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Analytic pseudo-rotations

Pierre Berger*

October 15, 2022

Abstract

We construct analytic symplectomorphisms of the cylinder or the sphere with zero or exactly two periodic points and which are not conjugate to a rotation. In the case of the cylinder, we show that these symplectomorphisms can be chosen ergodic or to the contrary with local emergence of maximal order. In particular, this disproves a conjecture of Birkhoff (1941) and solves a problem of Herman (1998). One aspect of the proof provides a new approximation theorem, it enables in particular to implement the Anosov-Katok scheme in new analytic settings.

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1 Statement of the main theorems

In low dimensional analytic dynamics, a fundamental question is wether far from periodic points, the dynamics is rigid. In real or complex dimension 1, dynamics without periodic point are rigid [Yoc84, Sul85]: the dynamics restricted to an invariant domain without periodic point is either a rotation or in the basin of periodic points. In dimension 2, for real analytic volume preserving diffeomorphisms of the cylinder or the real sphere, the Birkhoff rigidity conjecture for pseudo-rotations [Bir41] states that such are also conjugate to rotations. We are going to disprove this conjecture. More precisely we are going to show two examples of entire real symplectomorphisms of the annulus $\mathbb{A} = \mathbb{R}/\mathbb{Z} \times \mathbb{R}$, without periodic points and such that outside a region $\mathbb{A}_0'$ of $\mathbb{A}$ bounded by two disjoint analytic curves, the dynamics is analytically conjugate to a rotation, while inside $\mathbb{A}_0'$ the dynamics is not rigid. In the first example, the restriction $f|\mathbb{A}_0'$ will be ergodic – see Theorem A – while in the second example it will be extremely far from being ergodic – see Theorem C. More precisely the ergodic decomposition of the latter map will be infinite dimensional and even of maximal local order, i.e. with local emergence of maximal order 2. This confirms a conjecture on typicality of high emergence in many categories [Ber17].

From this we will deduce a corollary disproving the Birkhoff rigidity conjecture. The proof of the main theorems are shown by developing the Anosov-Katok method [AK70], together with a new approximation theorem for entire symplectomorphisms developing a recent work with Turaev [BT22]. The corollary regarding the sphere is obtained by using a blow-down technique introduced in [Ber22].

Finally we will remark that these analytic constructions give examples of entires symplectomorphisms of $\mathbb{C}/\mathbb{Z} \times \mathbb{C}$ without periodic point and with a non-empty instability region J.

1.1 Ergodic analytic and symplectic pseudo-rotation

Let S be an orientable analytic surface. We recall a diffeomorphism of S is **symplectic** if it preserves the orientation and the volume.

Conjecture (Birkhoff [Bir41, Pb 14-15])**.** *An analytic symplectomorphism of the sphere with only two fixed points and no other periodic point is necessarily topologically conjugate to a rotation.*

Analogously, an analytic symplectomorphism of a compact cylinder without periodic points is topologically conjugate to a rotation.

While following Birkhoff "considerable evidence was adducted for this conjecture", Anosov-Katok [AK70] gave examples of smooth symplectomorphisms of the sphere or the annulus with resp. 2 or 0 periodic points which are ergodic. Anosov and Katok proved their theorems by introducing the **approximation by conjugacy method**, that we will recall in the sequel. Also, in his famous list of open problems, Herman wrote that a positive answer to the following question would disprove the Birkhoff rigidity conjecture:

Question 1.1 (Herman [Her98][Q.3.1])**.** *Does there exist an analytic symplectomorphism of the cylinder or the sphere with a finite number of periodic points and a dense orbit?*

Corollary B will disprove both Birkhoff's conjectures and answer positively to Herman's question in the cylinder case. It will also bring a new analytic case of application of the approximation by conjugacy method, as wondered by Fayad-Katok in [FK04, §7]. To state the main theorems, we set:

$$\mathbb{T} := \mathbb{R}/\mathbb{Z}, \quad \mathbb{I} := (-1, 1), \quad \mathbb{A} := \mathbb{T} \times \mathbb{R}, \quad \text{and} \quad \mathbb{A}_0 = \mathbb{A} \times \mathbb{I}\,.$$

We recall that a map of $\mathbb{A}$ is entire if it extends to an analytic map on $\mathbb{C}/\mathbb{Z} \times \mathbb{C}$. An **entire symplectomorphism** is a symplectomorphism which is entire and whose inverse is entire.

An **analytic cylinder** of $\mathbb{A}_0'$ of $\mathbb{A}$ is a subset of the form $\{(\theta, y) \in \mathbb{A} : \gamma^-(\theta) < y < \gamma^+(\theta)\}$, for two analytic functions $\gamma^- < \gamma^+$.

Theorem A. *There is an entire symplectomorphism F of $\mathbb{A}$ which leaves invariant an analytic cylinder $\mathbb{A}_0' \subset \mathbb{A}$, whose restriction to $\mathbb{A}_0'$ is ergodic and whose restriction to $\mathbb{A} \backslash cl(\mathbb{A}_0')$ is analytically conjugate to a rotation. Moreover the complex extension of F has no periodic point in $\mathbb{C}/\mathbb{Z} \times \mathbb{C}$.*

The following consequence of Theorem A is the counter example of both Birkhoff conjectures and a positive answer to Herman's question in the case of the cylinder:

Corollary B.

1. *There is an analytic symplectomorphism of $cl(\mathbb{A}_0)$ which is ergodic and has no periodic point.*
2. *There is an analytic symplectomorphism of the sphere, whose restriction to a sub-cylinder is ergodic and which displays only two periodic points.*

This corollary is proved in Section 2.4 using the blow up techniques introduced in [Ber22]. It seems that the proof of Theorem A might be pushed forward to obtain a positive answer to the following:

Problem 1.2. *In Theorem A, show that we can replace $\mathbb{A}_0'$ by a cylinder with whose boundary is formed by two pseudo-circles.*

Theorem A will be proved by proving a new approximation Theorem 1.8 which enables to implement the approximation by conjugacy method to analytic maps of the cylinder. Analytic symplectomorphisms of the 2-torus which are ergodic and without periodic points are known since Furstenberg [Fur61, Thm 2.1] (one of the explicit examples is $(\theta_1, \theta_2) \in \mathbb{T}^2 \mapsto (\theta_1 + \alpha, \theta_1 + \theta_2)$ for any α irrational). Actually the approximation by conjugacy method is known to provide examples of analytic symplectomorphisms of $\mathbb{T}^2$ which are isotopic to the identity, see [BK19] for stronger results. Up to now, the only other known analytic realization of approximation by conjugacy method was Fayad-Katok's theorem [FK14] showing the existence of analytic uniquely ergodic volume preserving maps on odd spheres. As a matter of fact, we also answer a question of Fayad-Katok [FK04, §7.1] on wether analytic realization of the approximation by conjugacy method may be done on other manifolds than tori or odd spheres .

A way to repair the Birkhoff conjecture might be to ask rigidity for dynamics without periodic point[1] whose rotation number satisfies a diophantine condition, as wondered by Herman in [Her98][Q.3.2]. The approximation by conjugacy method has been been useful to construct many other interesting examples of dynamical systems with special property, see for instance the survey [FH78, Cro06]. Certainly the new approximation Theorem 1.8 enables to adapt these examples to the case of analytic maps of the cylinder. In the next subsection, we will study a new property. We show how to use this scheme to construct diffeomorphisms with high local emergence. The readers only interested by the Birkhoff conjecture can skip the next subsection and go directly to Section 1.3.

1.2 Analytic and symplectic pseudo-rotation with maximal local emergence

While an ergodic dynamics might sound complicated, the description of the statistical behavior of its orbits is by definition trivial. We recall that by Birkhoff's ergodic theorem, given a symplectic

[1] These are called pseudo-rotation in [BCLR06].

map f of a compact surface S, for Leb. a.e. point x the following limit is a well defined probability measure called the **empirical measure** of x.

$$\mathsf{e}(x) := \frac{1}{n}\sum_{k=1}^{n} \delta_{f^k(x)} \ .$$

The measure $\mathsf{e}(x)$ describes the statistical behavior of the orbit of x. The **empirical function** $\mathsf{e} : x \in S \mapsto \mathsf{e}(x)$ is a function with value in the space $\mathcal{M}(S)$ of probability measures on S. Note that e is a measurable function.

A natural question is how complex is the diversity of statistical behaviors of the orbits of points in a Lebesgue full set. To study this, we shall look at the size of the push forward $\mathsf{e}_*\mathrm{Leb}$ of the Lebesgue probability measure Leb of S by e. The measure is $\mathsf{e}_*\mathrm{Leb}$ is called the **ergodic decomposition**; it is a probability measure on the space $\mathcal{M}(S)$ of probability measures of S. The ergodic decomposition describes the distribution of the statistical behaviors of the orbits.

To measure the size of the ergodic decomposition, for every compact metric space X, we endow the space $\mathcal{M}(X)$ of probability measures on X with the **Kantorovitch-Wasserstein metric** d:

$$\forall \mu_1, \mu_2 \in \mathcal{M}(X), \quad \mathsf{d}(\mu_1, \mu_2) := \inf\left\{ \int_{X^2} d(x_1, x_2) d\mu : \mu \in \mathcal{M}(X^2) \text{ s.t. } p_{i*}\mu = \mu_i \ \forall i \in \{1,2\} \right\},$$

where $p_i : (x_1, x_2) \mapsto X^2 \to x_i \in X$ for $i \in \{1,2\}$. This distance induces the weak $\star$ topology on $\mathcal{M}(X)$ which is compact. Also it holds:

Proposition 1.3. *For any compact metric spaces X, Y, any $\mu \in \mathcal{M}(X)$ and $f, g \in C^0(X, Y)$ it holds:*

$$\mathsf{d}(f_*\mu, g_*\mu) \leq \max_{x \in X} d(f(x), g(x)).$$

Proof. Let $\hat{\mu}$ be the measure on the diagonal of $X \times X$ which is pushed forward by the 1st and 2sd coordinate projection to μ. Then observe that the pushforward ν of $\hat{\mu}$ by the product (f, g) is a transport from $f_*\mu$ to $g_*\mu$; its cost $\int_{Y \times Y} d(y, y')d\nu$ is at most $\max_{x \in X} d(f(x), g(x))$. □

The following has been proved several times (see [BB21, Thm 1.3] for references):

Theorem 1.4. *The box order of $(\mathcal{M}(S), \mathsf{d})$ is 2:*

$$\lim_{\epsilon \to 0} \frac{\log \log \mathcal{N}(\epsilon)}{|\log \epsilon|} = 2 \quad \textit{with } \mathcal{N}(\epsilon) \textit{ the } \epsilon\textit{-covering number of } (\mathcal{M}(S), \mathsf{d}) \ .$$

In contrast, up to now, all the bounds on the emergence of symplectic and analytic dynamics are finite dimensional:

Example 1.5 (Case study).

1. If the measure Leb is ergodic, then the ergodic decomposition is a Dirac measure at the Lebesgue measure: $\mathsf{e}_*\mathrm{Leb} = \delta_{\mathrm{Leb}}$.
2. For a straight irrational rotation of the annulus $\mathbb{A}_0 = \mathbb{T} \times \mathbb{I}$, we have $\mathsf{e}_*\mathrm{Leb} = \int_{\mathbb{I}} \delta_{\mathrm{Leb}_{\mathbb{T} \times \{y\}}} d\mathrm{Leb}\,(y)$ with $\mathrm{Leb}_{\mathbb{T} \times \{y\}}$ the one-dimensional Lebesgue measure on $\mathbb{T} \times \{y\}$. Hence the ergodic rotation of a straight irrational rotation of the annulus is one dimensional. The same occurs for an integrable twist map of $\mathbb{A}_0$.

A natural problem (see [Ber17, Ber20, BB21]) is to find dynamics for which $\mathsf{e}_*\mathrm{Leb}$ is infinite dimensional in many categories. The notion of emergence has been introduced to precise this

problem. In the present conservative setting[2], the emergence describes the size of the ergodic decomposition. More precisely, the **emergence** of f at scale $\epsilon > 0$ is the minimum number $\mathcal{E}(\epsilon) \geq 1$ of probability measures $(\mu_i)_{1\leq i\leq \mathcal{E}(\epsilon)}$ such that:

$$\int \min_i d(\mathsf{e}(x), \mu_i) d\mathrm{Leb} < \epsilon \ .$$

The **order of the emergence** is

$$\overline{\mathcal{O}}\mathcal{E}_f = \limsup_{\epsilon\to 0} \frac{\log\log \mathcal{E}(\epsilon)}{|\log \epsilon|} \ .$$

In [BB21, ineq. (2.2) and Prop. 3.14], it has been shown that $\overline{\mathcal{O}}\mathcal{E}_f \leq \dim S = 2$. Also we constructed an example of C^∞-flows on the disk with maximal emergence order: $\overline{\mathcal{O}}\mathcal{E}_f = 2$. We do not know how to perform this example in the analytic setting[3]. Also this example has an ergodic decomposition of local dimension 1. Actually, in view of Theorem 1.4, on can hope for the existence of an ergodic decomposition of infinite local dimension and even of positive local order.

Let us denote $\hat{\mathsf{e}} := \mathsf{e}_*\mathrm{Leb}$ the ergodic decomposition of f. The **order of the local emergence** of f is[4]:

$$\overline{\mathcal{O}}\mathcal{E}_{loc}(f) = \limsup_{\epsilon\to 0} \int \frac{\log|\log \hat{\mathsf{e}}(B(\mu,\epsilon))|}{|\log \epsilon|} d\hat{\mathsf{e}} \ .$$

In [Hel22], Helfter showed that for $\hat{\mathsf{e}}$ a.e. $\mu \in \mathcal{M}(S)$ it holds:

$$\overline{\mathcal{O}}_{loc}\hat{\mathsf{e}}(\mu) := \limsup_{\epsilon\to 0} \frac{\log|\log \hat{\mathsf{e}}(B(\mu,\epsilon))|}{|\log \epsilon|} \leq \overline{\mathcal{O}}\mathcal{E}_f \ .$$

As $\overline{\mathcal{O}}\mathcal{E}_f \leq 2$ by Theorem 1.4, it comes that $\overline{\mathcal{O}}_{loc}\hat{\mathsf{e}}(\mu) \leq 2$ a.e. Thus if $\overline{\mathcal{O}}\mathcal{E}_{loc}(f) = 2$, then the local order $\overline{\mathcal{O}}_{loc}\hat{\mathsf{e}}(\mu)$ of $\hat{\mathsf{e}}$ is 2 for $\hat{\mathsf{e}}$-a.e. $\mu \in \mathcal{M}(S)$. In this work we give the first example of smooth symplectomorphism with infinite dimensional local emergence for smooth dynamics. Moreover our example is entire and of maximal order of local emergence:

Theorem C. *There is an entire symplectomorphism F of $\mathbb{A}$ which leaves invariant an analytic cylinder $\mathbb{A}_0' \subset \mathbb{A}$, whose restriction to $\mathbb{A}_0'$ has local emergence 2 and whose restriction to $\mathbb{A} \setminus cl(\mathbb{A}_0')$ is analytically conjugate to a rotation. Moreover the complex extension of F has no periodic point in $\mathbb{C}/\mathbb{Z} \times \mathbb{C}$.*

A more restrictive version of local emergence could be done by replacing the lim inf instead of lim sup. Then a natural open problem is:

Question 1.6. *Does there exist a smooth conservative map such that the following limit is positive:*

$$\underline{\mathcal{O}}\mathcal{E}_{loc}(f) = \liminf_{\epsilon\to 0} \int \frac{\log|\log \hat{\mathsf{e}}(B(\mu,\epsilon))|}{|\log \epsilon|} d\hat{\mathsf{e}} \ ?$$

[2]The notion of emergence is also defined for dissipative system.

[3]Yet in the dissipative setting, a locally dense set of area contracting polynomial automorphism of $\mathbb{R}^2$ has been show to have emergence of order 2 in [BB22].

[4]This definition of order of the local emergence is at most the one of [Ber20] were the limsup is inside the integral.

1.3 Sketch of proof and main approximation theorem

Let $\mathrm{Symp}^\infty(\mathbb{A})$ denote the space of symplectomorphisms of $\mathbb{A}$. Let $\mathrm{Ham}^\infty(\mathbb{A})$ be the subgroup of Hamiltonian C^∞-maps of $\mathbb{A}$: there is a smooth family $(H_t)_{t\in[0,1]}$ of Hamiltonians $H_t \in C^\infty_\delta(\mathbb{A})$ which defines a family $(f_t)_{t\in[0,1]}$ such that $f_0 = id$, $f_1 = f$ and $\partial_t f_t$ is the symplectic gradient of H_t. Let $\mathrm{Ham}^\infty_0(\mathbb{A})$ be the subgroup of $\mathrm{Ham}^\infty(\mathbb{A})$ formed by maps so that $(H_t)_{t\in[0,1]}$ can be chosen supported by $\mathbb{A}_0$. We have $\mathrm{Ham}^\infty(\mathbb{A}) \subsetneq \mathrm{Symp}^\infty(\mathbb{A})$. Indeed, the rotation $R_\alpha : (\theta, y) \in \mathbb{A} \mapsto (\theta+\alpha, y)$ of angle $\alpha \in \mathbb{R}$ belongs to $\mathrm{Symp}^\infty(\mathbb{A}) \setminus \mathrm{Ham}^\infty(\mathbb{A})$.

The celebrated approximation by conjugacy method introduced by Anosov-Katok gives the existence of a map which satisfies all the properties of Theorem A but the analyticity. Our first chalenge is to perform this construction among analytic maps of the cylinder. Let us recall:

Theorem 1.7 (Anosov-Katok [AK70])**.** *There exists a sequence $(H_n)_{n\in\mathbb{N}}$ of maps $H_n \in \mathrm{Ham}^\infty_0(\mathbb{A})$ and a sequence of rational numbers $(\alpha_n)_{n\geq 0}$ converging to $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ such that the sequence of maps $F_n := H_n \circ R_{\alpha_n} \circ H_n^{-1}$ converges to a map $F \in \mathrm{Symp}^\infty(\mathbb{A})$ whose restriction $F|\mathbb{A}_0$ is ergodic.*

We will give a complete proof of this theorem in Section 2.2.1. Let us here sketch its proof to understand the difficulty to adapt it to the analytic case (Theorem A).

Sketch of proof. We construct $\alpha_n = p_n/q_n$ and H_n by induction such that:

(1) for most points $y \in \mathbb{I}$, the push forward of the Lebesgue measure on $\mathbb{T} \times \{y\}$ by H_{n+1} is close to the Lebesgue measure on $\mathbb{A}_0$,

(2) $H_{n+1} = H_n \circ h_{n+1}$ with h_{n+1} which commutes with the rotation of angle α_n.

The first property is obtained by constructing a map h_{n+1} as depicted in Fig. 1, so that it sends most of the Lebesgue measure of $\mathbb{T} \times \{y\}$ nearby the Lebesgue measure of $\mathbb{A}_0$.

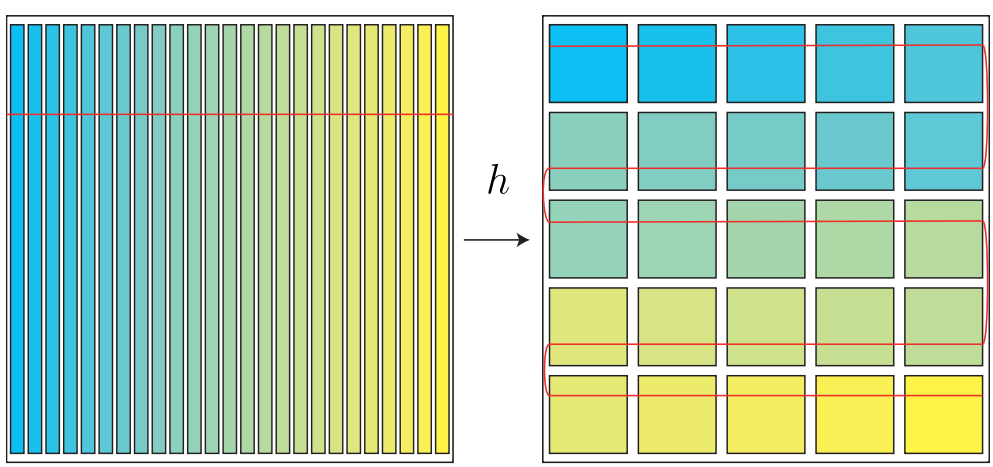

Figure 1: A conjugacy which sends horizontal curves to cuves close to the Lebesgue measure.

As H_n is area preserving and fixed before h_{n+1}, this property remains true for H_{n+1} as claimed in (1). To obtain (2), it suffices to consider a lift of h_{n+1} by the covering $(\theta, y) \mapsto (q_{n+1} \cdot \theta, y)$.

Finally we take $\alpha_{n+1} = p_{n+1}/q_{n+1}$ close to α_n so that F_{n+1} is close to:

$$H_n \circ h_{n+1} \circ R_{\alpha_n} \circ h_{n+1}^{-1} \circ H_n^{-1} = H_n \circ R_{\alpha_n} \circ H_n^{-1} = F_n \ .$$

as h_{n+1} commutes with R_{n+1}. This implies the convergence of $(F_n)_n$ to a map F.

Moreover, as q_{n+1} is large, all the points (θ, y) have their $R_{\alpha_{n+1}}$-orbits which are close to be equidistributed on $\mathbb{T} \times \{y\}$. Thus most of the points of $\mathbb{A}_0$ have their F_{n+1}-orbits close to be equidistributed on $\mathbb{A}_0$. This implies that F_n is close to be ergodic and thus F is ergodic. $\square$

To disprove the Birkhoff conjecture using the approximation by conjugation method, we shall find a sequence of analytic maps $(h_n)_n$ satisfying (1) and (2). Actually the whole property (1) is not needed, but we need at least that the sequence of compositions $H_n = h_1 \circ h_2 \cdots \circ h_n$ diverges on $\mathbb{A}_0$ without having a singularity in a uniform complex neighborhood of $\mathbb{A}_0 \subset \mathbb{A}_{\mathbb{C}} := \mathbb{C}/\mathbb{Z} \times \mathbb{C}$ and likewise for its inverse. An idea to tackle this problem is to define h_n as the time-one map of an entire Hamiltonian or to define it implicitly by an entire generating function. Then, in general, the map h_n is not an entire automorphism: it or its inverse has a singularity in $\mathbb{A}_{\mathbb{C}}$. So we have to check that after composition this singularity does not approach $\mathbb{A} \subset \mathbb{A}_{\mathbb{C}}$. This is not easy. Step (ii) is even more problematic: after a q_n-lifting of h_n to obtain (ii), a singularity of h_n becomes $1/q_n$-times closer to $\mathbb{A}$.

To avoid these problems, the idea is to work with **entire real automorphisms**. Then there are no singularity to take care of. The complex definition domains of the maps and of their inverses cannot shrink since it is maximal, that is equal to:

$$\mathbb{A}_{\mathbb{C}} := \mathbb{C}/\mathbb{Z} \times \mathbb{C} \, .$$

Horizontal twist maps which are Hamiltonian maps of the form $(\theta, y) \mapsto (\theta + \tau(y), y)$ and **vertical twist maps** which are Hamiltonian maps of the form $(\theta, y) \mapsto (\theta, y + v(\theta))$ provide examples of entire automorphisms. It suffices to take any τ and v entire. Horizontal twist maps leave $\mathbb{A}_0$ invariant while vertical twist maps do not. One can show that an entire symplectomorphism which leaves $\mathbb{A}_0$ invariant must be a horizontal twist map[5]. To overcome the lake of such entire symplectomorphisms, we relax the condition on the invariance of $\mathbb{A}_0$ and we ask instead that the entire map should be close to the identity on a large compact subset of

$$\mathbb{E}_{\mathbb{C}} := \{(\theta, y) \in \mathbb{C}/\mathbb{Z} \times \mathbb{C} : \mathrm{Re}(z) \notin \mathbb{I}\}.$$

We will see below (see the main approximation Theorem 1.8) that there are a lot of such mappings. Then the idea is to use them in the approximation by conjugacy method (as the h_n). Indeed the approximation by conjugacy method ensures the convergence of the $F_n = H_n \circ R_{\alpha_n} \circ H_n^{-1}$ on $\mathbb{C}/\mathbb{Z} \times \mathbb{C}$ to a map which is not conjugate to a rotation while we know that $(H_n)_n$ converges on a large compact subset of $\mathbb{E}_{\mathbb{C}}$ (a sequence of compositions of sufficiently close to the identity maps converges).

The main approximation Theorem 1.8 is done by adapting to the analytic setting a new result with Turaev [BT22], which states that compositions of horizontal and vertical twist maps are dense in $\mathrm{Ham}^{\infty}(\mathbb{A})$, see Theorem 3.1. It is easy to approximate these maps by entire automorphisms, yet these might be far from preserving $\mathbb{A}_0$. That is why, we will show that the set of commutators of vertical and horizontal twist maps with the set of horizontal twist maps generate a dense subgroup of $\mathrm{Ham}^{\infty}(\mathbb{A})$. Furthermore, we will notice that to approximate $\mathrm{Ham}_0^{\infty}(\mathbb{A})$ only horizontal twist maps or such commutators supported by $\mathbb{A}_0$ are necessary. See Theorem 3.3.

To pass to the analytic case, we will use Runge theorem to approximate the latter maps by entire automorphisms which are arbitrarily close to the identity on $\mathbb{A} \setminus \mathbb{A}_0$. This leads to the main main approximation Theorem 1.8 stated below. This apparently simple step is actually one of the main technical difficulty of this work.

[5]For every $y \in \mathbb{A}_0$, wee apply Picard's Theorem to the second coordinate projection of the entire symplectomorphism restricted to $\mathbb{C}/Z \times \{y\}$ to conclude that it must be constant and so equal to y

We need a few notations to state Theorem 1.8. For every $\delta \geq 0$, let $\mathbb{I}_\delta := [-1+\delta, 1-\delta]$, $\mathbb{A}_\delta = \mathbb{A} \times \mathbb{I}_\delta$ and for every $\delta \geq 0$, let $\mathrm{Ham}_\delta^\infty(\mathbb{A})$ be the subgroup of $\mathrm{Ham}_0^\infty(\mathbb{A})$ formed by flow maps of Hamiltonian $(H_t)_{t\in[0,1]}$ supported by $\mathbb{A}_\delta$. Observe that:

$$\mathrm{Ham}_0^\infty(\mathbb{A}) = \bigcup_{\delta>0} \mathrm{Ham}_\delta^\infty(\mathbb{A}) \ .$$

We denote $\mathrm{Ham}^\omega(\mathbb{A})$ be the subgroup of $\mathrm{Ham}^\infty(\mathbb{A})$ formed by Hamiltonian entire real automorphisms. For $\rho > 1$, put:

$$K_\rho := \mathbb{T}_\rho \times Q_\rho \quad \text{where } \mathbb{T}_\rho := \mathbb{T} + i[-\rho,\rho] \quad \text{and} \quad Q_\rho := [-\rho,-1] \sqcup [1,\rho] + i[-\rho,\rho] \ .$$

We denote $\mathrm{Ham}_\rho^\omega(\mathbb{A})$ the subset of $\mathrm{Ham}^\omega(\mathbb{A})$ formed by maps whose restriction to K_p is ρ^{-1}-close to the canonical inclusion $K_\rho \hookrightarrow \mathbb{A}_\mathbb{C}$:

$$\mathrm{Ham}_\rho^\omega(\mathbb{A}) := \left\{ h \in \mathrm{Ham}^\omega(\mathbb{A}) : \sup_{K_\rho} |h - id| < \rho^{-1} \right\} \ .$$

Theorem 1.8 (Main approximation result)**.** *Let $0 < \delta < 1$, let $h \in \mathrm{Ham}_\delta^\infty(\mathbb{A})$ and let $\mathcal{U}$ be a neighborhood of the restriction $h|\mathbb{A}_\delta$ in $C^\infty(\mathbb{A}_\delta, \mathbb{A})$. Then for any $\rho > 1$, there exists $\tilde{h} \in \mathrm{Ham}_\rho^\omega(\mathbb{A})$ such that the restriction $\tilde{h}|\mathbb{A}_\delta$ is in $\mathcal{U}$.*

Remark 1.9. By using Mergelyan's theorem instead of Runge's theorem, the proof of this theorem can be adapted to obtain an analogous statement for the surface $\mathbb{T}^2$ instead of $\mathbb{A}$.

A consequence of Theorem 1.8 regards $(1/q, 0)$-**periodic maps** for $g \geq 1$, which are maps h satisfying $h(\theta + 1/q, y) = h(\theta, y) + (1/q, 0)$ for every $(\theta, y) \in \mathbb{A}$:

Corollary 1.10. *Let $0 < \delta < 1$, let $h \in \mathrm{Ham}_\delta^\infty(\mathbb{A})$ be $(1/q, 0)$-periodic and let $\mathcal{U}$ be a C^∞-nghd of the restriction $h|\mathbb{A}_\delta$. Then for any $\rho > 1$, there exists $\tilde{h} \in \mathrm{Ham}_\rho^\omega(\mathbb{A})$ which is $(1/q, 0)$-periodic and whose restriction $\tilde{h}|\mathbb{A}_\delta$ is in $\mathcal{U}$.*

Proof. The map h induces an entire automorphism $[h]$ on the quotient $\mathbb{R}/q^{-1}\mathbb{Z} \times \mathbb{R}$. Note that $\mathbb{R}/q^{-1}\mathbb{Z} \times \mathbb{R}$ is diffeomorphic to $\mathbb{A}$ via the map $\psi : (\theta, y) \in \mathbb{R}/q^{-1}\mathbb{Z} \times \mathbb{R} \to (q \cdot \theta, y) \in \mathbb{A}$ and that $\psi \circ [h] \circ \psi^{-1}$ satisfies the assumptions of Theorem 1.8. Hence there exists a map $\tilde{g} \in \mathrm{Ham}_{q\cdot\rho}(\mathbb{A})$ such that the restrictions $\tilde{g}|\mathbb{A}_\delta$ and $\psi \circ [h] \circ \psi^{-1}$ are close. Then note that $\psi^{-1} \circ \tilde{g} \circ \psi$ is an entire automorphism on the quotient $\mathbb{R}/q^{-1}\mathbb{Z} \times \mathbb{R}$. It defines a map $\tilde{h} \in \mathrm{Ham}_\rho^\omega(\mathbb{A})$ which is $1/q$-periodic and such that the restriction $\tilde{h}|\mathbb{A}_\delta$ is close to $h|\mathbb{A}_\delta$. □

We will plug this corollary into the approximation by conjugation method to prove Theorem A in §2.2.2.

To prove Theorem C, we will first implement the approximation by conjugation method to prove the following new result:

Theorem 1.11. *There is $F \in \mathrm{Symp}^\infty(\mathbb{A})$ which leaves $\mathbb{A}_0$ invariant, without periodic point and whose order of local emergence of $F|\mathbb{A}_0$ is 2. Moreover F is C^0-conjugate to an irrational rotation via a conjugacy supported by $\mathbb{A}_0$.*

We will give a complete proof of this theorem in Section 2.3.1. Let us sketch its proof.

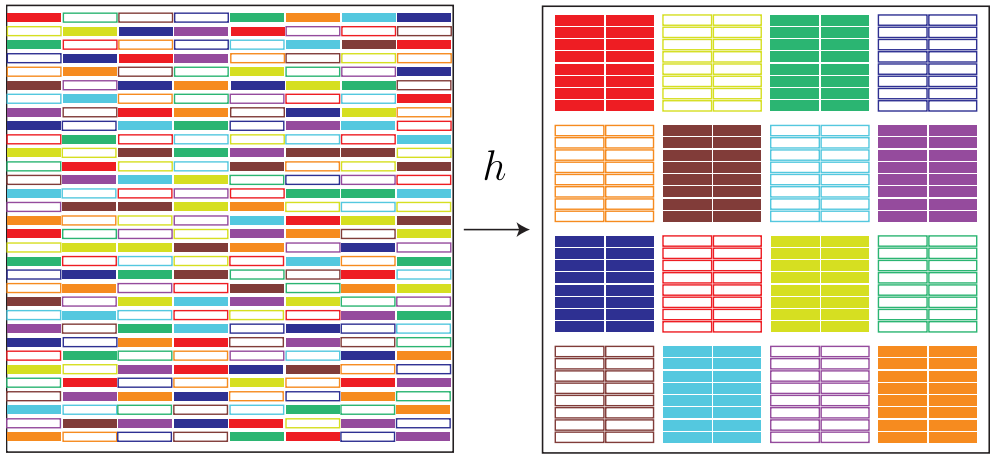

Figure 2: A conjugacy which sends horizontal curve in different strips to distant ones in law.

Sketch of proof. The proof follows the same lines as the one of Anosov-Katok, but differs at one point: we will not assume that the map h_{n+1} pushes forward of the measure on $\mathrm{Leb}_{\mathbb{T}\times\{y\}}$ nearby $\mathrm{Leb}_{\mathbb{A}_0}$, for most y. Instead, we will take the map h_{n+1} so that for most of the point y, the measure of the set of y' such that $h_{n+1*}\mathrm{Leb}_{\mathbb{T}\times\{y'\}}$ is ϵ_n-close to $h_{n+1*}\mathrm{Leb}_{\mathbb{T}\times\{y\}}$ is smaller than some $\exp(-\epsilon^{2-\eta_n})$ for some $\epsilon_n, \eta_n \to 0$, see Fig. 2 and Lemma 2.7. This lemma is proved using Moser's trick (see Theorem 4.1) and a combinatorial Lemma 4.5 develloping [BB21, Prop 4.2]. □

Again we will plug Corollary 1.10 into this proof to prove of Theorem C in §2.3.2.

1.4 A complex analytic consequence of the main results

Given a complex dynamics f of several complex variables, there are two sets of special interest. The closure J^* of the set of hyperbolic periodic points of f and the set J of points z with bounded and Lyapunov unstable backward and forward orbits. In the setting of polynomial automorphisms of $\mathbb{C}^2$, a natural problem which goes back to the early $90's$ is wether $J = J^*$. In the conservative setting, this problem can be seen as a complex counterpart of Birkhoff's conjecture, since on each component of the J's complement, the dynamics escapes to infinity or is analytically conjugate to a rotation, see [Bed18, Thm 2.1]. In the dissipative setting, arguments in favor of $J = J^*$ are the recent results of Crovisier-Pujals [CP18] and Dujardin [Duj20] which assert that for every real and resp. complex Hénon map satisfying $|\det Df| \cdot (\deg f)^2 < 1$, then $J = J^*$ a.e. for any invariant probability measure. In the conservative setting, we observe that the complexified Furstenberg symplectomorphism $(\theta_1, \theta_2) \in \mathbb{C}^2/\mathbb{Z}^2 \mapsto (\theta_1 + \alpha, \theta_2 + \theta_1)$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ satisfies $J = \mathbb{R}/\mathbb{Z} \times \mathbb{C}/\mathbb{Z} \neq \emptyset = J^*$. We bring the following new cases:

Corollary D. *The real entire symplectomorphisms F of $\mathbb{C}/\mathbb{Z} \times \mathbb{C}$ of Theorems A and C have non-empty Julia set J which contains $\mathbb{A}_0'$ and none periodic point. In particular $J \neq J^*$ in each of these examples.*

This corollary is proved in Section 2.4. This leads to the following natural question:

Question 1.12. *Does there exist a conservative entire map of $\mathbb{C}^2$ which satisfies $J \neq J^*$?*

Acknowlegments. *I am grateful to Vincent Humilière and Sobhan Seyfaddini for advices on how to write the proof of Moser-Krygin's Theorem 4.1. I am thankful to Romain Dujardin for several*

discussions around the $J = J^*$ problem. I am grateful to Bassam Fayad for his bibliographical and general advices. I indebted to Raphael Krikorian for many comments on the redaction of the proof of the main theorem.

2 Proof of the main theorems

2.1 Notations for probability measures

Let $(\mathcal{M}, \mathsf{d})$ denote the space of probability measures on $\mathbb{A}$ endowed with the Kantorovitch-Wasserstein distance (see §1.2). We denote by Leb the Lebesgue measure on $\mathbb{A}$ and by $\mathrm{Leb}_{\mathbb{T}}$ the one-dimensional Lebesgue measure on $\mathbb{T}$. For every cylinder $\mathbb{A}_0' \subset \mathbb{A}$, we set:

$$\mathrm{Leb}_{\mathbb{A}_0'} := \tfrac{1}{\mathrm{Leb}\,\mathbb{A}_0'}\mathrm{Leb}\,|\mathbb{A}_0',$$

which is a probability measure on $\mathbb{A}_0'$. For every $y \in \mathbb{R}$, we denote by $\mathrm{Leb}_{\mathbb{T}\times\{y\}}$ the probability measure $\mathrm{Leb}_{\mathbb{T}} \otimes \delta_y$. For every $x \in \mathbb{A}$, $n \geq 1$ and $F \in \mathrm{Symp}^\infty(\mathbb{A})$, the nth**-empirical measure** is:

$$\mathsf{e}_n^F(x) := \frac{1}{n}\sum_{k=1}^{n} \delta_{F^k(x)} \ .$$

By Birkhoff's ergodic theorem, if a cylinder $\mathbb{A}_0'$ is left invariant by F then for Leb. a.e. $x \in \mathbb{A}_0'$, the sequence $(\mathsf{e}_n^F(x))_n$ converges to a probability measure $\mathsf{e}^F(x)$. We recall that $F|\mathbb{A}_0'$ is ergodic iff $\mathsf{e}^F(x) = \mathrm{Leb}_{\mathbb{A}_0'}$ for a.e. $x \in \mathbb{A}_0'$.

2.2 Proof of Theorem A on ergodic pseudo-rotations

To prove that Theorem 1.8 implies the main theorems, we shall first recall the proof of Anosov-Katok's Theorem 1.7.

2.2.1 Existence of ergodic smooth pseudo-rotations

The proof of Theorem 1.7 can be done by induction using a sequence $(\epsilon_n)_n$ of positive numbers s.t.:

$$\sum_n \epsilon_n < 1 \ .$$

We recall that for $\alpha \in \mathbb{R}$, the rotation of angle α is denoted by $R_\alpha : (\theta, y) \in \mathbb{A} \mapsto (\theta + \alpha, y)$. Also for $\epsilon > 0$, we denote $\mathbb{A}_\epsilon := \mathbb{T} \times [-1+\epsilon, 1-\epsilon]$. We prove below:

Claim 2.1. *There are sequences of rational numbers $\alpha_n = \frac{p_n}{q_n}$ and of Hamiltonian maps $H_n \in \mathrm{Ham}_0^\infty(\mathbb{A})$ such that $F_n := H_n \circ R_{\alpha_n} \circ H_n^{-1}$ satisfies for every $n \geq 1$:*

(P_1) *for every $x \in H_n(\mathbb{A}_{\epsilon_n})$, the empirical measure $\mathsf{e}^{F_n}(x) = \mathsf{e}_{q_n}^{F_n}(x)$ is ϵ_n-close to $\mathrm{Leb}_{\mathbb{A}_0}$.*

(P_2) *The C^0-distance between $\mathsf{e}_k^{F_n}$ and $\mathsf{e}_k^{F_{n-1}}$ is smaller than ϵ_n for every $k \leq q_{n-1}$.*

(P_3) *The C^n-distance between F_n and F_{n-1} is smaller than ϵ_n.*

Proof that Claim 2.1 implies Theorem 1.7. By (P_3) and the convergence $\sum_n \epsilon_n$, the sequence $(F_n)_n$ converges to a C^∞-map $F \in \mathrm{Symp}^\infty(\mathbb{A})$.

Again by convergence of $\sum \epsilon_n$ and Borel Canteli-Lemma, the subset $B := \bigcap_{n\geq 0}\bigcup_{k\geq n} H_n(\mathbb{A}_{\epsilon_n})$ of $\mathbb{A}_0$ has full Lebesgue measure in $\mathbb{A}_0$. Also by (P_1) and (P_2), for every $x \in B$, the measure $\mathrm{Leb}_{\mathbb{A}_0}$ is

an accumulation point of the sequence of empirical measures $(\mathsf{e}_n^F(x))_n$. Thus by Birkhoff's ergodic theorem, a.e. point in B satisfies that $\mathsf{e}_n^F(x) \to \mathrm{Leb}_{\mathbb{A}_0}$. As B has full Lebesgue measure in $\mathbb{A}_0$ it comes that $F|\mathbb{A}_0$ is ergodic. □

Proof of Claim 2.1. Now let us show the Claim by induction on n. Put $q_0 = 1$, $\alpha_0 = 0$ and $H_0 = id$. Let $n \geq 1$, assume $(H_k)_{k<n}$ and $(\alpha_k = \frac{p_k}{q_k})_{k<n}$ constructed and satisfying the induction hypothesis. We will show in Section 4.2 the following:

Lemma 2.2. *For every $\eta > 0$, there exists a map $h_n \in \mathrm{Ham}_0^\infty(\mathbb{A})$ such that:*

1. *the map h_n is $(1/q_{n-1}, 0)$-periodic.*
2. *for every $(\theta, y) \in \mathbb{A}_{\epsilon_n}$, the map h_n pushes forward the one-dimensional Lebesgue probability measure $\mathrm{Leb}_{\mathbb{T}\times\{y\}}$ to a measure which is η-close to $\mathrm{Leb}_{\mathbb{A}_0}$.*

   Then for η sufficiently small, as H_{n-1} leaves invariant $\mathbb{A}_0$, we obtain:

2'. *for every $(\theta, y) \in \mathbb{A}_{\epsilon_n}$, the map $H_n := H_{n-1} \circ h_n$ pushes forward the one-dimensional Lebesgue probability measure $\mathrm{Leb}_{\mathbb{T}\times\{y\}}$ to a measure which is $\epsilon_n/2$-close to $\mathrm{Leb}_{\mathbb{A}_0}$.*

   Then by *1.*, we have:

$$H_n \circ R_{\alpha_{n-1}} \circ H_n^{-1} = H_{n-1} \circ h_n \circ R_{\alpha_{n-1}} \circ h_n^{-1} \circ H_{n-1}^{-1} = F_{n-1},$$

and so for every α_n sufficiently close to α_{n-1}, the map $F_n = H_n \circ R_{\alpha_n} \circ H_n^{-1}$ is sufficiently close to F_{n-1} to satisfy (P_2) and (P_3). Also observe that when $\alpha_n = \frac{p_n}{q_n}$ is close to α_{n-1}, then q_n is large (with $q_n \wedge p_n = 1$). For every $x = H_n(\theta, y) \in H_n(A_n)$, it holds that:

$$\mathsf{e}^{F_n}(x) = \frac{1}{q_n}\sum_{k=1}^{q_n} \delta_{H_n \circ R_{k\cdot p_n/q_n}(\theta,y)} = H_{n*}\left(\frac{1}{q_n}\sum_{k=1}^{q_n} \delta_{R_{k\cdot p_n/q_n}(\theta,y)}\right) = H_{n*}\left(\frac{1}{q_n}\sum_{k=1}^{q_n} \delta_{(\theta+kp_n/q_n,y)}\right)$$

is close to the pushed forward by H_n of $\mathrm{Leb}_{\mathbb{T}\times\{y\}}$ as q_n is large. Thus by *2'*, assumption (P_1) holds true with $B_n := H_n(A_n)$. □

Remark 2.3. We can develop the construction to obtain immediately that F has no periodic point. To this end, it suffices to add the following induction hypothesis[6]:

(P_4) for every $x \in \mathbb{A}_0$ and every $0 < k < q_{n-1}$, it holds:

$$d(F_n^k(x), x) > d(F_{n-1}^k(x), x) \cdot (1 - 2^{-n}) \ .$$

such an assumption is immediately verified by taking α_n sufficiently close to α_{n-1}. Also it implies that for every $k \geq 0$, with n such that $k < q_n$, it holds:

$$d(F^k(x), x) > d(F_n^k(x), x) \cdot \prod_{m\geq n} (1 - 2^{-m}) > 0 \ .$$

Consequently F has no periodic point.

[6] Actually using rotation number theory, one can show F has no periodic point without this extra assumption, but this argument seems difficult to adapt to the complex settting.

2.2.2 Existence of ergodic analytic pseudo-rotations

Let $\mathbb{E} := \mathbb{A} \setminus \mathbb{A}_0$ and $\mathbb{E}_{\mathbb{C}} := \mathbb{C}/\mathbb{Z} \times \{y \in \mathbb{C} : \mathrm{Re}(y) \notin \mathbb{I}\}$. We recall that:

$$K_\rho := \mathbb{T}_\rho \times Q_\rho \quad \text{where } \mathbb{T}_\rho := \mathbb{T} + i[-\rho, \rho] \quad \text{and} \quad Q_\rho := [-\rho, -1] \sqcup [1, \rho] + i[-\rho, \rho] \quad \forall \rho > 1 \ . \tag{2.1}$$

Similarly to the proof of Theorem 1.7, we are going to show:

Claim 2.4. *Let $\rho > 1$. There exist of a sequence of entire automorphisms $H_n \in \mathrm{Ham}^\omega_\rho(\mathbb{A})$ and a sequence of rational numbers α_n such that:*

- *(i) The sequence $(\alpha_n)_n$ converges to a number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and the sequence of restrictions $H_n|\mathbb{E}_{\mathbb{C}}$ converges to a map H which is analytic on $int(\mathbb{E}_{\mathbb{C}})$ and on $\mathbb{C}/\mathbb{Z} \times \{-1, 1\}$ with $\sup_{K_\rho} |H(x) - x| < \rho^{-1}$.*
- *(ii) The sequence $(F_n)_n$ of maps $F_n := H_n \circ R_{\alpha_n} \circ H_n^{-1}$ converges to an entire automorphism $F \in \mathrm{Symp}^\omega(\mathbb{A})$ which is ergodic on $\mathbb{A}' := \mathbb{A} \setminus H(\mathbb{E})$.*
- *(iii) The map F has no periodic point in $\mathbb{C}/\mathbb{Z} \times \mathbb{C}$.*

Proof that Claim 2.4 implies Theorems A. It remains only to show that $\mathbb{A}' := \mathbb{A} \setminus H(\mathbb{E})$ is an analytic cylinder. Note that the boundary of $cl(\mathbb{A}')$ is equal to the one of $H(\mathbb{E})$, and the latter is $H(\partial\mathbb{E}) = H(\mathbb{T} \times \{-1, 1\})$. By (i), the restriction $H|\partial\mathbb{E}$ is analytic and by Cauchy inequality, it is C^1-close to the canonical inclusion $\mathbb{T} \times \{-1, 1\} \hookrightarrow \mathbb{A}$ when ρ is large. Thus we can chose ρ large enough so that $H(\partial\mathbb{E}) = H(\mathbb{T} \times \{-1, 1\})$ is the union of two analytic graphs and so that $\mathbb{A}'$ is an analytic cylinder. □

Similarly to Anosov-Katok's proof, Claim 2.4 is proved by induction using a sequence $(\epsilon_n)_n$ of positive numbers such that:

$$\sum_n \epsilon_n < 1 \ .$$

For $\rho = n \geq 1$, Eq. (2.1) defines sets $\mathbb{T}_n$, Q_n and $K_n = \mathbb{T}_n \times Q_n$. Put:

$$\mathbb{D}(n) := \{x \in \mathbb{C} : |x| < n\}.$$

We are going to prove:

Claim 2.5. *For any $N \geq 1$, there are sequences of rational numbers $\alpha_n = \frac{p_n}{q_n} \to \alpha \in \mathbb{R} \setminus \mathbb{Q}$ and of entire automorphisms $H_n \in \mathrm{Ham}^\omega(\mathbb{A})$ such that with $\alpha_0 = 1$, $H_0 = id$ and $F_n := H_n \circ R_{\alpha_n} \circ H_n^{-1}$ for every $n \geq 1$, it satisfies:*

- *(P_0) The restrictions $H_n|K_{n+N}$ and $H_{n-1}|K_{n+N}$ are ϵ_{n+N}-C^0-close with $H_0 = id$.*
- *(P_1) for every $x \in H_n(\mathbb{A}_{\epsilon_n})$, the empirical measure $\mathsf{e}^{F_n}(x) = \mathsf{e}^{F_n}_{q_n}(x)$ is ϵ_n-close to $H_{n-1*}\mathrm{Leb}_{\mathbb{A}_0}$.*
- *(P_2) The distance between $\mathsf{e}^{F_n}_k(x)$ and $\mathsf{e}^{F_{n-1}}_k(x)$ is smaller than ϵ_n for every $k \leq q_{n-1}$ and $x \in \mathbb{T}_n \times \mathbb{D}(n)$.*
- *(P_3) The C^0-distance between $F_n|\mathbb{T}_n \times \mathbb{D}(n)$ and $F_{n-1}|\mathbb{T}_n \times \mathbb{D}(n)$ is smaller than ϵ_n. The C^0-distance between $F_n^{-1}|\mathbb{T}_n \times \mathbb{D}(n)$ and $F_{n-1}^{-1}|\mathbb{T}_n \times \mathbb{D}(n)$ is smaller than ϵ_n.*
- *(P_4) For every $x \in \mathbb{T}_n \times \mathbb{D}(n)$ and every $0 < k < q_{n-1}$, it holds:*

$$d(F_n^k(x), x) > d(F_{n-1}^k(x), x) \cdot (1 - 2^{-n}) \ .$$

Proof that Claim 2.5 implies Claim 2.4. Let $\rho > 1$ be given by Claim 2.4. Let $N > \rho$ be large enough such that $\sum_{n\geq N} \epsilon_n < \rho^{-1}$.

Let us prove (i). Observe that for every $x \in \mathbb{E}_{\mathbb{C}}$, there exists $N_0 \geq N$ such that $x \in K_{N_0}$. As $K_{N_0} \subset K_n$ for every $n \geq N_0$, by (P_0), the sequence $H_n(x)$ converges. Also if x is in $int(\mathbb{E}_{\mathbb{C}})$ (resp. in $\mathbb{C}/\mathbb{Z} \times \{-1, 1\}$), then there is a bidisk D (resp. disk) containing x and included in K_{N_0+1} (resp. $K_{N_0+1} \cap \mathbb{C}/\mathbb{Z} \times \{-1, 1\}$). So $(H_n(D))_n$ is bounded by (P_0). Thus H is analytic on $int(\mathbb{E}_{\mathbb{C}})$ and $\mathbb{C}/\mathbb{Z} \times \{-1, 1\}$. Moreover, observe that:

$$\sup_{x\in K_\rho} \|H(x) - x\| \leq \sup_{x\in K_N} \|H(x) - x\| \leq \sum_{n\geq 0} \sup_{x\in K_N} \|H_{n+1}(x) - H_n(x)\| \leq \sum_{n\geq N} \epsilon_n \leq \rho^{-1} .$$

Let us prove (ii). First observe that for the same reason as in the proof that Claim 2.4 implies Theorems A, for N large enough, the set $H_n(\mathbb{A}_0)$ is an analytic cylinder for every n and the sequence $(H_n(\mathbb{A}_0))_n$ converges to an analytic cylinder $\mathbb{A}_0' := \mathbb{A} \setminus H(\mathbb{E})$. Thus $H_{n*}\mathrm{Leb}_{\mathbb{A}_0}$ converges to the normalized Lebesgue measure $\mathrm{Leb}_{\mathbb{A}_0'}$ on the cylinder $\mathbb{A}_0'$ which equals $\bigcap_{n\geq 0} \bigcup_{k\geq n} H_k(\mathbb{A}_0)$:

$$\mathbb{A}_0' = \bigcap_{n\geq 0} \bigcup_{k\geq n} H_k(\mathbb{A}_0) \quad \text{and} \quad H_{n*}\mathrm{Leb}_{\mathbb{A}_0} \rightharpoonup \mathrm{Leb}_{\mathbb{A}_0'} .$$

By (P_3), we observe that $(F_n)_n$ converges to an entire real automophism F from $\mathbb{A}$ onto $\mathbb{A}$. As its Jacobean is constantly equal to 1, so it is actually an entire automorphism in $\mathrm{Symp}^\omega(\mathbb{A})$. By (P_0), the map F is integrable on $H(int\,\mathbb{E})$. By the Borel-Cantelli lemma, the subset $C := \bigcap_{n\geq 0} \bigcup_{k\geq n} H_k(\mathbb{A}_{\epsilon_k})$ has full Lebesgue measure in $\mathbb{A}_0' = \bigcap_{n\geq 0} \bigcup_{k\geq n} H_k(\mathbb{A}_0)$. Also by (P_1) and (P_2), for every $x \in C$, the measure $\mathrm{Leb}_{\mathbb{A}_0'}$ is an accumulation point of the sequence of empirical measures $(\mathsf{e}_n^F(x))_n$. Thus a.e. point in $\mathbb{A}_0'$ satisfies that $\mathsf{e}_n^F(x) = \mathrm{Leb}_{\mathbb{A}_0'}$ and so $F|\mathbb{A}_0'$ is ergodic.

Finally, using the same argument as in Remark 2.3, we obtain that (P_4) implies (iii). □

Proof of Claim 2.5. Let us show the Claim by induction on $n \geq 0$. Put $q_0 = 1$, $\alpha_0 = 0$ and $H_0 = id$. Let $n \geq 1$ and assume $(H_k)_{k<n}$ and $(\alpha_k = \frac{p_k}{q_k})_{k<n}$ constructed and satisfying the induction hypothesis. By Lemma 2.2, there exists $\mathring{h}_n \in \mathrm{Ham}_0^\infty(\mathbb{A})$ such that:

(1) the map $\mathring{h}_n$ is $(1/q_{n-1}, 0)$-periodic.

(2) for every $(\theta, y) \in A_{\epsilon_n}$, the map $\mathring{H}_n := H_{n-1} \circ \mathring{h}_n$ pushes forward the one-dimensional Lebesgue probability measure $\mathrm{Leb}_{\mathbb{T}\times\{y\}}$ to a measure which is $\epsilon_n/2$-close to $H_{n-1*}\mathrm{Leb}_{\mathbb{A}_0}$.

Let $\delta > 0$ be such that the support of $\mathring{h}_n$ is included in $\mathbb{A}_\delta$. By Corollary 1.10, for any C^∞-nghd $\mathcal{U}$ of the restriction $\mathring{h}_n|\mathbb{A}_\delta$, for any $\rho > 1$, there exists $h_n \in \mathrm{Ham}_\rho^\omega(\mathbb{A})$ which is $(1/q_n, 0)$-periodic and such that the restriction $h_n|\mathbb{A}_\delta$ is in $\mathcal{U}$. Now put:

$$H_n := H_{n-1} \circ h_n .$$

Observe that if we fixed $\rho > 1$ large enough depending on H_{n-1} and ϵ_{n+N}, then the restrictions $H_n|K_{n+N}$ and $H_{n-1}|K_{n+N}$ are ϵ_{n+N}-close, as stated by (P_0). Furthermore the measure $H_{n*}\mathrm{Leb}_{\mathbb{A}_0} = \mathrm{Leb}_{H_n(\mathbb{A}_0)}$ is $\epsilon_n/4$-close to $H_{n-1*}\mathrm{Leb}_{\mathbb{A}_0}$. We will fix $\mathcal{U}$ in the sequel. Again, by $(1/q_n, 0)$-periodicity of h_n, we have:

$$H_n \circ R_{\alpha_{n-1}} \circ H_n^{-1} = H_{n-1} \circ h_n \circ R_{\alpha_{n-1}} \circ h_n^{-1} \circ H_{n-1}^{-1} = F_{n-1},$$

and so for every α_n sufficiently close to α_{n-1}, $F_n = H_n \circ R_{\alpha_n} \circ H_n^{-1}$ is sufficiently close to F_{n-1} to satisfy (P_2), (P_3) and (P_4). Also observe that when $\alpha_n = \frac{p_n}{q_n}$ is close to α_{n-1}, then q_n is large (with $q_n \wedge p_n = 1$). Thus for every $x = H_n(\theta, y) \in H_n(\mathbb{A}_{\epsilon_n})$, it holds that

$$\mathsf{e}^{F_n}(x) = \frac{1}{q_n}\sum_{k=1}^{q_n} \delta_{H_n \circ R_{k\cdot p_n/q_n}(\theta,y)} = H_{n*}\left(\frac{1}{q_n}\sum_{k=1}^{q_n} \delta_{R_{k\cdot p_n/q_n}(\theta,y)}\right) = H_{n*}\left(\frac{1}{q_n}\sum_{1}^{q_n} \delta_{(\theta+kp_n/q_n,y)}\right)$$

is close to the push forward by H_n of $\mathrm{Leb}_{\mathbb{T}\times\{y\}}$. Thus by *(2)*, assuming $\mathcal{U}$ small enough, we obtain that $\mathsf{e}^{F_n}(x)$ is close to the push forward by $\mathring{H}_n$ of the one-dimensional Lebesgue probability measure $\mathrm{Leb}\,|\mathbb{T}\times\{y\}$, and so $\frac{3}{4}\epsilon_n$-close to $H_{n-1*}\mathrm{Leb}_{\mathbb{A}_0}$. As $H_{n*}\mathrm{Leb}_{\mathbb{A}_0}$ and $H_{n-1*}\mathrm{Leb}_{\mathbb{A}_0}$ are $\epsilon_n/4$-close, Property (P_1) holds true. □

2.3 Proof of Theorem C on pseudo-rotation with maximal local emergence

The proof of Theorem C follows the same lines as the one of Theorem A. We shall first give the proof of Theorem 1.11, which is the smooth version of Theorem C.

2.3.1 Existence of smooth pseudo-rotations with maximal local emergence

The proof of Theorem 1.11 is also done by induction using a sequence $(\epsilon_n)_n$ of positive numbers s.t.:

$$\sum_n \epsilon_n < 1\ .$$

We prove below:

Claim 2.6. *There are sequences of rational numbers $\alpha_n = \frac{p_n}{q_n}$ converging to $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, a decreasing sequence of positive numbers $(\eta_n)_n$ converging to 0 and a sequence of Hamiltonian maps $H_n \in \mathrm{Ham}_0^\infty(\mathbb{A})$ such that $H_0 = id$, $\alpha_0 = 1$ and for $n \geq 1$ it holds:*

(P_1) *There is a family $(I_{n,i})_{1\leq i\leq M_n}$ of $M_n \geq \exp(\eta_n^{-2+\epsilon_n})$ disjoint segments of $\mathbb{I}$ s.t.:*

- *(a) for any i, it holds $(1-\epsilon_n)\frac{2}{M_n} < \mathrm{Leb}\, I_{n,i} < \frac{2}{M_n}$; hence $\mathrm{Leb}\,(\mathbb{I}\setminus\bigcup_i I_{n,i}) < \frac{2}{M_n}\epsilon_n$,*
- *(b) for any $y \in I_{n,i}$ and $y' \in I_{n,j}$, $j \neq i$, the distance between $H_{n*}\mathrm{Leb}_{\mathbb{T}\times\{y\}}$ and $H_{n*}\mathrm{Leb}_{\mathbb{T}\times\{y'\}}$ is $> 2\eta_n$.*

(P_2) *It holds:*

$$\sup_{x\in\mathbb{A}} \|H_n(x) - H_{n-1}(x)\| < \epsilon_{n-1}\eta_{n-1} \quad \textit{and} \quad \sup_{x\in\mathbb{A}} \|H_n^{-1}(x) - H_{n-1}^{-1}(x)\| < \epsilon_{n-1}\eta_{n-1}\ .$$

(P_3) *The C^n-distance between $F_n := H_n \circ R_{\alpha_n} \circ H_n^{-1}$ and F_{n-1} is smaller than ϵ_n.*

Proof that Claim 2.6 implies Theorem 1.11. Let $(F_n)_n$ be given by the claim. By (P_3) and the convergence $\sum_n \epsilon_n$, the sequence $(F_n)_n$ converges to a C^∞-map $F \in \mathrm{Symp}^\infty(\mathbb{A})$. Let us show that F satisfies the properties of Theorem 1.11.

By (P_2), the sequence $(H_m)_m$ converges to a homeomorphism H. As each H_n coincides with the identity on the complement of $\mathbb{A}_0$, so does H and so F coincides with R_α on $\mathbb{A}\setminus\mathbb{A}_0$. Note also that H preserves the volume. Also by continuity of the composition it holds:

$$F = H \circ R_\alpha \circ H^{-1},$$

and so F is indeed C^0-conjugate to an irrational rotation.

Now let us show that F has maximal local emergence. For every n, i, let $B_{n,i} := \mathbb{T} \times I_{n,i}$ and $B_n := \bigcup_i B_{n,i}$. By (P_3), the set $C := \bigcap_{n\geq 0} \bigcup_{k\geq n} H_n(B_n)$ is included in $\mathbb{A}_0$. By $(P_1) - (a)$, the measure of $\bigcup_{k\geq n} H_n(B_n)$ is equal to the one of $\mathbb{A}_0$ and so the subset $C := \bigcap_{n\geq 0} \bigcup_{k\geq n} H_n(B_n)$ has full Lebesgue measure in $\mathbb{A}_0$.

Now let $x \in C$. Then there exist n arbitrarily large and i such that x belongs to $H_n(B_{n,i})$. Put $x = H_n(\theta, y)$. Then by $(P_1) - (a - b)$, we have:

$$\text{Leb}\left\{y' \in \mathbb{I} : d(H_{n*}(\text{Leb}_{\mathbb{T}\times\{y\}}), H_{n*}(\text{Leb}_{\mathbb{T}\times\{y'\}})) < 2\eta_n\right\} < \text{Leb}\, I_{n,i} + \text{Leb}\,(\mathbb{I} \setminus \cup_j I_{n,j}) \leq \frac{2(1+\epsilon_n)}{M_n}\,.$$

By (P_2) and Proposition 1.3, for every $m > n$, it holds:

$$\text{Leb}\left\{y' \in \mathbb{I} : d(H_{m*}(\text{Leb}_{\mathbb{T}\times\{y\}}), H_{m*}(\text{Leb}_{\mathbb{T}\times\{y'\}})) < (2 - \sum_{k=n}^{m-1} \epsilon_k)\eta_n\right\} < \frac{2(1+\epsilon_n)}{M_n}\,.$$

Taking the limit as $m \to \infty$ and infering that $\sum_k \epsilon_k < 1$, it comes:

$$\text{Leb}\left\{y' \in \mathbb{I} : d(H_*(\text{Leb}_{\mathbb{T}\times\{y\}}), H_*(\text{Leb}_{\mathbb{T}\times\{y'\}})) < \eta_n\right\} < \frac{2(1+\epsilon_n)}{M_n}\,.$$

As $F = H \circ R_\alpha \circ H^{-1}$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the empirical measure $\mathsf{e}^F(x')$ of $x' = H(\theta', y') \in \mathbb{A}_0$ is equal to $H_*(\text{Leb}_{\mathbb{T}\times\{y'\}})$. Thus:

$$\text{Leb}\left\{(\theta', y') \in \mathbb{A}_0 : d(\mathsf{e}^F \circ H(\theta, y), \mathsf{e}^F \circ H(\theta', y')) < \eta_n\right\} < \frac{2(1+\epsilon_n)}{M_n}\,.$$

Now we infer that $x = H(\theta, y)$ and that H is area preserving to obtain:

$$\text{Leb}\left\{x' \in \mathbb{A}_0 : d(\mathsf{e}^F(x), \mathsf{e}^F(x')) < \eta_n\right\} < \frac{2(1+\epsilon_n)}{M_n}\,.$$

Thus

$$\frac{\log|\log \text{Leb}\,\{x' \in \mathbb{A}_0 : d(\mathsf{e}^F(x), \mathsf{e}^F(x')) \leq \eta_n\}|}{-\log \eta_n} \geq \frac{\log|\log(2(1+\epsilon_n)) - \log M_n|}{-\log \eta_n}\,.$$

As $M_n > \exp(\eta_n^{-2+\epsilon_n})$, we obtain:

$$\frac{\log|\log \text{Leb}\,\{x' \in \mathbb{A}_0 : d(\mathsf{e}^F(x), \mathsf{e}^F(x')) \leq \eta_n\}|}{-\log \eta_n} \geq \frac{\log(\eta_n^{-2+\epsilon_n} - \log(2(1+\epsilon_n)))}{-\log \eta_n} \sim 2\,.$$

So the local order is a.e. ≥ 2. By Hefter's theorem [Hel22] and Theorem 1.4, the local order is at most 2 and so equal to 2. □

Proof of Claim 2.6. The proof is done by induction on $n \geq 0$. The step $n = 0$ is obvious. Assume $n \geq 1$, with H_{n-1} and $\alpha_{n-1} = \frac{p_{n-1}}{q_{n-1}}$ constructed.

We are going to construct H_n of the form $H_n = H_{n-1} \circ h_n$ with $h_n \in \text{Ham}_0^\infty(\mathbb{A})$ which is $(1/q_{n-1}, 0)$-periodic. Then, for α_n sufficiently close α_{n-1} we observe that (P_3) is satisfied (by Anosov-Katok's trick). Consequently, to prove the claim, it suffices to find a $(1/q_{n-1}, 0)$-periodic $h_n \in \text{Ham}_0^\infty(\mathbb{A})$ such that $H_n = H_{n-1} \circ h_n$ satisfies (P_1) and (P_2).

Let Q be a large multiple of q_{n-1} and let $(T_{k,\ell})_{0\le k,\ell<Q}$ be the tiling of $\mathbb{A}_0$ defined by:

$$T_{k,\ell} := \left\{\theta\in\mathbb{T} : \tfrac{k}{Q} < \theta < \tfrac{k+1}{Q}\right\} \times \left(-1+\tfrac{2\ell}{Q}, -1+\tfrac{2\ell+2}{Q}\right) .$$

When Q is large and if $h_n \in \mathrm{Ham}_0^\infty(\mathbb{A})$ leaves invariant the tiling, i.e. $h_n(T_{k,\ell}) = T_{k,\ell}$ for every (k,ℓ), then h_n and h_n^{-1} are Q^{-1}-C^0 close to the identity. We fix Q sufficiently large such that $H_n = H_{n-1}\circ h_n$ and its inverse are $\epsilon_{n-1}\eta_{n-1}$-C^0-close to respectively H_{n-1} and its inverse, as claimed in (P_2). Consequently, to prove the claim, it suffices to find a $(1/q_{n-1},0)$-periodic $h_n\in\mathrm{Ham}_0^\infty(\mathbb{A})$ which leaves invariant $(T_{k,\ell})_{0\le k,\ell<Q}$ and such that (P_1) is satisfied. In order to do so, we use the following shown in Section 4.3:

Lemma 2.7. *For every $1>\epsilon>0$, there exist $\eta>0$ arbitrarily small, an integer $M\ge\exp(\eta^{-2+\epsilon})$, a map $h\in\mathrm{Ham}^\infty(\mathbb{T}\times\mathbb{R}/2\mathbb{Z})$ and a family $(J_{\epsilon,i})_{1\le i\le M}$ of disjoint compact segments of $\mathbb{R}/2\mathbb{Z}$ such that:*

(a) it holds $\mathrm{Leb}\,(\mathbb{I}\setminus\bigcup_i J_{\epsilon,i}) = \frac{2}{M}\epsilon$ and $\mathrm{Leb}\,J_{\epsilon,i} = (1-\epsilon)\frac{2}{M}$ for every i,

(b) for any $j\ne i$, $y\in J_{\epsilon,i}$ and $y'\in J_{\epsilon,j}$, the distance between the measures $h_*\mathrm{Leb}_{\mathbb{T}\times\{y\}}$ and $h_*\mathrm{Leb}_{\mathbb{T}\times\{y'\}}$ is greater than 2η.

(c) h coincides with the identity on neighborhoods of $\{0\}\times\mathbb{R}/2\mathbb{Z}$ and $\mathbb{T}\times\{0\}$.

Let us apply Lemma 2.7 with $\epsilon=\epsilon_n/2$ fixed and some $\eta\in(0,\eta_{n-1})$ small enough so that

$$M \ge \exp(\eta^{-2+\epsilon_n/2}) \ge \exp(\eta_n^{-2+\epsilon_n}) \qquad \text{with } \eta_n := \eta/(Q\cdot\|DH_{n-1}\|_{C^0}) . \tag{2.2}$$

This is possible since the power $-2+\epsilon_n/2$ and $-2+\epsilon_n$ are different in the latter inequality.

Let $(J_{\epsilon,i})_{1\le i\le M}$ be the family of disjoint segments of $\mathbb{I}$ and let h be the map provided by the lemma. For every $0\le k,\ell<Q$, let:

$$\phi_{k,\ell} : (\theta,y)\in T_{k,\ell} \mapsto (Q\cdot\theta, Q\cdot y + Q - 2\ell) \in \mathbb{T}\times\mathbb{R}/2\mathbb{Z} .$$

Note that $\phi_{k,\ell}$ is a bijection from $T_{k,\ell}$ onto $(\mathbb{T}\setminus\{0\})\times(\mathbb{R}/2\mathbb{Z}\setminus\{0\})$. We define:

$$h_n := (\theta,y)\mapsto \begin{cases}(\theta,y) & \text{if } (\theta,y)\notin\bigcup_{0\le k,\ell<Q}T_{k,\ell}\,,\\ \phi_{k,\ell}^{-1}\circ h\circ\phi_{k,\ell}(\theta,y) & \text{if } (\theta,y)\in T_{k,\ell}\,.\end{cases} \tag{2.3}$$

Remark that h_n leaves invariant the tiling $(T_{i,j})_{0\le i,j<Q}$. Now let us verify (P_1). First let $M_n := Q\cdot M$. For $1\le i\le M_n$, put $i =: mM+\ell$ with $1\le\ell\le M$ and $0\le m<Q$. By identifying $J_{\epsilon,\ell}$ to a subset of $(0,2)$ we define:

$$I_{n,i} := J_{\epsilon,\ell}/Q - 1 + 2m/Q \subset (-1,1) = \mathbb{I} .$$

First observe that by Eq. (2.2), it holds:

$$M_n := Q\cdot M \ge Q\cdot\exp(\eta^{-2+\epsilon_n/2}) \ge \exp(\eta_n^{-2+\epsilon_n}),$$

as claimed in the begining of (P_1). Moreover by Lemma 2.7 (a), Property $(P_1)-(a)$ is verified as $\epsilon_n\ge\epsilon$. Also for any $y\in I_{\epsilon,i}\ne I_{\epsilon,j}\ni y'$, the distance between the measures $h_{n*}\mathrm{Leb}_{\mathbb{T}\times\{y\}}$ and $h_{n*}\mathrm{Leb}_{\mathbb{T}\times\{y'\}}$ is greater than $2\eta/Q$ and so the distance between the measures $H_{n*}\mathrm{Leb}_{\mathbb{T}\times\{y\}}$ and $H_{n*}\mathrm{Leb}_{\mathbb{T}\times\{y'\}}$ is greater than $2\eta/(Q\cdot\|DH_{n-1}\|_{C^0}) = 2\eta_n$ as claimed in $(P_1)-(b)$. □

2.3.2 Existence of analytic pseudo-rotations with maximal local emergence

For the same reasons, Theorem C is an immediate consequence of the following counterpart of Claim 2.4 where only (ii) has been changed.

Claim 2.8. *Let $\rho > 1$. There exist of a sequence of entire automorphisms $H_n \in \mathrm{Ham}^\omega(\mathbb{A})$ and a sequence of rational numbers α_n such that:*

- (i) *The sequence $(\alpha_n)_n$ converges to a number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and the sequence of restrictions $H_n|\mathbb{E}_\mathbb{C}$ converges to a map H which is analytic on $int(\mathbb{E}_\mathbb{C})$ and on $\mathbb{C}/\mathbb{Z} \times \{-1, 1\}$ with $\sup_{K_\rho} |H(x) - x| < \rho^{-1}$.*
- (ii) *the sequence $(F_n)_n$ of maps $F_n := H_n \circ R_{\alpha_n} \circ H_n^{-1}$ converges to an entire automorphism $F \in \mathrm{Symp}^\omega(\mathbb{A})$ whose restriction to $\mathbb{A}' := \mathbb{A} \setminus H(\mathbb{E})$ has local emergence 2.*
- (iii) *The map F has no periodic point in $\mathbb{C}/\mathbb{Z} \times \mathbb{C}$.*

Similarly, Claim 2.8 is proved below using some series $\sum_n \epsilon_n < 1$ of positive numbers and the following counterpart of Claim 2.5:

Claim 2.9. *For any $N \geq 1$, there are sequences of rational numbers $\alpha_n = \frac{p_n}{q_n} \to \alpha \in \mathbb{R} \setminus \mathbb{Q}$, of positive numbers $(\eta_n)_n$ converging to 0 and of entire automorphisms $H_n \in \mathrm{Ham}^\omega(\mathbb{A})$ such that $\alpha_0 = 1$, $H_0 = id$ and $F_n := H_n \circ R_{\alpha_n} \circ H_n^{-1}$, it satisfies for every $n \geq 1$:*

- (P_0) *the restrictions $H_n|K_{n+N}$ and $H_{n-1}|K_{n+N}$ are ϵ_{n+N}-C^0-close, with $H_0 = id$.*
- (P_1) *There is a family $(I_{n,i})_{1 \leq i \leq M_n}$ of $M_n \geq \exp(\eta_n^{-2+\epsilon_n})$ disjoint subsegments of $[-1+\delta_n, 1-\delta_n]$, with $\delta_n = \frac{1}{2}\epsilon_n/M_n$ s.t.:*
  - *(a) for any i, it holds $(1-\epsilon_n)\frac{2}{M_n} < \mathrm{Leb}\, I_{n,i} < \frac{2}{M_n}$; hence $\mathrm{Leb}\,(\mathbb{I} \setminus \bigcup_i I_{n,i}) < \frac{2}{M_n}\epsilon_n$,*
  - *(b) for any $y \in I_{n,i}$ and $y' \in I_{n,j}$, $j \neq i$, the distance between $H_{n*}\mathrm{Leb}_{\mathbb{T}\times\{y\}}$ and $H_{n*}\mathrm{Leb}_{\mathbb{T}\times\{y'\}}$ is $> \eta_n$.*
- (P_2) *It holds $\sup_{x \in \mathbb{A}_{\epsilon_n/M_n}} \|H_n(x) - H_{n-1}(x)\| < \epsilon_{n-1}\eta_{n-1}$.*
- (P_3) *The C^0-distance between $F_n|\mathbb{T}_n \times \mathbb{D}(n)$ and $F_{n-1}|\mathbb{T}_n \times \mathbb{D}(n)$ is smaller than ϵ_n. The C^0-distance between $F_n^{-1}|\mathbb{T}_n \times \mathbb{D}(n)$ and $F_{n-1}^{-1}|\mathbb{T}_n \times \mathbb{D}(n)$ is smaller than ϵ_n.*
- (P_4) *For every $x \in \mathbb{T}_n \times \mathbb{D}(n)$ and every $0 < k < q_{n-1}$, it holds:*
$$d(F_n^k(x), x) > (1 - 2^{-n}) \cdot d(F_{n-1}^k(x), x)\ .$$

Proof that Claim 2.9 implies Claim 2.8. Properties (i) and (iii) are proved as in the proof of that Claim 2.5 implies Claim 2.4, and likewise for the fact that $(F_n)_n$ converges to an entire automorphism $F \in \mathrm{Symp}^\omega(\mathbb{A})$. Using verbatim the same argument as in the proof of Claim 2.6 we obtain that the local emergence of $F|\mathbb{A}'_0$ is 2 as stated in (ii). □

Remark 2.10. Note that (P_2) implies that $(H_n|\mathbb{A}_0)_n$ converges for the C^0-compact-open topology. With slightly more work, one can also obtain that this limit is a homeomorphism onto its image. However, the construction does not imply that this limit extends $H : \mathbb{E} \hookrightarrow \mathbb{A}$ continuously (the union might be discountinuous at $\partial\mathbb{A}_0$), contrarily to the smooth case.

Proof of Claim 2.8. The proof is done basically as for Claim 2.5. The only change is that the map $\mathring{h}_n$ defined by Lemma 2.2 is replaced by the map defined in Eq. (2.3) using Lemma 2.7. Then we use Corollary 1.10 to obtain an analytic map H_n whose restriction to $\mathbb{A}_{\delta_n}$ is close to the one of $H_{n-1} \circ \mathring{h}_n$. This enables to obtain statements (P_1) and (P_2) of Claim 2.8 from the proof of statements (P_1) and (P_2) in Claim 2.6. □

2.4 Proof of the corollaries

Proof of Corollary B. By Theorem A, there is an entire symplectomorphism F of $\mathbb{A}$ which leaves invariant an analytic cylinder $\mathbb{A}_0' \subset \mathbb{A}$.

1.) For $\pm \in \{-,+\}$, let $\gamma^{\pm} \in C^{\omega}(\mathbb{T},\mathbb{R})$ be such that $\mathbb{A}_0' = \{(\theta,y) \in \mathbb{A} : \gamma^-(\theta) < y < \gamma^+(\theta)\}$. Note that up to a conjugacy with a map of the form $(\theta,y) \mapsto (\theta, \alpha \cdot y)$ with $\alpha > 0$, we can assume that the mean of $\gamma^+ - \gamma^-$ is 2. Now let $\ell(\theta) = (\gamma^+ - \gamma^-)(\theta)/2$. Note that $\int_{\mathbb{T}} \ell d\theta = 1$.

Thus there exists $L \in C^{\omega}(\mathbb{T},\mathbb{T})$ such that $\partial_\theta L = \ell$ and $L(0) = 0$. Note that $\partial_\theta L$ does not vanish and so L is a diffeomorphism. Set:

$$\psi : (\theta, y) \mapsto (L(\theta), y/\ell(\theta)).$$

Note that ψ is analytic and symplectic. Furthermore there is $\gamma^0 \in C^{\omega}(\mathbb{T},\mathbb{R})$ such that the image of $\mathbb{A}_0'$ by ψ is $\{(\theta,y) \in \mathbb{A} : \gamma^0(\theta) - 1 \leq y \leq \gamma^0(\theta) + 1\}$. Let h be the composition of ψ with the map $(\theta,y) \mapsto (\theta, y - \gamma^0(\theta))$. Observe that h sends $\mathbb{A}_0'$ to $\mathbb{A}_0$. Then $h \circ f \circ h^{-1}$ satisfies the sought properties.

2) Let $\hat{\mathbb{A}}_0$ be a neighborhood of $\mathbb{A}_0'$ which is bounded by two integrable curves $\Gamma_1 := \{(\theta, \gamma_1(\theta)) : \theta \in \mathbb{T}\}$ and $\Gamma_2 := \{(\theta, \gamma_2(\theta)) : \theta \in \mathbb{T}\}$. By [Ber22, Thm 2.14], as the system is integrable nearby these analytic curves, we can blow down them to transform $\hat{\mathbb{A}}_0$ to a sphere, so that the dynamics is pushed forward to an analytic and symplectic diffeomorphism of the sphere.

Let us give a short independent (sketch of) proof on how to complement $\hat{\mathbb{A}}_0$ to obtain a sphere. For $i \in \{1,2\}$, let V_i be a small neighborhood of Γ_i in which the dynamics is analytically conjugate to the rotation $(\theta,y) \in \mathbb{R}/\mathbb{Z} \times [-\epsilon,\epsilon] \mapsto (\theta+\alpha, y)$ via a map $\phi_i : V_i \to \mathbb{R}/2\pi\mathbb{Z} \times [-\epsilon,\epsilon]$ so that $\phi_i(\Gamma_i) = \mathbb{R}/2\pi\mathbb{Z} \times \{0\}$. Up to a composition with $-id$, we can assume that $\phi_i(V_i \cap \hat{\mathbb{A}}_0)$ is $\mathbb{R}/2\pi\mathbb{Z} \times [0,\epsilon]$. Now we compose ϕ_i with the analytic symplectomorphism $(\theta,r) \in \mathbb{R}/2\pi\mathbb{Z} \times [0,\epsilon] \mapsto (\sqrt{1+2r}\cos\theta, \sqrt{1+2r}\sin\theta)$. From this we obtain that the composition ψ_i is an analytic symplectomorphism from $V_i \cap \hat{\mathbb{A}}_0$ onto the annulus C_i of radii $(1, \sqrt{1+2\epsilon})$ and which conjugates $f|V_i \cap \hat{\mathbb{A}}_0$ to a map which coincides with the rotation of angle α on the disk of radius $\sqrt{1+2\epsilon}$. Now we glue the cylinder $\hat{\mathbb{A}}_0$ to two copies $\mathbb{D}_1$ and $\mathbb{D}_2$ of the disk of radius $\sqrt{1+2\epsilon}$ at C_1 and C_2 via resp. ψ_1 and ψ_2. This forms a sphere on which the dynamics extends canonically to the rotation of angle α on the inclusions of $\mathbb{D}_1$ and $\mathbb{D}_2$. This extension is ergodic on the inclusion of $\mathbb{A}_0'$ and have no more than two periodic points. □

Proof of Corollary D. Let F be given by Theorems A or C. We recall that F does not have periodic point in $\mathbb{C}/\mathbb{Z} \times \mathbb{C}$ and so that $J^* = \emptyset$. Hence it suffices so show that its Julia set J is nonempty. More precisely we are going to show that the cylinder $\mathbb{A}_0'$ is included in the Julia set of F. Let $x \in \mathbb{A}_0'$. Both the forward and backward orbits of x are bounded.

For the sake of contradiction, assume that there is an open neighborhood U of $x \in \mathbb{C}/\mathbb{Z} \times \mathbb{C}$ such that the following set is bounded:

$$U^+ := \bigcup_{n \in \mathbb{N}} F^n(U) \ .$$

Note that $F^{-1}(U^+) \supset U^+$. As F preserves the volume, $F^{-1}(U^+)$ and U^+ have the same volume. As $F^{-1}(U^+)$ is open, it cannot contains a point which is not in the closure of U^+. So $F^{-1}(U^+) \subset cl(U^+)$. Likewise we have $F^{-n}(U^+) \subset cl(U^+)$ for every n and so:

$$\hat{U} := \bigcup_{n \geq 0} F^{-n}(U^+) = \bigcup_{n \in \mathbb{Z}} F^n(U)$$

is a bounded F-invariant invariant open set. Then one proves as in [BS06, Appendix] or [Bed18, Thm 2.1], that $G := cl(\{F^n|\hat{U} : n \in \mathbb{Z}\}$ is a compact Abelian subgroup of complex automorphisms

of the bounded domain $\hat{U}$. Then by the Cartan Theorem [Nar71], the set G is a Lie group equal to an i-dimensional torus $\mathbb{T}^i$. As the set iterates of $F|\hat{U}$ is dense in G, the element $F|\hat{U}$ must act on G as an irrational rotation. Thus the restriction of F to the closure of the orbit of any $x \in \hat{U} \cap \mathbb{A}$ is semi-conjugate via an analytic map to an irrational rotation on $\mathbb{T}^i$. The rank of the semi-conjugacy must be invariant by the irrational rotation and so constant. Therefore, $G \cdot x = cl\{F^n(x) : n \in \mathbb{Z}\}$ must be a torus analytically embedded into $\mathbb{A}$. This torus cannot be of dimension ≥ 2 (as $\mathbb{T}^2$ cannot be embedded in $\mathbb{A}$) neither of dimension 0 (as F has no periodic point). Thus the orbit of x is included in an analytic circle (on which F acts as an irrational rotation). This implies that there is an analytic fibration by circles on $\mathbb{A}'_0$ which is left invariant by the dynamics. This is in contradiction to the case of Theorem A which asserts that a typical point of $\mathbb{A}'_0$ has the closure of its orbit equal to $\mathbb{A}'_0$. This is also in contradiction to the case of Theorem C which asserts that the local order of the emergence is 2, while by Example 1.5.2, a dynamics leaving invariant such a differentiable fibration should have an order of emergence 0. □

3 Approximation Theorems

In order to prove Theorem 1.8 we are going to study the generators of $\mathrm{Ham}_0^\infty(\mathbb{A})$ and $\mathrm{Ham}^\omega(\mathbb{A})$, see Theorems 3.3 and 3.4. In Section 3.4, we will prove that Theorem 3.4 implies Theorem 1.8.

3.1 Generators of Ham

The main theorem of [BT22] implies that $\mathrm{Ham}^\infty(\mathbb{A})$ is spanned by the following subgroups:

- Let $\mathcal{V}$ be the subgroup of $\mathrm{Ham}^\infty(\mathbb{A})$ of the form $(\theta, y) \mapsto (\theta, y + v(\theta))$ with $\int v d\theta = 0$.
- Let $\mathcal{T}$ be the subgroup of $\mathrm{Ham}^\infty(\mathbb{A})$ of the form $(\theta, y) \mapsto (\theta + \tau(y), y)$.

Theorem 3.1 (with Turaev)**.** *Any maps of* $\mathrm{Ham}^\infty(\mathbb{A})$ *can be arbitrarily well* C^∞*-approximated by a composition of maps in* $\mathcal{T}$ *or* $\mathcal{V}$

Interestingly, the proof is constructive. Also, as a consequence of the proof using a Lie bracket technique and Fourier's Theorem inspired from [BGH22], we will show that any maps of $\mathrm{Ham}^\infty(\mathbb{A})$ can be arbitrarily well approximated by a finite composition of maps in $\mathcal{T}$ or $[\mathcal{V}, \mathcal{T}]$, where:

$$[\mathcal{V}, \mathcal{T}] := \{[V, T] = V^{-1} \circ T^{-1} \circ V \circ T : V \in \mathcal{V} \ \& \ T \in \mathcal{T}\} .$$

Also the subgroup of entire maps of $\mathcal{T}$ is dense in $\mathcal{T}$ and the subgroup of entire maps of $\mathcal{V}$ is dense in $\mathcal{V}$. Yet to prove the main approximation Theorem 1.8, we should take care of the supports of the maps in this decomposition. First, we will see that any maps of $\mathrm{Ham}_0^\infty(\mathbb{A})$ can be arbitrarily well C^∞-approximated by a composition of maps in $\mathcal{T} \cap \mathrm{Ham}_0^\infty(\mathbb{A})$ or $[\mathcal{V}, \mathcal{T} \cap \mathrm{Ham}_0^\infty(\mathbb{A})] \cap \mathrm{Ham}_0^\infty(\mathbb{A})$. Indeed, the set $[\mathcal{V}, \mathcal{T} \cap \mathrm{Ham}_0^\infty(\mathbb{A})]$ is not included in $\mathrm{Ham}_0^\infty(\mathbb{A})$, so we shall study more carefully the supports of these decompositions by introducing with the following notations:

- For $\epsilon > 0$, let $\mathcal{V}_\epsilon := \{V \in \mathcal{V} : \sup_{x \in \mathbb{A}} \|V(x) - x\| < \epsilon\}$ be the C^0-ϵ-nghbd of id in $\mathcal{V}$.
- For $\delta > 0$, let $\mathcal{T}_\delta := \mathrm{Ham}_\delta^\infty(\mathbb{A}) \cap \mathcal{T}$, where $\mathrm{Ham}_\delta^\infty(\mathbb{A})$ is defined in Section 1.3.
- Let $[\mathcal{V}_\epsilon, \mathcal{T}_\delta] := \{[V, T] = V^{-1} \circ T^{-1} \circ V \circ T : V \in \mathcal{V}_\epsilon \ \& \ T \in \mathcal{T}_\delta\}$.

Most of the maps of $\mathcal{V}_\epsilon$ are not compactly supported. In contrast $\mathcal{T}_\delta$ and $[\mathcal{V}_\epsilon, \mathcal{T}_\delta]$ are formed by compactly supported maps if $\delta > \epsilon > 0$:

Fact 3.2. *For every $\delta > \epsilon > 0$, the set $[\mathcal{V}_\epsilon, \mathcal{T}_\delta]$ is formed by maps in* $\mathrm{Ham}^\infty_{\delta-\epsilon}(\mathbb{A})$.

Proof. Recall that $\mathbb{A} \setminus \mathbb{A}_{\delta-\epsilon} = \mathbb{T} \times (-\infty, -1+\delta-\epsilon) \sqcup \mathbb{T} \times (1-\delta+\epsilon, \infty)$. Let:

$$[V, T] = V^{-1} \circ T^{-1} \circ V \circ T \in [\mathcal{V}_\epsilon, \mathcal{T}_\delta].$$

Every z in $\mathbb{A} \setminus \mathbb{A}_{\delta-\epsilon}$ is fixed by T and sent by V into $\mathbb{A} \setminus \mathbb{A}_\delta$ on which T^{-1} is the identity. Thus $T^{-1} \circ V \circ T(z) = V(z)$ and so $[V, T](z) = z$. □

Here is a compactly supported counterpart of Theorem 3.1, that we will prove in Section 3.2:

Theorem 3.3. *For any $0 < \epsilon < \delta$, any map $F \in \mathrm{Ham}^\infty_\delta(\mathbb{A})$ and any C^∞-neighborhood $\mathcal{U}$ of F, there is a composition $\tilde{F} := F_1 \circ \cdots \circ F_M$ of maps F_j in $\mathcal{T}_\delta$ or in $[\mathcal{V}_\epsilon, \mathcal{T}_\delta]$ such that $\tilde{F}$ is in $\mathcal{U}$.*

It is indeed a compactly supported counterpart of Theorem 3.1 since by Fact 3.2, each map F_j belongs to $\mathrm{Ham}^\infty_0(\mathbb{A})$. We shall prove Theorem 1.8 by introducing the analytic counterpart of Theorem 3.3, which requires the following notations. First recall that for $\rho > 1$:

$$K_\rho := \mathbb{T}_\rho \times Q_\rho \quad \text{where } \mathbb{T}_\rho := \mathbb{T} + i[-\rho, \rho] \quad \text{and} \quad Q_\rho := [-\rho, -1] \sqcup [1, \rho] + i[-\rho, \rho] \ .$$

Recall that $\mathrm{Ham}^\omega(\mathbb{A})$ denotes the space of entire Hamiltonian maps of $\mathbb{A}$ and $\mathrm{Ham}^\omega_\rho(\mathbb{A})$ is its subset formed by maps whose restrictions to K_ρ are ρ^{-1}-C^0-close to the identity.

We will use the following generators for $\epsilon, \eta > 0$ and $\rho > 1$:

- Let $\mathcal{T}^\omega_\rho := \mathcal{T} \cap \mathrm{Ham}^\omega_\rho(\mathbb{A})$. In other words, it consists of maps of the form:

$$(\theta, y) \mapsto (\theta + \tau(y), y) \text{ where } \tau \text{ is an entire function satisfying } \sup_{Q_\rho} |\tau| < \rho^{-1}.$$

- Let $\mathcal{V}^\omega_\rho := \mathcal{V} \cap \mathrm{Ham}^\omega_\rho(\mathbb{A})$. In other words, it consists of maps of the form:

$$(\theta, y) \mapsto (\theta, y + v(\theta)) \text{ where } v \text{ is an entire function satisfying } \sup_{\mathbb{T}_\rho} |v| < \rho^{-1} \text{ and } \int_{\mathbb{T}} v \, d\mathrm{Leb} = 0 \ .$$

- For any $\rho_1, \rho_2 > 1$, let $[\mathcal{V}_{\rho_1}, \mathcal{T}_{\rho_2}] := \{[V, T] = V^{-1} \circ T^{-1} \circ V \circ T : V \in \mathcal{V}_{\rho_1} \ \& \ T \in \mathcal{T}_{\rho_2}\}$.

Here is the analytic counterpart of Theorem 3.3, that we will prove in Section 3.3:

Theorem 3.4. *For any $1 > \delta > 0$, for every sequence $(\rho_j)_{j \geq 0}$ of numbers > 1, for every map $F \in \mathrm{Ham}^\infty_\delta(\mathbb{A})$ and any neighborhood $\mathcal{U}$ of $F|\mathbb{A}_\delta$ in $C^\infty(\mathbb{A}_\delta, \mathbb{A})$, there is a composition $\tilde{F} := F_1 \circ \cdots \circ F_M$ of maps F_j in $\mathcal{T}^\omega_{\rho_j}$ or in $[\mathcal{V}^\omega_{\rho_0}, \mathcal{T}^\omega_{\rho_j}]$ whose restriction $\tilde{F}|\mathbb{A}_\delta$ is in $\mathcal{U}$.*

Let us now prove of the approximation Theorems 3.3 and 3.4.

3.2 The smooth case: proof of Theorem 3.3

For $\delta > 0$, let $C^\infty_\delta(\mathbb{A}, \mathbb{R})$ and $C^\infty_\delta(\mathbb{R}, \mathbb{R})$ be the spaces of smooth functions with support in respectively $\mathbb{A}_\delta$ and $[\delta - 1, 1 - \delta]$. We recall that the **symplectic gradient** of $H \in C^\infty(\mathbb{A}, \mathbb{R})$ is:

$$X_H := (\partial_y H, -\partial_\theta H) \ .$$

The proof of this theorem will use Poisson brackets. We recall that given two functions $f, g \in C^\infty(\mathbb{A}, \mathbb{R})$ the **Poisson bracket** $\{f, g\}$ is the function defined by:

$$\{f, g\} = \partial_\theta f \cdot \partial_y g - \partial_\theta g \cdot \partial_y f \ .$$

The Poisson bracket has the following well known property:

Proposition 3.5. *Let $H_1, H_2 \in C^\infty(\mathbb{A}, \mathbb{R})$ and $H = \{H_1, H_2\}$ be with respective symplectic gradients denoted by X_1, X_2 and X. Then X equals the Lie bracket of $[X_1, X_2] = DX_2(X_1) - DX_1(X_2)$:*

$$[X_1, X_2] = X \ .$$

In order to prove Theorem 3.3 we will use:

Lemma 3.6. *Let $\delta > 0$ and $H \in C^\infty_\delta(\mathbb{A}, \mathbb{R})$. For every neighborhood $\mathcal{U}$ of H in $C^\infty(\mathbb{A}, \mathbb{R})$, there exist a function $C \in C^\infty_\delta(\mathbb{R}, \mathbb{R})$ and $2N$ functions $A_1, \dots, A_N, B_1, \dots, B_N \in C^\infty(\mathbb{R}, \mathbb{R})$ such that the following is in $\mathcal{U}$:*

$$\tilde{H} : (\theta, y) \in \mathbb{A} \mapsto C(y) + \sum_{m=1}^{N} \left\{ \frac{\cos(2\pi m\theta)}{2\pi m}, A_m(y) \right\} + \left\{ \frac{\sin(2\pi m\theta)}{2\pi m}, B_m(y) \right\} \ .$$

Moreover the derivatives of the functions $A_1, \dots, A_N, B_1, \dots, B_N$ are in $C^\infty_\delta(\mathbb{R}, \mathbb{R})$.

Proof. Using Fourier's Theorem, there are $C \in C^\infty_\delta(\mathbb{R}, \mathbb{R})$ and $2N$ functions $a_1, \dots, a_N, b_1, \dots, b_N \in C^\infty_\delta(\mathbb{R}, \mathbb{R})$ such that the following is in $\mathcal{U}$:

$$\tilde{H}(\theta, y) \in \mathbb{A} \mapsto C(y) + \sum_{m=1}^{N} a_m(y) \cos(2\pi m\theta) + b_m(y) \sin(2\pi m\theta) \ .$$

Let $A_m := -\int_0^y b_m(t)dt$ and $B_m := \int_0^y a_m(t)dt$ whose derivatives are indeed in $C^\infty_\delta(\mathbb{R}, \mathbb{R})$. Note that:

$$\left\{ \frac{\cos(2\pi m\theta)}{2\pi m}, A_m(y) \right\} = b_m(y) \cdot \sin(2\pi m\theta) \quad \text{and} \quad \left\{ \frac{\sin(2\pi m\theta)}{2\pi m}, B_m(y) \right\} = a_m(y) \cdot \cos(2\pi m\theta) \ .$$

Thus $\tilde{H}$ has the sought form. □

A consequence of the later lemma and Proposition 3.5 is:

Lemma 3.7. *The map $\tilde{H}$ of Lemma 3.6 satisfies that there exist functions $(\tau_j)_{0 \le j \le 2N} \in C^\infty_\delta(\mathbb{R}, \mathbb{R})$ and $(v_j)_{1 \le j \le 2N} \in C^\infty(\mathbb{T}, \mathbb{R})$ such that with $\phi^t_{\tilde{H}}$ the time-t map of $\tilde{H}$ and with:*

$$T^t_j := (\theta, y) \mapsto (\theta + t\tau_j(y), y) \quad \text{and} \quad V^t_j := (\theta, y) \mapsto (\theta, y + tv_j(\theta))$$

it holds (at N fixed) in any C^r-topology when $t \to 0$:

$$\phi^{t^2}_{\tilde{H}} = [T^t_{2N}, V^t_{2N}] \circ \cdots \circ [T^t_j, V^t_j] \circ \cdots \circ [T^t_1, V^t_1] \circ T^{t^2}_0 + o(t^2) \ .$$

Moreover the C^0-norms of v_j are bounded by 1.

Proof. In the setting of Lemma 3.6, let $H_0 := (\theta, y) \in \mathbb{A} \mapsto C(y)$ and for $1 \le m \le N$, put $H_{2m} := (\theta, y) \in \mathbb{A} \mapsto \{\frac{1}{2\pi m} \cos(2\pi m\theta), A_m(y)\}$ and $H_{2m-1} := (\theta, y) \in \mathbb{A} \mapsto \{\frac{1}{2\pi m} \sin(2\pi m\theta), B_m(y)\}$. Let X_{H_j} be the symplectic gradient of H_j and $X_{\tilde{H}}$ the one of $\tilde{H}$. As $\tilde{H} = \sum_j H_j$, we have:

$$X_{\tilde{H}} = \sum_j X_{H_j} \ .$$

Hence, denoting $\phi^t_{H_j}$ the flot of X_{H_j}, in any C^r-topology $\infty > r \ge 1$, we have when $t \to 0$:

$$\phi^t_{\tilde{H}} = id + tX_{\tilde{H}} + O(t^2) = id + t \sum_{j=0}^{2N} X_{H_j} + O(t^2) \ . \tag{3.1}$$

Now observe that with $\tau_0 := C'(y)$ and $T_0^t(\theta, y) = (\theta + t\tau_0(y), y)$, it holds $id + tX_{H_0}(\theta, y) = T_0^t(\theta, y)$. Thus

$$\phi_{\tilde{H}}^t = (id + t\sum_{j=1}^{2N} X_{H_j}) \circ T_0^t + O(t^2) \ . \tag{3.2}$$

Also for $1 \leq j \leq 2N$, we have H_j of the form $H_j : (\theta, y) \in \mathbb{A} \mapsto \{f_j(\theta), g_j(y)\}$. Put:

$$v_j := -f_j' \quad \text{and} \quad \tau_j := g_j' \ .$$

Note that $\sup |f_j'| = 1$. With $T_j^t(\theta, y) = (\theta + t\tau_j(y), y)$ and $V_j^t(\theta, y) = (\theta, y + tv_j(\theta))$, the symplectic gradients of $(\theta, y) \mapsto f_j(\theta)$ and $(\theta, y) \mapsto g_j(y)$ are:

$$X_{f_j}(\theta, y) = (0, v_j(\theta)) = \partial_t V_j(\theta, y) \quad \text{and} \quad X_{g_j}(\theta, y) = (\tau_j(y), 0) = \partial_t T_j(\theta, y) \ .$$

Then by Proposition 3.5, the symplectic gradient of H_j is equal to the Lie backet of the symplectic gradients of f_j and g_j:

$$X_{H_j} = [\partial_t T_j^t, \partial_t V_j^t] \ .$$

Thus we have in any C^r-topology:

$$id + t^2 X_{H_j} = [T_j^t, V_j^t] + O(t^3) \ .$$

Injecting the latter into Eq. (3.2) at time t^2, it comes:

$$\phi_{\tilde{H}}^{t^2} = [T_{2N}^t, V_{2N}^t] \circ \cdots \circ [T_1^t, V_1^t] \circ T_0^{t^2} + O(t^3) \tag{3.3}$$

which is the sought result. □

Proof of Theorem 3.3. For any $f \in \mathrm{Ham}_\delta^\infty(\mathbb{A})$, there is a smooth family $(H_t)_{t\in[0,1]}$ of Hamiltonians $H_t \in C_\delta^\infty(\mathbb{A})$ which defines a family $(f_t)_{t\in[0,1]}$ such that $f_0 = id$, $f_1 = f$ and $\partial_t f_t$ is the symplectic gradient of H_t. Observe that for M large, we have:

$$f = (f \circ f_{(M-1)/M}^{-1}) \circ \cdots \circ (f_{i/M} \circ f_{(i-1)/M}^{-1}) \circ \cdots \circ f_{1/M} \ .$$

Each $(f_{j/M} \circ f_{(j-1)/M}^{-1})$ is $O(M^{-2})$-C^r-close to the time $\tau = M^{-1}$ map ϕ_j^τ of the vector field $\partial_t f_{j/M}$. Thus in the C^r-topology:

$$f = \phi_M^\tau \circ \cdots \circ \phi_1^\tau + O(M^{-1}) \ .$$

Note that ϕ_j^τ is the time τ of the symplectic gradient of $H_{j/M} \in C_\delta^\infty(\mathbb{A})$. By Lemmas 3.6 and 3.7, with $t^2 := \tau$, each ϕ_j^τ is equal to a composition of elements in $\mathcal{T}_\delta$ and $[\mathcal{V}_t, \mathcal{T}_\delta]$ up to precision $o(\tau) = o(M^{-1})$. Thus f is C^r-close to a composition of elements in $\mathcal{T}_\delta$ and $[\mathcal{V}_t, \mathcal{T}_\delta]$ for any $t > 0$. We conclude by choosing $t^2 < \epsilon$. □

3.3 The analytic case: proof of Theorem 3.4

The proof of Theorem 3.4 follows basically the same lines as the one of Theorem 3.3, although we have to extend the bounds to a complex compact set and use the Runge theorem.

In this subsection, we fix $1 > \delta > 0$ and a sequence $(\rho_j)_{j\geq 0}$ of numbers > 1 as in the statement of Theorem 3.4. For every $n \geq 1$, put:

$$M_n := \sup_{z\in\mathbb{T}_{\rho_0}} |2\pi n \cdot \exp(2\pi i \cdot n \cdot z)| \ .$$

Then observe that the same proof as Lemma 3.6 gives:

Lemma 3.8. *Let $\delta > 0$ and $H \in C^\infty_\delta(\mathbb{A}, \mathbb{R})$. For every C^∞-neighborhood of $\mathcal{U}$ of H, there exist a function $C \in C^\infty_\delta(\mathbb{R}, \mathbb{R})$ and $2N$ functions $A_1, \dots, A_N, B_1, \dots, B_N \in C^\infty(\mathbb{R}, \mathbb{R})$ such that the following is in $\mathcal{U}$:*

$$\tilde{H} : (\theta, y) \in \mathbb{A} \mapsto C(y) + \sum_{m=1}^{N} \left\{ \frac{\cos(2\pi m\theta)}{M_m}, A_m(y) \right\} + \left\{ \frac{\sin(2\pi m\theta)}{M_m}, B_m(y) \right\} .$$

Moreover the derivatives of the functions $A_1, \dots, A_N, B_1, \dots, B_N$ are in $C^\infty_\delta(\mathbb{R}, \mathbb{R})$.

Then by plugging Lemma 3.8 instead of Lemma 3.6 in the proof of Lemma 3.7 we obtain:

Lemma 3.9. *The map $\tilde{H}$ of Lemma 3.8 satisfies that there exist functions $(\tau_j)_{0 \le j \le 2N} \in C^\infty_\delta(\mathbb{R}, \mathbb{R})$ and $(v_j)_{1 \le j \le 2N} \in C^\infty(\mathbb{T}, \mathbb{R})$ such that with ϕ^t_H the time-t map of H and with:*

$$T^t_j := (\theta, y) \mapsto (\theta + t\tau_j(y), y) \quad \text{and} \quad V^t_j := (\theta, y) \mapsto (\theta, y + tv_j(\theta)) ,$$

it holds (at N fixed) in any C^r-topology, when $t \to 0$:

$$\phi^{t^2}_H = [T^t_{2N}, V^t_{2N}] \circ \cdots \circ [T^t_j, V^t_j] \circ \cdots \circ [T^t_1, V^t_1] \circ T^{t^2}_0 + o(t^2) .$$

Moreover each v_j is entire and satisfies $\sup\{|v_j(z)| : z \in \mathbb{T}_{\rho_0}\} \le 1$.

Using the latter lemma instead of Lemma 3.7 in the proof of Theorem 3.3 gives immediately:

Lemma 3.10. *Any $F \in \mathrm{Ham}^\infty_\delta(\mathbb{A})$ can be arbitrarily well C^∞-approximated by a composition $F_M \circ \cdots \circ F_1$ with each F_j in $\mathcal{T}_\delta$ or in $[\mathcal{V}^\omega_{\rho_0}, \mathcal{T}_\delta]$.*

We are now ready for the:

Proof of Theorem 3.4. Let $1 > \delta > 0$ and let $(\rho_j)_{j \ge 0}$ be a sequence of numbers > 1 as in the statement of the theorem. As the map $\rho \mapsto \mathcal{V}^\omega_\rho$ is decreasing, we can assume:

$$2\rho_0^{-1} < \delta . \tag{3.4}$$

Let $F \in \mathrm{Ham}^\infty_\delta(\mathbb{A})$ and let $\mathcal{U}$ be a neighborhood of the restriction of $F|\mathbb{A}_\delta$ in the C^∞-topology.

By Lemma 3.10, there exists a composition $F_M \circ \cdots \circ F_1$ whose restriction to $\mathbb{A}_\delta$ is in $\mathcal{U}$ and formed by maps F_j in $\mathcal{T}_\delta$ or in $[\mathcal{V}^\omega_{\rho_0}, \mathcal{T}_\delta]$. It suffices to change the maps F_j in $\mathcal{T}_\delta$ to maps in $\mathcal{T}^\omega_{\rho_j}$ and those in $[\mathcal{V}^\omega_{\rho_0}, \mathcal{T}_\delta]$ to maps in $[\mathcal{V}^\omega_{\rho_0}, \mathcal{T}^\omega_{\rho_j}]$ so that the restriction of their compositions to $\mathbb{A}_\delta = \mathbb{T} \times [-1+\delta, 1-\delta]$ remains in $\mathcal{U}$.

We recall that the support of each F_j is in $\mathbb{A}_{\delta - \rho_0^{-1}}$ by Fact 3.2. Thus F_j sends any small neighborhood of $\mathbb{A}_{\delta-\rho_0^{-1}}$ into another small neighborhood of $\mathbb{A}_{\delta-\rho_0^{-1}}$. Likewise, if $\tilde{F}_j$ is a map whose restriction to a neighborhood of $\mathbb{A}_{\delta-\rho_0^{-1}}$ is close to the one of F_j, then $\tilde{F}_j$ sends any small neighborhood of $\mathbb{A}_{\delta-\rho_0^{-1}}$ to another small neighborhood of $\mathbb{A}_{\delta-\rho_0^{-1}}$. Thus the composition $\tilde{F}_M \circ \cdots \circ \tilde{F}_1$ has its restriction to $\mathbb{A}_\delta \subset \mathbb{A}_{\delta-\rho_0^{-1}}$ close to the one of $F_M \circ \cdots \circ F_1$ and so close to $F|\mathbb{A}_\delta$. Hence it suffices to show:

Claim 3.11. *For every j, there exists $\tilde{F}_j$ in $\mathcal{T}^\omega_{\rho_j} \cup [\mathcal{V}^\omega_{\rho_0}, \mathcal{T}^\omega_{\rho_j}]$ whose restriction to a neighborhood of $\mathbb{A}_{\delta-\rho_0^{-1}}$ is C^∞-close to the one of F_j.*

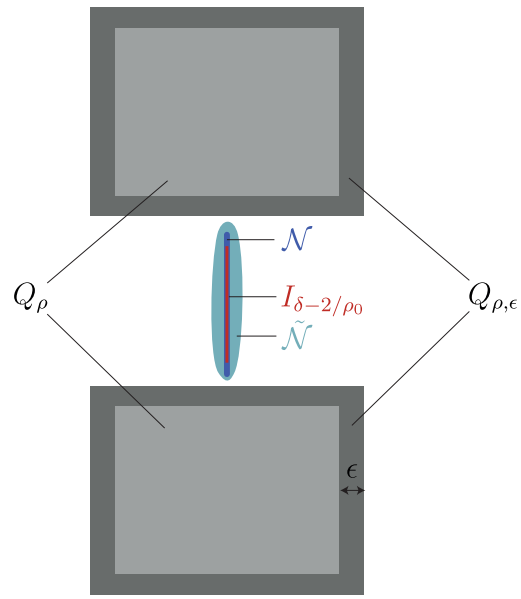

Figure 3: Domains of $\mathbb{C}$ involved in the application of the Runge theorem.

To show this claim, we fix a neighborhood $\mathcal{N}$ of $[-1+\delta-2\rho_0^{-1}, 1-\delta+2\rho_0^{-1}]$ which is disjoint from $\mathbb{R}\setminus(-1,1)$. Such a neighborhood exists by Eq. (3.4). There are two cases.

Case 1) If $F_j \in \mathcal{T}_\delta$, then there exists τ_j such that $F_j : (\theta, y) \mapsto (\theta + \tau_j(y), y)$. First, by the Weirstrass Theorem, there exists a polynomial C^∞-approximation $\hat{\tau}_j$ of $\tau_j|\mathcal{N}$. Now let $\tilde{\mathcal{N}}$ be a neighborhood of $cl(\mathcal{N})$ in $\mathbb{C}\setminus Q_{\rho_j}$ so that the complement of $\tilde{\mathcal{N}} \cup Q_{\rho_j}$ is connected. See Fig. 3.

Then using Runge's theorem, there exists an entire map $\tilde{\tau}_j$ which is C^0-close to $\hat{\tau}_j$ on $\tilde{\mathcal{N}}$ and which is ρ_j^{-1}-small on Q_{ρ_j}. As $\tilde{\tau}$ is real, up to considering $\frac{1}{2}\big(\tilde{\tau}(z) + \overline{\tilde{\tau}(\bar{z})}\big)$ instead of $\tilde{\tau}$, we can assume that $\tilde{\tau}$ is real. Then by the Cauchy inequality, the map $\tilde{\tau}_j$ restricted to $\mathcal{N}$ is C^∞-close to $\hat{\tau}_j|\mathcal{N}$ and so $\tau_j|\mathcal{N}$.

Then we observe that $\tilde{F}_j : (\theta, y) \mapsto (\theta + \tilde{\tau}_j(y), y)$ belongs to $\mathcal{T}^\omega_{\rho_j}$ and has its restriction to $\mathbb{T}\times\mathcal{N}$ close to the one of F_j.

Case 2) If $F_j \in [\mathcal{V}^\omega_{\rho_0}, \mathcal{T}_\delta]$, then there exist $V_j \in \mathcal{V}^\omega_{\rho_0}$ and $T_j : (\theta, y) \mapsto (\theta + \tau_j(y), y)$ such that $F_j = [V_j, T_j]$. As in the case (1), there is $\tilde{T}_j \in \mathcal{T}^\omega_{\rho_j}$ such that the restriction of T_j and $\tilde{T}_j$ to $\mathbb{T}\times\mathcal{N}$ are C^∞-close. Then put:

$$\tilde{F}_j = [V_j, \tilde{T}_j] = V_j^{-1} \circ \tilde{T}_j^{-1} \circ V_j \circ \tilde{T}_j \ .$$

Note that a small neighborhood of $\mathbb{A}_{\delta-\rho_0^{-1}}$ is sent by $\tilde{T}_j$ to a small neighborhood of $\mathbb{A}_{\delta-\rho_0^{-1}}$, which is then sent by V_j into a small neighborhood of $\mathbb{A}_{\delta-2\rho_0^{-1}}$, which is therefore included in $\mathbb{T}\times\mathcal{N}$. As $\tilde{T}_j^{-1}|\mathbb{T}\times\mathcal{N}$ is close to $T_j^{-1}|\mathbb{T}\times\mathcal{N}$, it comes that the restrictions of $\tilde{F}_j$ and F_j to a neighborhood of $\mathbb{A}_{\delta-\rho_0^{-1}}$ are C^∞-close. □

3.4 Proof that Theorem 3.4 implies Theorem 1.8

By Theorem 3.4, we know that every map $F \in \mathrm{Ham}^\infty_\delta(\mathbb{A})$ has its restriction to $\mathbb{A}_\delta$ which can be approximated by a compositon $\tilde{F} := F_1 \circ \cdots \circ F_M$ of maps F_j in $\mathcal{T}_{\rho_j} \cup [\mathcal{V}_{\rho_0}, \mathcal{T}_{\rho_j}]$, for any chosen sequence $(\rho_j)_j$ of numbers > 1. Yet we do not know if $\tilde{F}|K_\rho$ is small. Actually it might be not the case since the set K_ρ is a priori not left invariant by the maps F_j. Thus we shall first deduce

from Theorem 3.4 a similar statement (see Lemma 3.12) giving an approximation of $F|\mathbb{A}_\delta$ by the restriction of a composition of maps F_j which are moreover small on a neighborhood $K_{\rho,\epsilon}$ of K_ρ. Then we will show that the composition of these maps is indeed small on K_ρ.

This leads us to introduce a few technical notations.

Notations For $\epsilon > 0$, we denote $K_{\rho,\epsilon}$ and $Q_{\rho,\epsilon}$ the ϵ-neighborhoods of K_ρ and Q_ρ:

$$K_{\rho,\epsilon} = \mathbb{T}_{\rho+\epsilon} \times Q_{\rho,\epsilon} \quad \text{and} \quad Q_{\rho,\epsilon} := [-\rho-\epsilon, -1+\epsilon] \sqcup [1-\epsilon, \rho+\epsilon] + i[-\rho-\epsilon, \rho+\epsilon] \ .$$

Similarly, we denote $\mathrm{Ham}^\omega_{\rho,\epsilon}(\mathbb{A})$ the subset of $\mathrm{Ham}^\omega_\rho(\mathbb{A})$ formed by maps whose restrictions to $K_{\rho,\epsilon}$ is ρ^{-1}-C^0-close to the identity:

$$\mathrm{Ham}^\omega_{\rho,\epsilon}(\mathbb{A}) := \left\{ F \in \mathrm{Ham}^\omega_\rho(\mathbb{A}) : \sup_{x \in K_{\rho,\epsilon}} |F(x) - (x)| < \rho^{-1} \right\} \ .$$

Likewise we denote:

- Let $\mathcal{T}^\omega_{\rho,\epsilon} := \mathcal{T} \cap \mathrm{Ham}^\omega_{\rho,\epsilon}(\mathbb{A})$. In other words, it consists of maps of the form:

$$(\theta, y) \mapsto (\theta + \tau(y), y) \text{ where } \tau \text{ is an entire function satisfying } \sup_{Q_{\rho,\epsilon}} |\tau| < \rho^{-1}.$$

- Let $\mathcal{V}^\omega_{\rho,\epsilon} := \mathcal{V} \cap \mathrm{Ham}^\omega_{\rho,\epsilon}(\mathbb{A})$. In particular, its elements are of the form:

$$(\theta, y) \mapsto (\theta, y + v(\theta)) \text{ where } v \text{ is an entire function satisfying } \sup_{\mathbb{T}_{\rho+\epsilon}} |v| < \rho^{-1}.$$

- For any $\rho_1, \rho_2 > 1$, let $[\mathcal{V}^\omega_{\rho_1,\epsilon}; \mathcal{T}^\omega_{\rho_2,\epsilon}] := \{[V,T] = V^{-1} \circ T^{-1} \circ V \circ T : V \in \mathcal{V}^\omega_{\rho_1,\epsilon} \ \& \ T \in \mathcal{T}^\omega_{\rho_2,\epsilon}\}$.

Similarly to Theorem 3.4 we have:

Lemma 3.12. *For any $1 > \delta > \epsilon > 0$, for every sequence $(\rho_j)_{j\geq 0}$ of numbers > 1, for every map $F \in \mathrm{Ham}^\infty_\delta(\mathbb{A})$ and any C^∞-neighborhood $\mathcal{U}$ of $F|\mathbb{A}_\delta$, there is a composition $\tilde{F} := F_1 \circ \cdots \circ F_M$ of maps F_j in $\mathcal{T}^\omega_{\rho_j,\epsilon}$ or in $[\mathcal{V}^\omega_{\rho_0,\epsilon}, \mathcal{T}^\omega_{\rho_j,\epsilon}]$ such that $\tilde{F}|\mathbb{A}_\delta \in \mathcal{U}$.*

Proof. We can assume ρ_0 sufficiently large so that:

$$2\rho_0^{-1} + \epsilon < \delta \ . \tag{3.5}$$

We notice that $\mathcal{V}^\omega_{\rho_0+\epsilon} = \mathcal{V}^\omega_{\rho_0,\epsilon}$. Thus by Lemma 3.10 with $\rho_0 + \epsilon$ instead ρ_0, any $F \in \mathrm{Ham}^\infty_\delta(\mathbb{A})$ can be arbitrarily well C^∞-approximated by a composition $F_M \circ \cdots \circ F_1$ with each F_j in $\mathcal{T}_\delta$ or in $[\mathcal{V}^\omega_{\rho_0,\epsilon}, \mathcal{T}_\delta]$. Likewise it suffices to show:

Claim 3.13. *For every j, there exists $\tilde{F}_j$ in $\mathcal{T}^\omega_{\rho_j,\epsilon} \cup [\mathcal{V}^\omega_{\rho_0,\epsilon}, \mathcal{T}^\omega_{\rho_j,\epsilon}]$ whose restriction to a neighborhood of $\mathbb{A}_{\delta-\rho_0^{-1}}$ is C^∞-close to the one of F_j.*

To prove this claim, we proceed as for Claim 3.11. We chose a neighborhood $\mathcal{N}$ of $[-1+\delta-2\rho_0^{-1}, 1-\delta+2\rho_0^{-1}]$ which is disjoint from $\mathbb{R} \setminus (-1+\epsilon, 1-\epsilon)$. Such a neighborhood exists by Eq. (3.5). Then we distinguish two cases.

Case 1) If $F_j \in \mathcal{T}_\delta$, then it is of the form $F_j : (\theta, y) \mapsto (\theta + \tau_j(y), y)$. First, by the Weirstrass Theorem, there exists a polynomial C^∞-approximation $\hat{\tau}_j$ of $\tau_j|\mathcal{N}$. Now let $\tilde{\mathcal{N}}$ be a neighborhood of $cl(\mathcal{N})$ in $\mathbb{C} \setminus Q_{\rho_j,\epsilon}$ so that the complement of $\tilde{\mathcal{N}} \cup Q_{\rho_j,\epsilon}$ is connected. See Fig. 3. Then using

Runge's Theorem, there exists a real entire map $\tilde{\tau}_j$ which is C^0-close to $\hat{\tau}_j$ on $\tilde{\mathcal{N}}$ and which is ρ_j^{-1}-small on Q_{ρ_j}. Then by the Cauchy inequality, the map $\tilde{\tau}_j$ restricted to $\mathcal{N}$ is C^∞-close to τ_j.

Case 2) If $F_j \in [\mathcal{V}^\omega_{\rho_0,\epsilon}, \mathcal{T}_\delta]$, then there exist $V_j \in \mathcal{V}^\omega_{\rho_0,\epsilon}$ and $T_j : (\theta, y) \mapsto (\theta + \tau_j(y), y)$ such that $F_j = [V_j, T_j]$. As in the case (1), there is $\tilde{T}_j \in \mathcal{T}^\omega_{\rho_j,\epsilon}$ such that the restrictions of T_j and $\tilde{T}_j$ to a neighborhood of $\mathbb{T} \times \mathcal{N}$ are C^∞-close. Then we conclude exactly as in Claim 3.11. □

A second ingredient for the proof of Theorem 1.8 is the following counterpart of Fact 3.2:

Lemma 3.14. *For every $\epsilon > \epsilon' > 0$ and $\rho_k > \rho > 1$ such that $\epsilon > \epsilon' + 2\rho^{-1} > 0$, for all $[V, T] \in [\mathcal{V}^\omega_{\rho,\epsilon}\,;\, \mathcal{T}^\omega_{\rho_k,\epsilon}]$, it holds:*

$$\sup_{K_{\rho,\epsilon'}} \|[V,T] - id\| \leq \tfrac{1}{\rho'_k} \quad \textit{with } \rho'_k := \frac{\epsilon - \epsilon' - 2\rho^{-1}}{\epsilon - \epsilon' - \rho^{-1}} \cdot \rho_k \ .$$

Proof. Let $[V, T] \in [\mathcal{V}^\omega_{\rho,\epsilon}, \mathcal{T}^\omega_{\rho_k,\epsilon}]$. As $\rho_k > \rho$ and $\epsilon > \epsilon' + 2\rho^{-1}$, we have:

$$\mathbb{T}_{\rho+\epsilon'+2\rho_k^{-1}} \times Q_{\rho,\epsilon'+2\rho^{-1}} \subset K_{\rho,\epsilon'+2\rho^{-1}} \subset K_{\rho,\epsilon} \ . \tag{3.6}$$

Note that T sends $K_{\rho,\epsilon'} = \mathbb{T}_{\rho+\epsilon'} \times Q_{\rho,\epsilon'}$ into $\mathbb{T}_{\rho+\epsilon'+\rho_k^{-1}} \times Q_{\rho,\epsilon'}$ which is then sent by V into $\mathbb{T}_{\rho+\epsilon'+\rho_k^{-1}} \times Q_{\rho,\epsilon'+\rho^{-1}}$ and finally sent into $\mathbb{T}_{\rho+\epsilon'+2\rho_k^{-1}} \times Q_{\rho,\epsilon'+\rho^{-1}}$ by T^{-1}. Thus, Eq. (3.6), the following restrictions are equal:

$$[V,T]_{|K_{\rho,\epsilon'}} = (V^{-1})_{|K_{\rho,\epsilon'+2\rho^{-1}}} \circ (T^{-1})_{|K_{\rho,\epsilon'+2\rho^{-1}}} \circ V_{|K_{\rho,\epsilon'+2\rho^{-1}}} \circ T_{|K_{\rho,\epsilon'}} \ .$$

The restrictions to $K_{\rho,\epsilon}$ of V and its inverse are C^0-ρ^{-1} close to the canonical inclusion $K_{\rho,\epsilon} \hookrightarrow \mathbb{A}_{\mathbb{C}}$. Hence by the Cauchy inequality, the restriction of V^{-1} to $K_{\rho,\epsilon'+2\rho^{-1}}$ is C^1-$\rho^{-1}/(\epsilon - \epsilon' - 2\rho^{-1})$-close to the identity. Hence it holds:

$$\sup_{K_{\rho,\epsilon'}} \|V^{-1} \circ T^{-1} \circ V \circ T - T\| = \sup_{T(K_{\rho,\epsilon'})} \|V^{-1} \circ T^{-1} \circ V - id\| = \sup_{V \circ T(K_{\rho,\epsilon'})} \|V^{-1} \circ T^{-1} - V^{-1}\|$$

$$= \sup_{V \circ T(K_{\rho,\epsilon'})} \|(V^{-1})_{|K_{\rho,\epsilon'+2\rho^{-1}}} \circ T^{-1} - V^{-1}_{|K_{\rho,\epsilon'+2\rho^{-1}}}\| \leq \sup \|D(V^{-1})_{|K_{\rho,\epsilon'+2\rho^{-1}}}\| \cdot \sup_{V \circ T(K_{\rho,\epsilon'})} \|T^{-1} - id\|$$

$$\leq \frac{\rho^{-1}}{\epsilon - \epsilon' - 2\rho^{-1}} \cdot \sup_{V \circ T(K_{\rho,\epsilon'})} \|T^{-1} - id\| \leq \frac{\rho^{-1}}{\epsilon - \epsilon' - 2\rho^{-1}} \cdot \rho_k^{-1} \ .$$

Now we infere that $\sup_{K_{\rho_k,\epsilon}} \|T - id\| < \rho_k^{-1}$ to obtain:

$$\sup_{K_{\rho,\epsilon'}} \|[V,T]\| \leq \sup_{T(K_{\rho,\epsilon'})} \|V^{-1} \circ T^{-1} \circ V - id\| + \sup_{K_{\rho,\epsilon'}} \|T - id\| \leq \frac{\rho^{-1}}{\epsilon - \epsilon' - 2\rho^{-1}} \cdot \rho_k^{-1} + \rho_k^{-1}$$

which is $\rho_k'^{-1}$. □

Here is the last lemma needed for the proof of Theorem 1.8:

Lemma 3.15. *For any $\rho > 1$ and $\epsilon' > 0$, let $\rho'_k := 2^k \max\{\epsilon'^{-1}, \rho\}$ for every $k \geq 1$. Then for any sequence $(F_k)_{k\geq 1}$ of maps $F_k \in \mathrm{Ham}^\omega(\mathbb{A})$ such that $\sup_{K_{\rho,\epsilon'}} \|F_k - id\| < 1/\rho'_k$, it holds:*

$$F_1 \circ \cdots \circ F_n \in \mathrm{Ham}^\omega_\rho(\mathbb{A}) \textit{ for every } n \geq 1.$$

Proof. Let us first prove the case $n = 2$. First note that $F_2(K_\rho)$ is in $K_{\rho,1/\rho'_2} \subset K_{\rho,\epsilon'/4} \subset K_{\rho,\epsilon'}$. Thus:

$$\sup_{K_\rho} \|F_1 \circ F_2 - id\| \leq \sup_{F_2(K_\rho)} \|F_1 - id\| + \sup_{K_\rho} \|F_2 - id\| \leq \sup_{K_{\rho,\epsilon'}} \|F_1 - id\| + \sup_{K_\rho} \|F_2 - id\|$$

$$\leq 1/\rho'_1 + 1/\rho'_2 \leq \rho^{-1} \cdot (2^{-1} + 2^{-2}) < \rho^{-1} \ .$$

This proves that $F_1 \circ F_2$ belongs to $\mathrm{Ham}^\omega_\rho(\mathbb{A})$.

Now let us prove the general case for any $n \geq 2$. First note that $F_i(K_{\rho,\sum_{k>i} 2^{-k}\epsilon'})$ is included in $K_{\rho,\sum_{k\geq i} 2^{-k}\epsilon'}$. Thus by induction, $F_i \circ \cdots \circ F_n(K_\rho)$ is included in $K_{\rho,\sum_{k\geq i} 2^{-k}\epsilon'} \subset K_{\rho,\epsilon'}$. We have:

$$\sup_{K_\rho} \|F_1 \circ \cdots \circ F_n - id\| \leq \sum_{0\leq i\leq n-1} \sup_{F_i\circ\cdots\circ F_n(K_\rho)} \|F_{i+1} - id\| \leq \sum_{0\leq i\leq n-1} \sup_{K_{\rho,\epsilon'}} \|F_{i+1} - id\|$$

$$\leq \sum_{0\leq i\leq n-1} \rho'^{-1}_{i+1} \leq \sum_{0\leq i\leq n-1} 2^{-i-1} \cdot \rho^{-1} < \rho^{-1} \ .$$

This proves that $F_1 \circ \cdots \circ F_n$ belongs to $\mathrm{Ham}^\omega_\rho(\mathbb{A})$. □

Proof of Theorem 1.8. Let $0 < \delta < 1$ and $\rho > 1$. Let $F \in \mathrm{Ham}^\infty_\delta(\mathbb{A})$ and $\mathcal{U}$ a C^∞-nghd of the restriction $F|\mathbb{A}_\delta$. Let us show that there exists $\tilde{F} \in \mathrm{Ham}^\omega_\rho(\mathbb{A})$ such that $\tilde{F}|\mathbb{A}_\delta$ is in $\mathcal{U}$.

We can assume $\rho > 2/\delta$ and so that there are $\delta > \epsilon > \epsilon' > 0$ satisfying $\epsilon > \epsilon' + 2\rho^{-1}$. For every $k \geq 1$, let:

$$\rho'_k := 2^k \max\{\epsilon'^{-1}, \rho\} \quad \text{and} \quad \rho_k = \frac{\epsilon - \epsilon' - \rho^{-1}}{\epsilon - \epsilon' - 2\rho^{-1}} \cdot \rho'_k \ .$$

By Lemma 3.12, there is a composition $\tilde{F} := F_1 \circ \cdots \circ F_M$ of maps F_j in $\mathcal{T}^\omega_{\rho_j,\epsilon}$ or in $[\mathcal{V}^\omega_{\rho,\epsilon}, \mathcal{T}^\omega_{\rho_j,\epsilon}]$ such that $\tilde{F}|\mathbb{A}_\delta \in \mathcal{U}$. Thus by Lemma 3.14, each map F_j satisfies:

$$\sup_{K_{\rho,\epsilon'}} \|F_j - id\| < \rho'^{-1}_k \ .$$

Then by Lemma 3.15, we conclude that the composition $\tilde{F}$ is in $\mathrm{Ham}^\omega_\rho(\mathbb{A})$. □

4 Smooth Lemmas

4.1 A consequence of Moser's trick

Let (S, ω) be a surface of finite volume. We recall that $\mathrm{Symp}^\infty_0(S)$ denotes the space of compactly supported smooth symplectomorphisms of S. The following Folkloric theorem will be a key ingredient for the proof of Lemmas 2.2 and 2.7:

Theorem 4.1. *Let $(D_i)_{1\leq i\leq N}$ and $(D'_i)_{1\leq i\leq N}$ be two families of disjoints smooth closed disks in $\mathrm{int}\, S$, such that for every i, $\mathrm{Leb}\,(D_i) = D'_i$. Then there exists $f \in \mathrm{Symp}^\infty_0(S)$ such that $f(D_i) = D'_i$ for every $1 \leq i \leq N$.*

We were not able to find a proper reference for a proof of this lemma: Katok [Kat73, Basic Lemma §3] and Anosov-Katok [AK70, thm 1.3] show versions of this theorem which are weaker in dimension 2. Yet they wrote that A. B. Krygin had a proof of such result (without reference). Others would identify this theorem as a direct application of Moser's trick. Theorem 4.1 is proved by induction using:

Lemma 4.2. *For any subsets $D, D' \subset int\, S$, if D and D' are smooth closed disks and* $\mathrm{Leb}\,(D) = D'$, *then there exists $f \in \mathrm{Symp}_0^\infty(S)$ such that $f(D) = D'$.*

Proof. First we consider an isotopy $(g_t)_{t\in[0,1]}$ of S with support in the interior of S and such that $g_1(D) = D'$, this isotopy can be constructed by retracting the disks to a tiny ones and moving one to the other. Each g_t is a diffeomorphism that we shall transform to a symplectic map. Put $D_t := g_t(D)$. Using a tubular neighborhood of ∂D_t, it is easy to deform smoothly $(g_t)_{t\in[0,1]}$ so that g_t preserves the volume form nearby ∂D_t. Indeed it suffices to work in a tubular neighborhood of D_t and tune the size of the normal component of g_t.

So we can assume that $(g_t)_{t\in[0,1]}$ is a smooth isotopy of S such that $g_1(D) = D'$ and $g_t^*\omega = \omega$ nearby $\partial S \sqcup \partial D$. To prove the lemma it suffices to define a map ψ_1 so that $f := g_1 \circ \psi_1$ leaves invariant ω and ψ_1 and so f is equal to the identity nearby $\partial S \sqcup \partial D$.

We shall apply twice Moser's trick, one inside of D and resp. once outside D. Let $A_t := \mathrm{Leb}\, D_t/\mathrm{Leb}\, D$ and resp. $A_t := \mathrm{Leb}\,(S \setminus D_t)/\mathrm{Leb}\,(S \setminus D)$, where $D_t := g_t(D)$. Observe that $A_0 = A_1 = 1$. This defines a smooth homotopy $\omega_t := A_t^{-1} \cdot g_t^*\omega$ of symplectic forms that agree with ω on the neighborhood of ∂D (resp. $\partial D \sqcup \partial S$). Then by Moser's trick [MS17, Ex. 3.2.6], there exists a smooth isotopy $\psi_t : D \to D$ (resp. $S \setminus D \to S \setminus D$) such that: $\psi_0 = id$ and $\psi_t^*\omega_t = \omega$ and ψ_t is equal to the identity on a neighborhood of ∂D (resp. $\partial D \sqcup \partial S$). Then

$$(g_t \circ \psi_t)^*\omega = \psi_t^*(g_t^*\omega) = A_t^{-1}\psi_t^*\omega_t = A_t^{-1}\omega\ .$$

In particular $(g_1 \circ \psi_1)^*\omega = \omega$. The outer and inner constructions of ψ_1 are equal to the identity nearby ∂D , so they fit together to define a diffeomorphism ψ_1 of S which satisfies the thought properties. □

Proof of Theorem 4.1. By induction on $n \geq 1$, we construct a map $f_n \in \mathrm{Symp}_0^\infty(S)$ which sends $(D_i)_{1\leq i\leq n}$ to $(D'_i)_{1\leq i\leq n}$. The step $n = 1$ is the Moser theorem. Let $n > 1$. Let $S_n := S \setminus \bigcup_{i<n} D'_i$. By Lemma 4.2, there exists a map $f \in \mathrm{Symp}_0^\infty(S_n)$ which sends $f_{n-1}(D_n)$ to D'_n. We observe that $f_n := f \circ f_{n-1}$ satisfies the induction hypothesis. □

4.2 Lemma for Anosov-Katok's theorem

Lemma 2.2 states that:

Lemma 4.3. *For every $q \geq 1$ and $\epsilon > 0$, there exists a map $h \in \mathrm{Ham}_0^\infty(\mathbb{A})$ such that:*

1. *the map h is $(1/q, 0)$-periodic,*
2. *for every $(\theta, y) \in \mathbb{A}_\epsilon$ the map h sends the one-dimensional measure $\mathrm{Leb}_{\mathbb{T}\times\{y\}}$ to a measure which is ϵ-close to $\mathrm{Leb}_{\mathbb{A}_0}$.*

Proof. As both $\mathrm{Leb}_{\mathbb{A}_0}$ and each $\mathrm{Leb}_{\mathbb{T}\times\{y\}}$ are $(1/q, 0)$-periodic for every $q \geq 1$, it suffices to show the lemma in the case $q = 1$. Indeed then any q-covering of h will satisfy the lemma.

Hence for $\epsilon > 0$, it suffices to construct $h \in \mathrm{Ham}_0^\infty(\mathbb{A})$ such that for every $y \in \mathbb{I}_\epsilon := [-1+\epsilon, 1-\epsilon]$, the map h sends the one-dimensional Lebesgue probability measure $\mathrm{Leb}_{\mathbb{T}\times\{y\}}$ to a measure which is ϵ-close to $\mathrm{Leb}_{\mathbb{A}_0}$. The construction is depicted Fig. 1.

For $N = n^2$ large, we consider the following collection of boxes $(C_k)_{0\leq k\leq N-1}$:

$$C_k := [\tfrac{k+\epsilon/8}{N}, \tfrac{k+1-\epsilon/8}{N}] \times \mathbb{I}_\epsilon\ .$$

Note that the family $(C_k)_k$ is disjoint, does not intersect $\{0\} \times \mathbb{I}$ and that for every k we have:

$$\mathrm{Leb}_{\mathbb{A}_0} C_k = 2(1-\epsilon)(1-\epsilon/4)/\mathrm{Leb}\,\mathbb{A}_0 = (1-\epsilon)(1-\epsilon/4)\ .$$

Thus for every $v > (1-\epsilon)(1-\epsilon/4)$ close to enough to $(1-\epsilon)(1-\epsilon/4)$, there exists a disjoint family $(D_k)_{1\le k\le N}$ such that each D_k is a neighborhood C_k in $\mathbb{A}_0$ which does not intersect $\{0\}\times\mathbb{I}$, which is diffeomorphism to a close disk and whose volume is $\mathrm{Leb}_{A_0} C_k = v$.

Now recall that $N = n^2$. So there is a bijection $k \in \{0,\dots,N-1\} \to (x_k, y_k) \in \{0,\dots,n-1\}^2$. Let $z_k := (x_k/n, -1 + 2\cdot y_k/n) \in \mathbb{A}_0$. Note that for N large enough, the measure $N^{-1}\sum_{0\le k<N}\delta_{z_k}$ is $\frac{\epsilon}{2}$-close to $\mathrm{Leb}_{\mathbb{A}_0}$. Let $C'_k = z_k + [\epsilon/8n, (1-\epsilon/8)/n] \times [\epsilon/n, (2-\epsilon)/n]$. Note that the sets C'_k are disjoint, included in $\mathbb{A}_0 \setminus \{0\}\times\mathbb{I}$, have the same $\mathrm{Leb}_{\mathbb{A}_0}$-volume equal to $(1-\epsilon)(1-\epsilon/4)/n^2$ and their diameters are $< \sqrt{5}/n$. Likewise, for v close enough to $(1-\epsilon)(1-\epsilon/4)/n^2$, there exists a disjoint family $(D'_k)_{1\le k\le N}$ such that each $D'_k \subset \mathbb{A}_0 \setminus \{0\}\times\mathbb{I}$ is a neighborhood of C'_k which is diffeomorphic to a close disk and whose volume is v and whose diameter is $< \sqrt{5}/n$.

By Theorem 4.1, there exists $h \in \mathrm{Symp}^\infty(\mathbb{A}_0)$ which sends each D_k to D'_k and which is equal to the identity nearby $\partial\mathbb{A}_0$. Furthermore we can assume that it is equal to the identity nearby $\{0\}\times\mathbb{I}$. Thus $h \in \mathrm{Ham}^\infty(\mathbb{A}_0)$ and it can be extended to the identity to an element of $\mathrm{Ham}_0^\infty(\mathbb{A})$. Then observe that the Kantorovitch-Wasserstein distance between $h_*\mathrm{Leb}_{\mathbb{T}\times\{y\}}$ and $N^{-1}\sum_{k=1}^N \delta_{z_k}$ is smaller than:

$$\frac{1}{N}\sum_{k=1}^N \mathsf{d}(z_k, h(D_k)) + \mathrm{Leb}_{\mathbb{T}\times\{y\}}(\mathbb{T}\times\{y\} \setminus \bigcup_k C_k) \le \max_k \operatorname{diam} D'_k + \epsilon/4 \le \sqrt{5}/n + \epsilon/4$$

which is smaller than $\epsilon/2$ for n large enough. We recall that the distance between $N^{-1}\sum_{k=1}^N \delta_{z_k}$ and $\mathrm{Leb}_{\mathbb{A}_0}$ is $< \epsilon/2$ so the distance between $h_*\mathrm{Leb}_{\mathbb{T}\times\{y\}}$ and $\mathrm{Leb}_{\mathbb{A}_0}$ is $< \epsilon$. □

4.3 Lemma for emergence

Lemma 2.7 is:

Lemma 4.4. *For every $1 > \epsilon > 0$, there exist $\eta > 0$ arbitrarily small, an integer $M \ge \exp(\eta^{-2+\epsilon})$, a map $h \in \mathrm{Ham}^\infty(\mathbb{T}\times\mathbb{R}/2\mathbb{Z})$ and a family $(J_{\epsilon,i})_{1\le i\le M}$ of disjoint segments of $\mathbb{R}/2\mathbb{Z}$ such that:*

(a) it holds $\mathrm{Leb}\,(\mathbb{I}\setminus\bigcup_i J_{\epsilon,i}) = \frac{2}{M}\epsilon$ and $\mathrm{Leb}\, J_{\epsilon,i} = (1-\epsilon)\frac{2}{M}$ for every i,

(b) for any $j \ne i$, $y \in J_{\epsilon,i}$ and $y' \in J_{\epsilon,j}$, the distance between the measures $h_*\mathrm{Leb}_{\mathbb{T}\times\{y\}}$ and $h_*\mathrm{Leb}_{\mathbb{T}\times\{y'\}}$ is greater than η.

(c) h coincides with the identity on neighborhoods of $\{0\}\times\mathbb{R}/2\mathbb{Z}$ and $\mathbb{T}\times\{0\}$.

Proof. The proof of this lemma combines Theorem 4.1 and a development of [BB21, Prop 4.2]. The present construction is depicted Fig. 2. We fix $\epsilon > 0$ small enough to satisfy two inequalities (see Eqs. (4.1) and (4.2)) independent of a large integer $n \ge 3$ depending on ϵ. Let $N := 2\cdot n^2$ and let M be the integer part of $(2N)^{-1/4}e^{N/20}$:

$$N := 2\cdot n^2 \quad \text{and} \quad M = [(2N)^{-1/4}e^{N/20}]\,.$$

Now we consider the family of boxes $(B_{i,j})_{1\le i\le N/2, 1\le j\le M}$ defined by:

$$B_{i,j} := \left[(i+\epsilon^3/N)\frac{2}{N}, (i+1-\epsilon^3/N)\frac{2}{N}\right] \times J_{\epsilon,j} \quad \text{with } J_{\epsilon,j} := \left[(j+\epsilon/2)\frac{2}{M}, (j+1-\epsilon/2)\frac{2}{M}\right]\,.$$

The sets $(B_{i,j})_{i,j}$ are disjoint, included in $\mathbb{T}\times\mathbb{R}/2\mathbb{Z}\setminus(\{0\}\times\mathbb{R}/2\mathbb{Z}\cup\mathbb{T}\times\{0\})$ and have volume:

$$\mathrm{Leb}\, B_\bullet := \frac{2(1-2\epsilon^3/N)}{N}\cdot\frac{2(1-\epsilon)}{M} = \frac{2(1-\epsilon^3/n^2)}{n^2}\cdot\frac{(1-\epsilon)}{M}.$$

Note that $(J_{\epsilon,j})_j$ satisfies (a). We are going to define a conservative map h which sends each box $B_{i,j}$ into a union of N much larger boxes $(\tilde{B}_k)_{1\leq k\leq N}$.

Let us define $(\tilde{B}_k)_{1\leq k\leq N}$. As $N = 2\cdot n^2$, there is a bijection:

$$k \in \{1,\dots,N\} \to (x_k, y_k) \in \{0,\dots,n-1\}\times\{0,\dots,2n-1\}\ .$$

Let:

$$\tilde{B}_k = z_k + \left[\frac{\epsilon^2}{2}\frac{1}{n}, \left(1-\frac{\epsilon^2}{2}\right)\frac{1}{n}\right]^2 \quad \text{with } z_k := \tfrac{1}{n}(x_k, y_k) \in \mathbb{T}\times\mathbb{R}/2\mathbb{Z}\ .$$

The sets $(\tilde{B}_k)_k$ are ϵ^2/n^2-distant, included in $\mathbb{T}\times\mathbb{R}/2\mathbb{Z}\setminus(\{0\}\times\mathbb{R}/2\mathbb{Z}\cup\mathbb{T}\times\{0\})$ and have volume:

$$\operatorname{Leb}\tilde{B}_\bullet := (1-\epsilon^2)^2/n^2.$$

Now we shall determine in which $\tilde{B}_k$ each $B_{i,j}$ is sent to. To visualize this attribution, one can interpret each box $B_{i,j}$ as a colored pearl, each row $(B_{i,j})_{1\leq i\leq N/2}$ as a necklace and each $\tilde{B}_k$ as a pearl case of same color, see Fig. 2. Hence a map h which sends each $B_{i,j}$ into a certain $\tilde{B}_k$ define a coloring of each necklace. We are going to fix a coloring given by the lemma below. In the langage of colored necklace, it states that N colors suffices to color M necklaces of $N/2$ pearls each such that any pair of necklaces have at least a quarter of their pearls of different colors and such that each color is used approximately evenly among all the necklaces.

Lemma 4.5. *For every $N = 4\cdot n^2$ sufficiently large, there is a map:*

$$C : \{1,\dots,N/2\}\times\{1,\dots,M\} \to \{1,\dots,N\}$$

such that:

(1) For every $k\in\{1,\dots,N\}$, it holds $\#C^{-1}(\{k\}) \leq M(1+\epsilon^3/n^2)/2$.

(2) For every $j\neq j'$, it holds $\frac{2}{N}\#\{(i,i') : C(i,j) = C(i',j')\} < 3/4$.

This lemma is a development of the second step of [BB21, Prop 4.2]. It is proved below.

By (1), for every $1\leq k\leq N$, it holds:

$$\operatorname{Leb}\bigcup_{(i,j)\in C^{-1}(\{k\})} B_{i,j} = \#C^{-1}(\{k\})\cdot\operatorname{Leb}B_\bullet \leq \frac{M}{2}(1+\epsilon^3/n^2)\cdot\frac{2(1-\epsilon^3/n^2)}{n^2}\cdot\frac{(1-\epsilon)}{M} = \frac{(1-\epsilon)(1-\epsilon^6/n^4)}{n^2}\ .$$

On the other hand, we have $\operatorname{Leb}\tilde{B}_\bullet = (1-\epsilon^2)^2/n^2$. We assume $\epsilon>0$ sufficiently small so that

$$(1-\epsilon) < (1-\epsilon^2)^2\ . \tag{4.1}$$

Hence we have for every $k\in\{1,\dots,N\}$:

$$\operatorname{Leb}\bigcup_{(i,j)\in C^{-1}(\{k\})} B_{i,j} = \#C^{-1}(\{k\})\cdot\operatorname{Leb}B_\bullet < \operatorname{Leb}\tilde{B}_k\ .$$

So there exists v greater and arbitrarily close to $\operatorname{Leb}B_\bullet := \frac{2(1-\epsilon^3/n^2)}{n^2}\cdot\frac{(1-\epsilon)}{M}$ such that for every k:

$$\#C^{-1}(\{k\})\cdot v < \operatorname{Leb}\tilde{B}_k = (1-\epsilon^2)^2/n^2\ .$$

For v small enough, for every (i,j), there is a neighborhood $D_{i,j}$ of $B_{i,j}$ of volume v which is diffeomorphic to a closed disk, such that $(D_{i,j})_{i,j}$ are pairwise disjoint and disjoint from $\{0\}\times$

$\mathbb{R}/2\mathbb{Z} \cup \mathbb{T} \times \{0\}$. Also for v small enough, for every $1 \leq k \leq N$, there is a family $(\tilde{D}_{i,j})_{(i,j)\in C^{-1}(\{k\})}$ of disjoint disks which are included in $\tilde{B}_k$ and of volume v.

Now we apply Theorem 4.1 which asserts the existence of $h \in \mathrm{Ham}^\infty(\mathbb{T} \times \mathbb{R}/2\mathbb{Z})$ which sends each $D_{i,j}$ to $\tilde{D}_{i,j}$ and such that h coincides with the identity on neighborhoods of $\{0\} \times \mathbb{R}/2\mathbb{Z}$ and $\mathbb{T} \times \{0\}$. Hence Property (c) is satisfied. Let us verify Property (b).

For every i and $y \in J_{\epsilon,j}$, the circles $\mathbb{T}\times\{y\}$ intersects each $D_{i,j}$ at a set of length $> 2(1-\epsilon^3/n^2)/N$ and no more than a subset of $\mathbb{T} \times \{y\}$ of measure $< \epsilon^3/n^2$ is not included in some $D_{i,j}$. Thus for any $j' \neq j$ and $y' \in J_{\epsilon,j'}$, the distance between the measures $h_*\mathrm{Leb}_{\mathbb{T}\times\{y\}}$ and $h_*\mathrm{Leb}_{\mathbb{T}\times\{y'\}}$ is at least:

$$\min_{k\neq k'} d(\tilde{B}_k, \tilde{B}_{k'}) \cdot \big(\tfrac{N}{2} - \#\{(i,i') : C(i,j) = C(i',j')\}\big) \cdot \min_i \mathrm{Leb}\, D_{i,j} \cap \mathbb{T}\times\{y\} - \sqrt{2} \cdot \mathrm{Leb} \bigcup_i D_{i,j} \cap \mathbb{T}\times\{y\}$$

$$\geq \frac{\epsilon^2}{n} \cdot \frac{N}{8} \cdot \frac{2}{N}(1-\epsilon^3/n^2) - \sqrt{2} \cdot \epsilon^3/n^2 = \frac{\epsilon^2}{n} \cdot (\frac{1}{4} - \frac{1}{4}\epsilon^3/n^2 - \sqrt{2} \cdot \epsilon/n) \ .$$

We assume $\epsilon > 0$ sufficiently small so that

$$\frac{1}{4} - \frac{1}{4}\epsilon^3 - \sqrt{2} \cdot \frac{\epsilon}{n} > \frac{1}{5} \ . \tag{4.2}$$

Then the distance between the measures $h_*\mathrm{Leb}_{\mathbb{T}\times\{y\}}$ and $h_*\mathrm{Leb}_{\mathbb{T}\times\{y'\}}$ is at least:

$$\eta = \frac{\epsilon^2}{5 \cdot n},$$

as stated in (b). We recall that

$$M = [(2N)^{-1/4} e^{N/20}] = [(2n)^{-1/2} e^{n^2/10}] = \left[\left(\frac{2\epsilon^2}{5 \cdot \eta} \right)^{-1/2} e^{\frac{\epsilon^4}{250 \cdot \eta^2}} \right] \ .$$

As ϵ is fixed, when N is large, then η is small and so $\frac{\epsilon^4}{250\cdot\eta^2} > \eta^{\epsilon/2}$ and $\sqrt{\frac{5\cdot\eta}{2}} \cdot \epsilon^{-1} \cdot e^{\eta^{-2+\epsilon/2}} \geq e^{\eta^{-2+\epsilon}} + 1$. Thus:

$$M + 1 \geq \sqrt{\frac{5 \cdot \eta}{2}} \cdot \epsilon^{-1} \cdot e^{\frac{\epsilon^2}{50} \cdot \eta^{-2}} \geq \sqrt{\frac{5 \cdot \eta}{2}} \cdot \epsilon^{-1} \cdot e^{\eta^{-2+\epsilon/2}} \geq e^{\eta^{-2+\epsilon}} + 1.$$

This proves the lemma. □

Proof of Lemma 4.5. We consider the space:

$$\mathcal{F} := \{f : \{1, \ldots, N\} \to \{0,1\} : \sum_{i=1}^{N} f(i) = \tfrac{N}{2}\} \ .$$

We can associate to each element $f \in \mathcal{F}$ a coloring of a necklace formed by $N/2$ pearls. Indeed we can set that the color i is represented in the necklace iff $f(i) = 1$. This defines an increasing map:

$$C_f : \{1, \ldots, N/2\} \to \{1, \ldots, N\} \ .$$

We endow the space F with the Hamming distance:

$$\mathrm{Hamm}(f,g) := \#\{i \in \{1, \ldots, N\} | f(i) \neq g(i)\} \ .$$

Now chose randomly M elements $(f_j)_{1\le j\le M}$ of $\mathcal{F}$, for the equidistributed measure on the finite set $\mathcal{F}$ and we define:

$$C : (i,j) \in \{1,\dots,N/2\} \times \{1,\dots,M\} \mapsto C_{f_j}(i) \in \{1,\dots,N\} \ .$$

Let us show that the map C satisfies the statement of the lemma with positive probability.

For every $k \in \{1,\dots,N\}$, the probability of the event $\{f_j(k) = 1\}$ is $1/2$ for every j. Thus, with $\epsilon' := \epsilon^3/n^2$, by Bernstein's inequality the probability of the event $\{\#C^{-1}(\{k\}) \le M(1+\epsilon')/2\}$ is at most:

$$p_0 := 2 \cdot \exp\left(-\frac{M\epsilon'^2}{2(1+\frac{\epsilon'}{3})}\right) .$$

Thus the probability that Property *(1)* fails is $\le N \cdot p_0$ which is small when N is large.

Given $j \ne j'$, the probability p that $\#\{(i,i') : C(i,j) = C(i',j')\} \ge 3N/8$ is equal to $\#B/\#\mathcal{F}$ where B is a ball of radius $N/4$ in $\mathcal{F}$. The following bound is obtained by first relaxing the condition $\sum f(i) = N/2$ and then applying Berstein's inequality:

$$(4.3) \quad 2^{-N} \cdot \#B \le \mathrm{Prob}\left\{(X_n)_{1\le n\le N} \in \{0,1\}^N : \sum_n X_n \le \tfrac{N}{4}\right\}$$

$$= \mathrm{Prob}\left\{(X_n)_{1\le n\le N} \in \{-1,1\}^N : \tfrac{1}{N}\sum_n X_n > \tfrac{1}{2}\right\} \le \exp\left(-\frac{N\cdot(1/2)^2}{2(1+\frac{1}{3}\cdot\frac{1}{2})}\right) \le \exp(-\tfrac{3}{28N}) \le \exp(-\tfrac{N}{10}) \ .$$

On the other hand, the cardinality of $\mathcal{F}$ is $\binom{N}{N/2} \ge (2N)^{-1/2}2^N$ (by Stirling formula). Thus:

$$p = \frac{\#B}{\#\mathcal{F}} \le \sqrt{2N} \cdot \exp(-\tfrac{N}{10}) \ .$$

The probability to have obtained a map C which does not satisfy property *(2)* of the lemma is:

$$\sum_{k=1}^{M}(k-1)\cdot p \le \tfrac{1}{2}M^2 \cdot p \le \tfrac{1}{2}\left((2N)^{-1/4}e^{N/20}\right)^2 \cdot \sqrt{2N} \cdot \exp(-\tfrac{N}{10}) = \tfrac{1}{2} \ .$$

Hence the probability that both properties (1) and (2) fail is at most close to $1/2$. So there exists indeed C satisfying both properties. □

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