D_{i'} \subseteq D_{i''}$. Similarly, $D_{i''} \subseteq D_{i'}$, so $D_{i'} = D_{i''}$. But then $L_{i'} \cap L_{i''} \neq \emptyset$, contradiction.

Moreover, G acts on $\bigsqcup_{i\in I} D_i$, meaning $D := p(D_i) = p(\bigcup_{i\in I} D_i)$ is an open virtual disc defined over k and with end ω_1. By construction, $L \subseteq D$. Let us fix the integer m. By Proposition 6.5, there exists a field extension l/k such that $([l : k], m) = 1$ and for any $x \in D_l := D \times_k l$, we have $X(\mathscr{M}_{D_l,x}) \neq \emptyset$. Since $L_l \subseteq D_l$, the proof is concluded. □

Remark 6.11. We recall that if S is a triangulation of a k-analytic curve C, then the connected components of $C\backslash S$ are open virtual discs and open virtual annuli. There is, *a priori*, no embedding of an open virtual disc or an open virtual annulus into the projective analytic line.

We recall the notion of *strongly transitive action* in Definition 1.8 (see also Remark 1.9).

Theorem 6.12. *Let C/k be a proper normal geometrically connected and generically smooth analytic curve. Set $F := \mathscr{M}(C)$. Assume C has a triangulation S such that all of the connected components of $C\backslash S$ can be embedded in $\mathbb{P}^{1,\mathrm{an}}_k$. Let X be a k-variety. Suppose there exists a rational linear algebraic group G/F acting strongly transitively on the variety X_F. If $X(\mathscr{M}_s) \neq \emptyset$ for all $s \in S$, then*

1. *if $|k^\times|$ is dense in $\mathbb{R}_{>0}$, then $X(F) \neq \emptyset$;*
2. *if k is discretely valued, then X has a zero cycle of degree one over F.*

Proof. Let $s \in S$. As $X(\mathscr{M}_s) \neq \emptyset$, there exists a neighborhood U_s of s in C such that $X(\mathscr{M}(U_s)) \neq \emptyset$. Without loss of generality, we may assume that the boundary of U_s is a finite set of points (which are always of type 2 and 3). At the same time, $C\backslash S$ is a disjoint union of open virtual discs and open virtual annuli. Since $\bigcup_{s\in S} \partial U_s$ is finite, there are only finitely many connected components of $C\backslash S$ not entirely contained in $\bigcup_{s\in S} U_s$ (see also Remark 2.18). Let us denote them by $L_1, L_2, \ldots, L_n$.

(1) If $|k^\times|$ is dense in $\mathbb{R}_{>0}$, then by Propositions 6.5 and 6.10, $X(\mathscr{M}_{C,x}) \neq \emptyset$ for all $x \in C$, so by [29, Theorem 3.11], $X(F) \neq \emptyset$.

(2) By *loc.cit.*, there exists a finite field extension l_1/k for which $X(\mathscr{M}_{L_{1,l_1},x}) \neq \emptyset$ for all $x \in L_{1,l_1}$. Set $[l_1 : k] = m_1$. For $i \in \{2, 3, \ldots, n\}$, let l_i/k be a finite field extension such that $([l_i : k], \Pi_{j=1}^{i-1} m_j) = 1$, where $m_{i-1} := [l_{i-1} : k]$, and $X(\mathscr{M}_{L_{i,l_i},x}) \neq \emptyset$ for all $x \in L_{i,l_i}$.

Let l/k be the composite of $l_1, l_2, \ldots, l_n$ in $\overline{k}$. By construction, $m := [l : k] = \Pi_{i=1}^n [l_i : k] = m_1 m_2 \cdots m_n$. Set $C_l := C \times_k l$. Then $\mathscr{M}(C_l) = F \otimes_k l =: E$. As C is geometrically connected, $[E : F] = [l : k] = m$. Let us denote by p the projection $C_l \to C$. By assumption, for any $x \in C\backslash \bigcup_{i=1}^n L_i$, $X(\mathscr{M}_{C,x}) \neq \emptyset$. Hence, for any $x \in C_l\backslash \bigcup_{i=1}^n p^{-1}(L_i)$, we obtain that $X(\mathscr{M}_{C_l,x}) \neq \emptyset$. On the other hand, for any $x \in \bigcup_{i=1}^n L_i$, by construction, as for any $i \in \{1, 2, \ldots, n\}$, $\mathscr{M}_{C_{l_i},x} \subseteq \mathscr{M}_{C_l,x}$, we obtain that $X(\mathscr{M}_{C_l}, x) \neq \emptyset$. Hence, $X(\mathscr{M}_{C_l,x}) \neq \emptyset$ for all $x \in C_l$, implying by [29, Theorem 3.11] that $X(\mathscr{M}(C_l)) \neq \emptyset$.

By Propositions 6.5 and 6.10, there exists a finite field extension l'_1/k with $([l'_1, k], m) = 1$, and $X(\mathscr{M}_{L_{1,l'_1},x}) \neq \emptyset$ for all $x \in L_{1,l'_1}$. Set $m'_1 = [l'_1 : k]$. For $i \in \{2, 3, \ldots, n\}$, let l'_i/k be a finite field extension such that $([l'_i : k], m\Pi_{j=1}^{i-1} m'_j) = 1$, where $m'_{i-1} := [l'_{i-1} : k]$, and $X(\mathscr{M}_{L_{i,l'_i},x}) \neq \emptyset$ for all $x \in L_{i,l'_i}$.

Let l'/k be the composite of $l'_1, l'_2, \ldots, l'_n$ in $\overline{k}$. Set $m' := [l' : k]$. Then $(m', m) = 1$ and as in the case of l, $X(\mathscr{M}_{C_{l'},x}) \neq \emptyset$ for all $x \in C_{l'}$. By [29, Theorem 3.11], the latter implies that $X(\mathscr{M}(C_{l'})) \neq \emptyset$. Moreover, $E' := \mathscr{M}_{C_{l'}}(C_{l'})$ satisfies $[E' : F] = m'$.

Thus, $X(E) \neq \emptyset$ and $X(E') \neq \emptyset$, where $([E : F], [E' : F]) = 1$, so X has a zero cycle of degree one over F. □

Remark 6.13. In the proof of Theorem 6.12, we only used the fact that a *finite* number of the connected components of $C\backslash S$ can be embedded in $\mathbb{P}_k^{1,\mathrm{an}}$. They depend on the elements of the set $X(\mathscr{M}_s), s \in S$.

Remark 6.14. The case $C = \mathbb{P}_k^{1,\mathrm{an}}$ satisfies trivially the assumptions of Theorem 6.12 with respect to any triangulation or vertex set. By [11, 4.4], if S is a vertex set of C corresponding to a model $\mathscr{C}$ of C^{al} over k° that is *semi-stable*, then the assumptions of Theorem 6.12 are satisfied. Another example is given by Mumford curves, which can be locally embedded in $\mathbb{A}_k^{1,\mathrm{an}}$.

Remark 6.15. There are several families of varieties for which the existence of zero cycles of degree one implies the existence of a rational point (*e.g.* see [31]). In particular, by a theorem of Springer ([24, VI, Proposition 1.9]), this is true for quadratic forms (the case of residue characteristic 2 included by [13]).

Corollary 6.16. *Using the same notation as in Theorem 6.12, let q be a quadratic form defined over k. If q is isotropic over $\mathscr{M}_s$ for all $s \in S$, then q is isotropic over F.*

Remark 6.17. By Remark 2.2, in the statement of Theorem 6.12, $X(\mathscr{M}_s) \neq \emptyset$ is equivalent to $X(F_v) \neq \emptyset$, where F_v is the completion of F with respect to the discrete valuation v corresponding to s (which extends the norm on k).

7. Appendix: other examples

Remark 7.1. In the case of quadratic forms in Section 5, we used a theorem of Springer ([24, VI, Proposition 1.9]) to reduce to a case where we can apply the results of Section 2, and thus obtained Theorem 5.7–a Hasse principle for quadrics. Essentially, Springer's theorem allows us to reduce to quadrics which satisfy some strong smoothness assumptions over models of the curve. The latter are precisely the assumptions we encounter in Section 2.

By the same principle, the results of Section 2 should apply to any variety for which we can show a Springer-type theorem.

7.1. Unitary groups (by [36] and [33, Sect. 12]).

In [25], Larmour showed a Springer-type theorem for Hermitian forms. By Remark 7.1, it is expected that said theorem, in combination with the results of Section 2, gives rise to a Hasse principle for homogeneous varieties under unitary groups. To obtain this, we want to mimic the proof of Theorem 5.7. We recall that Hermitian forms are defined over division algebras, or, more generally, central simple algebras. Hence, a natural starting point is studying whether valuations on a field K (for us $K = \mathrm{Frac}\ \mathcal{O}^\circ(U_x)$, see the proof of Theorem 5.7) extend "well" to a division algebra over K.

This has already been studied in [36] under some restrictions. The results were then generalized using the same approach in [33]. In both cases, the authors show a Hasse principle for homogeneous varieties under unitary groups. We give here a brief summary of the interpretation of these results in our setting, without any claim to originality, starting with some background information. In a recent manuscript (see [15]), the authors generalize the main theorem (Theorem 7.10) in certain special cases.

For a detailed account on involutions on central simple algebras, see [23].

We will use Notation 3.1 throughout this section. Assume, moreover, that char $k \neq 2$.

7.1.1. *Homogeneous spaces under unitary groups.* Let $L := F(\sqrt{\lambda})$ be a quadratic field extension over F. Let A be a central simple algebra over L. Let σ be an involution of the second kind on A such that $L^\sigma = F$. Let (V, h) be a Hermitian form over (A, σ). We will denote by $G := U(A, \sigma, h)$ the associated unitary group (which is defined over F). By [5, Prop. 2.4], G is a rational reductive linear algebraic group.

We will say that h is *isotropic over* F if there exists $v \in V\backslash\{0\}$ such that $h(v, v) = 0$.

Definition 7.2. (1) The *degree* of A, denoted $\deg A$, is $\sqrt{\dim_L A}$. The *index* of A, denoted $\operatorname{ind} A$, is the degree of the division algebra Brauer equivalent to A. Both $\deg A$ and $\operatorname{ind} A$ are integers.

(2) Let W be a finitely generated A-module. The *reduced dimension* $\mathrm{rdim_A(W)}$ of W over A is defined to be $\dim_L W/\deg A$. By [23, pg. 6], $\mathrm{rdim_A(W)} \in \mathbb{N}$.

(3) Let W be an A-submodule of V. Then W is *totally isotropic* if for all $x \in W$, one has $h(x, x) = 0$.

(4) Let $0 < n_1 < n_2 \cdots < n_r < \deg A$ be integers. Let $X_h(n_1, n_2, \ldots, n_r)$ be the projective F-variety such that for any field extension K/F,

$$X_h(n_1, n_2, \ldots, n_r)(K) = \{(W_1, W_2, \ldots, W_r) :$$
$$0 \subsetneq W_1 \subsetneq W_2 \cdots \subsetneq W_r \subseteq V_K, W_i \text{ is totally isotropic, } \mathrm{rdim}_{A_K} W_i = n_i \ \forall i\},$$

where $A_K := A \otimes_F K, V_K := V \otimes_F K$.

The varieties $X_h(n_1, n_2, \ldots, n_r)$ are precisely the homogeneous spaces under G. Moreover, for a field extension K/F, there exists a simplified criterion for checking that $X_h(n_1, n_2, \ldots, n_r)(K) \neq \emptyset$.

Theorem 7.3 ([36, §2], [33, Thm. 12.1])**.** *(1) For any projective homogeneous F-variety X under G there exists an increasing sequence of integers $0 < n_1 < n_2 \cdots n_r \leqslant \deg A/2$ such that $X \cong X_h(n_1, n_2, \ldots, n_r)$.*

(2) Let K/F be a field extension. Then $X_h(n_1, n_2, \ldots, n_r)(K) \neq \emptyset$ if and only if the two following conditions are satisfied:

(A) *$X_h(n_r)(K) \neq \emptyset$,*
(B) *$\operatorname{ind} A$ divides n_i for all $i \in \{1, 2, \ldots, r\}$.*

(3) Let D be the division algebra Brauer equivalent to A. Then σ induces uniquely on D an involution of second kind τ; the Hermitian form (V, h) induces uniquely a Hermitian form (V', h') on (D, τ).

(4) There exists a bijection $X \mapsto X_0$ between the projective homogeneous F-varieties under $U(A, \sigma, h)$ and the projective homogeneous F-varieties under $U(D, \tau, h')$. Moreover, for any field extension K/F, $X(K) \neq \emptyset \iff X_0(K) \neq \emptyset$.

Remark 7.4. Similar results exist if σ is an involution of the first kind and $G = SU(A, \sigma, h)$- the special orthogonal group (see [36, Sect. 2]). See also Subsection 7.2.

Hypothesis 2. From now on, we assume that the base field k is a local field.

Proposition 7.5. *Let $X := X_h(n_1, n_2, \ldots, n_r)$ be a homogeneous space under $G = U(A, \sigma, h)$. Assume $X(F_v) \neq \emptyset$ for all $v \in V(F)$ discrete. Then* $\operatorname{ind} A$ *divides n_i for all $i \in \{1, 2, \ldots, r\}$.*

Proof. By Theorem 7.3 (2), for all $v \in V(F)$ discrete, $\operatorname{ind}(A \otimes_F F_v)$ divides n_i for all i. By [16, Thm. 5.5] and [32, Prop. 5.10] (see also first paragraph of proof for [33, Thm. 12.2]), then $\operatorname{ind} A$ also divides n_i for all i. □

7.1.2. *Maximal orders on division algebras (by* [33, Sect. 10]*).* Let R be a complete regular local ring of dimension 2 (in the setting of Section 2, the rings $\mathcal{O}^\circ(U_x)$ will play the role of R). Set $K := \mathrm{Frac}(R)$. Let us denote by $\widetilde{K}$ the residue field of R. Assume that $\widetilde{K}$ is a finite field and that char $\widetilde{K} \neq 2$. Let ω, δ be a regular system of parameters of R. We remark that $R_{(\omega)}$ and $R_{(\delta)}$ are discrete valuation rings. We will denote by $\widehat{R_{(\omega)}}$, resp. $\widehat{R_{(\delta)}}$, their completions.

Let $\lambda \in R$ be such that either $\lambda = u$ or $\lambda = u\omega$ for $u \in R^\times$. Set $L := K(\sqrt{\lambda})$. Let S denote the integral closure of R in L. Then S is also a regular local ring of dimension 2. Moreover, a regular system of parameters for S is given by ω_1, δ, where $\omega_1 = \omega$ if $\lambda = u$ and $\omega_1 = \sqrt{u\omega}$ otherwise.

Let D be a division algebra defined over L. Let τ be an involution on L of the second kind such that $L^\tau = K$. Assume D is unramified (see [33, pg. 4] for a definition) on S except possibly on (ω_1) and (δ).

Theorem 7.6 ([36, Lemma 3.7], [33, Lemma 9.3])**.** *Assume that* $(\deg D, \mathrm{char}\ \widetilde{K}) = 1$. *There exists an R-maximal order Λ in D such that $\tau(\Lambda) = \Lambda$, and*

(1) *there exist ω_D, $\delta_D \in \Lambda$ such that $\tau(\omega_D) = \omega$ and $\tau(\delta_D) = \delta$;*
(2) *if $a \in \Lambda$ is such that $\tau(a) = a$ and $Nrd_D(a) = u\omega^r\delta^s$ for $u \in R^\times, r, s \in \mathbb{Z}_{\geqslant 0}$, then, up to a square factor in Λ, $a = u\omega_D^{\epsilon_1}\delta_D^{\epsilon_2}$, where $\{\epsilon_1, \epsilon_2\} \in \{0, 1\}$. Here $\epsilon_1 \equiv r \mod 2$ and $\epsilon_2 \equiv s \mod 2$.*

We recall that the *reduced norm* on D is a multiplicative map $Nrd_D : D \to K$ (see [23, pg. 5]).

Remark 7.7. (1) Remark that if $h =< a_1, a_2, \ldots, a_n >$ is a Hermitian form over (D, τ), then $\tau(a_i) = a_i$ for all $i \in \{1, 2, \ldots, n\}$.

(2) If h is such a Hermitian form, then by Theorem 7.6, $h \cong h_1 \perp \omega_D h_2 \perp \omega_D h_3 \perp \omega_D \delta_D h_4$, where $h_j =< b_1^j, \cdots, b_{m_j}^j >$ with $b_s^j \in \Lambda^\times$ for all $s \in \{1, 2, \ldots, m_j\}$ and all $j \in \{1, 2, 3, 4\}$.

The following is a consequence of the Springer-type theorem of Larmour ([25]) for Hermitian forms.

Theorem 7.8 ([33, Cor. 9.5])**.** *Let h be a Hermitian form over (D, τ). If h is isotropic over* $\mathrm{Frac}(\widehat{R_{(\omega)}})$ *or* $\mathrm{Frac}(\widehat{R_{(\delta)}})$*, then it is isotropic over K.*

Remark 7.9. Thanks to Theorem 7.3(2), we translate the existence of rational points on $X := X_h(n_1, n_2, \ldots, n_r)$ to questions of isotropy of the Hermitian form h. By Remark 7.7(1) and Theorem 7.8 (compare also with the proof of Theorem 5.7), we can reduce to particular Hermitian forms h (*i.e.* ones of the type $< a_1, \ldots, a_n >$, where $a_i \in \Lambda^\times$). Then the corresponding variety X will satisfy the smoothness hypotheses of Section 2 (see also the proof of Theorem 7.10).

7.1.3. *The Hasse principle for G.* We continue using the same notation as in Subsect. 7.1.1. Additionally, we assume that k is a local field (Hypothesis 2). Let us denote by $\widetilde{k}$ its residue field. Let X/F be a projective homogeneous variety under $G = U(A, \sigma, h)$.

Theorem 7.10 ([36, Thm. 4.4], [33, Thm. 12.3])**.** *Suppose* $(2\mathrm{ind}(A), \mathrm{char}\ \widetilde{k}) = 1$. *Additionally, if $A = L$, then assume the rank of the Hermitian form h is at least 2. If $X(F_v) \neq \emptyset$ for all $v \in V(F)$ discrete, then $X(F) \neq \emptyset$.*

Proof. By Theorem 7.3, there exists a strictly increasing sequence of integers $0 < n_1 < \cdots < n_r < \deg A$ such that $X \cong X_h(n_1, n_2, \ldots, n_r)$. By *loc.cit.* and Proposition 7.5, we may assume that $X = X_h(n_r)$. It suffices to show $X_h(n_r)(F) \neq \emptyset$.

Let $Z \subseteq C^{\mathrm{an}}$ be a set of rigid points containing the ramification locus of A. For any $z \in Z$, let $V_z \subseteq C^{\mathrm{an}}$ be a strict affinoid neighborhood of z such that $X_h(n_r)(\mathscr{M}(V_z)) \neq \emptyset$ (Remarks 2.2(4) and 2.4). Let $\mathscr{C}$ be a proper sncd model of C corresponding to a vertex set S such that $\bigcup_{z\in Z} \partial V_z \subseteq S$ (Theorem 1.6). Let $\mathscr{C}_s$ be the special fiber of $\mathscr{C}$ and $\pi_{\mathscr{C}} : C^{\mathrm{an}} \to \mathscr{C}_s$ the specialization morphism.

Let $x \in C^{\mathrm{an}} \backslash (\bigcup_{z\in Z} V_z \cup S)$. Set $P_x := \pi_{\mathscr{C}}(x)$ (a closed point of $\mathscr{C}_s$) and $U_{\mathscr{C},x} := \pi_{\mathscr{C}}^{-1}(P_x)$ (a connected open subset of C^{an}). By Theorem 1.2, $\widehat{\mathcal{O}_{\mathscr{C},P_x}} = \mathcal{O}^\circ(U_{\mathscr{C},x})$, so $\mathcal{O}^\circ(U_{\mathscr{C},x})$ is a complete regular local ring of dimension 2. Let $\omega, \delta \in \mathcal{O}^\circ(U_{\mathscr{C},x})$ be a system of regular parameters satisfying the conditions of Corollary 5.6. Set $F_{\mathscr{C},x} := \mathrm{Frac}\ \mathcal{O}^\circ(U_{\mathscr{C},x})$.

Remark that $L \otimes_F F_{\mathscr{C},x} \cong F_{\mathscr{C},x}[\sqrt{\lambda}]$. For a fine enough model $\mathscr{C}$ (or, equivalently, large enough S), we may assume that:

(a) up to multiplication by a square, $\lambda \in \mathcal{O}^\circ(U_{\mathscr{C},x})$ is either of the form u or $u\omega$, where $u \in \mathcal{O}^\circ(U_{\mathscr{C},x})^\times$ ([36, Lemma 4.2]);
(b) the integral closure R' of $\mathcal{O}^\circ(U_{\mathscr{C},x})$ in $F_{\mathscr{C},x}[\sqrt{\lambda}]$ has regular parameters ω_1, δ, where $\omega_1 = \omega$ if $\lambda = u$ and $\omega_1 = \sqrt{u\omega}$ if $\lambda = u\omega$ ([33, Sect. 9]);
(c) A is unramified on R' except possibly at (ω_1) and (δ) (Corollary 5.6).

We remark that for any model $\mathscr{C}_1$ of C refining $\mathscr{C}$, $U_{\mathscr{C}_1,x} \subseteq U_{\mathscr{C},x}$, so $F_{\mathscr{C}_1,x} \subseteq F_{\mathscr{C},x}$.

As in the proof of [33, Thm. 12.2], we prove by induction on $\mathrm{ind}(A \otimes_F F_{\mathscr{C},x})$ the existence of a proper sncd model $\mathscr{C}_1$ of C refining $\mathscr{C}$ such that $X_h(n_r)(F_{\mathscr{C}_1,x}) \neq \emptyset$.

1) Suppose $\mathscr{C}$ is a proper sncd model of C satisfying the conditions (a), (b), (c) above. If $\mathrm{ind}(A \otimes_F F_{\mathscr{C},x}) = 1$, h corresponds to a quadratic form q_h over $F_{\mathscr{C},x}$, and one is isotropic if and only if the other is. One reduces to a question of isotropy of the quadratic form q_h over $F_{\mathscr{C},x}$ (see proof of [33, Thm. 12.2]). Then, by [17, Cor. 4.7], $X_h(n_r)(F_{\mathscr{C},x}) \neq \emptyset$.

2) Let $s \in \mathbb{N}$ be such that $s \geqslant 2$. Let $\mathscr{C}$ be a proper sncd model of C satisfying properties (a), (b), (c) above, and $\mathrm{ind}(A \otimes_F F_{\mathscr{C},x}) < s$. Assume then that there exists a proper sncd model $\mathscr{C}_1$ refining $\mathscr{C}$ such that $X_h(n_r)(F_{\mathscr{C}_1,x}) \neq \emptyset$.

3) Let $\mathscr{C}$ be a proper sncd model of C satisfying conditions (a), (b), (c) above and such that $\mathrm{ind}(A \otimes_F F_{\mathscr{C},x}) = s$. Let D_x be the division algebra Brauer equivalent to $A \otimes_F F_{\mathscr{C},x}$ over $F_{\mathscr{C},x}$. Then $\deg D_x \geqslant 2$. Let (D_x, τ_x, h_x) be the structure induced on D_x by $(A \otimes_F F_{\mathscr{C},x}, \sigma, h)$ (Theorem 7.3(3)). By Theorem 7.3(4), there exists $X_{h_x}(m)/F_{\mathscr{C},x}$ homogeneous under $U(D_x, \tau_x, h_x)$ such that $X_{h_x}(m)(K) \neq \emptyset \iff X_h(n_r)(K) \neq \emptyset$ for all field extensions $K/F_{\mathscr{C},x}$.

Let $\Lambda_x, \omega_x, \delta_x$ be as in Theorem 7.6. Let $< a_1, a_2, \ldots, a_n >$ be a diagonal form of h_x with $a_i \in \Lambda_x$. Then $\tau(a_i) = a_i$ for all i. Set $b_i := Nrd_{D_x}(a_i) \in F_{\mathscr{C},x}^\times$. Let $\mathscr{C}_1$ be a proper sncd model refining $\mathscr{C}$ constructed via [36, Lemma 4.3] for $b_1, b_2, \ldots, b_n$. We may assume that it still satisfies conditions (a), (b), (c) from above. By Theorem 1.6, $\mathscr{C}_1$ corresponds to a vertex set $S_1 \supseteq S$ of C^{an}. We know that $U_{\mathscr{C}_1,x} \subseteq U_{\mathscr{C},x}$ and $F_{\mathscr{C}_1,x} \subseteq F_{\mathscr{C},x}$.

By construction, $\mathcal{O}^\circ(U_{\mathscr{C}_1,x})$ has a regular system of parameters $\omega_{1,x}, \delta_{1,x}$ such that $b_i = Nrd_{D_x}(a_i) = u_i \omega_{1,x}^{r_i} \delta_{1,x}^{s_i}$, $u_i \in \mathcal{O}^\circ(U_{\mathscr{C}_1,x})^\times$, $r_i, s_i \in \mathbb{Z}$, $i \in \{1, 2, \ldots, n\}$.

If $D_x \otimes_{F_{\mathscr{C},x}} F_{\mathscr{C}_1,x}$ is *not* a division algebra, then $\mathrm{ind}(A \otimes_F F_{\mathscr{C}_1,x}) < \mathrm{ind}(A \otimes_F F_{\mathscr{C},x})$, so we may conclude by the inductive assumption that there exists a proper sncd model $\mathscr{C}_2$ of C refining $\mathscr{C}_1$ such that $X_h(n_r)(F_{\mathscr{C}_2,x}) \neq \emptyset$.

If $D_x \otimes_{F_{\mathscr{C},x}} F_{\mathscr{C}_1,x}$ *is* a division algebra, $h'_x :=< a_1, a_2, \ldots, a_n >$ is the Hermitian form induced by h_x, and now the a_i satisfy the properties of Theorem 7.6(2). Let $X_{h'_x}(m')/F_{\mathscr{C}_1,x}$ be the homogeneous variety corresponding to $X_{h_x}(m) \times_{F_{\mathscr{C},x}} F_{\mathscr{C}_1,x}$. Set $F_1 := \mathrm{Frac}(\mathcal{O}^\circ(\widehat{U_{\mathscr{C}_1,x}})_{(\omega_{1,x})})$ and $F_2 := \mathrm{Frac}(\mathcal{O}^\circ(\widehat{U_{\mathscr{C}_1,x}})_{(\delta_{1,x})})$. These are both complete discretely valued fields containing F. Moreover, the restriction of the valuation on k is either trivial or discrete (Remark 2.11). Hence, $X_{h_x}(m)(F_j) \neq \emptyset$, implying $X_{h'_x}(m')(F_j) \neq \emptyset$ for $j = 1, 2$.

Using an induction argument (see proof of [33, Thm. 12.2]), the problem is reduced to one of isotropy of h'_x. By assumption, h'_x is isotropic over F_j, $j = 1, 2$, so, by Theorem 7.8, it is isotropic over $F_{\mathscr{C}_1,x}$. Hence $X_{h'_x}(m')(F_{\mathscr{C}_1,x}) \neq \emptyset$. Consequently, $X_{h_x}(m)(F_{\mathscr{C}_1,x}) \neq \emptyset$, so $X_h(n_r)(F_{\mathscr{C}_1,x}) \neq \emptyset$.

We have shown that there exists a proper sncd model $\mathscr{C}$ of C satisfying $X_h(n_r)(F_{\mathscr{C},x}) \neq \emptyset$. As $F_{\mathscr{C},x} \subseteq \mathscr{M}_x$, we obtain that $X_h(n_r)(\mathscr{M}_x) \neq \emptyset$.

Hence, for all $x \in C^{\mathrm{an}}$, $X(\mathscr{M}_x) \neq \emptyset$. By [29, Thm. 3.11], this implies $X_h(n_r)(F) \neq \emptyset$. □

7.2. **Special unitary groups (by** [36]**).** If we let A be a central simple algebra with an involution σ of the *first* kind, and h a Hermitian form on (A, σ), then the group $SU(A, \sigma, h)$ is a rational reductive linear algebraic group ([7, Lemma 5]). By using techniques very similar to those in the case of unitary groups (the difference being that now $L = F = Z(A)$), one can prove the Hasse principle holds also for homogeneous spaces under special unitary groups. This was done in [36]. We refrain from translating this case to our setting as it would amount to repeating the arguments from Subsection 7.1.

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