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Noncollapsed degeneration of Einstein 4-manifolds, II

TRISTAN OZUCH

In this second article, we prove that any desingularization in the Gromov-Hausdorff sense of an Einstein orbifold by smooth Einstein metrics is the result of a gluing-perturbation procedure that we develop. This builds on our first paper where we proved that a Gromov-Hausdorff convergence implied a much stronger convergence in suitable weighted Hölder spaces, in which the analysis of the present paper takes place.

The description of Einstein metrics as the result of a gluing-perturbation procedure sheds light on the local structure of the moduli space of Einstein metrics near its boundary. More importantly here, we extend the obstruction to the desingularization of Einstein orbifolds found by Biquard, and prove that it holds for any desingularization by trees of quotients of gravitational instantons only assuming a mere Gromov-Hausdorff convergence instead of specific weighted Hölder spaces. This is conjecturally the general case, and can at least be ensured by topological assumptions such as a spin structure on the degenerating manifolds. We also identify an obstruction to desingularizing spherical and hyperbolic orbifolds by general Ricci-flat ALE spaces.

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Introduction

An Einstein metric, g satisfies, for some real Λ , the equation

$$\text{Ric}(g) = \Lambda g.$$

In dimension 4, they are considered optimal for the homogeneity of their Ricci curvature, as critical points of the Einstein-Hilbert functional with fixed volume, $g \mapsto \int_M R_g \, d\text{vol}_g$, and more importantly as minimizers of the L^2 -norm of Riemann curvature tensor, $g \mapsto \int_M |\text{Rm}_g|^2 \, d\text{vol}_g$.

From dimension 4, even under natural assumptions of bounded diameter (compactness) and lower bound on the volume (noncollapsing) Einstein metrics can develop singularities. One major goal for 4-dimensional geometry is therefore to understand the moduli space of Einstein metrics on a differentiable manifold M^4 defined as

$$(1) \quad \mathbf{E}(M^4) := \{(M^4, g) \mid \exists \Lambda \in \mathbb{R}, \text{Ric}(g) = \Lambda g, \text{Vol}(M^4, g) = 1\} / \mathcal{D}(M^4).$$

and to compactify it with a useful structure. This has been done in an L^2 and then Gromov-Hausdorff (GH) sense in [And92, CT06]. More precisely, if we denote by $\overline{\mathbf{E}(M^4)}_{GH}$ the compactification of the moduli space $\mathbf{E}(M^4)$ for the (pointed) Gromov-Hausdorff distance, d_{GH} , we have a decomposition

$$(2) \quad \overline{\mathbf{E}(M^4)}_{GH} = \mathbf{E}(M^4) \cup \partial_o \mathbf{E}(M^4) \cup \partial_\infty \mathbf{E}(M^4),$$

where $\partial_\infty \mathbf{E}(M^4)$ consists in limits with infinite diameter, and $\partial_o \mathbf{E}(M^4)$ consists in singular limits with bounded diameter.

We will focus on local questions and for simplicity assume most of the time that we work on spaces with uniformly bounded diameter and therefore study the Gromov-Hausdorff neighborhood of $\partial_o \mathbf{E}(M^4)$. We therefore work on the d_{GH} -completion of $\mathbf{E}(M^4)$, which is $\mathbf{E}(M^4) \cup \partial_o \mathbf{E}(M^4)$. The metric spaces in $\partial_o \mathbf{E}(M^4)$ and the associated singularity blow-ups in the Gromov-Hausdorff sense have been understood for a long time in [And89, BKN89]: they are respectively *Einstein orbifolds* and *Ricci-flat ALE orbifolds*.

Anderson then asked the converse question for instance in [And10], namely, are all Einstein orbifolds limits of smooth Einstein? To answer this question one has to understand if the reverse of the degeneration, the *desingularization*, of Einstein orbifolds in $\partial_o \mathbf{E}(M^4)$ is possible. A natural way to desingularize an orbifold is by a gluing-perturbation technique.

The goal of the present paper is to develop a gluing-perturbation procedure which attains *any* noncollapsed Einstein 4-manifold which is sufficiently close to an Einstein orbifold in the Gromov-Hausdorff sense. This in particular elucidates the d_{GH} -neighborhood of the boundary $\partial_o \mathbf{E}(M^4)$ in $\mathbf{E}(M^4)$, and we will use this description in future works. In this paper, we will use it to prove that not all Einstein orbifolds can be desingularized by Einstein metrics in the Gromov-Hausdorff sense with an expected topology which partially answers the above question of Anderson.

Desingularization of Einstein 4-orbifolds and obstructions

A natural technique to desingularize orbifolds is the following procedure: we glue Ricci-flat ALE manifolds to the singularities of the orbifold to obtain an approximate Einstein metrics, and then try to perturb it into an actual Einstein metric. We will call such gluings, naïve desingularizations of the orbifold and often denote them g_t^D (see Definition 1.6), where t is the set of gluing parameters which are small positive real numbers. The main result of [Ozu19a] is that the Gromov-Hausdorff proximity of an Einstein metric to an Einstein orbifold implies the proximity to a naïve desingularization g_t^D in the sense of a weighted Hölder norm denoted $C_{\beta,*}^{2,\alpha}(g_t^D)$. This norm is bounded on symmetric 2-tensors decaying in the neck regions where the gluing takes place.

In the present paper, we will propose a partial converse by proving that any naïve desingularization can be perturbed to a metric which is Einstein modulo some *obstructions*, which are elements of an approximate cokernel of the linearization of the Einstein operator. We will call such a metric an *Einstein modulo obstructions* metric.

Theorem 0.1 (Theorem 4.6) *Let g_t^D be a naïve desingularization of an Einstein orbifold (M_o, g_o) with small enough gluing parameters.*

Then, there exists a small $C_{\beta, *}^{2, \alpha}(g_t^D)$ -neighborhood of g_t^D in which there exists a unique metric \hat{g}_t which is Einstein modulo obstructions while satisfying some gauge conditions with respect to g_t^D .

The proof relies on an inverse function theorem applied to the Einstein operator in well chosen coordinates.

These Einstein modulo obstructions metrics \hat{g}_t are not interesting for themselves when they aren't Einstein as they are not geometrically motivated. Their purpose is for instance different from the metrics of [GV16] which are critical for some geometric functionals obtained by perturbing a connected sum of Einstein metrics.

Let us note that the Ricci flow starting at Einstein modulo obstructions metrics however exhibits interesting behaviors with respect to the Ricci flow. Indeed, in [BK17], an obstruction is identified to a particular desingularization of $\mathbb{T}^4/\mathbb{Z}_2$ and an ancient solution to the Ricci flow smoothing out the orbifold $\mathbb{T}^4/\mathbb{Z}_2$ is constructed thanks to it.

Our construction however produces *every* smooth Einstein desingularization in a Gromov-Hausdorff sense. Indeed, together with the convergence of [Ozu19a], as a direct consequence, we have the following complete description of the Einstein metrics in a Gromov-Hausdorff neighborhood of an Einstein 4-orbifold.

Corollary 1 (Corollary 5) Let (M_o, g_o) be an Einstein 4-orbifold. Then, there exists $\delta > 0$ such that if $(M, g^{\mathcal{E}})$ is an Einstein manifold satisfying

$$d_{GH}((M, g^{\mathcal{E}}), (M_o, g_o)) \leq \delta,$$

then, $(M, g^{\mathcal{E}})$ is isometric to a result of the gluing-perturbation procedure of Theorem 0.1.

A premoduli space in the neighborhood of a singular metric

Classically, studying a moduli space requires understanding its compactification with a useful structure. The compactification (2) a priori does not carry a useful structure as it comes from the rough Gromov-Hausdorff distance. The moduli space $\mathbf{E}(M^4)$ however admits a real-analytic structure around smooth metrics.

Theorem 0.2 ([Koi83]) *Let (M, g_0) be an Einstein manifold. Then, there exists a d_{GH} -neighborhood of g_0 in $\mathbf{E}(M)$ which is the quotient by the isometry group of g_0 of a real-analytic subvariety of a finite dimensional real-analytic submanifold, W , of the space of smooth metrics on M .*

The finite dimensional real-analytic submanifold, W , consists in metrics which are Einstein modulo the cokernel of the linearization of the Einstein equation at g_0 as is usually obtained by Lyapunov-Schmidt reduction. The Einstein metrics are exactly the metrics for which these obstructions vanish. Our description extends this local description of $\mathbf{E}(M^4)$ to the boundary $\partial_o\mathbf{E}(M^4)$, and the set of Einstein modulo obstructions metrics \hat{g}_t of Theorem 0.1 is the analogue of the ambient space W of Theorem 0.2.

Theorem 0.2 is an important local result which implies for instance that $\mathbf{E}(M)$ is locally finite. Anderson asked in [And10] whether this structure extends to $\mathbf{E}(M^4) \cup \partial_o\mathbf{E}(M^4)$. The new description of the neighborhood of $\partial_o\mathbf{E}(M)$ in $(\overline{\mathbf{E}(M)}_{GH}, d_{GH})$ of Corollary 1 provides a promising setting in which one can tackle this question. In particular, in Section 4.3, we provide an adaptation to the singular setting of Koiso’s premoduli space around metrics of $\partial_o\mathbf{E}(M)$.

Degeneration of Kähler-Einstein manifolds

Even if our purpose here is to study the *real* Einstein equation and not Kähler-Einstein metrics, our analysis in weighted Hölder spaces extends the analysis leading to the gluing-perturbation theorems of [Ban90, Spo14, BR15, HV20] in the Kähler setting. Indeed, it allows us to glue and perturb multiple trees of singularities with arbitrary scales and Einstein deformations. It would therefore be interesting to extend the constructions of [Spo14, BR15] to remove the “generic” ([Spo14]) or “non degenerate” ([BR15]) conditions which correspond to restricting the gluing scales depending on the size of the Einstein deformation. We should also be able to allow general degenerations forming trees of singularities.

For instance, in Section 6.2, we precise the construction of [HV20] in the case of Kronheimer’s gravitational instantons and prove that any tree of Kähler Ricci-flat ALE spaces can be glued and perturbed to a single Kähler Ricci-flat ALE metric with uniform controls (in our weighted Hölder norms) only depending on the group at infinity.

Obstructions to the Gromov-Hausdorff desingularization of Einstein orbifolds

Our main application in this paper is a nonexistence result: there exists Einstein orbifolds which cannot be approached by smooth Einstein metrics with specific topologies in

the Gromov-Hausdorff sense. For this, it is enough to prove that the obstructions of Theorem 0.1 do not vanish.

The hyperkähler ALE spaces which are called *gravitational instantons* have been classified in [Kro89a] and their Kähler quotients have been classified in [Suv11]. It is a famous conjecture, [BKN89], that all Ricci-flat ALE spaces are Kähler.

Our first goal here is to prove that an obstruction holds for any Gromov-Hausdorff desingularization by trees of Kähler Ricci-flat ALE orbifolds, which are conjecturally the only possibilities. The obstruction to satisfy is $\det \mathbf{R} = 0$ at a singular point of the orbifold metric, where \mathbf{R} is the Riemannian curvature seen as an endomorphism on the space of 2-forms.

Theorem 0.3 (Theorem 6.7) *Let $(M_i, g_i)_i$ be a sequence of Einstein manifolds converging in the Gromov-Hausdorff sense to an Einstein orbifold (M_o, g_o) , and assume that there exists a subsequence $(M_i, g_i)_i$ whose possible blow-up limits are Kähler Ricci-flat ALE orbifolds glued in the same orientation.*

Then, at every singular point p of (M_o, g_o) , we have

$$\det \mathbf{R}_{g_o}(p) = 0.$$

This answers positively a question from [Biq13] and extends it to the case of several singularities and allows the formation of trees of singularities. It more precisely states that the obstruction of [Biq13] holds for any known possible configuration of singularity models, and that it holds even assuming the weakest possible convergence instead of a convergence in particular weighted Hölder spaces.

Remark 1 Note that this obstruction is very different in nature from the result of [OSS16] which shows that most compact Kähler-Einstein 4-orbifolds with positive Ricci-curvature cannot be limits of Kähler-Einstein manifolds. Indeed, our obstruction $\det \mathbf{R} = 0$ is always satisfied in this situation, and it remains unknown if these metrics can be desingularized by *real* Einstein metrics. It is also different from the obstruction found in [BK17] where the obstruction $\det \mathbf{R} = 0$ is also satisfied by the orbifold $\mathbb{T}^4/\mathbb{Z}_2$.

Under topological assumptions, it is known that the singularity models appearing are Kähler and glued in the same orientation, in particular we have the following illustration of our obstruction.

Example 1 (Corollary 15) Consider $\mathbb{S}^4 \subset \mathbb{R}^5$ and the quotient by \mathbb{Z}_2 given by $(x_1, x_2, x_3, x_4, x_5) \sim (x_1, -x_2, -x_3, -x_4, -x_5)$. We will denote this space $\mathbb{S}^4/\mathbb{Z}_2$ which is an Einstein orbifold with two $\mathbb{R}^4/\mathbb{Z}_2$ singularities. The minimal resolutions of the two singularities $\mathbb{R}^4/\mathbb{Z}_2 \approx \mathbb{C}^2/\mathbb{Z}_2$ has the topology $M := \mathbb{S}^4/\mathbb{Z}_2 \# T^*\mathbb{S}^2 \# T^*\mathbb{S}^2$, where $\#$ denotes the gluing of an ALE space to an orbifold along their asymptotic cone.

Then, for any $1 \leq p < \infty$, there exists a sequence of metrics $(M, g_i)_i$ with both

$$\|\text{Ric}(g_i) - 3g_i\|_{L^p(g_i)} \rightarrow 0 \text{ and } \text{Ric}(g_i) \geq 3g_i$$

while

$$(M, g_i) \xrightarrow{GH} (\mathbb{S}^4/\mathbb{Z}_2, g_{\mathbb{S}^4/\mathbb{Z}_2}),$$

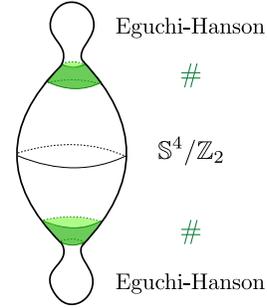
but there *does not* exist any sequence of Einstein metrics satisfying

$$\text{Ric}(g_i) = 3g_i,$$

and

$$(M, g_i) \xrightarrow{GH} (\mathbb{S}^4/\mathbb{Z}_2, g_{\mathbb{S}^4/\mathbb{Z}_2}).$$

In the same fashion, a conjecture of Anderson states that there is no sequence of asymptotically hyperbolic Einstein metrics on $T^*\mathbb{S}^2$ desingularizing the hyperbolic orbifold $\mathbb{H}^4/\mathbb{Z}_2$ obtained by antipodal identification in a global geodesic chart. It was proven in [Biq13] assuming among other things a convergence speed in weighted spaces towards the orbifold depending on the maximum of the curvature. We can prove it assuming a pointed Gromov-Hausdorff convergence together with a suitable control in weighted spaces at infinity, this time independent on the maximum of the curvature. It is again possible to desingularize $\mathbb{H}^4/\mathbb{Z}_2$ with Ricci pinched in any L^p , space for $1 \leq p < \infty$ or with Ric bounded above or below by -3 .



Hitchin-Thorpe inequality and degeneration of Einstein manifolds

The Hitchin-Thorpe inequality provides a topological obstruction to the existence of Einstein metrics on a given 4-dimensional differentiable manifold M ,

$$2\chi(M) \geq 3|\tau(M)|,$$

where χ is the Euler characteristic, and τ the signature. These topological invariants have definitions adapted to orbifolds and ALE spaces which we will denote $\tilde{\chi}$ and $\tilde{\tau}$, and any orbifold M_o admitting an Einstein metric satisfies

$$2\tilde{\chi}(M_o) \geq 3|\tilde{\tau}(M_o)|.$$

Any Gromov-Hausdorff desingularization damages this inequality, and the equality case implies the obstruction.

Theorem 0.4 (Theorem 7.2) *Let (M_o, g_o) be an Einstein orbifold, and assume that $(M, g_i)_i$ is a sequence of Einstein metrics converging to (M_o, g_o) in the Gromov-Hausdorff sense.*

Then, we have the following inequality,

$$2\chi(M) - 3|\tau(M)| \geq 2\tilde{\chi}(M_o) - 3|\tilde{\tau}(M_o)|.$$

Moreover, there is equality if and only if M is a desingularization of M_o by gluing of trees of Kähler Ricci-flat ALE orbifolds in the same orientation (with the same sign for $\tilde{\tau}$). In this equality case, we have the condition

$$\det \mathbf{R}(g_o) = 0$$

at every singular point.

Degeneration of Einstein metrics on spin manifolds

Another large class of manifolds on which we can prove our obstruction is the class of 4-manifolds admitting a spin structure.

Theorem 0.5 (Theorem 7.4) *Let $(M_i, g_i)_i$ be a sequence of spin Einstein 4-manifolds converging to an Einstein orbifold (M_o, g_o) in the Gromov-Hausdorff sense. Then, (M_o, g_o) is spin, and at any of its singular points whose group is in $SU(2)$, we have the obstruction*

$$\det \mathbf{R}_{g_o} = 0.$$

General obstructions for spherical and hyperbolic orbifolds. Our Theorem 5 holds for any singularity model which might be non-Kähler. We will use it lastly to identify an obstruction to desingularizing spherical or hyperbolic orbifolds by *any* Ricci-flat ALE manifold in Theorem 8.1. This provides an obstruction to any standard gluing-perturbation technique but will only imply an actual obstruction to the Gromov-Hausdorff desingularization by Ricci-flat ALE manifolds whose deformations are *integrable* (this is the case of all known examples).

Theorem 0.6 (Corollary 16) *Spherical and hyperbolic orbifolds cannot be desingularized in the Gromov-Hausdorff sense by Ricci-flat ALE spaces which are integrable (see Definition 5.1).*

Outline of the paper

In Section 1, we give the principal definitions, and in Section 2, we introduce and motivate the function spaces we will use throughout the paper, and moreover restate the results of [Ozu19a] thanks to them.

In Section 3, we prove that we can always pull-back an Einstein metric which is Gromov-Hausdorff close to an orbifold by a small diffeomorphism to ensure that it satisfies some gauge condition with respect to a naïve desingularization. The proof consists in a Lyapunov-Schmidt reduction in our weighted norms where the relevant operators are proven to be Fredholm.

In Section 4, we prove that any naïve desingularization can be perturbed to a metric which is Einstein modulo some obstruction, that is, an approximate cokernel of the linearization of the gauged Einstein operator. The point is that every possible Einstein metric is produced this way, and that whenever the obstructions do not vanish, it is impossible to perturb the naïve desingularization to an Einstein metric. The proof again relies on a Lyapunov-Schmidt reduction in our weighted Hölder spaces. We then extend Koiso's definition of a premoduli space in the neighborhood of a singular metric.

In Section 5, we estimate the obstructions to the above Einstein desingularization modulo obstructions. To obtain such an obstruction at *all* singular points, we need to use an analysis on partial desingularizations and produce better approximations of Einstein modulo obstructions metrics.

In Section 6, we test the above obstructions on degenerations of Einstein manifolds forming trees of Kähler Ricci-flat ALE orbifolds. By developing our analysis on trees of singularities, we prove that the obstruction of [Biq13] for the Eguchi-Hanson metric extends to any tree of quotients of gravitational instantons and holds under a mere Gromov-Hausdorff convergence. An important step is to prove that a gluing of gravitational instantons in the same orientation can be uniformly perturbed to an Einstein metric in our norms.

In Section 7, we investigate topological conditions which ensure that a sequence of Einstein manifolds degenerating will only produce trees of Kähler Ricci-flat ALE spaces. We mainly use the result of [Nak90] and consider the behavior of the Hitchin-Thorpe inequality as well as the degeneration of Einstein metrics on a spin manifold.

In Section 8, building on the notion of maximal volume for Ricci-flat ALE spaces of [BH19], we prove that even without assuming that the trees of singularities are Kähler, there is a non vanishing obstruction to the desingularization of spherical and hyperbolic orbifolds. We can however only prove that this is a Gromov-Hausdorff obstruction under the technical assumption that the Ricci-flat ALE spaces have integrable deformations.

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1 Orbifolds, ALE spaces and naïve desingularizations

Let us start by defining the objects we will use throughout this article.

1.1 Einstein orbifolds and ALE spaces

For Γ a finite subgroup of $SO(4)$ acting freely on \mathbb{S}^3 , let us denote $(\mathbb{R}^4/\Gamma, g_e)$ the flat orbifold obtained by the quotient by the action of Γ , and $r_e := d_e(\cdot, 0)$.

Definition 1.1 (Orbifold (with isolated singularities)) We will say that a metric space (M_o, g_o) is an orbifold of dimension $n \in \mathbb{N}$ if there exists $\epsilon_0 > 0$ and a finite number of points $(p_k)_k$ of M_o called *singular* such that we have the following properties:

- (1) the space $(M_o \setminus \{p_k\}_k, g_o)$ is a manifold of dimension n ,
- (2) for each singular point p_k of M_o , there exists a neighborhood of p_k , $U_k \subset M_o$, a finite subgroup acting freely on \mathbb{S}^{n-1} , $\Gamma_k \subset SO(n)$, and a diffeomorphism $\Phi_k : B_e(0, \epsilon_0) \subset \mathbb{R}^n/\Gamma_k \rightarrow U_k \subset M_o$ for which, the pull-back of $\Phi_k^* g_o$ on the covering \mathbb{R}^n is smooth.

Remark 2 Consequently, the analysis on an orbifold is exactly the same as the analysis on a manifold up to using finite local coverings at the singular points.

Definition 1.2 (The function r_o on an orbifold) We define r_o , a smooth function on M_o satisfying $r_o := (\Phi_k)_* r_e$ on each U_k , and such that on $M_o \setminus U_k$, we have $\epsilon_0 \leq r_o < 1$ (the different choices will be equivalent for our applications).

We will denote, for $0 < \epsilon \leq \epsilon_0$,

$$M_o(\epsilon) := \{r_o > \epsilon\} = M_o \setminus \left(\bigcup_k \Phi_k(\overline{B_e(0, \epsilon)}) \right).$$

Let us now turn to ALE Ricci-flat metrics.

Definition 1.3 (ALE orbifold (with isolated singularities)) An ALE orbifold of dimension $n \in \mathbb{N}$, (N, g_b) is a metric space for which there exists $\epsilon_0 > 0$, singular points $(p_k)_k$ and a compact $K \subset N$ for which we have:

- (1) (N, g_b) is a orbifold of dimension n ,
- (2) there exists a compact subset $K \subset N$ and a diffeomorphism $\Psi_\infty : (\mathbb{R}^n / \Gamma_\infty) \setminus \overline{B_e(0, \epsilon_0^{-1})} \rightarrow N \setminus K$ such that we have

$$r_e^l |\nabla^l (\Psi_\infty^* g_b - g_e)|_{g_e} \leq C_l r_e^{-n}.$$

Definition 1.4 (The function r_b on an ALE orbifold) We define r_b a smooth function on N satisfying $r_b := (\Psi_k)_* r_e$ on each U_k , and $r_b := (\Psi_\infty)_* r_e$ on U_∞ , and such that $\epsilon_0 \leq r_b \leq \epsilon_0^{-1}$ on the rest of N (the different choices are equivalent for our applications).

For $0 < \epsilon \leq \epsilon_0$, we will denote

$$N(\epsilon) := \{\epsilon < r_b < \epsilon^{-1}\} = N \setminus \left(\bigcup_k \Psi_k(\overline{B_e(0, \epsilon)}) \cup \Psi_\infty((\mathbb{R}^4 / \Gamma_\infty) \setminus \overline{B_e(0, \epsilon^{-1})}) \right).$$

Now, consider a subset S_o of the singular points of M_o (respectively S of N).

Definition 1.5 (Functionals r_{o,S_o} and $r_{b,S}$) We define the functional r_{o,S_o} (respectively $r_{b,S}$) exactly like in Definitions 1.2 (respectively 1.4) by only considering the sets U_k containing points of S_o (respectively S).

1.2 Naïve desingularizations

Let us now recall the definition of a naïve desingularization of an orbifold from [Ozu19a].

Gluing of ALE spaces to orbifold singularities. Let $0 < 2\epsilon < \epsilon_0$ be a fixed constant, $t > 0$, (M_o, g_o) an orbifold and $\Phi : B_e(0, \epsilon_0) \subset \mathbb{R}^4 / \Gamma \rightarrow U$ a local chart of Definition 1.1 around a singular point $p \in M_o$. Let also (N, g_b) be an ALE orbifold asymptotic to \mathbb{R}^4 / Γ , and $\Psi_\infty : (\mathbb{R}^4 / \Gamma) \setminus \overline{B_e(0, \epsilon_0^{-1})} \rightarrow N \setminus K$ a chart at infinity of Definition 1.3.

Define $s > 0$, $\phi_s : x \in \mathbb{R}^4 / \Gamma \rightarrow sx \in \mathbb{R}^4 / \Gamma$. For $t < \epsilon_0^4$, we define $M_o \# N$ as N glued to M_o thanks to the diffeomorphism

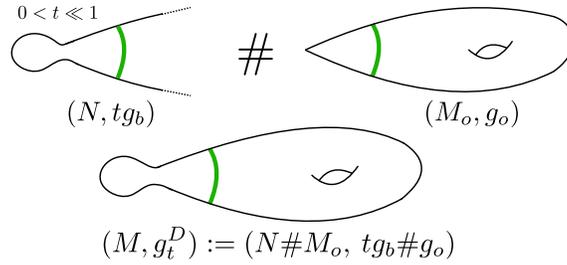
$$\Phi \circ \phi_{\sqrt{t}} \circ \Psi^{-1} : \Psi(A_e(\epsilon_0^{-1}, \epsilon_0 t^{-1/2})) \rightarrow \Phi(A_e(\epsilon_0^{-1} \sqrt{t}, \epsilon_0)).$$

Consider moreover $\chi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, a C^∞ cut-off function supported on $[0, 2]$ and equal to 1 on $[0, 1]$.

Definition 1.6 (Naïve gluing of an ALE space to an orbifold) We define a *naïve gluing* of (N, g_b) at scale $0 < t < \epsilon^4$ to (M_o, g_o) at the singular point p , which we will denote $(M_o \# N, g_o \#_{p,t} g_b)$ by putting $g_o \#_{p,t} g_b = g_o$ on $M \setminus U$, $g_o \#_{p,t} g_b = t g_b$ on K , and

$$g_o \#_{p,t} g_b = \chi(t^{-\frac{1}{4}} r_e) \Psi_\infty^* g_b + (1 - \chi(t^{-\frac{1}{4}} r_e)) \Phi^* g_o$$

on $\mathcal{A}(t, \epsilon) := A_e(\epsilon^{-1} \sqrt{t}, 2\epsilon)$.



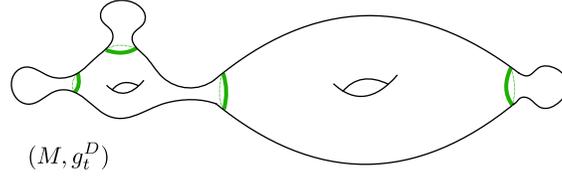
Remark 3 It is possible to compose $\phi_{\sqrt{t}}$ with any isometry of \mathbb{R}^4/Γ . This is equivalent to gluing a different Ricci-flat ALE metric.

More generally, it is possible to desingularize iteratively by trees of Ricci-flat ALE orbifolds. Consider (M_o, g_o) an Einstein orbifold (the index o stands for orbifold), and S_o a subset of its singular points and $(N_j, g_{b_j})_j$ (the index b_j stands for j -th bubble) a family of Ricci-flat ALE spaces asymptotic at infinity to \mathbb{R}^4/Γ_j and $(S_{b_j})_j$ a subset of their singular points. Let us finally assume that there is a one to one map $p : j \mapsto p_j \in S_o \cup \bigcup_k S_{b_k}$, where the singularity at p_j is \mathbb{R}^4/Γ_j . We will call $D := ((M_o, g_o, S_o), (N_j, g_{b_j}, S_{b_j})_j, p)$ a *desingularization pattern*.

Definition 1.7 (Naïve desingularization by a tree of singularities) Let $0 < 2\epsilon < \epsilon_0$, D be a desingularization pattern for (M_o, g_o) , and let $0 < t_j < \epsilon^4$ be relative gluing scales. The metric g_t^D is then the result of the following finite iteration:

- (1) start with a deepest bubble (N_j, g_{b_j}) , that is, j such that $S_j = \emptyset$,
- (2) if $p_j \in N_k$, replace (N_k, g_{b_k}, S_j) and $(N_j, g_{b_j}, \emptyset)$ in D by $(N_k \# N_j, g_{b_k} \#_{p_j, t_j} g_{b_j}, S_k \setminus \{p_j\})$ and restrict p as $l \mapsto p_l$ for $l \neq j$ in D and consider another deepest bubble. The same works if $p_j \in M_o$.
- (3) choose another deepest bubble and do the same.

For $t = (t_j)_j$, if N_j is glued to $p_j \in N_{j_1}$, and N_{j_1} is glued to $p_{j_1} \in N_{j_2}, \dots, N_{j_{k-1}}$ is glued to N_{j_k} , which is glued to M_o , we define $T_j := t_{j_1} t_{j_2} \dots t_{j_k}$. This way, on each N_j^{16t} , the metric is $T_j g_{b_j}$.



Let (M_o, g_o) be an Einstein orbifold, and (M, g^D) a naïve desingularization of (M_o, g_o) by a tree of ALE Ricci-flat orbifolds (N_j, g_{b_j}) glued at scales $T_j > 0$.

Here, the manifold M is also covered as $M = M_o^t \cup \bigcup_j N_j^t$, where

$$(3) \quad M_o^t := M_o \setminus \left(\bigcup_k \Phi_k(B_e(0, t_k^{\frac{1}{4}})) \right),$$

where $t_k > 0$ is the relative gluing scale of N_k at the singular point $p_k \in M_o$, and where

$$(4) \quad N_j^t := (N_j \setminus \Psi_\infty((\mathbb{R}^4/\Gamma_\infty) \setminus B_e(0, 2t_j^{-\frac{1}{4}}))) \setminus \left(\bigcup_k \Psi_k(B_e(0, t_k^{\frac{1}{4}})) \right).$$

On $M_o^{16t} \subset M_o^t$, we have $g^D = g_o$ and on each $N_j^{16t} \subset N_j^t$, we have $g^D = T_j g_{b_j}$. We also define $t_{\max} := \max_j t_j$. By Definition 1.6, on the intersection $N_j^t \cap M_o^t$ we then have $\sqrt{T_j} r_{b_j} = r_o$, and on the intersection $N_j^t \cap N_k^t$, we have $\sqrt{T_j} r_{b_j} = \sqrt{T_k} r_{b_k}$.

Definition 1.8 (Function r_D on a naïve desingularization) On a naïve desingularization (M, g^D) , we define a function r_D in the following way:

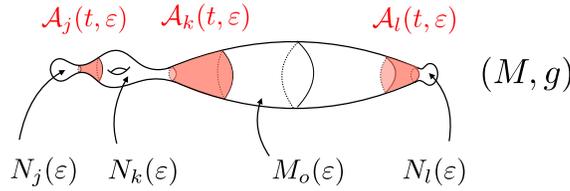
- (1) $r_D = r_o$ on M_o^t ,
- (2) $r_D = \sqrt{T_j} r_{b_j}$ on each N_j^t .

The function r_D is smooth on M .

Definition 1.9 (Neck regions, $\mathcal{A}_k(t, \epsilon)$) Let (N_k, g_{b_k}) be a Ricci-flat ALE orbifold of the above tree of singularities. We define $\mathcal{A}_k(t, \epsilon)$ as the connected region with $\epsilon^{-1} \sqrt{T_k} < \sqrt{T_k} r_{b_k} = r_D < \epsilon t_k^{-\frac{1}{2}} \sqrt{T_k}$ with a nonempty intersection with N_k^t .

Definition 1.10 (Cut-off functions $\chi_{M_o^t}$, $\chi_{N_j^t}$, $\chi_{\mathcal{A}_k(t, \epsilon)}$ and $\chi_{B(p_k, \epsilon)}$) We define the following cut-off functions thanks to the cut-off function χ used in Definition 1.6.

- $\chi_{M_o^t}$, equal to 1 on M_o^{16t} and equal to $1 - \chi(t_k^{-\frac{1}{4}} r_o)$ on each annulus $\mathcal{A}_k(t, \epsilon)$. It is supported on M_o^t .



- $\chi_{N_j^t}$, equal to 1 on N_j^{16t} and equal to $1 - \chi(t_k^{-\frac{1}{4}} r_{b_j})$ on each annulus $\mathcal{A}_k(t, \epsilon)$ at its singular points and $\chi(t_j^{\frac{1}{4}} r_{b_j})$ in a neighborhood of infinity. It is supported on N_j^t .
- $\chi_{\mathcal{A}_k(t, \epsilon)}$ equal to 1 on $\mathcal{A}_k(t, \frac{1}{2}\epsilon)$, and equal to $\chi(\epsilon^{-1} t_k^{\frac{1}{2}} r_{b_k}) - \chi(\epsilon r_{b_k})$. It is supported on $\mathcal{A}_k(t, \epsilon)$.
- $\chi_{B(p_k, \epsilon)}$ for $p_k \in M_o$ equal to 1 on $r_o < \epsilon$ equal to $\chi(\epsilon^{-1} r_o)$ around p_k . It is supported in supported in $r_o < 2\epsilon$ around p_k .

Note that since $t_k^{\frac{1}{2}} r_{b_k} = r_o$ on the gluing region, we have $\chi_{B(p_k, \epsilon)} - \chi_{\mathcal{A}_k(t, \epsilon)} = \chi(\epsilon r_{b_k}) - \chi(\epsilon t_k^{-\frac{1}{2}} r_o)$.

The definition extends to deeper Ricci-flat ALE orbifolds thanks to the iteration of Definition 1.7.

This in particular yields a partition of unity,

$$(5) \quad 1 = \chi_{M_o} + \sum_j \chi_{N_j^t}.$$

2 Weighted Hölder spaces and decoupling norms

We now present the spaces in which the analysis of the rest of the article takes place.

2.1 Weighted Hölder spaces

Let us construct weighted Hölder spaces adapted to our situation. Let (M, g_t^D) be a naïve desingularization of an orbifold (M_o, g_o) by Ricci-flat ALE orbifolds (N_j, g_{b_j}) at scales $T_j > 0$.

2.1.1 Weighted Hölder spaces on orbifolds and ALE spaces

Let us first define weighted spaces on manifolds asymptotic to cones or with conical singularities. For a tensor s , a point x , $\alpha > 0$ and a metric g , if we denote \exp_x the exponential map at x whose injectivity radius is $\text{inj}_g(x)$, we define the Hölder seminorm of s on M as

$$[s]_{C^\alpha(g)}(x) := \sup_{\{y \in T_x M, |y| < \text{inj}_g(x)\}} \left| \frac{\exp_x^* s(0) - \exp_x^* s(y)}{|y|^\alpha} \right|_{\exp_x^* g}.$$

For orbifolds, we will consider a norm which is bounded for tensors decaying at the singular points.

Definition 2.1 (Weighted Hölder norms on an orbifold) Let $\beta \in \mathbb{R}$, $k \in \mathbb{N}$, $0 < \alpha < 1$ and (M_o, g_o) an orbifold. Then, for all tensor s on M_o , we define

$$\|s\|_{C_\beta^{k,\alpha}(g_o)} := \sup_{M_o} r_o^{-\beta} \left(\sum_{i=0}^k r_o^i |\nabla_{g_o}^i s|_{g_o} + r_o^{k+\alpha} [\nabla_{g_o}^k s]_{C^\alpha(g_o)} \right).$$

Remark 4 The injectivity radius at a point $x \in M_o$ is equivalent to r_o .

For ALE orbifolds, we will consider a norm which is bounded for tensors decaying at infinity and at the singular points.

Definition 2.2 (Weighted norm on ALE orbifolds) For $\beta \in \mathbb{R}$, $k \in \mathbb{N}$ and $0 < \alpha < 1$ on an orbifold ALE (N, g_b) , we define

$$\|s\|_{C_\beta^{k,\alpha}(g_b)} := \sup_N \left\{ \max(r_b^\beta, r_b^{-\beta}) \left(\sum_{i=0}^k r_b^i |\nabla_{g_b}^i s|_{g_b} + r_b^{k+\alpha} [\nabla_{g_b}^k s]_{C^\alpha(g_b)} \right) \right\}.$$

Remark 5 The injectivity radius at a point $x \in N$ is equivalent to r_b .

Remark 6 We similarly define the norms for $C_\beta^{k,\alpha}(g_o, S_o)$ and $C_\beta^{k,\alpha}(g_b, S)$ by replacing r_o and r_b by r_{o,S_o} and $r_{b,S}$ of Definition 1.5.

2.1.2 Weighted Hölder spaces on trees of singularities

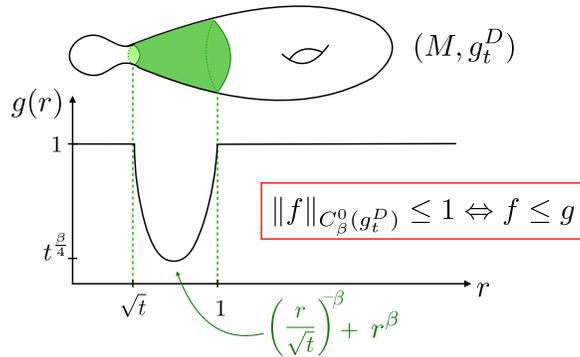
Let us assume that (M, g_t^D) is a naïve desingularization of (M_o, g_o) by a tree of singularities (N_j, g_{b_j}) . For $t_{\max} := \max_j t_j < \epsilon_0^4$, for $\epsilon_0 > 0$ the constant of Subsection 1.2 only depending on g_o and the g_{b_j} , we define the global weighted norm in the following way.

Definition 2.3 (Weighted Hölder norm on a naïve desingularization) Let $\beta \in \mathbb{R}$ and $k \in \mathbb{N}$, $0 < \alpha < 1$. We define for $s \in TM^{\otimes l_+} \otimes T^*M^{\otimes l_-}$ a tensor of type $(l_+, l_-) \in \mathbb{N}^2$, with $l := l_+ - l_-$ the associated conformal weight.

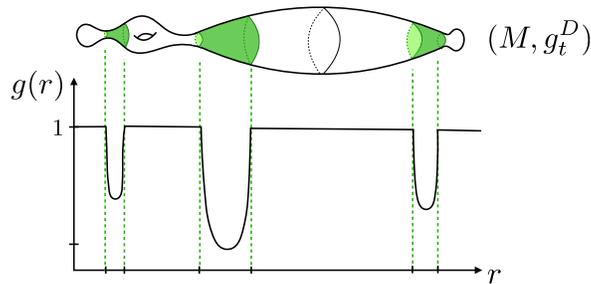
$$(6) \quad \|s\|_{C_\beta^{k,\alpha}(g^D)} := \|\chi_{M_o} s\|_{C_\beta^{k,\alpha}(g_o)} + \sum_j T_j^{\frac{l}{2}} \|\chi_{N_j^l} s\|_{C_\beta^{k,\alpha}(g_{b_j})}.$$

Remark 7 The factor $T_j^{\frac{l}{2}}$ in (6) comes from the fact that on N_j^l , the metric g^D is close to $T_j g_{b_j}$. For a tensor s of conformal weight l , we have $|s|_{T_j g_{b_j}} = T_j^{\frac{l}{2}} |s|_{g_{b_j}}$, and therefore $T_j^{\frac{l}{2}} \|\chi_{N_j^l} s\|_{C_\beta^{k,\alpha}(g_{b_j})} = \|\chi_{N_j^l} s\|_{C_\beta^{k,\alpha}(T_j g_{b_j})}$.

Remark 8 For a function, being bounded for this norm means being bounded in C_β^0 -norm means being bounded everywhere and having a particular decay in the neck regions.



Its main advantage is that it is totally adapted to trees of singularities.



Remark 9 If (M, g^D) is a partial desingularization at the singular points of S_o and S_j , we define the $C_\beta^{k,\alpha}(g^D)$ -norm similarly thanks to $C_\beta^{k,\alpha}(g_o, S_o)$ and $C_\beta^{k,\alpha}(g_b, S)$ of Remark 6.

Thanks to this norm, we can for example rewrite and extend the statement of [Ozu19a, Theorem 6.4].

Corollary 2 Let $D_0, v_0 > 0$, $l \in \mathbb{N}$, and $\beta = \beta(v_0, D_0) > 0$ obtained in [Ozu19a, Theorem 6.4]. Then, for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon, D_0, v_0, l) > 0$ such that if $(M, g^\mathcal{E})$ is an Einstein manifold satisfying

- the volume is bounded below by $v_0 > 0$,
- the diameter is bounded above by D_0 ,
- the Ricci curvature is bounded $|\text{Ric}| \leq 3$.

and for an Einstein orbifold (M_o, g_o) ,

$$d_{GH}((M, g^\mathcal{E}), (M_o, g_o)) \leq \delta,$$

then, there exists a naïve desingularization (M, g^D) of (M_o, g_o) by a tree of singularities and a diffeomorphism $\phi : M \rightarrow M$ such that

$$\|\phi^* g^\mathcal{E} - g^D\|_{C_\beta^l(g^D)} \leq \epsilon.$$

Proof Let $l \in \mathbb{N}$. Let us give a proof by contradiction and consider a sequence of counter examples, that is a sequence of Einstein manifolds $(M_i, g_i)_i$ such that

- $\text{Vol}(g_i) \geq v_0 > 0$,
- $\text{diam}(g_i) < D_0$ and
- $|\text{Ric}(g_i)|_{g_i} \leq 3$

converging in the Gromov-Hausdorff sense to an Einstein orbifold (M_o, g_o) , but such that there exists $\epsilon > 0$ for which, for all $i \in \mathbb{N}$ and any naïve desingularization (M_i, g_i^D) of (M_o, g_o) , and all diffeomorphism $\Phi_i : M_i \rightarrow M_i$, we have $\|\Phi_i^* g_i - g_i^D\|_{C_\beta^l(g_i^D)} > \epsilon$.

According to [Ozu19a, Theorem 6.4], this implies that there exists a subsequence $(M, g_i)_i$ with fixed topology, and a sequence $(M, g_i^D)_i$ contradicting the assumption for i large enough by definition of the weighted norm. \square

On the annuli of low curvature $\mathcal{A}_k(t, \epsilon_0)$ pulled back on flat annuli $A_e(\rho_1, \rho_2) \subset \mathbb{R}^4/\Gamma$, the weighted norm on (M, g^D) is equivalent to a particular norm which allows us to control *independently of the radii* the sum of tensors decaying at the center of the annulus and of tensors decaying at infinity.

Definition 2.4 (Weighted norm adapted to an annulus) Let $0 < \rho_1 < \rho_2$, $\beta \in \mathbb{R}$, $k \in \mathbb{N}$, $0 < \alpha < 1$ and a tensor s on $(A_e(\rho_1, \rho_2), g_e)$. We define

$$(7) \quad \eta(r_e) := \max \left(\left(\frac{\rho_1}{r_e} \right)^\beta, \left(\frac{r_e}{\rho_2} \right)^\beta \right) \leq 1,$$

and the norm,

$$\|s\|_{C_\beta^{k,\alpha}(A_e(\rho_1,\rho_2))} := \sup_{A_e(\rho_1,\rho_2)} \left[\eta^{-1}(r_e) \left(\sum_{i=0}^k r_e^i |\nabla_{g_e}^i s|_{g_e} + r_e^{k+\alpha} [\nabla_{g_e}^k s]_{C^\alpha(g_e)} \right) \right].$$

In the rest of this article, we will often use spaces denoted $fC_\beta^{k,\alpha}$ for a positive function f . They will always be equipped with the following norm

$$\|s\|_{fC_\beta^{k,\alpha}} := \left\| \frac{s}{f} \right\|_{C_\beta^{k,\alpha}}.$$

Remark 10 By definition of r_D , for all m , there exists a constant $C > 0$ only depending on the cut-off functions of Definition 1.10 such that

$$(8) \quad \frac{1}{C} \|s\|_{r_D^m C_\beta^{k,\alpha}(g^D)} \leq \|\chi_{M_o^t} s\|_{r_o^m C_\beta^{k,\alpha}(g_o)} + \sum_j T_j^{\frac{l-m}{2}} \|\chi_{N_j^t} s\|_{r_{b_j}^m C_\beta^{k,\alpha}(g_{b_j})} \leq C \|s\|_{r_D^m C_\beta^{k,\alpha}(g^D)}.$$

Remark 11 The metric g^D is equal to g_o on $\{r_D > \epsilon\} \cap M_o^t$, and to $T_j g_{b_j}$ on $\{\sqrt{T_j} \epsilon < r_D < 2\sqrt{T_j} \epsilon^{-1}\} \cap N_j^t$. Since on the $\mathcal{A}_k(t, \epsilon)$ between N_k and N_j (resp. M_o) identified with $A_e(\epsilon^{-1} \sqrt{t_k T_j}, \epsilon \sqrt{T_j})$ (resp. $A_e(\epsilon^{-1} \sqrt{t_k}, \epsilon)$), g^D is arbitrarily close to g_e , we see that defining $\tilde{\eta} : M \mapsto \mathbb{R}^+$ a function equal to

- 1 on $\{r_D > \epsilon\} \cap M_o^t$ and on $\{\sqrt{T_j} \epsilon < r_D < 2\sqrt{T_j} \epsilon^{-1}\} \cap N_j^t$ and
- equal to the function $\eta \leq 1$ defined in (7) on the associated euclidean annulus $A_e(\epsilon^{-1} \sqrt{t_k T_j}, \epsilon \sqrt{T_j})$ (or $A_e(\epsilon^{-1} \sqrt{t_k}, \epsilon)$).

Then, the norm $r_D^m C_\beta^{k,\alpha}(g^D)$ is equivalent (independently of t) to the norm which to a tensor s associates

$$(9) \quad \|s\| = \sup_M r_D^{-m} \tilde{\eta}(r_D)^{-1} \left(\sum_{i=0}^k r_D^i |\nabla_{g^D}^i s|_{g^D} + r_D^{k+\alpha} [\nabla_{g^D}^k s]_{C^\alpha(g^D)} \right).$$

Remark 12 Let $\beta \leq \beta'$, $k + \alpha \leq k' + \alpha'$, and $m, m' \in \mathbb{Z}$.

For all the previously mentioned weighted Hölder spaces generically denoted $r^m C_\beta^{k,\alpha}$, we have the following properties: for any tensors s and s'

- $\|s\|_{r^m C_\beta^{k,\alpha}} \leq \|s\|_{r^m C_{\beta'}^{k',\alpha'}}$,

- $\|\nabla^k s\|_{r^m C_\beta^{k'-k, \alpha}} \leq \|s\|_{r^{m+k} C_\beta^{k', \alpha}}$
- if $*$ is a composition, a product of a contraction of tensors, there exists $C = C(*, k, \alpha) > 0$ such that

$$(10) \quad \|s * s'\|_{r^{m+m'} C_{\beta+\beta'}^{k, \alpha}} \leq C \|s\|_{r^m C_\beta^{k, \alpha}} \|s'\|_{r^{m'} C_{\beta'}^{k, \alpha}}.$$

Let us give an explanation for that last inequality (10) for bilinear operations and assume first that $k = 0$, $\alpha = 0$ and consider depending on the situation

- (1) $(w, w') = (r_o^{-\beta}, r_o^{-\beta'})$ on (M_o, g_o) ,
- (2) $(w, w') = (\max(r_b^\beta, r_b^{-\beta}), \max(r_b^{\beta'}, r_b^{-\beta'}))$ on (N, g_b)
- (3) or $(w, w') = (\tilde{\eta}^{-1}, \tilde{\eta}'^{-1})$ with the weights $\tilde{\eta}$ and $\tilde{\eta}'$ used in (9) above respectively associated to β and β' on (M, g^D) .

The goal is to bound $w \cdot w' r^{-m-m'} |s * s'|$ uniformly and this is done using the definitions of the norms which yield $w r^{-m} |s| \leq \|s\|_{r^m C_\beta^0}$ and $w' r^{-m'} |s'| \leq \|s'\|_{r^{m'} C_{\beta'}^0}$ for any of the above spaces. The derivatives are treated thanks to the second above inequality and Leibniz rule.

2.1.3 Weighted Schauder estimates

Weighted Schauder estimates hold in these norms for the operator $P := \frac{1}{2} \nabla^* \nabla - \mathring{R}$.

Proposition 1 For all $\beta > 0$ and $0 < \alpha < 1$, there exists $C > 0$ and $\epsilon > 0$ such that if h is a symmetric 2-tensor on (M, g^D) , and g a metric on M satisfying

$$\|g - g^D\|_{C_\beta^{2, \alpha}(g^D)} \leq \epsilon,$$

then, we have

$$\|h\|_{C_\beta^{2, \alpha}(g^D)} \leq C (\|P_g h\|_{r_D^{-2} C_\beta^\alpha(g^D)} + \|h\|_{C_\beta^0(g^D)}).$$

Proof Let g be a metric on M satisfying $\|g - g^D\|_{C_\beta^{2, \alpha}(g^D)} \leq \epsilon$, for $\epsilon > 0$ which we will choose small enough along the proof. On the compacts $M_o(\epsilon_0)$ of the orbifold and $N_j(\epsilon_0)$ of the ALE orbifolds minus their singular points, we have an elliptic estimate for the operators P_{g_o} and $P_{g_{b_j}}$: there exists $C_1 > 0$ such that for any symmetric 2-tensors h_o on $M_o(\epsilon_0)$ and h_j on $N_j(\epsilon_0)$, we have

$$\|(h_o)_{|M_o(2\epsilon_0)}\|_{C^{2, \alpha}(g_o)} \leq C_1 (\|(P_{g_o} h_o)_{|M_o(\epsilon_0)}\|_{C^\alpha(g_o)} + \|(h_o)_{|M_o(\epsilon_0)}\|_{C^0(g_o)}),$$

and

$$\|(h_j)_{|N_j(2\epsilon_0)}\|_{C^{2,\alpha}(g_{b_j})} \leq C_1 \left(\|(P_{g_{b_j}} h_j)_{|N_j(\epsilon_0)}\|_{C^\alpha(g_{b_j})} + \|(h_j)_{|N_j(\epsilon_0)}\|_{C^0(g_{b_j})} \right).$$

By assumption, there exists $C > 0$ only depending on g_o and g_{b_j} such that $\|g - g_o\|_{C^{2,\alpha}(g_o)} \leq C\epsilon$ on $M_o(\epsilon_0)$ and $\|\frac{g}{T_j} - g_{b_j}\|_{C^{2,\alpha}(g_{b_j})} \leq C\epsilon$ on $N_j(\epsilon_0)$. We conclude that for ϵ small enough, the operators P_g and $P_{\frac{g}{T_j}}$, which are close to the operators P_{g_o} and $P_{g_{b_j}}$, satisfy for all h_o on $M_o(\epsilon_0)$ and h_j on $N_j(\epsilon_0)$,

$$\|(h_o)_{|M_o(2\epsilon_0)}\|_{C^{2,\alpha}(g_o)} \leq 2C_1 \left(\|(P_g h_o)_{|M_o(\epsilon_0)}\|_{C^\alpha(g_o)} + \|(h_o)_{|M_o(\epsilon_0)}\|_{C^0(g_o)} \right),$$

and

$$\|(h_j)_{|N_j(2\epsilon_0)}\|_{C^{2,\alpha}(g_{b_j})} \leq 2C_1 \left(\|(P_{\frac{g}{T_j}} h_j)_{|N_j(\epsilon_0)}\|_{C^\alpha(g_{b_j})} + \|(h_j)_{|N_j(\epsilon_0)}\|_{C^0(g_{b_j})} \right).$$

On each almost flat annulus $\mathcal{A}_k(t, \epsilon_0)$, let us denote $\mathcal{A}(\rho, \rho') := \{\rho \leq r_D \leq \rho'\}$. There exists a diffeomorphism $\phi_\rho : A_\epsilon(1/2, 4) \rightarrow \mathcal{A}(\rho/2, 4\rho)$ such that

$$\left\| \frac{\phi_\rho^* g}{\rho^2} - g_e \right\|_{C^{1,\alpha}(A_\epsilon(1/2,4))} \leq C\eta(\rho)\epsilon.$$

then, by ellipticity, for ϵ small enough, there exists $C_2 > 0$, such that for all symmetric 2-tensor h on $A_\epsilon(1/2, 4)$ we have,

$$\|h\|_{C^{2,\alpha}(A_\epsilon(1,2))} \leq 2C_2 \left(\|h\|_{C^0(A_\epsilon(1/2,4))} + \|P_{\frac{\phi_\rho^* g}{\rho^2}} h\|_{C^\alpha(A_\epsilon(1/2,4))} \right).$$

Coming back to (M, g) , this implies that for ϵ small enough, we have for h a symmetric 2-tensor on $\mathcal{A}(\rho/2, 4\rho)$,

$$\|h\|_{C^{2,\alpha}(\mathcal{A}(\rho,2\rho), \frac{g}{\rho^2})} \leq 4C_2 \left(\|h\|_{C^0(\mathcal{A}(\rho/2,4\rho), \frac{g}{\rho^2})} + \|P_{\frac{g}{\rho^2}} h\|_{C^\alpha(\mathcal{A}(\rho/2,4\rho), \frac{g}{\rho^2})} \right).$$

The norm of a symmetric 2-tensor s behaves in the following way by rescaling, for $t > 0$

$$|s|_{\frac{g}{t}} = t|s|_g,$$

and the operator P behaves in the following way by rescaling, for $t > 0$: $P_{\frac{g}{t}} = tP_g$. Multiplying both sides of the equality by r_D^{-2} , we get

$$\|h\|_{C^{2,\alpha}((\rho,2\rho),g)} \leq 4C_2 \left(\|h\|_{C^0((\rho/2,4\rho),g)} + r_D^2 \|P_g h\|_{C^\alpha((\rho/2,4\rho),g)} \right).$$

Given the controls on the derivatives of r_D , we deduce the stated result by definition of the weighted norms by multiplying both sides of the inequality by the weight of the norm. \square

Analogous estimates also hold for the elliptic operator $\delta\delta^*$ with the same proof.

Proposition 2 For all $\beta > 0$ and $0 < \alpha < 1$ there exists $C > 0$ and $\epsilon > 0$ such that if X is a vector field on (M, g^D) , and g a metric on M satisfying

$$\|g - g^D\|_{C_\beta^{2,\alpha}(g^D)} \leq \epsilon,$$

then, we have

$$\|X\|_{r_D C_\beta^{3,\alpha}(g^D)} \leq C(\|\delta_g \delta_g^* X\|_{r_D^{-1} C_\beta^{1,\alpha}(g^D)} + \|X\|_{r_D C_\beta^0(g^D)}).$$

2.2 Decoupling norms

We will see here that to expect good controls for the operators P and $\delta\delta^*$ in the annular regions of our manifold, we need to consider separately the influence of traceless constant 2-tensors for P and linear vector fields of the kernel of $\delta_e \delta_e^*$ for $\delta\delta^*$.

2.2.1 Estimates on annuli $A_e(\epsilon, \epsilon^{-1})$ of (\mathbb{R}^4, g_e) .

Let us start by studying the situation on flat annuli to motivate our new norms.

Proposition 3 Let $0 < \beta < 1$, $0 < \alpha < 1$, and $P = \frac{1}{2}\nabla^*\nabla - \mathring{R}$. There exists $C_e > 0$, and $\epsilon_e > 0$ such that for any symmetric 2-tensor h on an annulus of radii $0 < \epsilon < \epsilon_e$ and $\frac{1}{\epsilon}$, there exists a constant symmetric 2-tensor H_0 and a symmetric 2-tensor H_* satisfying

$$\nabla_e^* \nabla_e H_* = 0,$$

$$(11) \quad \|H_*\|_{C_1^{2,\alpha}(A_e(2\epsilon, (1/2)\epsilon^{-1}))} \leq C_e \|h - H_0\|_{C_\beta^0(A_e(\epsilon, \epsilon^{-1}))},$$

(notice the norm $C_1^{2,\alpha}$ for the left hand side) and,

$$(12) \quad \|h - H_0 - H_*\|_{C_\beta^{2,\alpha}(A_e(2\epsilon, (1/2)\epsilon^{-1}))} \leq C_e \|P_{g_e} h\|_{r_e^{-2} C_\beta^\alpha(A_e(\epsilon, \epsilon^{-1}))}.$$

This implies in particular the following control: for all $x \in A_e(1/2, 2)$,

$$(13) \quad \begin{aligned} & |h - H_0(x)|_{g_e} + |\nabla h(x)|_{g_e} + |\nabla^2 h(x)|_{g_e} + [\nabla^2 h]_{C^\alpha(g)}(x) \\ & \leq C_e \left((2\epsilon)^\beta \|P_{g_e} h\|_{r_e^{-2} C_\beta^\alpha(A_e(\epsilon, \epsilon^{-1}))} + 2\epsilon \|h - H_0\|_{C_\beta^{2,\alpha}(A_e(\epsilon, \epsilon^{-1}))} \right). \end{aligned}$$

Remark 13 This is a strictly better estimate than the elliptic estimates of Proposition 2 which would only have given

$$(14) \quad \begin{aligned} & |h - H_0(x)|_{g_e} + |\nabla h(x)|_{g_e} + |\nabla^2 h(x)|_{g_e} + [\nabla^2 h]_{C^\alpha(g_e)}(x) \\ & \leq C_e \left((2\epsilon)^\beta \|P_{g_e} h\|_{r_e^{-2} C_\beta^\alpha(A_e(\epsilon, \epsilon^{-1}))} + (2\epsilon)^\beta \|h - H_0\|_{C_\beta^{2,\alpha}(A_e(\epsilon, \epsilon^{-1}))} \right). \end{aligned}$$

The difference will be crucial in the proof of Proposition 11.

Proof Let us start by noting that (13) is a consequence of (11) and (12). Indeed, $h - H_0 = (h - H_0 - H_*) + H_*$, and we have therefore, denoting

$$\|s\|_{C^{2,\alpha}(1/2,2)} := \sup_{x \in A_e(1/2,2)} |s(x)|_{g_e} + |\nabla s(x)|_{g_e} + |\nabla^2 s(x)|_{g_e} + [\nabla^2 s]_{C^\alpha(g_e)}(x),$$

$$\begin{aligned} \|h - H_0\|_{C^{2,\alpha}(1/2,2)} & \leq \|h - H_0 - H_*\|_{C^{2,\alpha}(1/2,2)} + \|H_*\|_{C^{2,\alpha}(1/2,2)} \\ & \leq (2\epsilon)^\beta \|h - H_0 - H_*\|_{C_\beta^{2,\alpha}(A_e(\epsilon, \epsilon^{-1}))} + 2\epsilon \|H_*\|_{C_1^{2,\alpha}(A_e(\epsilon, \epsilon^{-1}))} \\ & \leq C_e \left((2\epsilon)^\beta \|P_{g_e} h\|_{r_e^{-2} C_\beta^\alpha(A_e(\epsilon, \epsilon^{-1}))} + 2\epsilon \|h - H_0\|_{C_\beta^{2,\alpha}(A_e(\epsilon, \epsilon^{-1}))} \right), \end{aligned}$$

by definition of the weighted norms and assuming, for $C_e > 0$, the inequalities (12) and (11).

On $\mathbb{R}^4 \setminus \{0\}$, the harmonic symmetric 2-tensors are sum of homogeneous harmonic symmetric 2-tensors whose coefficients in the canonical basis of \mathbb{R}^4 are proportional to r^j for $j \in \mathbb{Z} \setminus \{-1\}$. These harmonic symmetric 2-tensors are more precisely of the form $r_e^k H_k$ or $r_e^{-2-k} H_k$ for $k \in \mathbb{N}$, where H_k is a homogeneous symmetric 2-tensor with $|H_k|_{g_e} \sim r^0$ whose coefficients, once restricted to the unit sphere are eigenfunctions of the spherical Laplacian with eigenvalue $-k(k+2)$.

For any symmetric 2-tensor h on $A_e(\epsilon, \epsilon^{-1})$, let us define \tilde{H} the solution on $A_e(\epsilon, \epsilon^{-1})$ of the following Dirichlet problem, denoting for $r > 0$, $S_e(r) := \{r_e = r\}$,

$$\begin{cases} \nabla_e^* \nabla_e \tilde{H} = 0, \\ \tilde{H} = h \text{ on } S_e(\epsilon) \cup S_e(\epsilon^{-1}). \end{cases}$$

More precisely, $\tilde{H} = \sum_{k \geq 0} (\epsilon r_e)^k \tilde{H}_k^+ + (\epsilon^{-1} r_e)^{-2-k} \tilde{H}_k^-$ where the \tilde{H}_k^\pm are homogeneous with $|\tilde{H}_k^+|_{g_e} \sim r_e^0$ and which, once restricted to the sphere are eigenvectors associated to $-k(k+2)$. If we decompose in spherical harmonics $h|_{S_e(\epsilon)} =: \sum_k H_k(\epsilon)$ and $h|_{S_e(\epsilon^{-1})} =: \sum_k H_k(\epsilon^{-1})$, we have the system

$$(15) \quad \begin{cases} H_k(\epsilon^{-1}) = \tilde{H}_k^+ + \epsilon^{4+2k} \tilde{H}_k^-, \\ H_k(\epsilon) = \epsilon^{2k} \tilde{H}_k^+ + \tilde{H}_k^-, \end{cases}$$

and therefore,

$$(16) \quad \begin{cases} \tilde{H}_k^+ = \frac{1}{1 - \epsilon^{4+4k}} (H_k(\epsilon^{-1}) - \epsilon^{4+2k} H_k(\epsilon)), \\ \tilde{H}_k^- = \frac{1}{1 - \epsilon^{4+4k}} (H_k(\epsilon) - \epsilon^{2k} H_k(\epsilon^{-1})), \end{cases}$$

Denote $\tilde{H}_* := \tilde{H} - \tilde{H}_0^+$. Since $\nabla_e^* \nabla_e \tilde{H}_* = 0$, by elliptic regularity on the annulus $A(\rho/2, 4\rho) \subset A_e(\epsilon, \epsilon^{-1})$, there exists a constant $C > 0$ independent of h such that we have,

$$(17) \quad \|\tilde{H}_*\|_{C^0(A(\rho, 2\rho))} \leq \frac{C}{\rho^2} \|\tilde{H}_*\|_{L^2(A_e(\rho/2, 4\rho))},$$

so to control the norm $C_1^0(A_e(\epsilon, \epsilon^{-1}))$ of \tilde{H}_* , we just have to control the L^2 -norm of \tilde{H}_* on the different annuli $A_e(\rho, 2\rho) \subset A(\epsilon, \epsilon^{-1})$. Since the harmonic decompositions are $L^2(S_e(1))$ -orthogonal, we have for a constant $C > 0$ which may change from line to line

$$(18) \quad \begin{aligned} \int_{A_e(\rho, 2\rho)} |\tilde{H}_*|_{g_e}^2 dv_{g_e} &= \epsilon^4 \int_{\rho}^{2\rho} \int_{S_e(1)} |\tilde{H}_0^-|_{g_e}^2 r^{-4} dv_{S_e(1)} r^3 dr \\ &\quad + \sum_{k \geq 1} \int_{\rho}^{2\rho} \int_{S_e(1)} |\epsilon^k r^k \tilde{H}_k^+ + \epsilon^{2+k} r^{-2-k} \tilde{H}_k^-|_{g_e}^2 dv_{S_e(1)} r^3 dr \\ &\leq C \epsilon^4 \int_{S_e(1)} |\tilde{H}_0^-|_{g_e}^2 dv_{S_e(1)} \\ &\quad + C \sum_{k \geq 1} \epsilon^{2k} \int_{\rho}^{2\rho} \int_{S_e(1)} |\tilde{H}_k^+|_{g_e}^2 dv_{S_e(1)} r^{2k+3} dr \\ &\quad + \epsilon^{4+2k} \int_{\rho}^{2\rho} \int_{S_e(1)} |\tilde{H}_k^-|_{g_e}^2 dv_{S_e(1)} r^{-1-2k} dr \\ &\leq C \epsilon^4 \int_{S_e(1)} |\tilde{H}_0^-|_{g_e}^2 dv_{S_e(1)} \\ &\quad + C \sum_{k \geq 1} \epsilon^{2k} \rho^{2k+4} \int_{S_e(1)} |\tilde{H}_k^+|_{g_e}^2 dv_{S_e(1)} + \epsilon^{4+2k} \rho^{-2k} \int_{S_e(1)} |\tilde{H}_k^-|_{g_e}^2 dv_{S_e(1)} \\ &\leq C \epsilon^4 \epsilon^{-3} \int_{S_e(\epsilon)} |\tilde{H}_0^-|_{g_e}^2 dv_{S_e(\epsilon)} \\ &\quad + C \sum_{k \geq 1} \epsilon^{2k} \rho^{2k+4} \epsilon^3 \int_{S_e(\epsilon^{-1})} |\tilde{H}_k^+|_{g_e}^2 dv_{S_e(\epsilon^{-1})} \\ &\quad + \epsilon^{4+2k} \rho^{-2k} \epsilon^{-3} \int_{S_e(\epsilon)} |\tilde{H}_k^-|_{g_e}^2 dv_{S_e(\epsilon)} \end{aligned}$$

Now, the equalities (16) and the fact that the decompositions in spherical harmonics are orthogonal imply that for a constant $C > 0$ which may change from line to line we have

$$\begin{aligned}
 \sum_{k \geq 0} \int_{S_e(\epsilon)} |\tilde{H}_k^-|_{g_e}^2 dv_{S_e(\epsilon)} &\leq C \sum_k \int_{S_e(\epsilon)} |H_k(\epsilon)|_{g_e}^2 dv_{S_e(\epsilon)} + C\epsilon^{6+2k} \int_{S_e(\epsilon^{-1})} |H_k(\epsilon^{-1})|_{g_e}^2 dv_{S_e(\epsilon^{-1})} \\
 &\leq C \int_{S_e(\epsilon)} |h - \tilde{H}_0^+|_{g_e}^2 dv_{S_e(\epsilon)} + C\epsilon^6 \int_{S_e(\epsilon^{-1})} |h - \tilde{H}_0^+|_{g_e}^2 dv_{S_e(\epsilon^{-1})} \\
 &\leq C\epsilon^3 \|(h - \tilde{H}_0^+)_{|S_e(\epsilon)}\|_{C^0(g_e)}^2 + C\epsilon^3 \|(h - \tilde{H}_0^+)_{|S_e(\epsilon^{-1})}\|_{C^0(g_e)}^2 \\
 (19) \quad &\leq C\epsilon^3 \|h - \tilde{H}_0^+\|_{C_\beta^0(A_e(\epsilon, \epsilon^{-1}))}^2
 \end{aligned}$$

because $|(h - \tilde{H}_0^+)_{|S_e(\epsilon)}|_{g_e} \leq \|h - \tilde{H}_0^+\|_{C_\beta^0(A_e(\epsilon, \epsilon^{-1}))}$ and $|(h - \tilde{H}_0^+)_{|S_e(\epsilon^{-1})}|_{g_e} \leq \|h - \tilde{H}_0^+\|_{C_\beta^0(A_e(\epsilon, \epsilon^{-1}))}$ by definition of the norm, and similarly

$$(20) \quad \sum_{k \geq 1} \int_{S_e(\epsilon^{-1})} |\tilde{H}_k^+|_{g_e}^2 dv_{S_e(\epsilon^{-1})} \leq C\epsilon^{-3} \|h - \tilde{H}_0^+\|_{C_\beta^0(A_e(\epsilon, \epsilon^{-1}))}^2.$$

Together with (18), (19) and (20), and since on $A_e(\epsilon, \frac{1}{\epsilon})$ for any $k \geq 1$, $\epsilon(\rho + \rho^{-1}) \geq \epsilon^k \rho^{\pm k}$, this yields the following estimate for ϵ small enough and a constant $C > 0$,

$$(21) \quad \|\tilde{H}_*\|_{L^2(A_e(\rho, 2\rho))}^2 \leq C\rho^4 \epsilon^2 (\rho^1 + \rho^{-1})^2 \|h - \tilde{H}_0^+\|_{C_\beta^0(A_e(\epsilon, \epsilon^{-1}))}^2.$$

Combining (17) and (21), we get

$$(22) \quad \|\tilde{H}_*\|_{C_1^0(A_e(\epsilon, \epsilon^{-1}))} \leq C \|h - \tilde{H}_0^+\|_{C_\beta^0(A_e(\epsilon, \epsilon^{-1}))}.$$

Let us fix $x_0 \in S_e(1)$ and modify our symmetric 2-tensor \tilde{H} to get a symmetric 2-tensor H such that $h - H$ vanishes at x_0 and on $S_e(\epsilon)$ while being constant on $S_e(\epsilon^{-1})$. The only possible choice with harmonic symmetric 2-tensors is

$$H := \left(\tilde{H}_0^+ - \frac{c_0}{1 - \epsilon^2} \right) + \left(\tilde{H}_* + \frac{\epsilon^2}{1 - \epsilon^2} \frac{c_0}{r_e^2} \right),$$

where $c_0 = (\tilde{H} - h)(x_0)$. We will show that $h - H$ satisfies the estimate (12), but let us start by proving the control (11) stated. For this, denote $H_0 := \tilde{H}_0^+ - \frac{c_0}{1 - \epsilon^2}$ the constant part of H , and $H_* := \tilde{H}_* + \frac{1}{1 - \epsilon^2} \frac{c_0}{(\epsilon^{-1} r_e)^2}$ its varying part.

According to (21), the part $\sum_{k \geq 1} (\epsilon r_e)^k \tilde{H}_k^+ + (\epsilon^{-1} r_e)^{-2-k} \tilde{H}_k^-$ is well controlled in L^2 -norm by the varying parts of $h - H_0$ on $S_e(\epsilon)$ and $S_e(\epsilon^{-1})$: they are the same as the varying parts of $h - \tilde{H}_0$. There remains to control the part in r_e^{-2} , that is

$$\frac{1}{1 - \epsilon^2} \frac{(1 - \epsilon^2) \tilde{H}_0^- + c_0}{\epsilon^{-2} r_e^2}.$$

In order to control this part, let us look at the mean values of $h - H_0$ on $S_e(\epsilon)$ and $S_e(\epsilon^{-1})$. On $S_e(\epsilon)$, we have $(h - H)|_{S_e(\epsilon)} = 0$, and therefore

$$(h - H_0)|_{S_e(\epsilon)} = (h - H) + \tilde{H}_* + \frac{c_0}{1 - \epsilon^2} \frac{\epsilon^2}{r_e^2}$$

and its mean value on $S_e(\epsilon)$ is then

$$(23) \quad \tilde{H}_0^- + \frac{c_0}{1 - \epsilon^2}.$$

Similarly, since $h - \tilde{H} = 0$ on $S_e(\epsilon^{-1})$ and

$$(h - H_0)|_{S_e(\epsilon^{-1})} = (h - \tilde{H}) + \tilde{H}_* + \frac{c_0}{1 - \epsilon^2},$$

and its mean value is therefore

$$(24) \quad \epsilon^4 \tilde{H}_0^- + \frac{c_0}{1 - \epsilon^2}.$$

By considering linear combinations of (23) and (24), we control both $|c_0|_{g_e}$ and $|\tilde{H}_0|_{g_e}$ thanks to the mean values of $h - H_0$ on $S_e(\epsilon)$ and $S_e(\epsilon^{-1})$, and we consequently have for some constant $C > 0$,

$$|c_0|_{g_e} + |\tilde{H}_0^-|_{g_e} \leq C \|h - H_0\|_{C_\beta^0(A_e(\epsilon, \epsilon^{-1}))}.$$

Hence we finally have the existence of a constant $C > 0$ such that we have, going from L^2 -controls to C^0 -controls thanks to (17) applied to H_* ,

$$\|H_*\|_{C_1^0} \leq C \|h - H_0\|_{C_\beta^0},$$

and therefore the stated inequality (11).

Let us prove the estimate (12), and assume towards a contradiction that there exists a sequence of positive numbers $\epsilon_i \rightarrow 0$, and a sequence of symmetric 2-tensors h_i on annuli $A_e(\epsilon_i, \epsilon_i^{-1})$ satisfying, $\|h_i - H_i\|_{C_\beta^0(A_e(\epsilon_i, \epsilon_i^{-1}))} = 1$, and $\|P_e h_i\|_{r_e^{-2} C_\beta^0(A_e(\epsilon_i, \epsilon_i^{-1}))} \leq \frac{1}{i}$.

Remark 14 The failure of these properties will indeed yield the estimate (12) since by elliptic regularity we will get higher order estimates on the smaller domain $A_e(2\epsilon, (2\epsilon)^{-1})$.

Let then $(x_i)_i$ be a sequence of points of $A_e(\epsilon_i, \epsilon_i^{-1})$ where the $C_\beta^0(A_e(\epsilon_i, \epsilon_i^{-1}))$ -norm of $\bar{h}_i := h_i - H_i$ is reached. We can extract a subsequence with one of the following behaviors:

- (1) $r_e(x_i) \rightarrow +\infty$, and $\epsilon_i r_e(x_i) \rightarrow 0$,

- (2) $r_e(x_i) \rightarrow +\infty$, and $\epsilon_i r_e(x_i) \rightarrow c > 0$,
- (3) $r_e(x_i) \rightarrow 0$ and $\epsilon_i^{-1} r_e(x_i) \rightarrow +\infty$,
- (4) $r_e(x_i) \rightarrow 0$ and $\epsilon_i^{-1} r_e(x_i) \rightarrow c > 0$,
- (5) $r_e(x_i) \rightarrow c > 0$.

In all cases, we rescale to fix $r_e(x_i) = 1$ by defining, for all x ,

$$h'_i(x) := \frac{\bar{h}_i(r_e(x_i)x)}{\epsilon_i^\beta (r_e(x_i)^\beta + r_e(x_i)^{-\beta})},$$

which satisfies

$$(\nabla_e^* \nabla_e h'_i)(x) = r_e(x_i)^2 (\nabla_e^* \nabla_e \bar{h}_i)(r_e(x_i)x).$$

Since we had by assumption the controls

$$\bar{h}_i(x) \leq \epsilon_i^\beta (r_e(x)^\beta + r_e(x)^{-\beta}),$$

and

$$|\nabla_e^* \nabla_e \bar{h}_i|_{g_e}(x) \leq \frac{1}{i} r_e(x)^{-2} \epsilon_i^\beta (r_e(x)^\beta + r_e(x)^{-\beta}),$$

our new symmetric 2-tensor h'_i vanishes at $\frac{x_0}{r_e(x_i)}$ and on $S_e(\epsilon_i r_e(x_i)^{-1})$, and is constant on $S_e(\epsilon_i^{-1} r_e(x_i)^{-1})$. It moreover satisfies

$$|h'_i|_{g_e}(x) \leq \frac{((r_e(x_i)r_e(x))^\beta + (r_e(x_i)r_e(x))^{-\beta})}{r_e(x_i)^\beta + r_e(x_i)^{-\beta}}$$

with equality at x_i and

$$|\nabla_e^* \nabla_e h'_i|_{g_e}(x) \leq \frac{1}{i} r_e(x)^{-2} \frac{((r_e(x_i)r_e(x))^\beta + (r_e(x_i)r_e(x))^{-\beta})}{r_e(x_i)^\beta + r_e(x_i)^{-\beta}}.$$

In the different situations, up to extracting a subsequence, we finally get one of the following limits

- (1) on $\mathbb{R}^4 \setminus \{0\}$, a solution h'_∞ of $P_e h'_\infty = \frac{1}{2} \nabla_e^* \nabla_e h'_\infty = 0$, and $\sup r^{-\beta} h'_\infty = 1$, but there does not exist such a solution because the harmonic symmetric 2-tensors decay at least as $\mathcal{O}(r)$ at 0 if they vanish at 0 and must therefore grow at this rate at infinity. This is a contradiction.
- (2) on $B_e(1/c) \setminus \{0\}$, a solution h'_∞ of $P_e h'_\infty = 0$, and $\sup r^{-\beta} h'_\infty = 1$, and such that $(h'_\infty)|_{S_e(1/c)}$ is constant. The unique solution to the Dirichlet problem with the zero condition at 0 and a constant condition on $S_e(1/c)$ is $h'_\infty = 0$. This is a contradiction.

- (3) on $\mathbb{R}^4 \setminus \{0\}$, a solution h'_∞ of $P_e h'_\infty = 0$, and $\sup r^\beta h'_\infty = 1$, but there does not exist such a solution because the harmonic symmetric 2-tensors decaying at infinity decay at least like $\mathcal{O}(r^{-2})$, and therefore blow up at least at this rate at 0, and finally, $h'_\infty = 0$. This is a contradiction.
- (4) on $\mathbb{R}^4 \setminus B_e(1/c)$, a solution h'_∞ of $P_e h'_\infty = 0$, and $\sup r^\beta h'_\infty = 1$ and $(h'_\infty)|_{S_e(1/c)} = 0$. The unique solution to the Dirichlet problem on $\mathbb{R}^4 \setminus B_e(1/c)$, decaying at infinity and vanishing on $S_e(1/c)$ being zero, we have $h_\infty = 0$. This is a contradiction.
- (5) on $\mathbb{R}^4 \setminus \{0\}$, a solution h'_∞ of $P_e h'_\infty = 0$, and $\sup(r^\beta + r^{-\beta})h'_\infty = 1$ satisfying $h'_\infty(\frac{x_0}{c}) = 0$. The conditions $P_e h'_\infty = 0$, and $\sup(r^\beta + r^{-\beta})h'_\infty = 1$ imply that h'_∞ is constant, since h'_∞ vanishes at $\frac{x_0}{c}$, we have $h'_\infty = 0$. This is a contradiction.

We therefore deduce that there exists $\epsilon_e > 0$ and $C_e > 0$ such that for all $0 < \epsilon < \epsilon_e$ and all symmetric 2-tensor h on the annulus $A_e(\epsilon, \epsilon^{-1})$, we have

$$\|h - H_0\|_{C_\beta^0(A_e(\epsilon, \epsilon^{-1}))} \leq C_e \|P_g h\|_{r_e^{-2} C_\beta^\alpha(A_e(\epsilon, \epsilon^{-1}))}.$$

In order to prove the estimate (12) and go from a $C_\beta^0(A_e(\epsilon, \epsilon^{-1}))$ -controls to $C_\beta^{2,\alpha}(A_e(2\epsilon, (2\epsilon)^{-1}))$ -controls, we use elliptic estimates which are satisfied on the flat annuli according to the end of the proof of Proposition 1. \square

With a completely analogous proof using the harmonic decomposition of 1-forms on a cone of [CT94, (2.16)-(2.19)] (see also Section 3.1 for particular case of flat cones), we have the same result for vector fields, but this time, we treat the linear kernel of $\delta\delta^*$ on \mathbb{R}^4/Γ separately. On \mathbb{R}^4 , the elliptic operator

$$\delta_e \delta_e^* = \nabla_e^* \nabla_e - \frac{1}{2} d_e^* d = dd_e^* + \frac{1}{2} d_e^* d,$$

has its kernel equal to the linear vector fields of the kernel of $\delta_e \delta_e^*$ among the vector fields of order $\mathcal{O}(r_e^{1-\beta} + r_e^{1+\beta})$ for $0 < \beta < 1$, see Lemma 3.1 for a proof of this and Section 3.1 for a description of the kernel.

Proposition 4 Let $0 < \beta < 1$. There exists $C_e > 0$, and $\epsilon_e > 0$ such that for any vector field X on an annulus of radii $0 < \epsilon < \epsilon_e$ and $\frac{1}{\epsilon}$, there exists Y_0 , a linear vector field of the kernel of $\delta_e \delta_e^*$, and an element Y_* of the kernel of $\delta_e \delta_e^*$ satisfying

$$\begin{aligned} \|Y_*\|_{r_e C_1^{3,\alpha}(A_e(2\epsilon, (2\epsilon)^{-1}))} &\leq C_e \|X - Y_0\|_{r_e C_\beta^{3,\alpha}(A_e(\epsilon, \epsilon^{-1}))}, \\ \|X - Y_0 - Y_*\|_{r_e C_\beta^{3,\alpha}(A_e(2\epsilon, (2\epsilon)^{-1}))} &\leq C_e \|\delta_{g_e} \delta_{g_e}^* X\|_{r_e^{-1} C_\beta^{1,\alpha}(A_e(\epsilon, \epsilon^{-1}))}. \end{aligned}$$

In particular, this implies the following control, for all $x \in A_e(1/2, 2)$,

$$(25) \quad \begin{aligned} & |(X - Y_0)(x)|_{g_e} + |\nabla(X - Y_0)(x)|_{g_e} + |\nabla^2(X - Y_0)(x)|_{g_e} + [\nabla^2(X - Y_0)]_{C^\alpha(g)}(x) \\ & \leq C_e \left((2\epsilon)^\beta \|\delta_e \delta_e^* X\|_{r_e^{-1} C_\beta^{1,\alpha}(A_e(\epsilon, \epsilon^{-1}))} + 2\epsilon \|X - Y_0\|_{r_e C_\beta^{3,\alpha}(A_e(\epsilon, \epsilon^{-1}))} \right). \end{aligned}$$

2.2.2 Approximate kernels

Let (M, g^D) be a naïve desingularization of an Einstein orbifold. For each annulus $\mathcal{A}_k(t, \epsilon)$ (see Definition 1.9) between N_k and N_j or N_k and M_o , by construction there exists a diffeomorphism

$$\Phi_k : A_e(\epsilon^{-1} \sqrt{T_j} \sqrt{t_k}, \epsilon \sqrt{T_j}) \subset \mathbb{R}^4 / \Gamma_k \rightarrow \mathcal{A}_k(t, \epsilon) \subset M,$$

such that there exists $C > 0$ for which, for all $0 < \beta < 1$,

$$(26) \quad \|\Phi_k^* g^D - g_e\|_{C_\beta^{2,\alpha}(A_e(\epsilon^{-1} \sqrt{T_j} \sqrt{t_k}, \epsilon \sqrt{T_j}))} \leq C \epsilon^{2-\beta}.$$

Because of the above constant symmetric 2-tensors and the linear vector fields, we cannot expect estimates independent of the gluing scales in the definition of (M, g^D) of the type $\|h\|_{C_\beta^{2,\alpha}(g^D)} \leq C \|P_{g^D} h\|_{r_D^{-2} C_\beta^\alpha(g^D)}$ which are needed to apply an inverse function theorem. Indeed, we have the following estimates according to Proposition 7 (which is proven below). Recall that the cut-off functions are in Definition 1.10.

Remark 15 In most of the rest of this article, we will often abusively forget the diffeomorphism Φ_k to simplify the notations. For instance, a symmetric 2-tensor $\Phi_k^*(\chi_{\mathcal{A}_k(t,\epsilon)} H)$ will be denoted $\chi_{\mathcal{A}_k(t,\epsilon)} H$ on M .

Proposition 5 On a naïve desingularization (M, g_t^D) , for all $0 < \beta < 1$, there exists $C > 0$ such that for H_k a constant symmetric 2-tensor, and $\chi_{\mathcal{A}_k(t,\epsilon)}$ the cut-off function defined in Definition 1.10,

$$\|P_{g^D}(\chi_{\mathcal{A}_k(t,\epsilon)} H_k)\|_{r_D^{-2} C_\beta^\alpha(g^D)} \leq C |H_k|_{g_e},$$

but

$$\|\chi_{\mathcal{A}_k(t,\epsilon)} H_k\|_{C_\beta^0(g^D)} \geq \frac{1}{2} t_{\max}^{-\frac{\beta}{4}} |H_k|_{g_e}.$$

□

Linear vector fields in the kernel of $\delta \delta^*$ also rule out the existence of estimates independent of t for the operator $\delta \delta^*$ according to Proposition 7 proven below.

Proposition 6 On a naïve desingularization (M, g_t^D) , for all $0 < \beta < 1$, there exists $C > 0$ such that for X_k a linear vector field in the kernel of $\delta_e \delta_e^*$,

$$\|\delta_{g^D} \delta_{g^D}^* (\chi_{\mathcal{A}_k(t, \epsilon)} X_k)\|_{r_D^{-1} C_{\beta}^{1, \alpha}(g^D)} \leq C \|X_k\|_{r_e C_0^0(g_e)},$$

but

$$\|\chi_{\mathcal{A}_k(t, \epsilon)} X_k\|_{C_{\beta}^0(g^D)} \geq \frac{1}{2} t^{\frac{-\beta}{4}} \|X_k\|_{r_e C_0^0(g_e)}.$$

□

Weighted decoupling norms. Propositions 3 and 4 actually show that we can control the inverses of our operators once we solve our equations modulo constant symmetric 2-tensors and the linear vector fields of the kernel of $\delta_e \delta_e^*$ on \mathbb{R}^4 and Propositions 5 and 6 show that we cannot expect better. We therefore introduce new norms to reflect this. They are similar to the norms introduced in [Bam12] for similar reasons.

Definition 2.5 (Norm $\|\cdot\|_{C_{\beta, *}^{k, \alpha}}$ on symmetric 2-tensors) Let h be a symmetric 2-tensor on (M, g^D) , (respectively (M_o, g_o) or (N, g_b)). We define its $C_{\beta, *}^{k, \alpha}$ -norm by

$$\|h\|_{C_{\beta, *}^{k, \alpha}} := \inf_{h_*, H_k} \|h_*\|_{C_{\beta}^{k, \alpha}} + \sum_k |H_k|_{g_e},$$

where the infimum is taken among the couples (h_*, H_k) satisfying $h = h_* + \sum_k \chi_{\mathcal{A}_k(t, \epsilon)} H_k$ (respectively $h = h_* + \sum_k \chi_{B_o(\epsilon)} H_k$ or $h = h_* + \sum_k \chi_{B_b(\epsilon)} H_k$), for H_k a constant traceless symmetric 2-tensor on \mathbb{R}^4/Γ_k .

Definition 2.6 (Norm $\|\cdot\|_{r C_{\beta, *}^{k, \alpha}}$ on vector fields) Let X a vector field on (M, g^D) (respectively (M_o, g_o) or (N, g_b)). We define its $r C_{\beta, *}^{k, \alpha}$ -norm, where r is the function r_D (respectively r_o or r_b) by

$$\|X\|_{r C_{\beta, *}^{k, \alpha}} := \inf_{X_*, X_k} \|X_*\|_{r C_{\beta}^{k, \alpha}} + \sum_k \|X_k\|_{r C_0^0(g_e)},$$

where the infimum is taken among the couples (X_*, X_k) satisfying $X = X_* + \sum_k \chi_{\mathcal{A}_k(t, \epsilon)} X_k$ (respectively $X = X_* + \sum_k \chi_{B_o(\epsilon)} X_k$ or $X = X_* + \chi_{B_b(\epsilon)} X_k$).

Remark 16 By definition of the weighted norms, on an orbifold or orbifold ALE, the decompositions $h = h_* + \sum_k \chi_{B(\epsilon)} H_k$ and $X = X_* + \sum_k \chi_{B(\epsilon)} X_k$ are unique and determined respectively by the limits of h and of $\frac{X}{r}$ when $r \rightarrow 0$ (where $r = r_o$ or $r = r_b$). Indeed, in other cases, the expression we minimize is infinite.

Remark 17 By definition, we have

$$\|\cdot\|_{C_{\beta,*}^{k,\alpha}} \leq \|\cdot\|_{C_{\beta}^{k,\alpha}}, \text{ and } \|\cdot\|_{rC_{\beta,*}^{k,\alpha}} \leq \|\cdot\|_{rC_{\beta}^{k,\alpha}},$$

and the spaces $(C_{\beta,*}^{k,\alpha}, \|\cdot\|_{C_{\beta,*}^{k,\alpha}})$ and $(rC_{\beta,*}^{k,\alpha}, \|\cdot\|_{rC_{\beta,*}^{k,\alpha}})$ are clearly Banach spaces.

2.2.3 Estimates in the decoupling norms

Let us show that it is possible to control thanks to the $r^m C_{\beta}^{k,\alpha}$ -norm the images by the operators P and $\delta\delta^*$ of elements of $r^{m+2}C_{\beta,*}^{k+2,\alpha}$.

Proposition 7 Let $0 < \beta < 1$, and (M_\bullet, g_\bullet) one of the spaces (M_o, g_o) , (N_j, g_{b_j}) or (M, g^D) , g a metric, h a symmetric 2-tensor, and X a vector field on M_\bullet . We then have, the following controls:

$$\|P_{g_\bullet} h\|_{r_\bullet^{-2}C_{\beta}^{\alpha}(g_\bullet)} \leq C \|h\|_{C_{\beta,*}^{2,\alpha}(g_\bullet)},$$

$$\|\delta_{g_\bullet} \delta_{g_\bullet}^* X\|_{r_\bullet^{-1}C_{\beta}^{1,\alpha}(g_\bullet)} \leq C \|X\|_{r_\bullet C_{\beta,*}^{3,\alpha}(g_\bullet)},$$

$$\|P_{g_\bullet}(h) - P_{g_\bullet}(h)\|_{r_\bullet^{-2}C_{\beta}^{\alpha}(g_\bullet)} \leq C \|g - g_\bullet\|_{C_{\beta,*}^{2,\alpha}(g_\bullet)} \|h\|_{C_{\beta,*}^{2,\alpha}(g_\bullet)},$$

and

$$\|\delta_g \delta_g^*(X) - \delta_{g_\bullet} \delta_{g_\bullet}^*(X)\|_{r_\bullet^{-1}C_{\beta}^{1,\alpha}(g_\bullet)} \leq C \|g - g_\bullet\|_{C_{\beta,*}^{2,\alpha}(g_\bullet)} \|X\|_{r_\bullet C_{\beta,*}^{3,\alpha}(g_\bullet)}.$$

Proof Let us show the result for g^D , the proof for other spaces is very similar. For the two first inequalities, consider h a symmetric 2-tensor and X a vector field on M , and some decompositions $h = h_* + \sum_k \chi_{\mathcal{A}_k(t,\epsilon)} H_k$ and $X = X_* + \sum_k \chi_{\mathcal{A}_k(t,\epsilon)} X_k$. Remark 12 implies that we have the following controls for h_* and X_* ,

$$(27) \quad \|P_{g^D} h_*\|_{r_D^{-2}C_{\beta}^{\alpha}(g^D)} \leq C \|h_*\|_{C_{\beta}^{2,\alpha}(g^D)},$$

and

$$(28) \quad \|\delta_{g^D} \delta_{g^D}^* X_*\|_{r_D^{-1}C_{\beta}^{1,\alpha}(g^D)} \leq C \|X_*\|_{r_D C_{\beta}^{3,\alpha}(g^D)}.$$

On \mathbb{R}^4/Γ , we have $P_e H_k = 0$ and $\delta_e \delta_e^* X_k = 0$, hence, since for all $l \in \mathbb{N}$, we have

$$(29) \quad |\nabla^l \chi_k|_{g^D} \leq C l r_D^{-l},$$

and thanks to the control (26), we have

$$(30) \quad \|P_{g^D}(\chi_{\mathcal{A}_k(t,\epsilon)} H_k)\|_{C_{\beta}^{\alpha}(g^D)} \leq C |H_k|_{g_e},$$

and

$$(31) \quad \|\delta_{g^D} \delta_{g^D}^* (\chi_{\mathcal{A}_k(t,\epsilon)} X_k)\|_{r_D^{-1} C_\beta^{1,\alpha}(g^D)} \leq C \|X_k\|_{r_e C_0^0(g_e)},$$

where we pulled-back thanks to the diffeomorphism

$$\Phi_k : A_e(\epsilon^{-1} \sqrt{T_j} \sqrt{t_k}, \epsilon \sqrt{T_j}) \subset \mathbb{R}^4 / \Gamma_k \rightarrow \mathcal{A}_k(t, \epsilon) \subset M.$$

Summing the controls (27) and (30) on the one hand, and the controls (28) and (31) on the other hand, yields the two first inequalities stated.

Let us now focus on the two last inequalities, which are more difficult to obtain. The control we want is local, let us therefore write down the expressions of our operators in local coordinates in an orthonormal basis (e_i) . For a symmetric 2-tensor h , denoting $h_{ij} = h(e_i, e_j)$ and R_{ijkl} the Riemannian curvature in coordinates, we have

$$(32) \quad P_g(h)_{ij} = \frac{1}{2} (\nabla_g^* \nabla_g h)_{ij} - g^{kp} g^{lq} R_{ikjl} h_{pq},$$

where ∇_i is the covariant derivative for g in the direction e_i . We directly see thanks to the estimates of Remark 12 that we have the controls:

$$\|P_g(h) - P_{g^D}(h)\|_{r_D^{-2} C_\beta^\alpha(g^D)} \leq C \|g - g^D\|_{C_\beta^{2,\alpha}(g^D)} \|h\|_{C_\beta^{2,\alpha}(g^D)}.$$

Let us now consider $\sum_k \chi_{\mathcal{A}_k(t,\epsilon)} H_k$ and $\sum_k \chi_{\mathcal{A}_k(t,\epsilon)} H'_k$. These tensors being all supported in the annuli $\mathcal{A}_k(t, \epsilon)$, we just need to restrict our attention to them. The crucial remark is that in (32), every term involves at least a derivative of h or of $g - g^D$. Hence, we have a more precise control on h a symmetric 2-tensor supported in $\mathcal{A}_k(t, \epsilon_0)$

$$(33) \quad \begin{aligned} \|P_g(h) - P_{g^D}(h)\|_{r_D^{-2} C_\beta^\alpha(g^D)} &\leq C \left(\|g - g^D\|_{C_0^{2,\alpha}(g^D)} \|\nabla^2 h\|_{r_D^{-2} C_\beta^\alpha(g^D)} \right. \\ &\quad + \|\nabla(g - g^D)\|_{r_D^{-1} C_\beta^{1,\alpha}(g^D)} \|\nabla h\|_{r_D^{-1} C_\beta^{1,\alpha}(g^D)} \\ &\quad \left. + \|\nabla^2(g - g^D)\|_{r_D^{-2} C_\beta^\alpha(g^D)} \|h\|_{C_0^{2,\alpha}(g^D)} \right), \end{aligned}$$

(notice the norms $C_0^{2,\alpha}(g^D)$ in which we have $\|\chi_{\mathcal{A}_k(t,\epsilon)} H_k\|_{C_0^{2,\alpha}(g^D)} \leq C |H_k|_{g_e}$ and $\|\chi_{\mathcal{A}_k(t,\epsilon)} H'_k\|_{C_0^{2,\alpha}(g^D)} \leq C |H'_k|_{g_e}$). There remains to control the derivatives of the tensors $\chi_{\mathcal{A}_k(t,\epsilon)} H_k$ and $\chi_{\mathcal{A}_k(t,\epsilon)} H'_k$. Since the H_k and H'_k are constant on \mathbb{R}^4 , and since the cut off functions are bounded in $C_0^2(g^D)$ by (29), for $i \in \{1, 2\}$, we have

$$\|\nabla^i (\chi_{\mathcal{A}_k(t,\epsilon)} H_k)\|_{r_D^{-i} C_\beta^{2-i,\alpha}(g^D)} \leq C |H_k|_{g_e},$$

and

$$\|\nabla^i (\chi_{\mathcal{A}_k(t,\epsilon)} H'_k)\|_{r_D^{-i} C_\beta^{2-i,\alpha}(g^D)} \leq C |H'_k|_{g_e},$$

which together with (33) let us conclude that the third estimate holds.

For the vector fields, we have the following rewriting for X a vector field supported in $\mathcal{A}_k(t, \epsilon_0)$

$$\begin{aligned} \|\delta_g \delta_g^*(X) - \delta_{g^D} \delta_{g^D}^*(X)\|_{r_D^{-1} C_\beta^{1,\alpha}(g^D)} &\leq C(\|(\delta_g - \delta_{g^D})(\mathcal{L}_X g)\|_{r_D^{-1} C_\beta^{1,\alpha}(g^D)} \\ &\quad + \|\delta_{g^D}(\mathcal{L}_X(g^D - g))\|_{r_D^{-1} C_\beta^{1,\alpha}(g^D)}). \end{aligned}$$

We moreover know that for X_k a linear vector field in the kernel of $\delta_e \delta_e^*$, then the symmetric 2-tensor $\mathcal{L}_{X_k} g_e$ is constant, and more generally, for H_k a constant symmetric 2-tensor, we have $\delta_{g_e}(\mathcal{L}_{X_k} H_k) = 0$ on \mathbb{R}^4 . Using these two facts and the controls of the cut-off functions, we conclude that the last estimate of the statement holds by an argument similar to the above one for 2-tensors and P . \square

2.2.4 Elliptic estimates for the decoupling norms

Some elliptic estimates are still satisfied in these norms.

Proposition 8 Let $0 < \beta < 1$, g a metric, h a symmetric 2-tensor and X a vector field on M_o (respectively N_j , or M). Then, there exists $\epsilon_* = \epsilon_*(g_o, g_{b_j}, g^D, \beta) > 0$ and $C > 0$ such that if we have $\|g - g_\bullet\|_{C_{\beta,*}^{2,\alpha}(g_\bullet)} \leq \epsilon_*$, where g_\bullet is one of the norms g_o , g_{b_j} or g^D , then,

$$\|h\|_{C_{\beta,*}^{2,\alpha}(g_\bullet)} \leq C(\|P_g h\|_{r^{-2} C_\beta^\alpha(g_\bullet)} + \|h\|_{C_{\beta,*}^0(g_\bullet)}),$$

and

$$\|X\|_{r C_{\beta,*}^{3,\alpha}(g_\bullet)} \leq C(\|\delta_g \delta_g^* X\|_{r^{-1} C_\beta^{1,\alpha}(g_\bullet)} + \|X\|_{r C_{\beta,*}^0(g_\bullet)}).$$

Proof Let g_\bullet be one of the metrics g_o , g_{b_j} or g^D , and for all k , H_k a traceless constant symmetric 2-tensor on \mathbb{R}^4/Γ_k , and X_k a Killing vector field on \mathbb{R}^4/Γ_k . Let moreover h_* be a symmetric 2-tensor of $C_\beta^{2,\alpha}(g_\bullet)$ and X_* be a vector field of $r C_\beta^{3,\alpha}(g_\bullet)$, and define $h = h_* + \sum_k \chi_\bullet X_k$ and $X = X_* + \sum_k \chi_\bullet X_k$, where χ_\bullet is $\chi_{\mathcal{A}_k(t,\epsilon)}$ or $\chi_{B_{g_\bullet}(\epsilon)}$ (of Definition 1.10) depending on the metric.

We then have the following controls:

$$\|P_{g_\bullet} \chi_\bullet H_k\|_{r^{-2} C_\beta^\alpha(g_\bullet)} \leq C |H_k|_{g_e},$$

and

$$\|\delta_{g_\bullet} \delta_{g_\bullet}^* \chi_\bullet X_k\|_{r^{-1} C_\beta^{1,\alpha}(g_\bullet)} \leq C \|X_k\|_{r_e C_\beta^0(g_e)}.$$

Hence, for h_* , we have

$$\|P_{g_\bullet} h_*\|_{r_\bullet^{-2} C_\beta^\alpha} \leq C(\|P_{g_\bullet} h\|_{r_\bullet^{-2} C_\beta^\alpha(g_\bullet)} + \sum_k |H_k|_{g_e}),$$

and the expected estimate for $g = g_\bullet$ is then a consequence of the elliptic estimates in the weighted spaces of Lemma 1 which give

$$\|h_*\|_{C_\beta^{2,\alpha}(g_\bullet)} \leq C(\|P_{g_\bullet} h_*\|_{r_\bullet^{-2} C_\beta^\alpha(g_\bullet)} + \|h_*\|_{C_\beta^0(g_\bullet)}),$$

and imply therefore that

$$\|h\|_{C_{\beta,*}^{2,\alpha}(g_\bullet)} \leq 2C^2(\|P_{g_\bullet} h\|_{r_\bullet^{-2} C_\beta^\alpha(g_\bullet)} + \|h\|_{C_{\beta,*}^0(g_\bullet)}).$$

The same argument works for the operator $\delta\delta^*$ on the vector fields thanks to the elliptic estimates of Lemma 2.

Proposition 7 finally lets us go from the metric g_\bullet to a metric g satisfying $\|g - g_\bullet\|_{C_{\beta,*}^{2,\alpha}(g_\bullet)} \leq \epsilon_*$. \square

3 Reduced divergence-free gauge

When the Einstein orbifold which we approximate has nonpositive scalar curvature, we can always put our Einstein metrics in Bianchi gauge with respect to a naïve desingularization (see [Biq13, Lemme 8.2] adapted to our norms). When the Ricci curvature of our Einstein manifolds is positive, this is not necessarily true, but we can still use the divergence-free gauge. This is the goal of this section whose main result is Proposition 10. To show this, we will use a Banach fixed point theorem approach which necessitates the study of the linearized equation:

$$\delta\delta^* X = -\delta h,$$

where X is a vector field, and h a symmetric 2-tensor.

In our degenerating situation, we want to obtain estimates in our weighted norms which are independent of the gluing scales. A difficulty is that our limit orbifold might have more symmetries than the Ricci-flat ALE spaces (for example, $\mathbb{S}^4/\mathbb{Z}_2$ desingularized by Eguchi-Hanson metrics). The associated Killing vector fields would give an approximate kernel for $\delta\delta^*$ which would not be an actual kernel or cokernel. We will need to define a reduced divergence-free gauge to obtain uniform estimates as the gluing scales go to zero.

Remark 18 All along this section, if nothing is precised, an Einstein orbifold (M_o, g_o) will be either compact or ALE.

3.1 Kernel of the linearization

Let us focus on the operator $\delta\delta^*$ on a flat cone $(\mathbb{R}^4/\Gamma, g_e)$, on an orbifold (M_o, g_o) , and on Ricci-flat ALE orbifolds (N_j, g_{b_j}) .

On a flat cone. On the flat cone $(\mathbb{R}^4/\Gamma, g_e) = (\mathbb{R}^+ \times \mathbb{S}^3/\Gamma, dr^2 + r^2 g_{\mathbb{S}^3/\Gamma})$, according to [CT94, Section 2], any 1-form on \mathbb{R}^4/Γ is a countable sum of 1-forms of one of the following types which are preserved by $\delta\delta^*$:

- (1) $p(r)\psi$, where $\delta_{\mathbb{S}^3/\Gamma}\psi = 0$, and ψ is eigenvector of the Hodge Laplacian of \mathbb{S}^3/Γ ,
- (2) $r^{-1}l(r)\phi dr + u(r)r d_{\mathbb{S}^3/\Gamma}\phi$, and ϕ is eigenfunction of the Hodge Laplacian of \mathbb{S}^3/Γ ,

where $p, l, u : \mathbb{R}^+ \rightarrow \mathbb{R}$ and $\phi : \mathbb{S}^3/\Gamma \rightarrow \mathbb{R}$ are functions, and where ψ is a 1-form on \mathbb{S}^3/Γ .

According to [AV12, Section 4.1], thanks to the computation of the eigenvalues of the Laplacian and of the Hodge Laplacian on the 1-forms of the sphere [Fol89, Theorem C], the solutions to $\delta_e\delta_e^*\omega = 0$ are countable sums of 1-forms of the following types

- (1) $r^{a_j^\pm}\psi$ with $a_j^\pm := \pm(1+j)$, $j \in \mathbb{N}^*$, where ψ is an eigenvector the Hodge Laplacian,
- (2) $r^{b_j^\pm} d_{\mathbb{S}^3/\Gamma}\phi + b_j^\pm r^{b_j^\pm-1}\phi dr$, or $2r^{b_j^\pm+2} d_{\mathbb{S}^3/\Gamma}\phi + b_j^\mp r^{b_j^\pm+1}\phi dr$, with $b_j^\pm = -1 \pm (1+j)$, $j \in \mathbb{N}$ and where ϕ is an eigenfunction the Hodge Laplacian.

Since we are interested in solving an equation

$$\delta\delta^*X = -\delta h,$$

where X is a vector field, and h a symmetric 2-tensor is in $C_{\beta,*}^{2,\alpha}$, we are naturally looking for X in $r_D C_{\beta,*}^{3,\alpha}$. The *exceptional values* of $\delta_e\delta_e^*$ are the values $\gamma \in \mathbb{R}$ such that there exists a homogeneous 1-form whose norm is proportional to r_e^γ in the kernel of $\delta_e\delta_e^*$. We are interested in the exceptional values around the exceptional value 1 associated to the linear vector fields of the kernel of $\delta_e\delta_e^*$.

Lemma 3.1 *On $(\mathbb{R}^4/\Gamma) \setminus \{0\}$ for $\Gamma \neq \{e\}$, 1 is the only exceptional value between -3 and 2 .*

Proof According to the above discussion, the exceptional values are a priori of the form $a_j^\pm - 1 = -1 \pm (1+j)$ for $j \in \mathbb{N}^*$, $b_j^\pm - 1 = -2 \pm (1+j)$ with $j \in \mathbb{N}$, or

$b_j^\pm + 1 = \pm(1 + j)$ with $j \in \mathbb{N}$. Let us first note that $a_j^\pm - 1 \in (-3, 2)$ for $j \in \mathbb{N}^*$ implies that $a_j^\pm - 1 = 1$, and therefore that no other exceptional value between -3 and 2 come from the first type of 1-form.

For $b_j^\pm + 1$, the values 0 and -1 are a priori possible, and for $b_j^\pm - 1$, -1 and -2 are a priori possible. However, these values cannot appear on a flat cone \mathbb{R}^4/Γ for $\Gamma \neq \{e\}$. Indeed, the values $b_j^\pm - 1 = 0$ and $b^\pm + 1 = -2$ only appear if -3 is an eigenvalue of the Laplacian on the link of the cone, but this is not the case for \mathbb{S}^3/Γ because there does not exist any non zero Γ -invariant linear function on \mathbb{R}^4 .

For the values $b_j^\pm - 1 = -1$ and $b_j^\pm + 1 = -1$, we use the form of the solutions. In the first case, $b_j^\pm = 0$ gives

$$r^{b_j^\pm} d_{\mathbb{S}^3/\Gamma} \phi + b_j^\pm r^{b_j^\pm - 1} \phi dr = d_{\mathbb{S}^3/\Gamma} \phi,$$

for $\Delta_{\mathbb{S}^3/\Gamma} \phi = 0$, therefore ϕ is constant and finally $d_{\mathbb{S}^3/\Gamma} \phi = 0$. In the second case, the equality $b_j^\pm = -2$, that is $b_j^\mp = 0$, gives

$$2r^{b_j^\pm + 2} d_{\mathbb{S}^3/\Gamma} \phi + b_j^\mp r^{b_j^\pm + 1} \phi dr = d_{\mathbb{S}^3/\Gamma} \phi,$$

for $\Delta_{\mathbb{S}^3/\Gamma} \phi = 0$, therefore ϕ is constant and finally $d_{\mathbb{S}^3/\Gamma} \phi = 0$. \square

The 1-forms associated to the exceptional value 1 are sum of 1-forms of the three following types:

- (1) $r^2 \psi$, where ψ is the dual of a Killing vector field of \mathbb{S}^3 ,
- (2) $r dr$,
- (3) $2r \phi dr + r^2 d_{\mathbb{S}^3/\Gamma} \phi$.

On an orbifold or an ALE space. Since there is no exceptional value other than 1 in $(-3, 2)$, we have the following result on an orbifold ALE.

Proposition 9 Let (N_j, g_{b_j}) be a Ricci-flat ALE orbifold. For $0 < \beta < 1$, the operator

$$\delta_{g_{b_j}} \delta_{g_{b_j}}^* : r_{b_j} C_{\beta, *}^{3, \alpha} \rightarrow r_{b_j}^{-1} C_{\beta}^{1, \alpha}$$

is bijective.

Let (M_o, g_o) be a compact Einstein orbifold. For $0 < \beta < 1$, the operator

$$\delta_{g_o} \delta_{g_o}^* : r_o C_{\beta, *}^{3, \alpha} \rightarrow r_o^{-1} C_{\beta}^{1, \alpha}$$

is Fredholm and both its kernel and its cokernel are equal to \mathbf{K}_o , the set of Killing vector fields of (M_o, g_o) .

As a consequence, there exist $C_o > 0$ and $\epsilon_o > 0$ depending on g_o such that if $\|g - g_o\|_{C_{\beta,*}^{2,\alpha}(M_o)} \leq \epsilon_o$, then we have for any vector field $X \in \mathbf{K}_o^\perp$ on M_o

$$\|X\|_{r_o C_{\beta,*}^{3,\alpha}(g_o)} \leq C_o \|\delta_g \delta_g^* X\|_{r_o^{-1} C_{\beta}^{1,\alpha}(g_o)}.$$

There also exists $C_j > 0$ and $\epsilon_j > 0$ depending on g_{b_j} such that if $\|g - g_{b_j}\|_{C_{\beta,*}^{2,\alpha}(N_j)} \leq \epsilon_j$ then we have for any vector field X on N_j ,

$$\|X\|_{r_{b_j} C_{\beta,*}^{3,\alpha}(g_{b_j})} \leq C_j \|\delta_g \delta_g^* X\|_{r_{b_j}^{-1} C_{\beta}^{1,\alpha}(g_{b_j})}.$$

Proof For orbifold singularities, we will first authorize our tensors to behave like $r^{1-\beta}$ for $0 < \beta < 1$ at the singularities, instead of being in $r C_{\beta,*}^{3,\alpha}$ to use the theory of elliptic operators in weighted Hölder spaces, see for instance [PR78, Chapter 2] of [LM85]. Let us start by considering an Einstein orbifold (M_o, g_o) and the operator $\delta_{g_o} \delta_{g_o}^* : r_o C_{-\beta}^{3,\alpha} \rightarrow r_o^{-1} C_{-\beta}^{1,\alpha}$ (notice the $-\beta$). Its kernel is composed of Killing vector fields of g_o . Indeed, if for $X \in r_o C_{-\beta}^{3,\alpha}$ we have $\delta_{g_o} \delta_{g_o}^* X = 0$, integrating by parts yields,

$$\begin{aligned} 0 &= \int_{M_o} \langle \delta_{g_o} \delta_{g_o}^* X, X \rangle dv_o \\ &= \int_{M_o} |\delta_{g_o}^* X|_{g_o}^2 dv_o + \lim_{r \rightarrow 0} \int_{\{r_o=r\}} \delta_{g_o}^* X(n, X) \\ &= \int_{M_o} |\delta_{g_o}^* X|_{g_o}^2 dv_o, \end{aligned}$$

where $n = \frac{\nabla r_o}{|\nabla r_o|}$, because the boundary term which is schematically $\lim_{r \rightarrow 0} (\mathcal{O}(r^{-\beta+1-\beta+3}))$ vanishes. Similarly, its cokernel is equal to the kernel of $\delta_{g_o} \delta_{g_o}^*$ on $r_o^{-3} C_{\beta}^{1,\alpha}$ which is also reduced to \mathbf{K}_o because there is no exceptional value between -3 and 1 .

On an ALE orbifold (N, g_b) , let us assume that a vector field $X \in r_b^{1-\beta} C_0^{3,\alpha}$ satisfies $\delta_{g_b} \delta_{g_b}^* X = 0$. Since there is no exceptional value between -3 and 1 , we actually have $X = \mathcal{O}(r_b)$ when $r_b \rightarrow 0$ at the singular points of (N, g_b) , and $X = \mathcal{O}(r_b^{-3})$ at infinity. Let us then consider the following integration by parts,

$$\begin{aligned} 0 &= \int_N \langle \delta_{g_b} \delta_{g_b}^* X, X \rangle dv_b \\ &= \int_N |\delta_{g_b}^* X|_{g_b}^2 dv_b - \lim_{\rho \rightarrow \infty} \int_{\{r_b=\rho\}} \delta_{g_b}^* X(n, X) + \lim_{r \rightarrow 0} \int_{\{r_b=r\}} \delta_{g_b}^* X(n, X) \\ &= \int_N |\delta_{g_b}^* X|_{g_b}^2 dv_b, \end{aligned}$$

where the boundary term vanishes because it is the sum of the limit for $r_b \rightarrow \infty$ of a $\mathcal{O}(|X|_{g_b} |\nabla X|_{g_b} r_b^3) = \mathcal{O}(r_b^{-4})$ and of the limit when $r_b \rightarrow 0$ of a $\mathcal{O}(|X|_{g_b} |\nabla X|_{g_b} r_b^3) = \mathcal{O}(r_b^4)$. Hence, we have $\delta_{g_b}^* X = 0$, and since g_b is Ricci-flat, $(\delta_b + \frac{1}{2} d\text{tr}_b) \delta_{g_b}^* X = \nabla_b^* \nabla_b X = 0$, which implies that $\nabla_b X = 0$ by integration by parts against X , and finally, that X is parallel on N . Since X tends to 0 at infinity, we have $X = 0$. The operator $\delta_{g_b} \delta_{g_b}^* : r_b^{1-\beta} C_0^{3,\alpha} \rightarrow r_b^{-1-\beta} C_0^{1,\alpha}$ is therefore injective.

The cokernel of the self adjoint operator $\delta_{g_b} \delta_{g_b}^* : r_b^{1-\beta} C_0^{3,\alpha} \rightarrow r_b^{-1-\beta} C_0^{1,\alpha}$ is equal to the kernel of $\delta_{g_b} \delta_{g_b}^*$ on $r_b^{-3+\beta} C_0^{1,\alpha}$ which is also reduced to $\{0\}$ because there is no exceptional values between $-3 + \beta$ and $1 - \beta$. The operator $\delta_{g_b} \delta_{g_b}^* : r_b^{1-\beta} C_0^{3,\alpha} \rightarrow r_b^{-1-\beta} C_0^{1,\alpha}$ is therefore bijective.

Let us finally work in the norms we are interested in and study the operators $\delta_{g_o} \delta_{g_o}^* : r_o C_{\beta,*}^{3,\alpha} \rightarrow r_o^{-1} C_\beta^{1,\alpha}$ and $\delta_{g_b} \delta_{g_b}^* : r_b C_{\beta,*}^{3,\alpha} \rightarrow r_b^{-1} C_\beta^{1,\alpha}$. Since the spaces $r_b C_{\beta,*}^{3,\alpha}$ and $r_o C_{\beta,*}^{3,\alpha}$ are respectively only the direct sum of $r_b C_\beta^{3,\alpha}$ and $r_o C_\beta^{3,\alpha}$ with a space of finite dimension composed of cut-off of linear vector fields, the image remains closed and of finite codimension. We can be more precise by noticing that

$$\delta_{g_o} \delta_{g_o}^* (r_o C_{\beta,*}^{3,\alpha}) = \delta_{g_o} \delta_{g_o}^* (r_o C_{-\beta}^{3,\alpha}) \cap r_o^{-1} C_\beta^{1,\alpha}.$$

Indeed, we have $\delta_{g_o} \delta_{g_o}^* (r_o C_{\beta,*}^{3,\alpha}) \subset \delta_{g_o} \delta_{g_o}^* (r_o C_{-\beta}^{3,\alpha}) \cap r_o^{-1} C_\beta^{1,\alpha}$ because $r_o C_{\beta,*}^{3,\alpha} \subset r_o C_{-\beta}^{3,\alpha}$ and thanks to Proposition 7. Conversely, if for $X \in r_o C_{-\beta}^{3,\alpha}$ we have $\delta_{g_o} \delta_{g_o}^* X \in r_o^{-1} C_\beta^{1,\alpha}$, then, since the only exceptional value between $1 - \beta$ and $1 + \beta$ is 1 and corresponds to the linear kernel of $\delta_e \delta_e^*$, we have $X \in r_o C_{\beta,*}^{3,\alpha}$. Similarly, we conclude that

$$\delta_{g_b} \delta_{g_b}^* (r_b C_{\beta,*}^{3,\alpha}) = \delta_{g_b} \delta_{g_b}^* (r_b^{1-\beta} C_0^{3,\alpha}) \cap r_b^{-1} C_\beta^{1,\alpha},$$

and finally, $\delta_{g_o} \delta_{g_o}^* : r_o C_{\beta,*}^{3,\alpha} \rightarrow r_o^{-1} C_\beta^{1,\alpha}$ is Fredholm with \mathbf{K}_o as kernel and cokernel, and $\delta_{g_b} \delta_{g_b}^* : r_b C_{\beta,*}^{3,\alpha} \rightarrow r_b^{-1} C_\beta^{1,\alpha}$ is bijective. We finally conclude by the open mapping theorem between Banach spaces which is stable by small perturbation of the operator. \square

3.2 Controls on the inverse of the linearization

These controls will help us treat the case of trees of singularities with small enough gluing parameters.

For this, we approximate the kernel \mathbf{K}_o on our naïve desingularization (M, g^D) in the following way. Note that $\mathbf{K}_o = 0$ for an ALE Ricci-flat orbifold (M_o, g_o) by Proposition

9. For all $\mathbf{X}_o \in \mathbf{K}_o$, according to Remark 16, on an orbifold, there exists a unique decomposition

$$\mathbf{X}_o = \mathbf{X}_{o,*} + \sum_k \chi_{B(p_k, \epsilon_0)} \mathbf{X}_{o,k},$$

such that $\|\mathbf{X}_o\|_{r_o C_{\beta,*}^{3,\alpha}} = \|\mathbf{X}_{o,*}\|_{r_o C_{\beta}^{3,\alpha}} + \sum_k \|\mathbf{X}_{o,k}\|_{r_\epsilon C_0^0}$ (other decompositions make the value infinite). We then define $\tilde{\mathbf{K}}_o$ as the space of the following vector fields on M

$$\tilde{\mathbf{X}}_{o,t} := \chi_{M_o^t} \mathbf{X}_{o,*} + \sum_k \chi_{\mathcal{A}_k(t, \epsilon_0)} \mathbf{X}_{o,k},$$

for $\mathbf{X}_o \in \mathbf{K}_o$. Note that we therefore have $\tilde{\mathbf{X}}_{o,t} = \mathbf{X}_o$ on M_o^{16t} .

Remark 19 By elliptic regularity on M_o , the norms $L^2(g_o)$, $r_D C_{\beta,*}^{3,\alpha}(g_o)$ and $r_D^{-1} C_{\beta}^{1,\alpha}(g_o)$ are equivalent on the finite-dimensional space \mathbf{K}_o . Since the C_0^4 -norms of the cut-off functions are bounded, we conclude that for ϵ and t_{\max} small enough, the norms $L^2(g^D)$, $r_D C_{\beta,*}^{3,\alpha}(g^D)$ and $r_D^{-1} C_{\beta}^{1,\alpha}(g^D)$ are equivalent on $\tilde{\mathbf{K}}_o$.

Definition 3.2 (Reduced divergence-free gauge) We define the *reduced divergence* operator, $\tilde{\delta}_g := \pi_{\tilde{\mathbf{K}}_o^\perp} \delta_g$, where $\pi_{\tilde{\mathbf{K}}_o^\perp}$ is the $L^2(g^D)$ -orthogonal projection on $\tilde{\mathbf{K}}_o^\perp$. We will say that a metric g_1 is in *reduced divergence-free* gauge with respect to a metric g_2 if $\tilde{\delta}_{g_2} g_1 = 0$.

Let us start by noticing that the operator $\tilde{\delta}_{g^D}$ is actually very close to δ_{g^D} for a naïve desingularization g^D with small enough gluing parameters.

Lemma 3.3 *There exists $C > 0$ such that for any symmetric 2-tensor $h \in C_{\beta,*}^{2,\alpha}(g^D)$, we have,*

$$(34) \quad \|(\tilde{\delta}_{g^D} - \delta_{g^D})h\|_{r_D^{-1} C_{\beta}^{1,\alpha}(g^D)} \leq C t_{\max} \|h\|_{C_{\beta,*}^{2,\alpha}(g^D)}.$$

Proof If (M_o, g_o) is ALE, then, one has $\mathbf{K}_o = \{0\}$ and therefore $\tilde{\delta}_{g^D} = \delta_{g^D}$. Let us focus on the case when M_o is compact.

Thanks to the equivalence of the different norms, see Remark 19, it is enough to show that the $L^2(g^D)$ -projection on $\tilde{\mathbf{K}}_o$ of $\delta_{g^D} h$ is small to show the result. We naturally proceed by integration by parts. Let $\tilde{\mathbf{X}}_{o,t} \in \tilde{\mathbf{K}}_o$ for $\mathbf{X}_o \in \mathbf{K}_o$ be an approximate Killing vector field as above. We have,

$$\left| \int_M (\delta_{g^D} h)(\tilde{\mathbf{X}}_{o,t}) dv_{g^D} \right| = \left| \int_M \langle h, \delta_{g^D}^*(\tilde{\mathbf{X}}_{o,t}) \rangle_{g^D} dv_{g^D} \right|,$$

and,

$$\begin{aligned} \delta_{g^D}^*(\tilde{\mathbf{X}}_{o,t}) &= \delta_{g_o}^* \mathbf{X}_o + \delta_{g_o}^* ((\chi_{M_o^l} - 1) \mathbf{X}_{o,*}) + \delta_{g_o}^* \left((\chi_{\mathcal{A}_k(t,\epsilon_0)} - \chi_{B_o(p_k,\epsilon_0)}) \sum_k \mathbf{X}_{o,k} \right) \\ &\quad + (\delta_{g_o}^* - \delta_{g^D}^*) \left(\chi_{\mathcal{A}_k(t,\epsilon_0)} \sum_k \mathbf{X}_{o,k} \right), \end{aligned}$$

where by definition $\delta_{g_o}^* \mathbf{X}_o = 0$. Thanks to Definition 1.10, on M_o^l , $\chi_{M_o^l} \mathbf{X}_{o,*}$ is equal to $\mathbf{X}_{o,*}$ except on the annuli of radii $t_k^{\frac{1}{4}}$ and $2t_k^{\frac{1}{4}}$, and $\chi_{\mathcal{A}_k(t,\epsilon_0)} - \chi_{B_o(p_k,\epsilon_0)}$ is supported in $\epsilon_0^{-1} \sqrt{t_k} < r_D < 2\epsilon_0^{-1} \sqrt{t_k}$ and well-defined on M . For any $l \in \mathbb{N}$, the cut off functions are moreover uniformly bounded in C_0^l . If we denote $\mathbb{1}_A$ the indicator function of A , for the vector fields, we therefore have

$$r_D^l \left(|\nabla^l \mathbf{X}_{o,*}|_{g_o} + \left| \nabla^l \sum_k \mathbf{X}_{o,k} \right|_{g_o} \right) \leq C_l \|\mathbf{X}_o\|_{r_o C_{\beta,*}^l} r_D,$$

and

$$r_D^l |\nabla^l (g^D - g_o)|_{g_o} \leq C_l \mathbb{1}_{\{r_D < t_k^{\frac{1}{4}}\}} (r_D^2 + t_k^2 r_D^{-4}).$$

As a consequence, because of the properties of the norms detailed in Remark 12, we have

$$|\delta_{g_o}^* ((\chi_{M_o^l} - 1) \mathbf{X}_{o,*})|_{g_o} \leq C_l \mathbb{1}_{\{t_k^{\frac{1}{4}} < r_D < 2t_k^{\frac{1}{4}}\}} \|\mathbf{X}_o\|_{r_o C_0^1(g_o)},$$

on the annulus of radii $\epsilon_0^{-1} \sqrt{t_k} < r_D < 2\epsilon_0^{-1} \sqrt{t_k}$, we have

$$\left| \delta_{g_o}^* \left((\chi_{\mathcal{A}_k(t,\epsilon_0)} - \chi_{B_o(p_k,\epsilon_0)}) \sum_k \mathbf{X}_{o,k} \right) \right| \leq C_l \mathbb{1}_{\{\epsilon_0^{-1} \sqrt{t_k} < r_D < 2\epsilon_0^{-1} \sqrt{t_k}\}} \|\mathbf{X}_o\|_{r_o C_0^1(g_o)},$$

and

$$\left| (\delta_{g_o}^* - \delta_{g^D}^*) \left(\chi_{\mathcal{A}_k(t,\epsilon_0)} \sum_k \mathbf{X}_{o,k} \right) \right| \leq C_l \mathbb{1}_{\{\epsilon_0^{-1} \sqrt{t_k} < r_D < t_k^{\frac{1}{4}}\}} (r_D^2 + t_k^2 r_D^{-4}) \|\mathbf{X}_o\|_{r_o C_0^1(g_o)}.$$

Finally, since we have $\text{Vol}(A_e(t_k^{\frac{1}{4}}, 2t_k^{\frac{1}{4}})) \approx t_k$, $\text{Vol}(A_e(\epsilon_0^{-1} \sqrt{t_k}, 2\epsilon_0^{-1} \sqrt{t_k})) \approx t_k^2$, and also $\int_{\{\epsilon_0^{-1} \sqrt{t_k} < r_D < t_k^{\frac{1}{4}}\}} (r_D^2 + t_k^2 r_D^{-4}) dv_{g^D} \approx t_k^{\frac{3}{2}} + t_k^2 |\log t_k|$, we have

$$\left| \int_M \langle \delta_{g^D} h, \tilde{\mathbf{X}}_{o,t} \rangle_{g^D} dv_{g^D} \right| \leq C t_{\max} \|h\|_{C_0^0(g^D)} \|\mathbf{X}_o\|_{r_o C_0^1(g_o)}.$$

Finally, let us denote $\tilde{\mathbf{Y}}_{o,t} = \pi_{\mathbf{K}_o} \delta_{g^D} h$, we have $\tilde{\delta}_{g^D} h = \pi_{\tilde{\mathbf{K}}_o} \delta_{g^D} h = \delta_{g^D} h - \tilde{\mathbf{Y}}_{o,t}$ with

$$\|\tilde{\mathbf{Y}}_{o,t}\|_{r_D^{-1} C_{\beta}^{1,\alpha}(g^D)} \leq C t_{\max} \|h\|_{C_0^0(g^D)},$$

by the equivalence of the norms of Remark 19. \square

Lemma 3.4 *Let $0 < \beta < 1$, $0 < \alpha < 1$ and (M, g^D) a naïve desingularization of a compact or ALE Einstein orbifold by a tree of singularities. Then, there exists $\tau_D > 0$ and $\epsilon_D > 0$ and $C_D > 0$, only depending on β and the constants of Proposition 9, such that for $t_{\max} < \tau_D$, and any metric g satisfying $\|g - g_t^D\|_{C_{\beta, *}(g^D)} \leq \epsilon_D$, the operator*

$$\tilde{\delta}_g \delta_g^* : \tilde{\mathbf{K}}_o^\perp \cap r_D C_{\beta, *}(g^D) \rightarrow \tilde{\mathbf{K}}_o^\perp \cap r_D^{-1} C_\beta^{1, \alpha}(g^D)$$

is invertible and we have for any vector field $X \perp \tilde{\mathbf{K}}_o$ on M ,

$$\|X\|_{r_D C_{\beta, *}(g^D)} \leq C_D \|\tilde{\delta}_g \delta_g^* X\|_{r_D^{-1} C_\beta^{1, \alpha}(g^D)}.$$

Proof Let $0 < \epsilon < \epsilon_D^{\frac{1}{2-\beta}} < \epsilon_0$ for ϵ_D and ϵ which we will choose small enough along the proof, and assume that $t_{\max} < \epsilon^4$. Therefore, by construction, on each annulus $\mathcal{A}_k := \mathcal{A}_k(t, \epsilon)$ between N_k and N_j or N_k and M_o (in which case, we will fix $T_o = 1$), we have a diffeomorphism

$$\Phi_k : A_e(\epsilon^{-1} \sqrt{T_j} \sqrt{t_k}, \epsilon \sqrt{T_j}) \subset \mathbb{R}^4 / \Gamma_k \rightarrow \mathcal{A}_k(t, \epsilon) \subset M,$$

such that for all $0 < \beta < 1$, there exists $C > 0$, for which we have

$$(35) \quad \|\Phi_k^* g^D - g_e\|_{C_\beta^{2, \alpha}(A_e(\epsilon^{-1} \sqrt{T_j} \sqrt{t_k}, \epsilon \sqrt{T_j}))} \leq C \epsilon^{2-\beta} < C \epsilon_D.$$

Until the end of the proof, we will use the notation

$$A_k := A_e(\epsilon^{-1} \sqrt{T_j} \sqrt{t_k}, \epsilon \sqrt{T_j}).$$

According to the estimate (34), for t_{\max} small enough, it is enough to have

$$\|X\|_{r_D C_{\beta, *}(g^D)} \leq \frac{C_D}{2} \|\delta_{g^D} \delta_{g^D}^* X\|_{r_D^{-1} C_\beta^{1, \alpha}(g^D)}$$

to obtain the stated result.

The diffeomorphisms $\Phi_k : A_k \rightarrow \mathcal{A}_k$ allow us to pull the situation back on \mathbb{R}^4 , where the ratio of the annuli A_k is $\epsilon^2 t_k^{-1/2}$ which is arbitrarily large for t_{\max} arbitrarily small. According to the estimate (25) of Proposition 4 and thanks to the controls of Proposition 7, for t_{\max} and ϵ_D small enough, then, there exist linear vector fields X_k of the kernel of $\delta_e \delta_e^*$ such that the vector fields $\chi_{\mathcal{A}_k} X_k$ in the annuli \mathcal{A}_k satisfy

$$(36) \quad \begin{aligned} \|\Phi_k^* X - X_k\|_{r_e C_0^{3, \alpha}(A(T_j^{1/2} t_k^{1/4}))} &\leq C_e T_j^{\frac{1}{2}} t_k^{\frac{1}{4}} \left(\epsilon^{-\beta} t_k^{\frac{\beta}{4}} \|\delta_{g_e} \delta_{g_e}^* \Phi_k^* X\|_{r_e^{-1} C_\beta^{1, \alpha}(A_k)} \right. \\ &\quad \left. + 2\epsilon^{-1} t_k^{\frac{1}{4}} \|\Phi_k^* X - X_k\|_{r_e C_\beta^{3, \alpha}(A_k)} \right) \\ &\leq 2C_e T_j^{\frac{1}{2}} t_k^{\frac{1}{4}} \left(\epsilon^{-\beta} t_k^{\frac{\beta}{4}} \|(\delta_{g^D} \delta_{g^D}^* X)|_{\mathcal{A}_k}\|_{r_D^{-1} C_\beta^{1, \alpha}(g^D)} \right. \\ &\quad \left. + 2\epsilon^{-1} t_k^{\frac{1}{4}} \|(X - \chi_{\mathcal{A}_k} X_k)|_{\mathcal{A}_k}\|_{r_D C_\beta^{3, \alpha}(g^D)} \right) \end{aligned}$$

on $A(\sqrt{T_j t_k^{1/4}}) := A_e((1/2)\sqrt{T_j t_k^{1/4}}, 4\sqrt{T_j t_k^{1/4}})$. Let us then consider the decomposition

$$(37) \quad X = X_* + \sum_k \chi_{\mathcal{A}_k} X_k$$

with the above linear vector fields X_k in the kernel of $\delta_e \delta_e^*$ for the rest of the proof.

The objective is now to show that there exists a constant $C_D > 0$ such that

$$\|X_*\|_{r_D C_\beta^{3,\alpha}(g^D)} + \sum_k \|X_k\|_{r_e C_0^0(g_e)} \leq C_D \|\delta_{g^D} \delta_{g^D}^* X\|_{r_D^{-1} C_\beta^{1,\alpha}(g^D)}.$$

In order to do this, we will reduce our situation to M_o and to the N_j where such controls have been shown in Proposition 9.

On $M_o^{t/16}$, $g^D - g_o$ is supported in $M_o^{t/16} \setminus M_o^{16t}$, that is where $\frac{1}{2}t_k^{1/4} \leq r_D < 2t_k^{1/4}$ on each annulus \mathcal{A}_k , and for all $l \in \mathbb{N}$, there exists $C_l > 0$ such that in these regions, we have

$$(38) \quad t_k^{l/4} |\nabla^l (g^D - g_o)|_{g_o} \leq C_l t_k^{l/2}.$$

Consider the cut-off function $\chi_{M_o^{t/16}}$ of Definition 1.10 supported in $M_o^{t/16}$ such that $\chi_{M_o^{t/16}} \equiv 1$ on M_o^t and such that for all $l \in \mathbb{N}$, there exists $C_l > 0$ for which in each \mathcal{A}_k ,

$$(39) \quad t_k^{l/4} |\nabla^l \chi_{M_o^{t/16}}|_{g^D} \leq C_l.$$

We then define a vector field X_o on M_o by

$$X_o := \chi_{M_o^{t/16}} X_* + \sum_{k \in K_o} \chi_{B_o(p_k, \epsilon)} X_k,$$

where K_o is the set of k such that the annulus \mathcal{A}_k has a nonempty intersection with M_o^t .

By construction, $X_* = X_{o,*}$ on M_o^t and we therefore have the following obvious control. Denoting $X_{o,*} := \chi_{M_o^{t/16}} X_*$, we have

$$(40) \quad \|X_{o,*}\|_{r_o C_\beta^{3,\alpha}(g_o)} \geq \|X_{o,*}\|_{r_o C_\beta^0(g_o)} \geq \|(X_*)|_{M_o^t}\|_{r_D C_\beta^0(g^D)}.$$

On $M_o^{t/16}$, we have,

$$(41) \quad \delta_{g^D} \delta_{g^D}^* X = \delta_{g_o} \delta_{g_o}^* X_o + \delta_{g_o} \delta_{g_o}^* (X - X_o) + (\delta_{g^D} \delta_{g^D}^* - \delta_{g_o} \delta_{g_o}^*) X.$$

Since the cut off functions are bounded in norm $C_0^{3,\alpha}(g^D)$ and $C_0^{3,\alpha}(g_o)$ by (39), and since their derivatives are supported in $M_o^{t/16} \setminus M_o^t$, we have the following bound on the last two terms of (41): for $C > 0$ depending on the cut off function, we have

- $\|\delta_{g_o} \delta_{g_o}^* (X - X_o)\|_{r_D^{-1} C_\beta^{1,\alpha}(g^D)} \leq C \|(X_*)_{|M_o^{t/16} \setminus M_o^t}\|_{r_D C_\beta^{3,\alpha}(g^D)}$ since the difference between X and X_o on M_o^{16t} only comes from the cut-off on X_* , and
- $\|((\delta_{g^D} \delta_{g^D}^* - \delta_{g_o} \delta_{g_o}^*)X)_{|M_o^{t/16}}\|_{r_D^{-1} C_\beta^{1,\alpha}(g^D)} \leq C \sum_{k \in K_o} t_k^{\frac{1}{2}} \|X\|_{r_D C_{\beta,*}^{3,\alpha}}$ thanks to (38).

Consequently, by (41), and using (36), for $C > 0$ depending on the above constants, we have

$$\begin{aligned}
\|(\delta_{g^D} \delta_{g^D}^* X)_{|M_o^{t/16}}\|_{r_D^{-1} C_\beta^{1,\alpha}(g^D)} &\geq \|\delta_{g_o} \delta_{g_o}^* X_o\|_{r_o^{-1} C_\beta^{1,\alpha}(g_o)} \\
&\quad - C \|(X_*)_{|M_o^{t/16} \setminus M_o^t}\|_{r_D C_\beta^{3,\alpha}(g^D)} \\
&\quad - C \sum_{k \in K_o} t_k^{\frac{1}{2}} \|X\|_{r_D C_{\beta,*}^{3,\alpha}} \\
&\geq \|\delta_{g_o} \delta_{g_o}^* X_o\|_{r_o^{-1} C_\beta^{1,\alpha}(g_o)} \\
&\quad - 2C \left(\|(\delta_{g^D} \delta_{g^D}^* X)_{|\mathcal{A}_k}\|_{r_D^{-1} C_\beta^{1,\alpha}(g^D)} \right. \\
&\quad \left. + \sum_{k \in K_o} \epsilon^{\beta-1} t_k^{\frac{1-\beta}{4}} \|(X - \chi_{\mathcal{A}_k(t,\epsilon)} X_k)_{\mathcal{A}_k}\|_{r_D C_\beta^{3,\alpha}(g^D)} \right) \\
(42) \quad &\quad - C \sum_{k \in K_o} t_k^{\frac{1}{2}} \|X\|_{r_D C_{\beta,*}^{3,\alpha}}.
\end{aligned}$$

Now, when $t_{\max} \rightarrow 0$, we have

$$\frac{\|\pi_{\mathbf{K}_o^\perp} X_o\|_{r_o C_{\beta,*}^{3,\alpha}(g_o)}}{\|X_o\|_{r_o C_{\beta,*}^{3,\alpha}(g_o)}} \rightarrow 0$$

because $X \perp \tilde{\mathbf{K}}_o$. Proposition 9 therefore yields, for t_{\max} small enough,

$$(43) \quad \|X_o\|_{r_o C_{\beta,*}^{3,\alpha}(g_o)} \leq 2C_o \|\delta_{g_o} \delta_{g_o}^* X_o\|_{r_o^{-1} C_\beta^{1,\alpha}(g_o)}.$$

Therefore, thanks to (40) and (42), for t_{\max} small enough, and denoting by $C' > 0$ a constant that may change from line to line but only depending on the previous ones of this proof, and therefore only on g_o and the g_{b_i} and $\gamma(t_{\max}) := \sum_k t_{\max}^{\frac{1}{2}} + t_{\max}^{\frac{1-\beta}{4}}$, we have

$$\begin{aligned}
&\|(X_*)_{|M_o^t}\|_{r_D C_\beta^0(g^D)} + \sum_{k \in K_o} \|X_k\|_{r_e C_\beta^0(g_e)} - C' \gamma(t_{\max}) \|X\|_{r_D C_{\beta,*}^0(g^D)} \\
&\leq \|X_o\|_{r_o C_{\beta,*}^{3,\alpha}(g_o)} - C' \gamma(t_{\max}) \|X\|_{r_D C_{\beta,*}^0(g^D)} \\
&\leq 2C_o \|\delta_{g_o} \delta_{g_o}^* X_o\|_{r_o^{-1} C_\beta^{1,\alpha}(g_o)} - C' \gamma(t_{\max}) \|X\|_{r_D C_{\beta,*}^0(g^D)} \\
(44) \quad &\leq C' \|\delta_{g^D} \delta_{g^D}^* X\|_{r_D^{-1} C_\beta^{1,\alpha}(g^D)},
\end{aligned}$$

where we successively used (40), (43) and (42). Indeed, on an orbifold (M_o, g_o) , the vector fields X_k of the decomposition (37) reaching the infimum of the definition of the norm $\|\cdot\|_{r_o C_{\beta,*}^{3,\alpha}}$ are determined by the limit of $r_o^{-1}X_o$ at each singular point according to Remark 16. Here, the infimum is therefore reached with the X_k of the decomposition (37).

We next consider the vector field $X_1 := X - \sum_{k \in K_o} \chi_{\mathcal{A}_k} X_k$ which satisfies for a constant $C > 0$,

$$(45) \quad \|\delta_{g^D} \delta_{g^D}^* X_1\|_{r_D^{-1} C_{\beta}^{1,\alpha}(g^D)} \leq C \left(\|\delta_{g^D} \delta_{g^D}^* X\|_{r_D^{-1} C_{\beta}^{1,\alpha}(g^D)} + \gamma(t_{\max}) \|X\|_{r_D C_{\beta,*}^0(g^D)} \right)$$

thanks to the control (44) of $\sum_{k \in K_o} \|X_k\|_{r_e C_{\beta}^0(g_e)}$.

Given $j \in K_o$, the Ricci-flat ALE orbifold (N_j, g_{b_j}) is glued to M_o and we can extend the vector field $X_1 = X_* + \sum_{k \notin K_o} \chi_{\mathcal{A}_k} X_k$ to N_j by

$$X_j := \chi_{N_j^{t/16}} X_* + \sum_{l \in K_j} \chi_{B_j(p_l, \epsilon)} X_l,$$

where K_j is the set of $k \neq j$ such that \mathcal{A}_k has a nonempty intersection with N_j^t .

Remark 20 By considering X_1 instead of X , we do not have a linear vector field of the kernel of $\delta_e \delta_e^*$ to extend at infinity of N_j . The vector field X_j is therefore well controlled in $r_{b_j} C_{\beta,*}^{3,\alpha}(g_{b_j})$.

The difference $\frac{g^D}{T_j} - g_{b_j}$ is supported in $N_j^{t/16} \setminus N_j^{16t}$ and there exists for all $l \in \mathbb{N}$, $C_l > 0$ such that we have the following controls. Around the singular points where

$$\frac{1}{2} \sqrt{T_j} t_k^{\frac{1}{4}} < r_D = \sqrt{T_j} r_{b_j} < 2 \sqrt{T_j} t_k^{\frac{1}{4}},$$

we have

$$(46) \quad t_k^{\frac{1}{4}} \left| \nabla^l \left(\frac{g^D}{T_j} - g_{b_j} \right) \right|_{g_{b_j}} \leq C_l t_k^{\frac{1}{2}}.$$

and at infinity, where

$$\frac{1}{2} \sqrt{T_j} t_j^{-\frac{1}{4}} < r_D = \sqrt{T_j} r_{b_j} < 2 \sqrt{T_j} t_j^{-\frac{1}{4}}$$

we have

$$(47) \quad t_j^{-\frac{1}{4}} \left| \nabla^l \left(\frac{g^D}{T_j} - g_{b_j} \right) \right|_{g_{b_j}} \leq C_l t_j^{\frac{1}{2}}.$$

Denoting $X_{j,*} := \chi_{N_j^{t/16}} X_*$, we have

$$(48) \quad \|X_{j,*}\|_{r_{b_j} C_{\beta}^{3,\alpha}(g_{b_j})} \geq \|X_{j,*}\|_{r_{b_j} C_{\beta}^0(g_{b_j})} \geq \|(X_*)_{N_j^t}\|_{r_D C_{\beta}^0(g^D)},$$

and thanks to (36) and the inequalities (46) and (47), we have

$$\delta_{g^D} \delta_{g^D}^* X_1 := \delta_{g_{b_j}} \delta_{g_{b_j}}^* X_j + \delta_{g_{b_j}} \delta_{g_{b_j}}^* (X_1 - X_j) + (\delta_{g^D} \delta_{g^D}^* - \delta_{g_{b_j}} \delta_{g_{b_j}}^*) X_1,$$

analogously to (42), we find for $C > 0$ depending on the above constants such that

$$(49) \quad \begin{aligned} \|(\delta_{g^D} \delta_{g^D}^* X_1)_{|N_j^{t/16}}\|_{r_D^{-1} C_{\beta}^{1,\alpha}(g^D)} &\geq \|\delta_{g_{b_j}} \delta_{g_{b_j}}^* X_j\|_{r_{b_j}^{-1} C_{\beta}^{1,\alpha}(g_{b_j})} \\ &\quad - C \left(\|(\delta_{g^D} \delta_{g^D}^* X_1)_{|\mathcal{A}_k}\|_{r_D^{-1} C_{\beta}^{1,\alpha}(g^D)} \right. \\ &\quad \left. + \sum_{k \in K_j} e^{\beta-1} t_k^{\frac{1-\beta}{4}} \|(X_*)_{\mathcal{A}_k}\|_{r_D C_{\beta}^{3,\alpha}(g^D)} \right) \\ &\quad - C t_k^{\frac{1}{2}} \|X_1\|_{r_D C_{\beta,*}^{3,\alpha}} - C t_j^{\frac{1}{2}} \|X_1\|_{r_D C_{\beta,*}^{3,\alpha}}, \end{aligned}$$

where we remark that $\|X_1\|_{r_D C_{\beta,*}^{3,\alpha}} \leq \|X_*\|_{r_D C_{\beta}^{3,\alpha}} + \sum_{k \in K_j} \|X_k\|_{r_e C_0^0(g_e)}$.

Proposition 9 then yields

$$\|X_j\|_{r_j C_{\beta,*}^{3,\alpha}(g_{b_j})} \leq C_j \|\delta_{g_{b_j}} \delta_{g_{b_j}}^* X_j\|_{r_j^{-1} C_{\beta}^{1,\alpha}(g_{b_j})},$$

and thanks to the control (45), we then have for a constant $C > 0$ only depending on the constants of Propositions 9 and 4, the control

$$(50) \quad \begin{aligned} \|(X_*)_{|N_j^t}\|_{r_D C_{\beta}^0(g^D)} + \sum_{k \in K_j} \|X_k\|_{r_e C_0^0(g_e)} - C\gamma(t_{\max}) \|X\|_{r_D C_{\beta,*}^0(g^D)} \\ \leq C \|\delta_{g^D} \delta_{g^D}^* X\|_{r_D^{-1} C_{\beta}^{1,\alpha}(g^D)}, \end{aligned}$$

similar to (44).

Iterating this to the other Ricci-flat ALE orbifolds of the tree of singularities, we get controls similar to (50) on all the N_j which, with (44) on M_o , give the following control on the whole manifold only depending on g_o and the g_{b_j}

$$\begin{aligned} \|X_*\|_{r_D C_{\beta}^0(g^D)} + \sum_k \|X_k\|_{r_e C_0^0(g_e)} - C\gamma(t_{\max}) \|X\|_{r_D C_{\beta,*}^0(g^D)} \\ \leq C \|\delta_{g^D} \delta_{g^D}^* X\|_{r_D^{-1} C_{\beta}^{1,\alpha}(g^D)} \end{aligned}$$

and for t_{\max} small enough. Together with the elliptic estimates of Proposition 8, this shows the stated result for $g = g^D$ because $M = M_0^t \cup \bigcup_j N_j^t$.

To get the estimate for another metric g close to g^D , we just use Proposition 7 to ensure that for $\|g - g^D\|_{C_{\beta,*}^{2,\alpha}(g^D)}$ arbitrarily small, $\tilde{\delta}_g \delta_g^*$ is arbitrarily close to $\delta_{g^D} \delta_{g^D}^*$ for the operator norm on $r_D C_{\beta,*}^{3,\alpha}(g^D)$.

Finally, notice that $(\tilde{\delta}_{g^D})^* = \delta_{g^D}^* \pi_{\tilde{\mathbf{K}}_o^\perp}$, and therefore that $\tilde{\delta}_{g^D} \delta_{g^D}^*$ is self adjoint on $\tilde{\mathbf{K}}_o^\perp$. Its injectivity implies its surjectivity by integration by parts on the compact manifold (M, g^D) . \square

We can finally prove the main result of the section by fixed point theorem.

Proposition 10 Let $0 < \beta < \frac{1}{2}$, and $(M, g^D) = (M, g_t^D)$ be a naïve desingularization of a compact Einstein orbifold, (M_o, g_o) . Then, there exist $\epsilon_D, \tau_D, C_D > 0$ which only depends on the constants of Lemma 4.3 such that for $t_{\max} \leq \tau_D$ and for any metric g satisfying $\|g - g^D\|_{C_{\beta,*}^{2,\alpha}(g^D)} \leq \epsilon_D$, there exists a unique vector field $X \perp \tilde{\mathbf{K}}_o$ on M for which,

$$\tilde{\delta}_{g^D}(\exp_X^* g) = 0,$$

where \exp_X is the diffeomorphism $\exp_X : x \in M \mapsto \exp_x^{g^D}(X(x))$.

We moreover have, $\|X\|_{r_D C_{\beta,*}^{3,\alpha}(g^D)} \leq C_D \|\tilde{\delta}_{g^D}(g - g^D)\|_{r_D^{-1} C_{\beta}^{1,\alpha}(g^D)}$, and therefore, there exists $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\lim_0 \eta = 0$ such that we have

$$\|\exp_X^* g - g\|_{C_{\beta,*}^{2,\alpha}(g^D)} \leq \eta(\|g - g^D\|_{C_{\beta,*}^{2,\alpha}(g^D)}).$$

Proof Let us fix g a metric on M , such that $\|g - g^D\|_{C_{\beta,*}^{2,\alpha}} \leq \epsilon$ for $\epsilon > 0$ which we will choose small enough along the proof and define the operator $F_g : r_D C_{\beta,*}^{3,\alpha}(g^D) \rightarrow r_D^{-1} C_{\beta}^{1,\alpha}(g^D)$ which to a vector field X associates

$$F_g(X) := \tilde{\delta}_{(\exp_{g^D} X)_* g^D} g.$$

The objective is therefore to find X such that $F_g(X) = 0$, which will imply that $\tilde{\delta}_{g^D}(\exp_{g^D} X)^* g = 0$ because for any diffeomorphism ϕ , $\phi_*(\delta_{g^D} \phi^* g) = \delta_{\phi_* g^D} g$ (by applying ϕ_* to g^D which is C^∞ , we do not loose regularity). The map $g \mapsto F_g$ is linear, and the linearization of the operator F_{g^D} around zero is $\tilde{\delta}_{g^D} \delta_{g^D}^*$ which is invertible between the orthogonals of $\tilde{\mathbf{K}}_o$ according to Lemma 3.4.

There remains to control the nonlinear terms in our norms. Let us denote them $Q = F_{g^D} - \tilde{\delta}_{g^D} \delta_{g^D}^*$. We schematically have that $Q(X)$ is a converging sum of terms of the form $(\nabla_{g^D}^2 X * X) * X * \dots * X$, $(\nabla_{g^D} X * \nabla_{g^D} X) * X * \dots * X$, $(\text{Rm}(g^D) * X * X) * \dots * X$

which are at least quadratic in X , where $*$ denotes various multilinear operations or contractions.

Using (10) of Remark 12, since on a compact manifold $\|r_D\|_{C_0^{3,\alpha}}$ is bounded, there exists $C > 0$ such that

$$(51) \quad \|\cdot\|_{r_D^{-1}C_\beta^{1,\alpha}(g^D)} \leq C \|\cdot\|_{C_\beta^{1,\alpha}(g^D)},$$

and for any vector field X , assuming $\beta < \frac{1}{2}$, we have

$$(52) \quad \|\nabla_{g^D} X\|_{r_D^{-1/2}C_0^{2,\alpha}(g^D)} \leq \|\nabla_{g^D} X\|_{r_D^{-1/2}C_\beta^{2,\alpha}(g^D)} \leq \|X\|_{r_D^{1/2}C_\beta^{3,\alpha}(g^D)} \leq C \|X\|_{r_D C_{\beta,*}^{3,\alpha}(g^D)}.$$

We therefore find

$$\begin{aligned} & \|Q(X) - Q(X')\|_{r_D^{-1}C_\beta^{1,\alpha}(g^D)} \\ & \leq C \left(\|X - X'\|_{r_D C_0^{1,\alpha}(g^D)} \left(\|\nabla^2 X\|_{r_D^{-1}C_\beta^{1,\alpha}(g^D)} + \|\nabla^2 X'\|_{r_D^{-1}C_\beta^{1,\alpha}(g^D)} \right) \right. \\ & \quad + \left(\|X\|_{r_D C_0^{1,\alpha}(g^D)} + \|X'\|_{r_D C_0^{1,\alpha}(g^D)} \right) \|\nabla^2(X - X')\|_{r_D^{-1}C_\beta^{1,\alpha}(g^D)} \\ & \quad + \left(\|X\|_{r_D C_0^{1,\alpha}(g^D)} + \|X'\|_{r_D C_0^{1,\alpha}(g^D)} \right) \|X - X'\|_{r_D C_0^{1,\alpha}(g^D)} \|\text{Rm}(g^D)\|_{r_D^{-2}C_\beta^{1,\alpha}(g^D)} \\ & \quad \left. + \|\nabla(X - X')\|_{r_D^{-1/2}C_0^{2,\alpha}(g^D)} \left(\|\nabla X\|_{r_D^{-1/2}C_\beta^{2,\alpha}(g^D)} + \|\nabla X'\|_{r_D^{-1/2}C_\beta^{2,\alpha}(g^D)} \right) \right) \\ & \leq 3C \left(\|X\|_{r_D C_{\beta,*}^{3,\alpha}(g^D)} + \|X'\|_{r_D C_{\beta,*}^{3,\alpha}(g^D)} \right) \|X - X'\|_{r_D C_{\beta,*}^{3,\alpha}(g^D)}, \end{aligned}$$

notice the different norms with β and 0 for the weight power. We controlled the $C_\beta^{1,\alpha}(g^D)$ -norm (which is larger than the $r_D^{-1}C_\beta^{1,\alpha}(g^D)$ -norm by (51)) in the first three lines and the $r_D^{-1}C_\beta^{1,\alpha}(g^D)$ -norm in the last one (notice the $r_D^{-1/2}$ -norms controlled by (52)).

Remark 21 Using $r_D^{-1/2}$ -norms was necessary because the first derivatives of the linear element of the kernel of $\delta_{g_e} \delta_{g_e}^*$ do not decay in the neck regions, that is, $\nabla_{g^D} \chi_{\mathcal{A}_k(t,\epsilon)} X_k \notin C_\beta^{2,\alpha}(g^D)$.

The crucial reason for such a control of the nonlinear terms, already noted in [Biq13, Proof of Lemma 8.2], is that our norm is equivalent to a norm $C^{k,\alpha}(g^D)$ weighted by a function uniformly bounded below by 1 independently on t , see (9). We can therefore finally put our metrics in gauge with respect to each other thanks to a fixed point theorem with explicit constant below, Lemma 3.5.

Lemma 3.5 *Let $\Phi : E \rightarrow F$, be a smooth map between Banach spaces and let $Q := \Phi - \Phi(0) - d_0 \Phi$.*

Assume that there exist $q > 0$, $r_0 > 0$ and $c > 0$ such that:

(1) for all x and y in $B(0, r_0)$, we have the following control on the nonlinear terms

$$\|Q(x) - Q(y)\| \leq q(\|x\| + \|y\|)\|x - y\|.$$

(2) the linearization $d_0\Phi$ is an isomorphism, and more precisely, we have

$$\|(d_0\Phi)^{-1}\| \leq c.$$

If $r \leq \min\left(r_0, \frac{1}{2qc}\right)$ and $\|\Phi(0)\| \leq \frac{r}{2c}$, then, the equation $\Phi(x) = 0$ admits a unique solution in $B(0, r)$.

Let us finally remark that for a linear vector field X_k in the kernel of $\delta_e\delta_e^*$, the symmetric 2-tensor $\delta_e^*X_k$ is constant. This lets us define for any metric g on M , a continuous map $\psi_g : r_D C_{\beta,*}^{3,\alpha} \rightarrow C_{\beta,*}^{2,\alpha}$ by

$$X \mapsto \psi_g(X) := \exp_X^* g.$$

It is indeed continuous since for any diffeomorphism $\phi : M \rightarrow M$, we have

$$(\phi^*g)(x)_{kl} = g(\phi(x))_{ij} \frac{\partial_i \phi}{\partial x^k} \frac{\partial_j \phi}{\partial x^l}$$

in local coordinates and therefore, for any vector field $X_* \in r_D C_{\beta}^{3,\alpha}$, the symmetric 2-tensor $\exp_X^* g - g$ is arbitrarily small for the $C_{\beta,*}^{2,\alpha}$ -norm. For the constant part, it is enough to note that for a linear vector field X_k in the kernel of $\delta_e\delta_e^*$, and for a constant symmetric 2-tensor H_k , the symmetric 2-tensor $\exp_{X_k}^* H_k$ is also constant and controlled $|\exp_{X_k}^* H_k|_{g_e} \leq C\|X_k\|_{r_e C^0} |H_k|_{g_e}$. \square

3.3 Einstein metrics in gauge

Let us now come back to Einstein metrics which can be characterized in dimension n thanks to the Bianchi identity as the zero set of

$$E(g) := \text{Ric}(g) - \frac{\mathbf{R}(g)}{2}g + \frac{n-2}{2n}\bar{\mathbf{R}}(g)g$$

on a compact manifold M , where $\bar{\mathbf{R}} := \frac{1}{\text{Vol}(M,g)} \int_M \mathbf{R}(g) dv_g$. Notice that $\delta_g E(g) = 0$, again by the Bianchi identity.

The equation $E(g) = 0$ is invariant by the action of diffeomorphisms and by scaling, we will therefore restrict our attention to deformations which are transverse to these actions in order to obtain an operator whose linearization is elliptic. More precisely, we will fix the volume and fix a gauge thanks to the reduced divergence-free condition.

It turns out that we can characterize the zeros of E in reduced divergence-free gauge as the zeros of a single operator ${}_{g^D}$ defined by

$${}_{g^D}(g) := E(g) + \delta_g^* \tilde{\delta}_{g^D} g.$$

Indeed, if we have $E(g) = 0$ and $\tilde{\delta}_{g^D} g = 0$, then we have ${}_{g^D}(g) = 0$. And conversely, if ${}_{g^D}(g) = 0$, then since $E(g)$ is divergence-free (for g) by the Bianchi identity, by taking the reduced divergence of ${}_{g^D}(g) = 0$, we get

$$\tilde{\delta}_{g^D}({}_{g^D}(g)) = (\tilde{\delta}_g \delta_g^*) \tilde{\delta}_{g^D} g.$$

Since for g close enough to g^D , $(\tilde{\delta}_g \delta_g^*)$ is invertible on the image of $\tilde{\delta}_{g^D}$ by Lemma 3.4, we finally have $\tilde{\delta}_{g^D} g = 0$ and $E(g) = 0$. In a $C_{\beta,*}^{2,\alpha}(g^D)$ -neighborhood of g^D the zero set of ${}_{g^D}$ is exactly the set of Einstein metrics in reduced divergence-free gauge with respect to g^D .

Corollary 3 Let $D_0, \nu_0 > 0$, $l \in \mathbb{N}$, and $\beta = \beta(\nu_0, D_0) > 0$ obtained in Corollary 2. Then, for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon, D_0, \nu_0, l) > 0$ such that if $(M, g^\mathcal{E})$ is an Einstein manifold satisfying

- the volume is bounded below by $\nu_0 > 0$,
- the diameter is bounded above by D_0 ,
- the Ricci curvature is bounded $|\text{Ric}| \leq 3$.

and such that for an Einstein orbifold (M_o, g_o) ,

$$d_{GH}((M, g^\mathcal{E}), (M_o, g_o)) \leq \delta,$$

then, there exists a naïve desingularization (M, g^D) of (M_o, g_o) by a tree of singularities, and a diffeomorphism $\psi : M \rightarrow M$ such that

$$\|\psi^* g^\mathcal{E} - g^D\|_{C_{\beta,*}^l(g^D)} \leq \epsilon,$$

and

$$\tilde{\delta}_{g^D}(\psi^* g^\mathcal{E}) = 0.$$

In particular, we have

$${}_{g^D}(\psi^* g^\mathcal{E}) = 0.$$

4 Resolution of the Einstein equation modulo obstructions

We will now show that it is always possible to produce metrics which are Einstein modulo some obstructions (which are elements of the cokernel of the linearization of the Einstein operator) in our weighted Hölder spaces. The main result of the section is Theorem 4.6 which allows us to perturb any naïve desingularization g^D to an Einstein modulo obstructions metric and in particular according to [Ozu19a] we produce all Einstein metrics close to an Einstein orbifold in the Gromov-Hausdorff sense by this procedure.

We have seen in Corollary 3 that up to a diffeomorphism, any Einstein metric g close to (M_o, g_o) in the Gromov-Hausdorff sense is a solution of

$$g^D(g) := E(g) + \delta_g^* \tilde{\delta}_{g^D} g = 0.$$

To study this equation, we will naturally start by studying its linearization on volume preserving deformations at g^D , that is, on symmetric 2-tensors h satisfying $\int_M \text{tr}_{g^D} h \, dv_{g^D} = 0$, for which we have the formula

$$\begin{aligned} \bar{P}_{g^D}(h) := d_{g^D} g^D(h) = & \frac{1}{2} \left(\nabla_{g^D}^* \nabla_{g^D} h - 2\mathring{R}_{g^D}(h) \right. \\ & - 2\delta_{g^D}^* \delta_{g^D} h + 2\delta_{g^D}^* \tilde{\delta}_{g^D} h - (\delta_{g^D} \delta_{g^D} h) g^D \\ & + (\Delta_{g^D} \text{tr}_{g^D} h) g^D - \nabla_{g^D}^2 \text{tr}_{g^D} h \\ & + \text{Ric}_{g^D} \circ h + h \circ \text{Ric}_{g^D} - \mathring{R}_{g^D} h + \langle \text{Ric}_{g^D}, h \rangle_{g^D} g^D \\ & \left. + \frac{1}{2} \bar{\mathring{R}}(g) h - \frac{1}{2 \text{Vol}(g)} \int_M \left\langle \text{Ric}(g^D) - \frac{\mathring{R}(g^D)}{2}, h \right\rangle_{g^D} dv_{g^D} \right), \end{aligned} \quad (53)$$

in dimension 4. If g^D were an Einstein metric and h a divergence-free symmetric 2-tensor, then the operator \bar{P}_{g^D} would reduce to

$$P_{g^D} := \frac{1}{2} \nabla_{g^D}^* \nabla_{g^D} - \mathring{R}_{g^D},$$

which is simpler to study. Since g^D is almost Einstein and h will be almost divergence-free, we will mostly study the operator P_{g^D} , and we will obtain results for \bar{P}_{g^D} by approximation.

4.1 Kernel and cokernel of the linearization on model spaces

Exceptional values for $P_e := \frac{1}{2} \nabla_e^* \nabla_e$ on $(\mathbb{R}^4/\Gamma, g_e)$ and gauge constraints. As described in the proof of Proposition 3, the elements of the kernel of P_e on \mathbb{R}^4/Γ are

sums of homogeneous symmetric 2-tensors whose coefficients in an orthonormal basis of \mathbb{R}^4 are homogeneous of order k or $-2 - k$ for $k \in \mathbb{N}$.

However, some of these tensors cannot appear in our developments because they are not trace-free or in divergence-free gauge on our nontrivial quotient of \mathbb{R}^4/Γ .

Lemma 4.1 ([CT94, Proposition 4.65]) *On \mathbb{R}^4/Γ for $\Gamma \neq \{e\}$, there is no harmonic homogeneous symmetric 2-tensor whose coefficients are of order 1, -2 or -3 in divergence-free gauge.*

Kernel of the operator P on the model spaces. Let us start by describing the kernel of P on our model spaces.

Lemma 4.2 *Let (N, g_b) be a Ricci-flat ALE orbifold, and denote $P_b := \frac{1}{2}\nabla_b^*\nabla_b - \mathring{\mathbf{R}}_b$, and $\mathbf{O}(g_b)$, the kernel of P_{g_b} on $C_{\beta,*}^{2,\alpha}(g_b)$.*

The elements of $\mathbf{O}(g_b)$ decay at least like r_b^{-4} at infinity, and for all $\mathbf{o}_b \in \mathbf{O}(g_b)$, we have the following development coordinates at infinity,

$$\mathbf{o}_b = O^4 + \mathcal{O}(r_b^{-5}),$$

with $O^4 \sim r_b^{-4}$ a harmonic homogeneous symmetric 2-tensor.

Let also (M_o, g_o) be a compact Einstein orbifold, we denote $\mathbf{O}(g_o)$ the kernel of P_o on $C_{\beta,}^{2,\alpha}$ for all $0 < \beta < 1$. An element $\mathbf{o}_o \in \mathbf{O}(g_o)$ has a development*

$$\mathbf{o}_o = O_0 + O_2 + \mathcal{O}(r_o^3),$$

for harmonic homogeneous symmetric 2-tensors $O_i \sim r_o^i$.

Proof Let us consider $\mathbf{o} \in \mathbf{O}(g_b)$, for which $P_b\mathbf{o} = 0$, and $\mathbf{o} = \mathcal{O}(r_b^{-\beta})$ for some $\beta > 0$. Such a symmetric 2-tensor is actually necessarily traceless and divergence-free. Indeed, we have $\delta_{g_b}P_{g_b} = \frac{1}{2}\nabla_{g_b}^*\nabla_{g_b}\delta_{g_b}$, and $\text{tr}_{g_b}P_{g_b} = \frac{1}{2}\nabla_{g_b}^*\nabla_{g_b}\text{tr}_{g_b}$. Therefore, if $P_{g_b}h = 0$ for $h = \mathcal{O}(r_b^{-\delta})$ for some $\delta > 0$, then $\delta_{g_b}h = 0$, and $\text{tr}_{g_b}h = 0$ by the maximum principle. We deduce from Lemma 4.1 that \mathbf{o} decays at least like r_b^{-4} and its principal term is a harmonic symmetric 2-tensor.

In the same way in the neighborhood of a singularity of an orbifold (M_o, g_o) or of a Ricci-flat ALE orbifold (N, g_b) , since there is no harmonic symmetric 2-tensor with linear growth because of the action of the nontrivial group Γ , an element of the kernel admits a development

$$\mathbf{o} = O_0 + O_2 + \mathcal{O}(r_b^3),$$

where O_0 and O_2 are harmonic homogeneous symmetric 2-tensors in r_o^0 and r_o^2 respectively. \square

Estimates on the inverses. Just like for the operator $\delta\delta^*$ in Proposition 9, the operators P_{g_o} and P_{g_b} are injective on the orthogonal of their respective kernels.

Lemma 4.3 *Let (N, g_b) be a Ricci-flat ALE orbifold and (M_o, g_o) an Einstein orbifold, and $0 < \beta < 1$.*

Then, the operators

$$P_b : \mathbf{O}(g_b)^\perp \cap C_{\beta,*}^{2,\alpha}(g_b) \rightarrow r_b^{-2}C_\beta^\alpha(g_b),$$

and

$$P_o : \mathbf{O}(g_o)^\perp \cap C_{\beta,*}^{2,\alpha}(g_o) \rightarrow r_o^{-2}C_\beta^\alpha(g_o)$$

are injective and there exist $C_o > 0$ and $C_b > 0$, such that we have for any symmetric 2-tensor $h_b \perp \mathbf{O}(g_b)$ on N and $h_o \perp \mathbf{O}(g_o)$ on M_o ,

$$(54) \quad C_b^{-1} \|P_b h_b\|_{r_b^{-2}C_\beta^\alpha(g_b)} \leq \|h_b\|_{C_{\beta,*}^{2,\alpha}(g_b)} \leq C_b \|P_b h_b\|_{r_b^{-2}C_\beta^\alpha(g_b)},$$

and

$$(55) \quad C_o^{-1} \|P_o h_o\|_{r_o^{-2}C_\beta^\alpha(g_o)} \leq \|h_o\|_{C_{\beta,*}^{2,\alpha}(g_o)} \leq C_o \|P_o h_o\|_{r_o^{-2}C_\beta^\alpha(g_o)}.$$

Moreover, their respective cokernels are $\mathbf{O}(g_b)$ and $\mathbf{O}(g_o)$.

Proof By standard theory of elliptic operators between weighted Hölder spaces (see for instance [PR78, Chapter 2]), the operators

$$P_o : C_{-\beta}^{2,\alpha}(g_o) \rightarrow r_o^{-2}C_{-\beta}^\alpha(g_o),$$

and

$$P_b : r_b^{-\beta}C_0^{2,\alpha}(g_b) \rightarrow r_b^{-2-\beta}C_0^\alpha(g_b)$$

are Fredholm for $0 < \beta < 1$ because we avoid the exceptional values close to zero: -2 and 1 . Let us study their kernels and cokernels.

Let us start by the case of an Einstein orbifold (M_o, g_o) and notice that $\ker_{C_{-\beta}^{2,\alpha}(g_o)} P_o \subset \mathbf{O}(g_o)$ because there is no exceptional value between 0 and $-\beta$. The kernel of $P_o : C_{-\beta}^{2,\alpha}(g_o) \rightarrow r_o^{-2}C_{-\beta}^\alpha(g_o)$ is therefore equal to $\mathbf{O}(g_o)$. Since P_o is self adjoint and since we are strictly between two exceptional values, its cokernel is the kernel of P_o on $r_o^{-2}C_\beta^\alpha(g_o)$ which is also reduced to $\mathbf{O}(g_o)$ by a similar argument.

Similarly, the kernel of $P_b : r_b^{-\beta}C_0^{2,\alpha}(g_b) \rightarrow r_b^{-2-\beta}C_0^\alpha(g_b)$ is $\mathbf{O}(g_b)$, and its cokernel is the kernel of P_b on $r_b^{-2+\beta}C_0^\alpha(g_b)$ which is reduced to $\mathbf{O}(g_b)$.

For the cokernels of

$$P_b : \mathbf{O}(g_b)^\perp \cap C_{\beta,*}^{2,\alpha}(g_b) \rightarrow r_b^{-2} C_\beta^\alpha(g_b),$$

and

$$P_o : \mathbf{O}(g_o)^\perp \cap C_{\beta,*}^{2,\alpha}(g_o) \rightarrow r_o^{-2} C_\beta^\alpha(g_o),$$

like in the end of the proof of Proposition 9, we use the fact that

$$P_o(C_{\beta,*}^{2,\alpha}(g_o)) = P_o(C_{-\beta}^{2,\alpha}(g_o)) \cap r_o^{-2} C_\beta^\alpha(g_o),$$

and

$$P_b(C_{\beta,*}^{2,\alpha}(g_b)) = P_b(r_b^{-\beta} C_0^{2,\alpha}(g_{b_j})) \cap r_b^{-2} C_\beta^\alpha(g_b).$$

□

Approximation of kernels and cokernels on a naïve desingularization. We wish to solve the equation $\text{Ric}(g) = \Lambda g$ for a metric g close to g^D modulo the kernel and the cokernel of the linearization of the Einstein operator. We will use approximate kernels and cokernels defined as the truncated infinitesimal deformations of each model space on the tree of singularities in order to obtain uniform controls as the singularities form, that is as the gluing parameters t tend to 0.

Let $\mathbf{o}_o \in \mathbf{O}(g_o)$ and $\mathbf{o}_j \in \mathbf{O}(g_{b_j})$, and define $\mathbf{o}_o = \mathbf{o}_{o,*} + \sum_k \chi_{B_o(p_k, \epsilon_0)} \mathbf{o}_{o,k}$ and $\mathbf{o}_j = \mathbf{o}_{j,*} + \sum_k \chi_{B_j(p_k, \epsilon_0)} \mathbf{o}_{j,k}$ their respective decompositions as a symmetric 2-tensor of $C_\beta^{2,\alpha}$ and constant symmetric 2-tensors truncated in the neighborhoods of the singular points. Thanks to the cut-off functions of Definition 5, we define on M the following symmetric 2-tensors

$$\tilde{\mathbf{o}}_{o,t} := \chi_{M_o} \mathbf{o}_{o,*} + \sum_k \chi_{\mathcal{A}_k(t, \epsilon_0)} \mathbf{o}_{o,k},$$

and

$$\tilde{\mathbf{o}}_{j,t} := \chi_{N_j} \mathbf{o}_{j,*} + \sum_k \chi_{\mathcal{A}_k} \mathbf{o}_{j,k}.$$

Remark 22 We have $\tilde{\mathbf{o}}_{o,t} = \mathbf{o}_o$ on M_o^{16t} , and $\tilde{\mathbf{o}}_{j,t} = \mathbf{o}_j$ on N_j^{16t} .

Definition 4.4 (Space of truncated obstructions) Let (M, g_t^D) be a naïve desingularization of a Einstein orbifold (M_o, g_o) . On M , we will denote

$$\tilde{\mathbf{O}}(g^D) := \left\{ \tilde{\mathbf{o}}_{o,t} + \sum_j \tilde{\mathbf{o}}_{j,t}, \mathbf{o}_o \in \mathbf{O}(g_o), \mathbf{o}_j \in \mathbf{O}(g_{b_j}) \right\},$$

the space of *truncated obstructions*.

Remark 23 For $0 < \beta < 2$, by elliptic regularity for the elements of $\mathbf{O}(g_o)$ and the $\mathbf{O}(g_{b_j})$, and the C_0^4 -control of the cut-off functions we have

$$\left\| \tilde{\mathbf{o}}_{o,t} + \sum_j T_j \tilde{\mathbf{o}}_{j,t} \right\|_{C^0(g^D)} \approx \left\| \tilde{\mathbf{o}}_{o,t} + \sum_j T_j \tilde{\mathbf{o}}_{j,t} \right\|_{C_{\beta,*}^{2,\alpha}(g^D)} \approx \sup (\|\mathbf{o}_o\|_{C^0(g_o)}, \|\mathbf{o}_j\|_{C^0(g_{b_j})}),$$

and

$$\left\| \tilde{\mathbf{o}}_{o,t} + \sum_j \tilde{\mathbf{o}}_{j,t} \right\|_{L^2(g^D)} \approx \left\| \tilde{\mathbf{o}}_{o,t} + \sum_j \tilde{\mathbf{o}}_{j,t} \right\|_{r_D^{-2} C_{\beta}^{\alpha}(g^D)} \approx \sup (\|\mathbf{o}_o\|_{L^2(g_o)}, \|\mathbf{o}_j\|_{L^2(g_{b_j})}).$$

We would like to produce Einstein metrics in reduced divergence-free gauge with respect to g^D . But the point is that it is not always possible because the space $\tilde{\mathbf{O}}(g^D)$ is an (approximate) obstruction space. We will show that we can perturb $g^D + v$ for parameters $t > 0$ and $v \in \tilde{\mathbf{O}}(g^D)$ small enough to obtain a metric $\hat{g}_v = \hat{g}_{t,v}$ which will be in gauge with respect to $g^D + v$ and solution of:

$$g^D(\hat{g}_v) \in \tilde{\mathbf{O}}(g^D),$$

hence the term Einstein *modulo obstructions*.

Control of the inverse of the linearization. We can first show that the linearization is invertible and that we can control its inverse *independently of the gluing scales* thanks to Lemma 4.3.

Let us start by showing that the operator $\pi_{\tilde{\mathbf{O}}(g^D) \perp} P_{g^D}$ is close to P_{g^D} for a sufficiently degenerate tree of singularities.

Lemma 4.5 *There exists $C > 0$ such that for any symmetric 2-tensor $h \in C_{\beta,*}^{2,\alpha}(g^D)$, we have*

$$\|(\pi_{\tilde{\mathbf{O}}(g^D) \perp} P_{g^D} - P_{g^D})h\|_{r_D^{-2} C_{\beta}^{\alpha}(g^D)} \leq C t_{\max}^{\frac{1}{2}} \|h\|_{C_{\beta,*}^{2,\alpha}(g^D)}.$$

Remark 24 Here, the constant C only depends on the constants C_o and C_{b_j} of Lemma 4.3.

Proof The proof is similar to that of the estimate (34). Thanks to the equivalence of the norms, see Remark 23, we only have to control the $L^2(g^D)$ -norm of the projection on $\tilde{\mathbf{O}}(g^D)$.

On M_o , since P_{g^D} is self adjoint, an integration by parts yields

$$\left| \int_M \langle P_{g^D} h, \tilde{\mathbf{o}}_{o,t} \rangle_{g^D} dv_{g^D} \right| = \left| \int_M \langle h, P_{g^D}(\tilde{\mathbf{o}}_{o,t}) \rangle_{g^D} dv_{g^D} \right|,$$

and since $g^D = g_o$ on M_o^{16t} and $P_{g_o}(\mathbf{o}_o) = 0$ we use the decomposition

$$\begin{aligned}
 P_{g^D}(\tilde{\mathbf{o}}_{o,t}) &= P_{g_o}((\chi_{M_o^{16t}} - 1)\mathbf{o}_{o,*}) \\
 &\quad + P_{g_o}\left(\sum_k (\chi_{\mathcal{A}_k(t,\epsilon_0)} - \chi_{B_o(p_k,\epsilon_0)})\mathbf{o}_{o,k}\right) \\
 (56) \quad &\quad + (P_{g_o} - P_{g^D})\left(\sum_k \chi_{\mathcal{A}_k(t,\epsilon_0)}\mathbf{o}_{o,k}\right)
 \end{aligned}$$

in order to obtain the following estimates (compare with the proof of (34)) thanks to the control of the cut-off functions of Definition 1.10 and to Remark 23

$$(57) \quad |P_{g_o}((\chi_{M_o^{16t}} - 1)\mathbf{o}_{o,*})|_{g_o} \leq C\|\mathbf{o}_o\|_{L^2(g_o)} \mathbb{1}_{\{2t_k^{\frac{1}{4}} < r_D < 4t_k^{\frac{1}{4}}\}} t_k^{-\frac{1}{2}},$$

$$(58) \quad \left|P_{g_o}\left(\sum_k (\chi_{\mathcal{A}_k(t,\epsilon_0)} - \chi_{B_o(p_k,\epsilon_0)})\mathbf{o}_{o,k}\right)\right| \leq C\|\mathbf{o}_o\|_{L^2(g_o)} \mathbb{1}_{\{\epsilon_0^{-1}t_k^{\frac{1}{2}} < r_D < 2\epsilon_0^{-1}t_k^{\frac{1}{2}}\}} t_k^{-1},$$

and

$$(59) \quad \left|(P_{g_o} - P_{g^D})\left(\sum_k \chi_{\mathcal{A}_k(t,\epsilon_0)}\mathbf{o}_{o,k}\right)\right| \leq C\|\mathbf{o}_o\|_{L^2(g_o)} \mathbb{1}_{\{\epsilon_0^{-1}t_k^{\frac{1}{2}} < r_D < 2t_k^{\frac{1}{4}}\}} (1 + t_k^2 r_D^{-6}).$$

Since for any $C > 0$ independent on t , we have

- $\text{Vol}_{g_e}(\{2t_k^{\frac{1}{4}} < r^D < 4t_k^{\frac{1}{4}}\}) \leq Ct_k$,
- $\text{Vol}_{g_e}(\{\epsilon_0^{-1}t_k^{\frac{1}{2}} < r_D < 2\epsilon_0^{-1}t_k^{\frac{1}{2}}\}) \leq C\epsilon_0^{-4}t_k^2$, and
- $\int_{\{\epsilon_0^{-1}t_k^{\frac{1}{2}} < r_D < 2t_k^{\frac{1}{4}}\}} (1 + t_k^2 r_D^{-6}) dv_{g^D} \leq Ct_{\max}$,

integrating the controls (57), (58) and (59) of the terms of (56) yields

$$(60) \quad \left|\int_M \langle P_{g^D} h, \tilde{\mathbf{o}}_{o,t} \rangle_{g^D} dv_{g^D}\right| \leq Ct_k^{\frac{1}{2}} \|h\|_{C_{\beta,*}^{2,\alpha}(g^D)} \|\mathbf{o}_o\|_{L^2(g_o)}.$$

Similarly, for the N_j , consider $\mathbf{o}_j \in \mathbf{O}(g_{b_j})$. By invariance of the L^2 -norm of 2-tensors in dimension 4 and since $P_{\frac{g}{t}} = tP_g$ for any metric g and $t > 0$, we have

$$\int_M \langle P_{g^D} h, \tilde{\mathbf{o}}_{j,t} \rangle_{g^D} dv_{g^D} = \int_M \left\langle \frac{h}{T_j}, P_{\frac{g^D}{T_j}} \tilde{\mathbf{o}}_{j,t} \right\rangle_{\frac{g^D}{T_j}} dv_{\frac{g^D}{T_j}}.$$

The control at the singular points is the same as in the case of M_o and at infinity we have

$$P_{\frac{g^D}{T_j}} \tilde{\mathbf{o}}_{j,t} = (P_{\frac{g^D}{T_j}} - P_{g_{b_j}}) \tilde{\mathbf{o}}_{j,t} + P_{g_{b_j}} \mathbf{o}_j + P_{g_{b_j}}((\chi_{N_j^{16t}} - 1)\mathbf{o}_{j,*}),$$

and therefore, since $\mathbf{o}_j = \mathcal{O}(r_{b_j}^{-4})$, we have

$$|P_{\frac{g^D}{T_j}} \tilde{\mathbf{o}}_{j,t}|_{g_{b_j}} \leq C\|\mathbf{o}_j\|_{L^2(g_{b_j})} \mathbb{1}_{\{\frac{1}{2}t_j^{-1/4} < r_{b_j} < t_j^{-1/4}\}} r_{b_j}^{-6} \approx C\|\mathbf{o}_j\|_{L^2(g_{b_j})} \mathbb{1}_{\{\frac{1}{2}t_j^{-1/4} < r_{b_j} < t_j^{-1/4}\}} t_j^{\frac{3}{2}},$$

and since $\text{Vol}(A_e(\frac{1}{2}t_j^{-\frac{1}{4}}, t_j^{-\frac{1}{4}})) \leq Ct_j^{-1}$ and at every point $|\chi_{N_j^{16t}} \frac{h}{T_j}|_{g_{b_j}} \leq \|h\|_{C_{\beta,*}^{2,\alpha}(g^D)}$, we have

$$(61) \quad \left| \int_M \langle P_{g^D} h, \tilde{\mathbf{o}}_{j,t} \rangle_{g^D} dv_{g^D} \right| \leq Ct_{\max}^{\frac{1}{2}} \|h\|_{C_{\beta,*}^{2,\alpha}(g^D)} \|\mathbf{o}_j\|_{L^2(g_{b_j})},$$

and finally,

$$\|(\pi_{\tilde{\mathbf{O}}(g^D)^\perp} P_g^D - P_g^D)h\|_{r_D^{-2}C_{\beta}^\alpha(g^D)} \leq Ct_{\max}^{\frac{1}{2}} \|h\|_{C_{\beta,*}^{2,\alpha}(g^D)}.$$

□

Proposition 11 Let $0 < \beta < 1$, $k \in \mathbb{N}$, $0 < \alpha < 1$ and let (M, g^D) be a naïve desingularization of a compact or ALE Einstein orbifold by a tree of singularities. Then, there exists $\tau_D > 0$ and $\epsilon_D > 0$ and $C_D > 0$ only depending on β , and the constants of Lemma 4.3 and of Proposition 3 such that for $t_{\max} < \tau_D$ and for any metric g such that $\|g - g_t^D\|_{C_{\beta,*}^{2,\alpha}(g^D)} \leq \epsilon_D$, the operator

$$\pi_{\tilde{\mathbf{O}}(g^D)^\perp} P_g : \tilde{\mathbf{O}}(g^D)^\perp \cap C_{\beta,*}^{2,\alpha}(g^D) \rightarrow \tilde{\mathbf{O}}(g^D)^\perp \cap r_D^{-2}C_{\beta}^\alpha(g^D),$$

where $\pi_{\tilde{\mathbf{O}}(g^D)^\perp}$ is the orthogonal projection for g^D on $\tilde{\mathbf{O}}(g^D)^\perp$, is invertible and we have for any symmetric 2-tensor $h \perp \tilde{\mathbf{O}}(g^D)$ on M ,

$$\|h\|_{C_{\beta,*}^{2,\alpha}(g^D)} \leq C_D \|\pi_{\tilde{\mathbf{O}}(g^D)^\perp} P_g h\|_{r_D^{-2}C_{\beta}^\alpha(g^D)}.$$

Proof The proof is similar to that of Lemma 3.4. The idea is again to extend the symmetric 2-tensors on the model spaces and to deduce a control on the whole tree of singularities.

Let $0 < \epsilon < \epsilon_D^{\frac{1}{2-\beta}} < \epsilon_0$ for ϵ_D and ϵ which we will choose small enough along the proof, and assume that $t_{\max} < \epsilon^4$ in order to have on each annulus $\mathcal{A}_k := \mathcal{A}_k(t, \epsilon)$ between N_k and N_j , the existence of a diffeomorphism

$$\Phi_k : A_e(\epsilon^{-1}\sqrt{T_j}\sqrt{t_k}, \epsilon\sqrt{T_j}) \subset \mathbb{R}^4/\Gamma_k \rightarrow \mathcal{A}_k \subset M,$$

such that for any $0 < \beta < 1$, there exists $C > 0$,

$$\|\Phi_k^* g^D - g_e\|_{C_{\beta}^{2,\alpha}(A_e(\epsilon^{-1}\sqrt{T_j}\sqrt{t_k}, \epsilon\sqrt{T_j}))} \leq C\epsilon^{2-\beta} < C\epsilon_D,$$

by definition of g^D . Until the end of the proof, we will denote

$$A_k := A_e(\epsilon^{-1}\sqrt{T_j}\sqrt{t_k}, \epsilon\sqrt{T_j}).$$

Let h be a symmetric 2-tensor on M . Thanks to the estimate (13) of Proposition 3 and to its generalization to metrics close to g_e by Proposition 7, for ϵ_D and t_{\max} small

enough, we can choose constant and traceless symmetric 2-tensors H_k on \mathbb{R}^4/Γ_k such that we have on $A(\sqrt{T_j t_k}^{1/4}) := A_e((1/2)\sqrt{T_j t_k}^{1/4}, 4\sqrt{T_j t_k}^{1/4})$

$$\begin{aligned}
\|\Phi_k^* h - H_k\|_{C_0^{2,\alpha}(A(T_j^{1/2} t_k^{1/4}))} &\leq C_e \left(\epsilon^{-\beta} t_k^{\frac{\beta}{4}} \|P_{g_e} \Phi_k^* h\|_{r_e^{-2} C_\beta^\alpha(A_k)} \right. \\
&\quad \left. + 4\epsilon^{-1} t_k^{\frac{1}{4}} \|\Phi_k^* h - H_k\|_{C_\beta^{2,\alpha}(A_k)} \right) \\
&\leq 2C_e \left(\epsilon^{-\beta} t_k^{\frac{\beta}{4}} \|(P_{g^D} h)|_{\mathcal{A}_k}\|_{r_D^{-2} C_\beta^\alpha(g^D)} \right. \\
(62) \quad &\quad \left. + 4\epsilon^{-1} t_k^{\frac{1}{4}} \|(h - \chi_{\mathcal{A}_k} H_k)\|_{C_\beta^{2,\alpha}(g^D)} \right).
\end{aligned}$$

Let us then consider the decomposition

$$h = h_* + \sum_k \chi_{\mathcal{A}_k} H_k.$$

We define a symmetric 2-tensor h_o extending h to M_o in the following way:

$$h_o := \chi_{M_o^{t/16}} h_* + \sum_{k \in K_o} \chi_{B_o(p_k, \epsilon)} H_k,$$

where K_o is the set of k such that $\mathcal{A}_k \cap M_o^t \neq \emptyset$. Denoting $h_{o,*} := \chi_{M_o^{t/16}} h_*$, we have

$$\|h_{o,*}\|_{C_\beta^{2,\alpha}(g_o)} \geq \|h_{o,*}\|_{C_\beta^0(g_o)} \geq \|(h_*)|_{M_o^t}\|_{C_\beta^0(g^D)},$$

because M_o^t is outside of the region damaged by the gluing.

Since for t_{\max} small enough, the metric g^D is close to the metric g_o on $M_o^{t/16}$ by (38) and

$$P_{g^D} h = P_{g_o} h_o + P_{g_o} (h - h_o) + (P_{g^D} - P_{g_o}) h,$$

we moreover have the following control thanks to (62),

$$\begin{aligned}
\|(P_{g^D} h)|_{M_o^{t/16}}\|_{r_D^{-2} C_\beta^\alpha(g^D)} &\geq \|P_{g_o} h_o\|_{r_o^{-2} C_\beta^\alpha(g_o)} \\
&\quad - CC_e \sum_{k \in K_o} \|(P_{g^D} h)|_{\mathcal{A}_k}\|_{r_D^{-2} C_\beta^\alpha(g^D)} \\
(63) \quad &\quad - \gamma(t_{\max}) \|h\|_{C_{\beta,*}^{2,\alpha}(g^D)},
\end{aligned}$$

where $\gamma(t_{\max}) \rightarrow 0$ when $t_{\max} \rightarrow 0$ (compare with (42) for the vector fields).

Since $h \perp \tilde{\mathbf{O}}(g^D)$, we have $\frac{\|\pi_{\mathbf{O}(g_o)} h_o\|_{C_{\beta,*}^{2,\alpha}}}{\|h_o\|_{C_{\beta,*}^{2,\alpha}}} \rightarrow 0$, and by Lemma 4.3, this implies, for t_{\max} small enough, the control

$$\|h_o\|_{C_{\beta,*}^{2,\alpha}(g_o)} \leq 2C_o \|P_o h_o\|_{r_o^{-2} C_\beta^\alpha(g_o)},$$

and the estimate (63) and Lemma 4.5 imply that for t_{\max} small enough, there exists $C > 0$ only depending on the previous constants such that we have

$$(64) \quad \|(h_*)|_{M_o} \|_{C_{\beta}^0(g^D)} + \sum_{k \in K_o} |H_k|_{g_e} - C\gamma(t_{\max}) \|h\|_{C_{\beta,*}^{2,\alpha}(g^D)} \leq C \|\pi_{\tilde{\mathbf{O}}(g^D)} \perp P_{g^D} h\|_{r_D^{-2} C_{\beta}^{\alpha}(g^D)}.$$

Indeed, on an orbifold (M_o, g_o) , the constant symmetric 2-tensors of the decomposition in the definition of the norm $\|\cdot\|_{C_{\beta,*}^{2,\alpha}}$ are determined, see Remark 16.

Let us then consider the symmetric 2-tensor $h_1 := h - \sum_{k \in K_o} H_k$ which satisfies for $C > 0$ depending on the previous constants,

$$(65) \quad \|\pi_{\tilde{\mathbf{O}}(g^D)} \perp P_{g^D} h_1\|_{r_D^{-2} C_{\beta}^{\alpha}(g^D)} \leq C \|\pi_{\tilde{\mathbf{O}}(g^D)} \perp P_{g^D} h\|_{r_D^{-2} C_{\beta}^{\alpha}(g^D)} + C\gamma(t_{\max}) \|h\|_{C_{\beta,*}^{2,\alpha}(g^D)}$$

thanks to the control (64) of $\sum_{k \in K_o} |H_k|_{g_e}$.

Given $j \in K_o$, the Ricci-flat ALE orbifold (N_j, g_{b_j}) is glued to M_o and we can extend the symmetric 2-tensor $h_1 = h_* + \sum_{l \notin K_o} \chi_{\mathcal{A}_l} H_l$ to N_j by

$$h_j := \chi_{N_j^{t/16}} h_* + \sum_{l \in K_j} \chi_{B_j(p_l, \epsilon)} H_l,$$

where K_j is the set of $l \notin K_o$ such that \mathcal{A}_l has a nonempty intersection with the neighborhood of a N_j^t .

Denoting $h_{j,*} := \chi_{N_j^{t/16}} h_*$, we have

$$(66) \quad \|h_{j,*}\|_{C_{\beta}^{2,\alpha}(g_{b_j})} \geq \|h_{j,*}\|_{C_{\beta}^0(g_{b_j})} \geq \|(h_*)|_{N_j^t}\|_{C_{\beta}^0(g^D)},$$

and by (62) and since $\frac{g^D}{T_j}$ is close to g_{b_j} on $N_j^{t/16}$ for t_{\max} small enough, we moreover have the following control thanks to (62),

$$(67) \quad \begin{aligned} \|(P_{g^D} h_1)|_{N_j^{t/16}}\|_{r_D^{-2} C_{\beta}^{\alpha}(g^D)} &\geq \|P_{g_{b_j}} h_j\|_{r_{b_j}^{-2} C_{\beta}^{\alpha}(g_{b_j})} \\ &\quad - CC_e \sum_{k \in K_j} \|(P_{g^D} h_1)|_{\mathcal{A}_k}\|_{r_D^{-2} C_{\beta}^{\alpha}(g^D)} \\ &\quad - \gamma(t_{\max}) \|h\|_{C_{\beta,*}^{2,\alpha}(g^D)}, \end{aligned}$$

where $\gamma(t_{\max}) \rightarrow 0$ when $t_{\max} \rightarrow 0$. We then have a control on $(h_*)|_{N_j^t}$ and on the H_k , for $k \in K_j$ thanks to Lemma 4.3 by using again the fact that $h \perp \tilde{\mathbf{O}}(g^D)$ which implies that for t_{\max} small enough, we have

$$\|h_j\|_{r_j C_{\beta,*}^{3,\alpha}(g_{b_j})} \leq 2C_j \|P_{g_{b_j}} h_j\|_{r_j^{-2} C_{\beta}^{\alpha}(g_{b_j})}.$$

The estimates (67), (62) and Lemma 4.5 then yield

$$(68) \quad \|(h_*)|_{N_j}\|_{C_\beta^0} + \sum_{k \in K_j} |H_k|_{g_e} - C\gamma(t_{\max})\|h\|_{C_{\beta,*}^{2,\alpha}(g^D)} \leq C\|\pi_{\tilde{\mathbf{O}}(g^D)^\perp} P_{g^D} h\|_{r_D^{-2} C_\beta^\alpha(g^D)}.$$

Iterating the above controls to the other Ricci-flat ALE orbifolds of the tree of singularities, we obtain controls similar to (68) on all of the N_j and with the control (64), we finally find

$$(1 - C\gamma(t_{\max}))\|h\|_{C_{\beta,*}^{2,\alpha}(g^D)} \leq C\|\pi_{\tilde{\mathbf{O}}(g^D)^\perp} P_{g^D} h\|_{r_D^{-2} C_\beta^\alpha(g^D)},$$

and therefore the stated result for $g = g^D$ for t_{\max} small enough.

To obtain the same result for a metric g close to g^D , we simply apply Proposition 7 to ensure that for $\|g - g^D\|_{C_{\beta,*}^{2,\alpha}(g^D)}$ arbitrarily small, P_g is arbitrarily close to P_{g^D} for the operator norm on $C_{\beta,*}^{2,\alpha}(g^D)$.

The operator P being self adjoint on a compact manifold, its injectivity implies its surjectivity by integration by parts. \square

4.2 Resolution modulo obstructions of the Einstein equation

Let us now show that we can always solve the Einstein equation *modulo obstructions*. Let us recall that being Einstein and in reduced divergence-free with respect to a naïve desingularization g^D is equivalent to being a zero of the operator

$$g^D : g \mapsto \text{Ric}(g) - \frac{\mathbf{R}(g)}{2}g + \lambda g + \delta_g^* \tilde{\delta}_{g^D} g.$$

Theorem 4.6 *Let (M_o, g_o) be a compact or ALE Einstein orbifold of dimension 4 such that $\text{Ric}(g_o) = \Lambda g_o$, for $\Lambda \in \mathbb{R}$, and let $(N_j, g_{b_j})_j$ be a tree of singularities desingularizing (M_o, g_o) with pattern D , and $0 < \beta < 1$.*

Then, there exists $\tau > 0$, $\epsilon > 0$ only depending on β and the constants of Lemma 4.3 and of Proposition 3 such that for any naïve desingularization $g^D := g_t^D$, with $t_{\max} < \tau$, and for all $v \in \tilde{\mathbf{O}}(g^D)$ satisfying $\|v\|_{C_{\beta,}^0(g^D)} < \epsilon$, there exists a unique solution $\hat{g}_v = \hat{g}_{t,v}$ to the equation*

$$g^D(\hat{g}_v) \in \tilde{\mathbf{O}}(g^D),$$

satisfying the following conditions:

$$(1) \quad \|\hat{g}_v - g^D\|_{C_{\beta,*}^{2,\alpha}} \leq 2\epsilon,$$

(2) $\hat{g}_v - (g^D + v)$ is $L^2(g^D)$ -orthogonal to $\tilde{\mathbf{O}}(g^D)$.

Proof Let $(M, g_t^D) = (M, g^D)$ be a naïve desingularization of an orbifold (M_o, g_o) by a tree of singularities $(N_j, g_{b_j})_j$.

Define the operator $\Psi : g^D + C_{\beta, *}^{2, \alpha}(g^D) \rightarrow \tilde{\mathbf{O}}(g^D)^\perp \cap r_D^{-2} C_\beta^\alpha(g^D)$ by

$$\Psi(g) := \pi_{\tilde{\mathbf{O}}(g^D)^\perp} g = \pi_{\tilde{\mathbf{O}}(g^D)^\perp} \left(\text{Ric}(g) - \frac{\mathbf{R}(g)}{2} g + \lambda g + \delta_g^* \tilde{\delta}_g^D(g) \right),$$

where $\pi_{\tilde{\mathbf{O}}(g^D)^\perp}$ is the $L^2(g^D)$ -orthogonal projection on $\tilde{\mathbf{O}}(g^D)^\perp$. The conclusion of the theorem for $v = 0$ then rewrites: there exists a unique solution $g \in g^D + (\tilde{\mathbf{O}}(g^D)^\perp \cap C_{\beta, *}^{2, \alpha})$ to the equation $\Psi(g) = 0$.

Let us apply the inverse function theorem, Lemma 3.5, to Ψ . The linearization of the operator Ψ at g^D for any symmetric 2-tensor h satisfying $\int_M \text{tr}_{g^D} h d\nu_{g^D} = 0$ is

$$d_{g^D} \Psi(h) = \pi_{\tilde{\mathbf{O}}(g^D)^\perp} \tilde{P}_{g^D}(h),$$

where \tilde{P}_{g^D} is explicited in (53). Let us show that this linearization is invertible at g^D and has an inverse which is uniformly bounded as $t \rightarrow 0$. We want to go back to the operator $P_g = \frac{1}{2} \nabla_g^* \nabla_g h - \mathring{\mathbf{R}}_g(h)$ for which the invertibility has been shown in Proposition 11.

First, by the estimate (34), we have

$$\|\tilde{\delta}_{g^D} h - \delta_{g^D} h\|_{r_D^{-1} C_{\beta, *}^{1, \alpha}(g^D)} \leq C t_{\max}^{\frac{1}{2}} \|h\|_{C_{\beta, *}^{2, \alpha}(g^D)},$$

and therefore the term $-2\delta_g^* \delta_g h + 2\delta_g^* \tilde{\delta}_g^D h$ of (53) is controlled in the following way

$$(69) \quad \left\| -2\delta_g^* \delta_g h + 2\delta_g^* \tilde{\delta}_g^D h \right\|_{r_D^{-2} C_\beta^\alpha(g^D)} \leq C t_{\max}^{\frac{1}{2}} \|h\|_{C_{\beta, *}^{2, \alpha}(g^D)}.$$

Next, notice that the Ricci curvature of g^D is almost constant:

$$(70) \quad \left\| \text{Ric}(g^D) - \Lambda g^D \right\|_{r_D^{-2} C_\beta^\alpha} \leq C t_{\max}^{\frac{2-\beta}{4}},$$

because $\text{Ric}(g^D) - \Lambda g^D = 0$ for $r_D > 2t_{\max}^{1/4}$, and on the rest of the manifold, $|\text{Ric}(g^D) - \Lambda g^D| \leq C$. Therefore, for t_{\max} arbitrarily small, $d_{g^D} \Psi$ is close (as an operator from $C_{\beta, *}^{2, \alpha}$ to $r_D^{-2} C_\beta^\alpha$) up a power of t_{\max} to the operator $\pi_{\tilde{\mathbf{O}}(g^D)^\perp} \hat{P}_{g^D}$, where for a symmetric 2-tensor h ,

$$\begin{aligned} \hat{P}_{g^D}(h) &:= \frac{1}{2} \left(\nabla_{g^D}^* \nabla_{g^D} h - 2\mathring{\mathbf{R}}_{g^D}(h) \right. \\ &\quad \left. - (\delta_{g^D} \delta_{g^D} h) g^D \right. \\ &\quad \left. - \nabla_{g^D}^2 \text{tr}_{g^D} h + (\Delta_{g^D} \text{tr}_{g^D} h) g^D + \frac{\mathbf{R}_{g^D}}{4} (\text{tr}_{g^D} h) g^D \right), \end{aligned}$$

where we neglected the difference of the divergence terms by (69), and simplified the terms involving the Ricci curvature by (70).

Let us use the following decomposition of a 2-tensor on (M, g_t^D) : for any symmetric 2-tensor $h \in C_{\beta, *}^{2, \alpha}(g^D)$, there exists a unique decomposition

$$(71) \quad h = h_T + \delta_{g^D}^* X$$

with

- (1) a symmetric 2-tensor $h_T \in C_{\beta, *}^{2, \alpha}(g^D)$ satisfying $\tilde{\delta}_{g^D} h_T = 0$,
- (2) a vector field $X \in r_D C_{\beta, *}^{3, \alpha}(g^D) \cap \tilde{\mathbf{K}}_o^\perp$.

Indeed, according to Lemma 3.4, for any 2-tensor $h \in C_{\beta, *}^{2, \alpha}(g^D)$, there exists a unique $X \in r_D C_{\beta, *}^{3, \alpha}(g^D) \cap \tilde{\mathbf{K}}_o^\perp$ such that

$$\tilde{\delta}_{g^D} \delta_{g^D}^* X = \tilde{\delta}_{g^D} h.$$

The decomposition (71) is then $h = (h - \delta_{g^D}^* X) + \delta_{g^D}^* X$ and the sum is $L^2(g^D)$ -orthogonal.

Remark 25 A simple integration by parts shows that a differentiable symmetric 2-tensor h on M satisfies $\tilde{\delta}_{g^D}$ if and only if it is $L^2(g^D)$ -orthogonal to $\delta_{g^D}^*(C^\infty(TM) \cap \tilde{\mathbf{K}}_o^\perp)$.

Similarly, for a symmetric 2-tensor $v \in r_D^{-2} C_\beta^\alpha$ we have a unique decomposition

$$(72) \quad v = v_T + \delta_{g^D}^* Y$$

with a symmetric 2-tensor v_T orthogonal to $\delta_{g^D}^*(C^\infty(TM) \cap \tilde{\mathbf{K}}_o^\perp)$ and a vector field $Y \perp \tilde{\mathbf{K}}_o$.

Now, we have the following properties:

- (1) our metric has almost constant Ricci curvature by (70),
- (2) the divergence of the obstructions is arbitrarily small, that is for $\tilde{\mathbf{o}} \in \tilde{\mathbf{O}}(g^D)$, we have

$$\frac{\|\delta_{g^D} \tilde{\mathbf{o}}\|_{r_D^{-1} C_\beta^{1, \alpha}}}{\|\tilde{\mathbf{o}}\|_{C_{\beta, *}^{2, \alpha}}} \rightarrow 0$$

and

$$\frac{\|\delta_{g^D} \tilde{\mathbf{o}}\|_{r_D^{-3} C_\beta^0}}{\|\tilde{\mathbf{o}}\|_{r_D^{-2} C_\beta^\alpha}} \rightarrow 0$$

as $t_{\max} \rightarrow 0$ by elliptic regularity .

These imply that for $X \in r_D C_{\beta,*}^{3,\alpha} \cap \tilde{\mathbf{K}}_o^\perp$, one has

$$(73) \quad \|\pi_{\tilde{\mathbf{O}}(g^D)^\perp} \hat{P}_{g^D}(\delta_{g^D}^* X) - \delta_{g^D}^* \tilde{\delta}_{g^D} \delta_{g^D}^* X\|_{r_D^{-2} C_\beta^\alpha} \leq \gamma(t_{\max}) \|X\|_{r_D C_{\beta,*}^{3,\alpha}},$$

for $\gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\lim_0 \gamma = 0$, and that

Consider the decompositions (71) and (72) where we identify the symmetric 2-tensor of the form $\delta_{g^D}^* X$ with the vector field X since

$$\delta_{g^D}^* : \tilde{\mathbf{K}}_o^\perp \cap r_D C_{\beta,*}^{3,\alpha} \rightarrow C_{\beta,*}^{2,\alpha}$$

and

$$\delta_{g^D}^* : \tilde{\mathbf{K}}_o^\perp \cap r_D^{-1} C_{\beta,*}^{1,\alpha} \rightarrow r_D^{-2} C_{\beta,*}^\alpha$$

are injective. Thanks to (73), the operator $\pi_{\tilde{\mathbf{O}}(g^D)^\perp} \hat{P}_{g^D} : C_{\beta,*}^{2,\alpha} \rightarrow r_D^{-2} C_{\beta,*}^\alpha$ decomposes blockwise

$$(74) \quad \pi_{\tilde{\mathbf{O}}(g^D)^\perp} \hat{P}_{g^D} = \begin{bmatrix} \tilde{\delta}_{g^D} \delta_{g^D}^* & 0 \\ 0 & \pi_{\tilde{\mathbf{O}}(g^D)^\perp} \hat{P}_{g^D} \end{bmatrix} + A_{g^D}$$

where A_{g^D} is an arbitrarily small (for t_{\max} arbitrarily small) operator between the same spaces, and

$$\begin{aligned} \hat{P}_{g^D}(h) &:= \frac{1}{2} \left(\nabla_{g^D}^* \nabla_{g^D} h - 2\mathring{\mathbf{R}}_{g^D}(h) \right. \\ &\quad \left. - \nabla_{g^D}^2 \text{tr}_{g^D} h + (\Delta_{g^D} \text{tr}_{g^D} h) g^D + \frac{\mathbf{R}_{g^D}}{4} (\text{tr}_{g^D} h) g^D \right). \end{aligned}$$

Given the shape of the matrix in (74), it is enough to show that

$$\tilde{\delta}_{g^D} \delta_{g^D}^* : \tilde{\mathbf{K}}_o^\perp \cap r_D C_{\beta,*}^{3,\alpha} \rightarrow \tilde{\mathbf{K}}_o^\perp \cap r_D^{-1} C_{\beta,*}^{1,\alpha}$$

and

$$\pi_{\tilde{\mathbf{O}}(g^D)^\perp} \hat{P}_{g^D} : \tilde{\mathbf{O}}(g^D)^\perp \cap C_{\beta,*}^{2,\alpha} \rightarrow \tilde{\mathbf{O}}(g^D)^\perp \cap r_D^{-2} C_\beta^\alpha$$

are invertible with inverse bounded independently of t_{\max} . This is already the case for $\tilde{\delta}_{g^D} \delta_{g^D}^*$ thanks to Lemma 3.4.

Let us now focus on $\pi_{\tilde{\mathbf{O}}(g^D)^\perp} \hat{P}_{g^D}$. For any symmetric 2-tensor v , we have an $L^2(g^D)$ -orthogonal decomposition into a conformal and traceless part

$$(75) \quad v = \frac{\text{tr}_{g^D} v}{4} g^D + \left(v - \frac{\text{tr}_{g^D} v}{4} g^D \right).$$

Now, we have

$$\text{tr}_{g^D} \left(\pi_{\tilde{\mathbf{O}}(g^D)^\perp} \hat{P}_{g^D}(h) \right) = \text{tr}_{g^D} \hat{P}_{g^D}(h) - \text{tr}_{g^D} \left(\pi_{\tilde{\mathbf{O}}(g^D)^\perp} \hat{P}_{g^D}(h) \right)$$

and since for an element $\tilde{\mathbf{o}}$ of $\tilde{\mathbf{O}}(g^D)$, we have

$$(76) \quad \frac{\|(\mathrm{tr}_{g^D} \tilde{\mathbf{o}})g^D\|_{r_D^{-2}C_\beta^\alpha}}{\|\tilde{\mathbf{o}}\|_{r_D^{-2}C_\beta^\alpha}} \rightarrow 0$$

and

$$(77) \quad \frac{\|(\mathrm{tr}_{g^D} \tilde{\mathbf{o}})g^D\|_{C_{\beta,*}^{2,\alpha}}}{\|\tilde{\mathbf{o}}\|_{C_{\beta,*}^{2,\alpha}}} \rightarrow 0$$

as $t_{\max} \rightarrow 0$, the operator $\mathrm{tr}_{g^D}(\pi_{\tilde{\mathbf{O}}(g^D)\perp} \hat{P}_{g^D})$ is arbitrarily close to the operator $\mathrm{tr}_{g^D} \hat{P}_{g^D}$ (as operators from $C_{\beta,*}^{2,\alpha}$ to $r_D^{-2}C_\beta^\alpha$) which is itself arbitrarily close to $\Delta_{g^D} + \Lambda$ for t_{\max} arbitrarily small because our metric is almost Einstein according to (70).

For the traceless part of $\pi_{\tilde{\mathbf{O}}(g^D)\perp} \hat{P}_{g^D}(h)$, recall that for any symmetric 2-tensor h , we have the following decomposition

$$(78) \quad \begin{aligned} \pi_{\tilde{\mathbf{O}}(g^D)\perp} \hat{P}_{g^D}(h) - \mathrm{tr}_{g^D}(\pi_{\tilde{\mathbf{O}}(g^D)\perp} \hat{P}_{g^D}(h)) \frac{g^D}{4} &= \pi_{\tilde{\mathbf{O}}(g^D)\perp} \left(\hat{P}_{g^D}(h) - \mathrm{tr}_{g^D}(\hat{P}_{g^D}(h)) \frac{g^D}{4} \right) \\ &\quad - \mathrm{tr}_{g^D}(\hat{P}_{g^D}(h)) \frac{\pi_{\tilde{\mathbf{O}}(g^D)\perp} g^D}{4} \\ &\quad + \left(\mathrm{tr}_{g^D}(\pi_{\tilde{\mathbf{O}}(g^D)\perp} \hat{P}_{g^D}(h)) \right) \frac{g^D}{4}. \end{aligned}$$

Using the control (70) once more together with (76) and (77), we neglect the last two terms of (78) and see that the traceless part of $\pi_{\tilde{\mathbf{O}}(g^D)\perp} \hat{P}_{g^D}(h)$ is arbitrarily close to the first term $\pi_{\tilde{\mathbf{O}}(g^D)\perp} \left(\hat{P}_{g^D}(h) - \mathrm{tr}_{g^D}(\hat{P}_{g^D}(h)) \frac{g^D}{4} \right)$, that is, to

$$\pi_{\tilde{\mathbf{O}}(g^D)\perp} \left(P_{g^D}(h) - (\nabla_{g^D}^2 \mathrm{tr}_{g^D} h - \Delta_{g^D}(\mathrm{tr}_{g^D} h)) \frac{g^D}{4} \right).$$

Let us therefore decompose h and the operator $\pi_{\tilde{\mathbf{O}}(g^D)\perp} \hat{P}_{g^D}(h)$ in conformal and traceless parts. Block-wise, we obtain the following operator:

$$\pi_{\tilde{\mathbf{O}}(g^D)\perp} \hat{P}_{g^D} = \begin{bmatrix} \Delta_{g^D} + \Lambda & 0 \\ \pi_{\tilde{\mathbf{O}}(g^D)\perp} (\Delta_{g^D} \mathrm{tr}_{g^D} h \frac{g^D}{4} - \nabla_{g^D}^2 \mathrm{tr}_{g^D} h) & \pi_{\tilde{\mathbf{O}}(g^D)\perp} P_{g^D} \end{bmatrix} + A'_{g^D}$$

where $A'_{g^D} : C_{\beta,*}^{2,\alpha} \rightarrow r_D^{-2}C_\beta^\alpha$ is an arbitrarily small operator for t_{\max} arbitrarily small. There remains to show that the operator

$$\begin{bmatrix} \Delta_{g^D} + \Lambda & 0 \\ \pi_{\tilde{\mathbf{O}}(g^D)\perp} (\Delta_{g^D} \mathrm{tr}_{g^D} h \frac{g^D}{4} - \nabla_{g^D}^2 \mathrm{tr}_{g^D} h) & \pi_{\tilde{\mathbf{O}}(g^D)\perp} P_{g^D} \end{bmatrix}$$

from $C_{\beta,*}^{2,\alpha}$ to $r_D^{-2}C_{\beta}^{\alpha}$ is invertible with a uniformly bounded inverse (independently of t) for t_{\max} small enough. Given the shape of the matrix, it is enough to show it for the two diagonal blocks since the operator at the bottom left is uniformly bounded. The operator $\Delta_{g^D} + \Lambda$ is invertible with a uniformly bounded inverse thanks to Lichnerowicz eigenvalue estimate, see [And10, Section 5] for instance, and the estimate (70). This is also the case for $\pi_{\mathbf{O}(g^D)^\perp} P_{g^D}$ thanks to Proposition 11.

We conclude that the linearization of Ψ at g^D is invertible with a bounded inverse as it is arbitrarily close to an invertible operator.

Remark 26 The operator \bar{P}_{g^D} itself is not self adjoint because the metric g^D is not Einstein. Indeed, all of the terms are self adjoint except $-\nabla_g^2 \text{tr}_g h$, $-(\delta_g \delta_g h)g$, and $\langle \text{Ric}_g, h \rangle_g g$, and we remark that the adjoint of $h \mapsto \nabla_g^2 \text{tr}_g h$ is $h \mapsto (\delta_g \delta_g h)g$. However, the term $h \mapsto \langle \text{Ric}_g, h \rangle_g g$ whose adjoint is $h \mapsto (\text{tr}_g h) \text{Ric}_g$ prevents \bar{P}_{g^D} to be self adjoint when g^D is not Einstein.

To apply the inverse function theorem, Lemma 3.5 to the operator Ψ , there remains to control the non-linear terms. Since the variations of the Ricci curvature for a variation h of a metric g , are schematically,

$$\text{Ric}(g+h) = \text{Ric}(g) + (g+h)^{-1} * \text{Rm}(g) + (g+h)^{-2} * \nabla^2 h + (g+h)^{-3} * \nabla h * \nabla h,$$

where $*$ refers to diverse multilinear operations and by Remark 12, the non-linear terms $Q_{g^D}(h) := \Psi(g^D + h) - \Psi(g^D) - d_{g^D} \Psi(h)$ satisfy the control

$$\begin{aligned} & \|Q_{g^D}(h) - Q_{g^D}(h')\|_{r_D^{-2}C_{\beta}^{\alpha}(g^D)} \\ & \leq C \left((\|h\|_{C_0^{\alpha}(g^D)} + \|h'\|_{C_0^{\alpha}(g^D)}) (\|h - h'\|_{C_0^{\alpha}(g^D)}^2) \|\text{Rm}(g^D)\|_{r_D^{-2}C_{\beta}^{\alpha}(g^D)} \right. \\ & \quad + \|h - h'\|_{C_0^{\alpha}(g^D)} (\|\nabla^2 h\|_{r_D^{-2}C_{\beta}^{\alpha}(g^D)} + \|\nabla^2 h'\|_{r_D^{-2}C_{\beta}^{\alpha}(g^D)}) \\ & \quad + (\|h\|_{C_0^{\alpha}(g^D)} + \|h'\|_{C_0^{\alpha}(g^D)}) \|\nabla^2(h - h')\|_{r_D^{-2}C_{\beta}^{\alpha}(g^D)} \\ & \quad \left. + 2\|\nabla(h - h')\|_{r_D^{-1}C_{\beta}^{\alpha}(g^D)} (\|\nabla h\|_{r_D^{-1}C_{\beta}^{\alpha}(g^D)} + \|\nabla(h')\|_{r_D^{-1}C_{\beta}^{\alpha}(g^D)}) \right) \\ & \leq 3C (\|h\|_{C_{\beta,*}^{2,\alpha}(g^D)} + \|h'\|_{C_{\beta,*}^{2,\alpha}(g^D)}) \|h - h'\|_{C_{\beta,*}^{2,\alpha}(g^D)}. \end{aligned}$$

We moreover have the control

$$\|\Psi(g^D)\|_{r_D^{-2}C_{\beta}^{\alpha}(g^D)} \leq C t_{\max}^{\frac{2-\beta}{4}}.$$

Hence, according to the inverse function theorem, Lemma 3.5, for t_{\max} small enough, there exists a unique solution \hat{g} with $\hat{g} - g^D \perp \mathbf{O}(g^D)$, to the equation $\Psi(\hat{g}) = 0$ satisfying moreover $\|g^D - \hat{g}\|_{C_{\beta,*}^{2,\alpha}(g^D)} \leq \|\Psi(g^D)\|_{r_D^{-2}C_{\beta}^{\alpha}(g^D)} \leq C t_{\max}^{\frac{2-\beta}{4}}$.

Now, we have only solved the equation in the neighborhood of g^D and on the orthogonal of $\tilde{\mathbf{O}}(g^D)$. For $v \in \tilde{\mathbf{O}}(g^D)$ we study the operator $g \mapsto \Psi(g + v)$. The control of the non-linear terms is exactly the same for this operator, and for v arbitrarily small, its linearization at g^D is arbitrarily close to $d_{g^D}\Psi$ which is invertible. As a consequence, there exists $\epsilon > 0$ such that for all $\|v\|_{C_\beta^0(g^D)} < \epsilon$, there exists a unique solution \hat{g}_v of $\Psi(\hat{g}_v) = 0$ with $\hat{g}_v - (g^D + v) \perp \tilde{\mathbf{O}}(g^D)$.

Remark 27 By adding v , we however damage the estimate on $\Psi(g^D + v)$ which becomes

$$(79) \quad \|\Psi(g^D + v)\|_{r_D^{-2}C_\beta^\alpha(g^D)} \leq C(\|v\|_{C_\beta^{2,\alpha}(g^D)}^2 + t_{\max}^{\frac{2-\beta}{4}}).$$

We will see later in Section 5.1 that without an integrability assumption, we cannot hope for a better estimate.

□

Corollary 4 With the notations of Theorem 4.6, the map $v \mapsto \hat{g}_v$ is analytic.

Proof This is a consequence of the implicit function theorem for analytic functions, see [Whi65] for instance. Let us define the map

$$(v, h) \mapsto \Psi(\hat{g}_0 + v + h)$$

from $\tilde{\mathbf{O}}(g^D) \times (\tilde{\mathbf{O}}(g^D)^\perp \cap C_{\beta,*}^{2,\alpha})$ to $\tilde{\mathbf{O}}(g^D)^\perp \cap r_D^{-2}C_\beta^\alpha(g^D)$ where we denote

$$\Psi(g) := \pi_{\tilde{\mathbf{O}}(g^D)^\perp} g^D(g) = \pi_{\tilde{\mathbf{O}}(g^D)^\perp} (E(g) + \delta_g^* \tilde{\delta}_{g^D}(g)),$$

like in the proof of Theorem 4.6 and where \hat{g}_0 is the solution of Theorem 4.6 for $v = 0$.

The map $(v, h) \mapsto \Psi(\hat{g}_0 + v + h)$ is analytic since E and $g \mapsto \delta_g^*$ are, and since $\tilde{\delta}_{g^D}$ and $\pi_{\tilde{\mathbf{O}}(g^D)^\perp}$ are linear. We know that $\|\hat{g}_0 - g^D\| \leq Ct_{\max}^{\frac{2-\beta}{4}}$, and that $d_{g^D}\Psi$ is invertible thanks to the proof of Theorem 4.6, hence $d_{\hat{g}_0}\Psi$ is also invertible.

The implicit function theorem for analytic functions, see [Whi65], then implies, by the uniqueness of the solution \hat{g}_v of Theorem 4.6 that for v small, the map $v \mapsto \hat{g}_v - (g^D + v)$ is analytic and that $v \mapsto \hat{g}_v$ is analytic too. □

Remark 28 The previous analysis of Theorem 4.6 extends to partial desingularizations.

More precisely, let S_o be a subset of the singularities of M_o and for each j , S_j a subset of the singularities of N_j and denote $S = (S_o, (S_j)_j)$ and $M_S := M_o \#_j N_j$ ($\#$ means gluing

the ALE spaces at orbifold singularities) where the gluings are given by some gluing pattern D .

For $t_{\max, S}$ and v small enough, the metric $g_S^D = g_{S,t}^D$ iteratively defined just like in Definition 1.6 can be perturbed to a *unique* solution $\hat{g}_{S,v} = \hat{g}_{S,t,v}$ to the equation

$$(80) \quad g^D(\hat{g}_{S,v}) \in \tilde{\mathbf{O}}(g_S^D),$$

satisfying the following conditions:

- (1) $\|\hat{g}_{S,v} - g_S^D\|_{C_{\beta, *}(g_S^D)}^{2, \alpha} \leq 2\epsilon$, where $C_{\beta, *}(g_S^D)$ is the partial desingularization norm of Remark 9,
- (2) $\hat{g}_{S,v} - g_S^D$ is $L^2(g_S^D)$ -orthogonal.

Thanks to Corollary 3, we have the following result.

Corollary 5 Let $D_0, v_0 > 0$, $l \in \mathbb{N}$, and $\beta = \beta(v_0, D_0) > 0$ obtained in Corollary 2. Then, for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon, D_0, v_0, l) > 0$ such that if $(M, g^\mathcal{E})$ is an Einstein manifold satisfying

- the volume is bounded below by $v_0 > 0$,
- the diameter is bounded above by D_0 ,
- the Ricci curvature is bounded $|\text{Ric}| \leq 3$.

and such that for an Einstein orbifold (M_o, g_o) , we have

$$d_{GH}((M, g^\mathcal{E}), (M_o, g_o)) \leq \delta,$$

then, there exists a naïve desingularization $(M, g_{t,v}^D)$ of (M_o, g_o) by a tree of singularities, and a diffeomorphism $\psi : M \rightarrow M$ such that

$$\psi^* g^\mathcal{E} = \hat{g}_{t,v},$$

where $\hat{g}_{t,v}$ is the perturbation of $g_t^D + v$ of Theorem 4.6.

4.3 Premoduli space of Einstein metrics around a singular one

Let us now explain how the Einstein modulo obstructions metrics of Theorem 4.6 are analogous to the metrics in the set W of Theorem 0.2.

Let M a differentiable 4-manifold and consider an orbifold $(M_o, g_o) \in \partial_o \mathbf{E}(M) \subset \overline{\mathbf{E}(M)}_{GH} \setminus \mathbf{E}(M)$. According to Corollary 5, the Einstein metrics which are sufficiently

close to (M_o, g_o) in the Gromov-Hausdorff are results of the gluing-perturbation procedure of Theorem 4.6.

By analogy with the definition of [Koi83] of the premoduli space of Einstein metrics around a smooth one, we define the *directional* premoduli space in the neighborhood of a singular metric in the following way.

Definition 4.7 (Directional premoduli space) We define $\mathbb{E}_{g_o}^D(M)$, the *premoduli space of Einstein metrics* on M in the neighborhood of (M_o, g_o) and *in the direction D* , as the set of metrics g on M for which there exists t such that we have $\|g - g_t^D\|_{C_{\beta,*}^{2,\alpha}(g_t^D)} < \epsilon$ for $\epsilon > 0$ the constant of Theorem 4.6 and $v \in \tilde{\mathbf{O}}(g_t^D)$ for which we have:

- (1) $E(g) = 0$, $\text{Vol}(g) = \text{Vol}(g_o)$,
- (2) $g - (g_t^D + v) \perp_{g_t^D} \tilde{\mathbf{O}}(g_t^D)$,
- (3) $\tilde{\delta}_{g_t^D} g = 0$.

Remark 29 The above premoduli space is directional in the sense that it does not cover all of the Gromov-Hausdorff desingularizations of (M_o, g_o) , but only the ones whose tree of singularities is D . Given an arbitrary sequence $(g_i)_i$ of smooth Einstein metrics on M d_{GH} -converging to (M_o, g_o) , then, up to considering a subsequence, all of the metrics g_i are in a *single* directional premoduli space $\mathbb{E}_{g_o}^D(M)$.

The result of [Ozu19a] together with Theorem 4.6 yield the following statement.

Corollary 6 The directional premoduli space $\mathbb{E}_{g_o}^D(M)$ is the zero-set of E on the set denoted \hat{W} of metrics $\hat{g}_{t,v}$ in the $C_{\beta,*}^{2,\alpha}(g_t^D)$ -neighborhoods of the metrics g_t^D of Theorem 4.6.

Since E is analytic, Koiso's proof of Theorem 0.2 reduces to proving that W is an analytic submanifold by implicit function theorem. Therefore the question of the regularity of the d_{GH} -completion of $\mathbf{E}(M^4)$, $\mathbf{E}(M^4) \cup \partial_o \mathbf{E}(M^4)$, reduces to the question of the regularity of the set \hat{W} of the Einstein modulo obstructions metrics of Theorem 4.6.

5 Obstructions to the Gromov-Hausdorff desingularization

Let us now come to the main application of this article, which is the obstruction to the desingularization of Einstein orbifolds.

5.1 Analysis of integrable Ricci-flat ALE spaces

In order to obtain an obstruction result we will need to assume that the Ricci-flat ALE metrics in the trees of singularities have integrable Ricci-flat deformations.

5.1.1 Integrable Ricci-flat ALE

Since Corollary 2 does not control the convergence speed towards the limit orbifold or the Ricci-flat ALE spaces, like in Theorem 4.6, we have to fix a gauge $v \in \tilde{\mathbf{O}}(g^D)$ on the approximate kernel of the operator \bar{P} .

Not to damage our controls, we cannot simply use $g^D + v$ as an approximate metric. We need to find a better approximation to extend Proposition 13 to the case when $v \neq 0$. It turns out that this will only be possible if we assume that the Ricci-flat ALE metrics are *integrable*.

Definition 5.1 (Integrable Ricci-flat ALE orbifold) We will say that a Ricci-flat metric ALE g_b is *integrable* if for all $v \in \mathbf{O}(g_b)$ small enough, there exists a Ricci-flat metric ALE $\bar{g}_{b,v}$ satisfying $\bar{g}_{b,v} - (g_b + v) \perp \mathbf{O}(g_b)$ and $\|\bar{g}_{b,v} - g_b\|_{L^2(g_b)} \leq 2\|v\|_{L^2}$, and such that $\delta_{g_b}\bar{g}_{b,v} = 0$.

Remark 30 All of the known examples of Ricci-flat ALE spaces are integrable since they are quotients of hyperkähler spaces. Moreover, any infinitesimal L^2 -deformation of ALE Ricci-flat orbifolds is automatically divergence-free and trace-free by the proof of Lemma 4.2.

5.1.2 Weighted Hölder spaces and asymptotics of Ricci-flat ALE spaces

Let us introduce yet another function space to control the asymptotics of our ALE metrics. This will be crucial to deduce obstructions in the following sections.

Definition 5.2 ($C_{\beta,**}^{2,\alpha}$ -norm on a ALE orbifold) Let (N, g_b) be an ALE orbifold, and let h be a symmetric 2-tensor on N , and assume that $h = H^4 + \mathcal{O}(r_b^{-4-\beta})$ for $\beta > 0$ and H^4 a homogeneous harmonic symmetric 2-tensor with $|H^4| \sim r_e^{-4}$. We define its $C_{\beta,**}^{2,\alpha}$ -norm by

$$\|h\|_{C_{\beta,**}^{2,\alpha}} := \sup r_e^4 |H^4|_{g_e} + \|(1 + r_b)^4 (h - \chi(\epsilon r_b) H^4)\|_{C_{\beta,*}^{2,\alpha}}.$$

Definition 5.3 ($r_b C_{\beta, **}^{3, \alpha}$ -norm on a ALE orbifold) Let (N, g_b) be an ALE orbifold, and let X be a vector field on N , and assume that $X = Y^3 + \mathcal{O}(r_b^{-3-\beta})$ for $\beta > 0$ and Y^3 a homogeneous element of the kernel of $\delta_e \delta_e^*$ with $|Y^3| \sim r_e^{-3}$. We define its $r_b C_{\beta, **}^{3, \alpha}$ -norm by

$$\|X\|_{r_b C_{\beta, **}^{3, \alpha}} := \sup r_e^3 |Y^3|_{g_e} + \|(1+r_b)^4 (X - \chi(\epsilon r_b) Y^3)\|_{r_b C_{\beta, *}^{3, \alpha}}.$$

These norms are motivated by the following Lemmata.

Lemma 5.4 *Let (N, g_b) be an ALE orbifold. Then, there exists $C > 0$ such that we have, for any $h \perp \mathbf{O}(g_b)$,*

$$(81) \quad C^{-1} \|(1+r_b)^4 \bar{P}_{g_b} h\|_{r_b^{-2} C_{\beta}^{\alpha}} \leq \|h\|_{C_{\beta, **}^{2, \alpha}} \leq C \|(1+r_b)^4 \bar{P}_{g_b} h\|_{r_b^{-2} C_{\beta}^{\alpha}}.$$

Proof By the theory of elliptic operators in weighted Hölder spaces (see for instance [PR78, Chapter 2]), the operator $\bar{P}_{g_b} : (1+r_b)^{-4} C_{-\beta}^{2, \alpha} \rightarrow (1+r_b)^{-4} r_b^{-2} C_{-\beta}^{\alpha}$ is Fredholm with kernel $\mathbf{O}(g_b)$ and cokernel $\mathbf{O}(g_b)$ because there is no other exceptional value than 0 between 2 and -4 . This implies that $\bar{P}_{g_b} : (1+r_b)^{-4} C_{-\beta}^{2, \alpha} \cap \mathbf{O}(g_b)^{\perp} \rightarrow (1+r_b)^{-4} r_b^{-2} C_{-\beta}^{\alpha} \cap \mathbf{O}(g_b)^{\perp}$ is invertible with a bounded inverse.

Moreover, we have $\bar{P}_{g_b}^{-1}((1+r_b)^{-4} r_b^{-2} C_{\beta}^{\alpha}) = C_{\beta, **}^{2, \alpha}$ since -4 is the first negative exceptional value for \bar{P} , and the stated inequality comes from the fact that the inverse is bounded. \square

Similarly for vector fields, we have the following result.

Lemma 5.5 *Let (N, g_b) be an ALE orbifold. Then, there exists $C > 0$ such that for any vector field X on N , we have*

$$(82) \quad \|X\|_{r_b C_{\beta, **}^{3, \alpha}} \leq C \|(1+r_b)^4 \delta_{g_b} \delta_{g_b}^* X\|_{r_b^{-1} C_{\beta}^{1, \alpha}}.$$

In particular the analysis of Theorem 4.6 and Proposition 10 extends to the case where (M_o, g_o) is a Ricci-flat ALE orbifold and where the norm $C_{\beta, *}^{2, \alpha}(g_o)$ is replaced by $C_{\beta, **}^{2, \alpha}(g_o)$ thanks to Lemma 5.4 and $r_o C_{\beta, *}^{3, \alpha}(g_o)$ is replaced by $r_o C_{\beta, **}^{3, \alpha}(g_o)$ thanks to Lemma 5.5. Indeed, all of the controls are local around the singular points or coming from an estimate on the inverse on the rest of the orbifold exactly like (81) and (82). For the operator \bar{P} , this yields the following control on the asymptotic terms on the ALE end.

Corollary 7 Let (N, g_b) be a Ricci-flat ALE orbifold, and denote (N^B, g_t^B) a naïve desingularization of (N, g_b) by a tree of Ricci-flat ALE orbifolds glued according to a pattern B with relative scales t .

For v and t small enough depending on the constants of Lemma 4.3, let $\bar{g}_{t,v}$ be the unique metric (according to Theorem 4.6) satisfying for $\epsilon > 0$ small enough:

- (1) $\|g_t^B - \bar{g}_{t,v}\|_{C_{\beta, **}^{2, \alpha}(g_t^B)} < 2\epsilon$,
- (2) $(g_t^B + v) - \bar{g}_{t,v}$ is $L^2(g_t^B)$ -orthogonal to $\tilde{\mathbf{O}}(g_t^B)$, and
- (3) $\pi_{\tilde{\mathbf{O}}(g_t^B)^\perp} g_t^B(\bar{g}_{t,v}) = 0$.

Then, for any $0 < \beta < 1$, we have

$$\bar{g}_{t,v} = g_e + \bar{H}_{t,v}^4 + \mathcal{O}(r_B^{-4-\beta}),$$

for $|\bar{H}_{t,v}^4| \sim r_B^{-4}$ and $\bar{H}_{t,v}^4 \rightarrow \bar{H}^4$, the asymptotic terms of (N, g_b) as $(t, v) \rightarrow (0, 0)$.

Similarly, using Lemma 5.5, we can put our ALE metrics in gauge with respect to each other. The nonlinear terms are taken care of like in Proposition 10 by noting that the weight of our norm, $(1 + r_b)^4 r_b^{-1}$, is this time also larger than 1 at infinity.

Corollary 8 Let $0 < \beta < 1$, let (N, g_b) be a Ricci-flat ALE orbifold, and denote (N^B, g_t^B) a naïve desingularization of (N, g_b) by a tree of Ricci-flat ALE orbifolds glued according to a pattern B with relative scales t .

Then, there exist $\epsilon_B, \tau_B, C_B > 0$ which only depend on the metrics g_b and the elements of B such that for $t_{\max} \leq \tau_B$ and for any metric g satisfying $\|g - g_t^B\|_{C_{\beta, **}^{2, \alpha}(g_t^B)} \leq \epsilon_B$, there exists a unique vector field X on M for which,

$$\tilde{\delta}_{g_t^B}(\exp_X^* g) = 0,$$

where \exp_X is the diffeomorphism $\exp_X : x \in M \mapsto \exp_x^{g_t^B}(X(x))$.

We moreover have, $\|X\|_{r_B C_{\beta, **}^{3, \alpha}(g_t^B)} \leq C_B \|(1 + r_B)^4 \tilde{\delta}_{g_t^B}(g - g_t^B)\|_{r_B^{-1} C_{\beta}^{1, \alpha}(g_t^B)}$, and therefore, there exists $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\lim_0 \eta = 0$ such that we have

$$\|\exp_X^* g - g\|_{C_{\beta, **}^{2, \alpha}(g_t^B)} \leq \eta(\|g - g_t^B\|_{C_{\beta, **}^{2, \alpha}(g_t^B)}).$$

5.2 Approximate Einstein modulo obstructions metric

Let (M_o, g_o) be an Einstein orbifold and let p be one of its singular points whose singularity model is \mathbb{R}^4/Γ .

Let us now construct a good approximation of the metric $\hat{g}_{t,v}$ of Theorem 4.6, which we will denote $g_{t,v}^A$. This is a crucial step to understand and approximate the obstructions coming from the Ricci-flat ALE spaces appearing at a singular point $p \in M_o$.

In all of this section, we will assume the following properties:

- (M, g_t^D) is a naïve desingularization of (M_o, g_o) ,
- at the singular point $p \in M_o$, there is only one Ricci-flat ALE manifold (N, g_b) glued, and therefore no tree of singularity,
- (N, g_b) is an *integrable* Ricci-flat ALE manifold.

Consider S the complement of $\{p\}$ among the singular points of M_o , let $(M_S, g_S^D = g_{S,t_S}^D)$ be a naïve partial desingularization of (M_o, g_o) which only leaves the point p singular and let $(M_S, \hat{g}_S = \hat{g}_{S,t_S,v_S})$ be the perturbation of $(M_S, g_S^D + v_S)$ orthogonally to $\tilde{\mathbf{O}}(g_S^D)$ of Theorem 4.6 satisfying

$$g_S^D(\hat{g}_S) = \mathbf{o}_S \in \tilde{\mathbf{O}}(g_S^D).$$

At p , the metric \hat{g}_S has the following development in local coordinates where it is in divergence-free gauge with respect to g_e ,

$$(83) \quad \hat{g}_S = g_e + \hat{H}_S + \mathcal{O}(r_o^3)$$

and we know that $g_S^D(\hat{g}_S) = \mathbf{o}_S = \mathbf{O}_S + \mathcal{O}(r_e^2)$ with $|\mathbf{O}_S|_{g_e} \sim r_e^0$, $\text{tr}_{g_e} \mathbf{O}_S = 0$ and $\delta_{g_e} \mathbf{O}_S = 0$. We therefore have

$$(84) \quad 0 = g_S^D(\hat{g}_S) - \mathbf{O}_S = \lambda g_e + \bar{P}_e(\hat{H}_S) - \mathbf{O}_S + \mathcal{O}(r_e^2),$$

where $\lambda = \frac{n-2}{2n} \bar{\mathbf{R}}(\hat{g}_S)$. Consequently $\bar{P}_e \hat{H}_S + \lambda g_e = \mathbf{O}_S$, where \mathbf{O}_S is the limit of \mathbf{o}_S at p .

For some small $v_p \in \mathbf{O}(g_b)$, we will glue a Ricci-flat deformation \bar{g}_{b,v_p} of g_b at a scale t_p to \hat{g}_S . To obtain a better estimate, we will extend the quadratic terms of \hat{g}_S in order to minimize the error in the gluing.

Proposition 12 Let (N, g_b) a Ricci-flat ALE orbifold asymptotic to \mathbb{R}^4/Γ , \hat{H}_S a quadratic symmetric 2-tensor on \mathbb{R}^4/Γ , $\lambda \in \mathbb{R}$ and \mathbf{O}_S a constant symmetric 2-tensor on \mathbb{R}^4/Γ such that we have:

$$\bar{P}_e \hat{H}_S + \lambda g_e = \mathbf{O}_S.$$

Then, there exists a 2-tensor \hat{h}_S , and for $(\mathbf{o}_j)_j$ an L^2 -orthonormal basis of $\mathbf{O}(g_b)$, real numbers $\hat{\lambda}_j$ such that $(\hat{h}_S, \hat{\lambda}_j)$ is a solution of

$$(85) \quad \begin{cases} \bar{P}_b \hat{h}_S + \lambda g_b = \chi \mathbf{O}_S + \sum_j \hat{\lambda}_j \mathbf{o}_j, \\ |\hat{h}_S - \chi \hat{H}_S|_{g_b} = \mathcal{O}(r_b^{-2}), \end{cases}$$

where χ is a cut-off function supported in a neighborhood of the infinity of (N, g_b) where the ALE coordinates are defined, and where

$$(86) \quad \hat{\lambda}_j := - \int_{\mathbb{S}^3/\Gamma} (3 \langle \hat{H}_S, O_j^4 \rangle_{g_e} + O_j^4 (\nabla_e \text{tr}_e \hat{H}_S, \partial_{r_e})) dv_{\mathbb{S}^3/\Gamma} + \int_N \chi \langle \mathbf{O}_S, \mathbf{o}_j \rangle_{g_b} dv_b.$$

The set of solutions to the above equation (85) is $(\hat{h}_S + \mathbf{O}(g_b), \hat{\lambda}_j)$.

Proof We have $\bar{P}_b(\chi \hat{H}_S) + \lambda g_b - \chi \mathbf{O}_S \in r_b^{-2} C_\beta^\alpha(g_b)$ for $0 < \beta < 1$ because in a neighborhood of infinity, $g_b - g_e = \mathcal{O}(r_b^{-4})$ together with the derivatives. Lemma 4.3 also holds by replacing the operator P_b by the operator \bar{P}_b as a consequence of Theorem 4.6 in the case where $(M_o, g_o) = (N, g_b)$ is a Ricci-flat ALE manifold. Consequently, there exists $\hat{h}' \in C_\beta^{2,\alpha}(g_b)$ such that $\bar{P}_b(\chi \hat{H}_S + \hat{h}') + \lambda g_b = \chi \mathbf{O}_S + \sum_j \hat{\lambda}_j \mathbf{o}_j$ with

$$(87) \quad \hat{\lambda}_j = - \int_{\mathbb{S}^3/\Gamma} (3 \langle \hat{H}_S, O_j^4 \rangle_{g_e} + \frac{1}{2} O_j^4 (\nabla \text{tr} \hat{H}_S, \partial_{r_e})) dv_{\mathbb{S}^3/\Gamma} + \int_N \chi \langle \mathbf{O}_S, \mathbf{o}_j \rangle_{g_b} dv_b,$$

where $O_j^4 \sim r_b^{-4}$ is the first term of the development of \mathbf{o}_j at infinity. Indeed, by integration by parts, and using the fact that $\text{tr}_b \mathbf{o}_j = \langle g_b, \mathbf{o}_j \rangle_{g_b} = 0$ and $\delta_b \mathbf{o}_j = 0$, we get,

$$\begin{aligned} \langle \bar{P}_b(\chi \hat{H}_S), \mathbf{o}_j \rangle_{L^2} &= \frac{1}{2} \lim_{\rho \rightarrow \infty} \int_{r_b \leq \rho} \langle \nabla_b^* \nabla_b(\chi \hat{H}_S) - \nabla_b^2(\text{tr}_b \chi \hat{H}_S), \mathbf{o}_j \rangle dv_{g_b} \\ &= \frac{1}{2} \lim_{\rho \rightarrow \infty} \int_{r_b = \rho} (\langle \chi \hat{H}_S, \nabla_n \mathbf{o}_j \rangle - \langle \nabla_n(\chi \hat{H}_S), \mathbf{o}_j \rangle \\ &\quad + \mathbf{o}_j(\nabla_b(\text{tr}_b \chi \hat{H}_S), \partial_{r_e})) dS_\rho \\ &= - \int_{\mathbb{S}^3/\Gamma} (3 \langle \hat{H}_S, O_j^4 \rangle_{g_e} + \frac{1}{2} O_j^4 (\nabla_e \text{tr}_e \hat{H}_S, \partial_{r_e})) dv_{\mathbb{S}^3/\Gamma}. \end{aligned}$$

Now, the integral $\int_N \chi \langle \mathbf{O}_S, \mathbf{o}_j \rangle_{g_b} dv_b$ converges even if $\langle \mathbf{O}_S, \mathbf{o}_j \rangle_{g_b} = \mathcal{O}(r_b^{-4})$. Indeed, in ALE coordinates, $r_b^4 \mathbf{o}_j = \phi_{ij} dx^i dx^j + \mathcal{O}(r_b^{-1})$, where the $\phi_{ij} : \mathbb{S}^3 \rightarrow \mathbb{R}$ are nonconstant eigenfunctions and therefore have zero mean values, hence, $\int_{\{r_b = \rho\}} \langle \mathbf{O}_S, \mathbf{o}_j \rangle_{g_b} dS_\rho = \mathcal{O}(\rho^{-2})$ and the integral converges. The values of $\hat{\lambda}_j$ from (87) therefore ensure that we have

$$\bar{P}_b(\chi \hat{H}_S) + \lambda g_b - \chi \mathbf{O}_S - \sum_j \hat{\lambda}_j \mathbf{o}_j \perp_{g_b} \mathbf{O}(g_b).$$

□

Choosing $t_S, t_p > 0$, $v_S \in \tilde{\mathbf{O}}(g_{S,t_S}^D)$ and $v_p \in \mathbf{O}(g_b)$, by gluing metric \bar{g}_{b,v_p} at scale t_p to \hat{g}_{S,v_S,t_S} , we reach all of the gauges of Theorem 4.6, that is we attain some element of $(g_t^D + v) + \tilde{\mathbf{O}}(g_t^D)^\perp$ for any small v and t .

We define $\tilde{\mathbf{O}}_{t_p}(\bar{g}_{b,v_p})$ as the cut-off of the elements of $\mathbf{O}(\bar{g}_{b,v_p})$ as in Definition 4.4 at scale t_p . We denote $\hat{h}_S(t, v)$ the 2-tensor satisfying $\hat{h}_S(t, v) \perp_{\bar{g}_{b,v_p}} \tilde{\mathbf{O}}_{t_p}(\bar{g}_{b,v_p})$ obtained in Proposition 12 with the Ricci-flat ALE metric \bar{g}_{b,v_p} and \hat{H}_S the quadratic terms of $\hat{g}_S = \hat{g}_{S,t_S,v_S}$.

Definition 5.6 (Approximate metric $g_{t,v}^A$) Let $v \in \tilde{\mathbf{O}}(g_t^D)$ for (M, g_t^D) a naïve desingularization of (M_o, g_o) .

The Riemannian manifold $(M, g^A = g_{t,v}^A)$ is obtained by naïve gluing (Definition 1.6) of $(N, \bar{g}_{b,v_p} + t_p \hat{h}_S(t, v))$ to $(M_S, \hat{g}_{S,t_S,v_S})$ at scale $t_p > 0$.

The numbers $t_p, t_S > 0$, $v_p \in \mathbf{O}(g_b)$ and $v_S \in \tilde{\mathbf{O}}(g_S^D)$ are chosen in order to have $g_{t,v}^A - (g_t^D + v) \perp \tilde{\mathbf{O}}(g_t^D)$.

5.3 Better approximation and obstructions

The obstruction will come from the better controls of $g_{t,v}^A$ and the following proposition.

Proposition 13 Let $0 < \alpha < 1$, and (M, g_t^D) be a naïve desingularization, and assume that for $t_{\max} < \tau$, the metric $(M, \hat{g} = \hat{g}_{t,v})$ obtained by Theorem 4.6 is an *Einstein metric* (without obstructions). Then, there exists $\epsilon > 0$ and $C > 0$ only depending on g_o and the g_{b_j} , such that, denoting $\Psi(g_{t,v}^A) := \pi_{\tilde{\mathbf{O}}(g^D)^\perp} g^D(g_{t,v}^A)$ and $\mathbf{o}_{t,v}^A := \pi_{\tilde{\mathbf{O}}_{t_p}(\bar{g}_{b,v_p})} g^D(g_{t,v}^A)$, we have

$$(88) \quad \|\hat{g}_{t,v} - g_{t,v}^A\|_{C_{\beta,*}^{2,\alpha}(g_t^D)} \leq C \|\Psi(g_{t,v}^A)\|_{r_D^{-2} C_\beta^\alpha(g_t^D)},$$

and

$$(89) \quad \|\mathbf{o}_{t,v}^A\|_{L^2(g^D)} \leq \left(\|\Psi(g_{t,v}^A)\|_{r_D^{-2} C_\beta^\alpha(g^D)} + t_p^{\frac{1}{2} + \frac{\beta}{4}} \right) \|\Psi(g_{t,v}^A)\|_{r_D^{-2} C_\beta^\alpha(g^D)}.$$

Proof Let us denote $g^A := g_{t,v}^A$, $\hat{g} := \hat{g}_{t,v}$ and $h^A := \hat{g} - g^A$ as well as $\bar{g}_b = \bar{g}_{b,v_p}$ for simplicity. The inequality (88) is a direct consequence of the proof of Theorem 4.6 because the proof uses an inverse function theorem. Indeed, at the linear level, by Lemma 4.3, we have

$$(90) \quad \|\hat{g} - g^A\|_{C_{\beta,*}^{2,\alpha}(g^D)} \leq C \|\pi_{\tilde{\mathbf{O}}(g^D)^\perp} \bar{P}_{g^D}(\hat{g} - g^A)\|_{r_D^{-2} C_\beta^\alpha(g^D)}.$$

Since $\Psi(\hat{g}) = 0$, the controls of the nonlinear terms of Ψ imply that

$$(91) \quad \begin{aligned} \|\Psi(g^A)\|_{r_D^{-2}C_\beta^\alpha(g^D)} &\geq \|\pi_{\tilde{\mathbf{o}}(g^D)^\perp} \bar{P}_{g^D}(\hat{g} - g^A)\|_{r_D^{-2}C_\beta^\alpha(g^D)} \\ &- C(\|\hat{g} - g^D\|_{C_{\beta,*}^{2,\alpha}(g^D)} + \|g^A - g^D\|_{C_{\beta,*}^{2,\alpha}(g^D)})\|\hat{g} - g^A\|_{C_{\beta,*}^{2,\alpha}(g^D)}. \end{aligned}$$

We moreover know that $\|g^A - g^D\|_{C_{\beta,*}^{2,\alpha}(g^D)} \leq 2\epsilon$ and $\|\hat{g} - g^D\|_{C_{\beta,*}^{2,\alpha}(g^D)} = \mathcal{O}(t_{\max}^{\frac{2-\beta}{4}})$ thanks to the proof of Theorem 4.6 and by assumption. Therefore, choosing ϵ and t_{\max} small enough and putting the estimates (90) and (91) together yields the expected control (88).

Let us focus on the zone N^t where the elements of $\tilde{\mathbf{O}}_{t_p}(\bar{g}_{b,v_p})$ are supported. For $\tilde{\mathbf{o}} \in \tilde{\mathbf{O}}_{t_p}(\bar{g}_{b,v_p})$, we have

$$(92) \quad \begin{aligned} |\langle \bar{P}_{g^A} h^A, \tilde{\mathbf{o}} \rangle_{L^2(g^D)}| &\leq |\langle (\bar{P}_{g^A} - \bar{P}_{t_p \bar{g}_b}) h^A, \tilde{\mathbf{o}} \rangle_{L^2(t_p \bar{g}_b)}| + |\langle \bar{P}_{t_p \bar{g}_b} h^A, \tilde{\mathbf{o}} \rangle_{L^2(t_p \bar{g}_b)}| \\ &= |\langle (\bar{P}_{g^A} - \bar{P}_{t_p \bar{g}_b}) h^A, \tilde{\mathbf{o}} \rangle_{L^2(t_p \bar{g}_b)}| + |\langle t_p^{-1} h^A, \bar{P}_{\bar{g}_b} \tilde{\mathbf{o}} \rangle_{L^2(\bar{g}_b)}|, \end{aligned}$$

since $\bar{P}_{\bar{g}_b}$ is self dual and since the L^2 -product of 2-tensor is invariant by rescaling in dimension 4 as well as the rescaling behavior of \bar{P} . Now, we have the following estimates on $N^t = \{r_b < 2t_p^{-\frac{1}{4}}\} \subset \{r_D < 2t_p^{\frac{1}{4}}\}$, for $k \in \{0, 1, 2\}$, there exists $C > 0$ such that:

- (1) $r_D^k |\nabla_{t_p \bar{g}_b}^k (g^A - t_p \bar{g}_b)|_{t_p \bar{g}_b} \leq C r_D^2$ since $g^A = t_p \bar{g}_b + t_p^2 \hat{h}_S$ on N^{16t} by construction and because of the controls of the cut-off functions,
- (2) $r_b^k |\nabla_{\bar{g}_b}^k \tilde{\mathbf{o}}| \leq C r_b^{-4}$ as well as $\bar{P}_{\bar{g}_b} \tilde{\mathbf{o}} = 0$ on $N^{16t} \subset \{r_D < t_p^{\frac{1}{4}}\}$, and
- (3) $r_D^k |\nabla_{t_p \bar{g}_b}^k h^A|_{t_p \bar{g}_b} \leq C (t_p^{-\frac{1}{2}} r_D)^{-\beta} \|\Psi(g^A)\|_{r_D^{-2}C_\beta^\alpha(g^D)}$ by (88).

We can therefore control the terms of (92) in the following way:

$$(93) \quad \begin{aligned} |\langle \bar{P}_{g^A} h^A, \tilde{\mathbf{o}} \rangle_{L^2(g^D)}| &\leq C \|\Psi(g^A)\|_{r_D^{-2}C_\beta^\alpha(g^D)} \int_{t_p^{\frac{1}{2}}}^{t_p^{\frac{1}{4}}} \left(\frac{r}{\sqrt{t_p}}\right)^{-\beta} \left(\frac{r}{\sqrt{t_p}}\right)^{-4} r^3 dr \\ &\quad + \|\Psi(g^A)\|_{r_D^{-2}C_\beta^\alpha(g^D)} \int_{t_p^{-\frac{1}{4}}}^{2t_p^{-\frac{1}{4}}} t_p^{-1} r^{-\beta} r^{-6} r^3 dr \\ &\leq C \|\Psi(g^A)\|_{r_D^{-2}C_\beta^\alpha(g^D)} \left(t_p^{2+\frac{\beta}{4}} + t_p^{\frac{1}{2}+\frac{\beta}{4}}\right) \\ &\leq C \|\Psi(g^A)\|_{r_D^{-2}C_\beta^\alpha(g^D)} t_p^{\frac{1}{2}+\frac{\beta}{4}}. \end{aligned}$$

Let $\mathbf{o}^A = \mathbf{o}_{t,v}^A := \pi_{\tilde{\mathbf{O}}_{t_p}(\bar{g}_{b,v_p})} g^D(g^A)$, since $g^D(\hat{g}) = 0$, and $d_{g^A} g^D = \bar{P}_{g^A}$. We have

$$(94) \quad -\|\mathbf{o}^A\|_{L^2(g^D)}^2 = \langle g^D(\hat{g}) - \mathbf{o}^A, \mathbf{o}^A \rangle = \langle \Psi(g^A) + \bar{P}_{g^A} h^A + Q_{g^A}(h^A), \mathbf{o}^A \rangle,$$

where the non-linear terms $Q_{g^A}(h^A) = {}_{g^D}(\hat{g}) - {}_{g^D}(g^A) - \bar{P}_{g^A}(h^A)$ satisfy

$$(95) \quad \|Q_{g^A}(h^A)\|_{r_D^{-2}C^0(M)} \leq C\|h^A\|_{C_{\beta,*}^0}^2 \leq C\|\Psi(g^A)\|_{r_D^{-2}C_{\beta}^0(g^D)}^2,$$

and by definition of the weighted norms, since $r_D^{-2}C_{\beta}^{\alpha}(g^D) \subset L^2(g^D)$, we therefore have for $\mathbf{o} \in \tilde{\mathbf{O}}(g^D)$,

$$|\langle \mathbf{o}, Q_{g^A}(h^A) \rangle_{L^2(g^D)}| \leq C\|\mathbf{o}\|_{L^2}\|\Psi(g^A)\|_{r_D^{-2}C_{\beta}^0(g^D)}^2.$$

Hence, since $\Psi(g^A) \perp_{g^D} \mathbf{o}^A$ by definition of Ψ , we have by (94), and thanks to (93) and (95) we therefore have

$$\|\mathbf{o}^A\|_{L^2(g^D)} \leq C(\|\Psi(g^A)\|_{r_D^{-2}C_{\beta}^{\alpha}(g^D)} + t_p^{\frac{1}{2} + \frac{\beta}{4}})\|\Psi(g^A)\|_{r_D^{-2}C_{\beta}^{\alpha}(g^D)}.$$

□

Remark 31 The inequality (88) means that if we are able to construct a metric g^A such that $\Psi(g^A)$ is small, then g^A is a good approximation of \hat{g} , the only zero of Ψ . This allows us to approximate the metrics \hat{g} , and therefore degenerating Einstein metrics with an arbitrarily good precision.

The inequality (89) is an obstruction result. Indeed, if we construct a good approximation g^A , for which $\Psi(g^A)$ is small, but without having \mathbf{o}^A small, then \hat{g} cannot be Einstein, and the metric g^A (and therefore g^D) cannot be perturbed to an Einstein metric orthogonally to $\tilde{\mathbf{O}}(g^D)$.

Let us now control the above quantities of (89) for $g_{t,v}^A$.

Proposition 14 For $(t_S, t_p) > 0$ small enough and $k \in \mathbb{N}$, denoting $(\tilde{\mathbf{o}}_j)_j$ an orthonormal basis of $\tilde{\mathbf{O}}_{t_p}(\bar{g}_{t_p, v_p})$ there exist real numbers $(\hat{\lambda}_j = \hat{\lambda}_j(t, v))_j$ and $C_k > 0$ such that we have

$$(96) \quad \|\pi_{\tilde{\mathbf{O}}(g^D) \perp g^D}(g_{t,v}^A)\|_{r_D^{-2}C_{\beta}^k(g^D)} \leq C_k t_p^{\frac{3-\beta}{4}},$$

and, for any i_0 , we have

$$(97) \quad \left\langle g^D(g_{t,v}^A) - t_p \sum_j \hat{\lambda}_j \tilde{\mathbf{o}}_j, \tilde{\mathbf{o}}_{i_0} \right\rangle_{L^2(g^D)} \leq C_0 t_p^{\frac{5}{4}}.$$

Remark 32 If we did not use the partial Einstein modulo obstructions desingularization \hat{g}_S , we would only have a control with powers of t_{\max} instead of t_p . In particular, we would not be able to later prove that an obstruction holds at *all* of the singular points, but just at one of them.

Proof Let us again use the following notations along this proof: $g^A := g_{t,v}^A$, $\bar{g}_b = \bar{g}_{b,v_p}$, $\hat{g}_S = \hat{g}_{S,t_S,v_S}$ for simplicity.

On M_S^{16t} , we have by assumption

$$(98) \quad g_S^D(\hat{g}_S) = \mathbf{o}_S \in \tilde{\mathbf{O}}(g_S^D),$$

and on N^{16t} , we have

$$(99) \quad t_p \bar{g}_b(t_p(\bar{g}_{b,v_p} + t_p \hat{h}_S)) = \chi \mathbf{O}_S + t_p \sum_j \hat{\lambda}_j \mathbf{o}_j + \mathcal{O}(r_D^2).$$

Now, since at the point p , the development of \hat{g}_S in local coordinates where the metric is divergence-free gauge with respect to g_e is

$$(100) \quad \hat{g}_S = g_e + \hat{H}_S + \mathcal{O}(r_o^3),$$

and since $t_p(\bar{g}_{b,v_p} + t_p \hat{h}_S)$ has the following development for $t_p^{-\frac{1}{4}} \leq r_b \leq 2t_p^{-\frac{1}{4}}$

$$(101) \quad t_p(\bar{g}_{b,v_p} + t_p \hat{h}_S) = g_e + \hat{H}_S + \mathcal{O}(t_p^2 r_b^{-2} + t_p r_b^{-4}),$$

with corresponding controls for the derivatives up to order 2. On the annulus of radii $r_D = t_p^{\frac{1}{4}}$ and $r_D = 2t_p^{\frac{1}{4}}$, recalling that $r_D = r_o = \sqrt{t_p} r_b$, for $k \in \mathbb{N}$, we have

$$\begin{aligned} r_D^k |\nabla_{g^D}^k (\hat{g}_S - (t_p \bar{g}_{b,v_p} + t_p \hat{h}_S))|_{g^D} &= \mathcal{O}(r_D^3 + t_p^2 (t_p^{-\frac{1}{2}} r_D)^{-2} + t_p (t_p^{-\frac{1}{2}} r_D)^{-4}) \\ &= \mathcal{O}(t_p^{\frac{3}{4}} + t_p^{\frac{3}{4}} + t_p) \\ &= \mathcal{O}(t_p^{\frac{3}{4}}) \end{aligned}$$

thanks to (100) and (101). By definition of the gluing, this yields

$$(102) \quad r_D^{2+k} |\nabla_{g_t^D}^k (\pi_{\tilde{\mathbf{O}}(g_t^D)}(g_{t,v}^A))| = \mathcal{O}(t_p^{\frac{3}{4}}),$$

According to (98), (99) and (102), we have the estimate (96).

Finally, we have the control (97) thanks to (101), (99) and (102):

$$t_p |\hat{\lambda}_j(t, v)| \leq C \left(t_p^{\frac{3-\beta}{4}} + t_p^{\frac{2+\beta}{4}} \right) t_p^{\frac{3-\beta}{4}} \leq C t_p^{\frac{5}{4}}$$

□

Lemma 5.7 *Let $\hat{\lambda}_j(t, v)$ be the real numbers of Proposition 12 obtained by extending \hat{H}_S , the quadratic terms of $\hat{g}_S = \hat{g}_{S,t_S,v_S}$ on (N, \bar{g}_{b,v_p}) , and let $\hat{\lambda}_j$ be the ones obtained by extending the divergence-free quadratic terms of g_o on (N, g_b) . Then as $t, v \rightarrow 0$, we have*

$$(103) \quad |\hat{\lambda}_j(t, v) - \hat{\lambda}_j| \xrightarrow{(t,v) \rightarrow (0,0)} 0.$$

Proof We have the expression

$$\begin{aligned} \hat{\lambda}_j(t, v) = & - \int_{\mathbb{S}^3/\Gamma} (3\langle \hat{H}_S(t_S, v_S), \mathcal{O}_j^4(v_p) \rangle_{g_e} + \mathcal{O}_j^4(v_p)(\nabla_e \operatorname{tr}_e \hat{H}_S(t_S, v_S), \partial_{r_e})) dv_{\mathbb{S}^3/\Gamma} \\ & + \int_N \chi \langle \mathbf{O}_S(t_S, v_S), \mathbf{o}_j(v_p) \rangle dv_b \end{aligned}$$

thanks to Proposition 12. Since we want to show that it converges, as $(t, v) \rightarrow 0$, to

$$\hat{\lambda}_j := - \int_{\mathbb{S}^3/\Gamma} \left(3\langle \hat{H}_S, \mathcal{O}_j^4 \rangle_{g_e} + \frac{1}{2} \mathcal{O}_j^4(\nabla_e \operatorname{tr}_e \hat{H}_S, \partial_{r_e}) \right) dv_{\mathbb{S}^3/\Gamma}.$$

We therefore just have to show that as $(t, v) \rightarrow 0$, we have

- (1) $\hat{H}_S(t_S, v_S) \rightarrow \hat{H}_S$ in $r_e^2 C_0^1(g_e)$,
- (2) $|\mathbf{O}_S(t_S, v_S)|_{g_e} \rightarrow 0$, and
- (3) $\mathcal{O}_j^4(v_p) \rightarrow \mathcal{O}_j^4$ in $r_e^{-4} C_0^0(g_e)$.

Thanks to Theorem 4.6, we know that \hat{g}_{S, t_S, v_S} converges smoothly to g_o on compact of $M_o \setminus S$ as $(t_S, v_S) \rightarrow (0, 0)$, in particular, we have smooth convergence on small neighborhoods of p . Therefore, the quadratic terms of the expansion of \hat{g}_{S, t_S, v_S} converge to those of g_o as $(t_S, v_S) \rightarrow (0, 0)$. Consequently, the obstruction $\mathbf{O}_S(t_S, v_S) = \bar{P}_e \hat{H}_S(t_S, v_S) + \lambda(t_S, v_S) g_e$ also converges to $\bar{P}_e \hat{H}_S + \lambda g_e = 0$.

Similarly, thanks to Corollary 7, the asymptotic term of \bar{g}_{b, v_p} converge to the asymptotic terms of g_b as $v_p \rightarrow 0$. \square

Remark 33 We needed to consider *partial* desingularizations to obtain these controls.

Remark 34 Thanks to the computations of [Biq17, Proposition 4], it turns out that generically (when the self-dual part of the curvature at p , \mathbf{R}_+ is of rank 2 and $\Lambda \neq 0$), the difference $\hat{\lambda}_j(t, v) - \hat{\lambda}_j = \langle v, \mathbf{o}_j \rangle (\mathbf{o}_j(\mathbf{R}_+) - \Lambda \mathbf{o}_j) + \mathcal{O}(\|v\|_{C_{\beta, *}}^{2, \alpha})$ does not vanish.

Remark 35 If there were non integrable infinitesimal deformations of g_b , we a priori could not expect to prove an obstruction result by the above techniques. Indeed, the metric has an expansion $\bar{g}_{b, v_p} = g_b + v_p + w + \mathcal{O}(|v_p|^3)$, where w satisfies:

$$\bar{Q}_{g_b}^{(2)}(v_p, v_p) + \bar{P}_{g_b}(w) = \pi_{\mathbf{O}(g_b)} \bar{Q}_{g_b}^{(2)}(v_p, v_p) \in \mathbf{O}(g_b),$$

and we potentially have $\pi_{\mathbf{O}(g_b)} \bar{Q}_{g_b}^{(2)}(v_p, v_p) = \mathcal{O}(|v_p|^2)$ if v_p is not integrable. By considering the metric $\frac{1}{t_p} g_{t, v}^A$, we have the following development of g_b on N^{16t} ,

$$g_b \left(\frac{1}{t_p} g_{t, v}^A \right) = g_b(\bar{g}_{b, v_p}) + t_p \bar{P}_b(\hat{h}_S) + t_p Q_{g_b}^{(2)}(v_p, \hat{h}_S) + \mathcal{O}(t_p^2).$$

Up to the order t_p^2 there are three different sources of obstructions:

- (1) the projection of ${}_{g_b}(\bar{g}_{b,v_p}) = \mathcal{O}(|v_p|^2)$ on $\mathbf{O}(g_b)$,
- (2) the projection of $t_p P_b(\hat{h}_S) = \mathcal{O}(t_p)$ on $\mathbf{O}(g_b)$, and
- (3) the projection of $t_p Q_{g_b}^{(2)}(v_p, \hat{h}_S) = \mathcal{O}(t_p |v_p|)$ on $\mathbf{O}(g_b)$.

Hence, we can only "see" the obstructions which are $\mathcal{O}(t_p)$ if $\text{Ric}(\bar{g}_{b,v_p}) = 0$, or if $|v_p|^2 \ll t_p$.

By adapting the end of the proof of [Biq13, Proposition 3.1], we get the following useful result.

Corollary 9 Let H_2 be a quadratic symmetric 2-tensor satisfying $d_e \text{Ric}(H_2) = \Lambda g_e$, and let $V \sim r_e^3$ be a homogeneous vector field which satisfies $\delta_e \delta_e^* V = -\delta_e H_2$, and define $\hat{H}_S := H_2 + \delta_e^* V$ which satisfies $\delta_e \hat{H}_S = 0$. Then, we have

$$\hat{\lambda}_j = \lambda_j,$$

where

$$\hat{\lambda}_j := - \int_{\mathbb{S}^3/\Gamma} \left(3 \langle \hat{H}_S, O_j^4 \rangle_{g_e} + \frac{1}{2} O_j^4 (\nabla_e \text{tr}_e \hat{H}_S, \partial_{r_e}) \right) dv_{\mathbb{S}^3/\Gamma},$$

and

$$\lambda_j := - \int_{\mathbb{S}^3/\Gamma} \left(3 \langle H_2, O_j^4 \rangle_{g_e} + O_j^4 (B_e H_2, \partial_{r_e}) \right) dv_{\mathbb{S}^3/\Gamma}$$

5.4 Obstruction to the Gromov-Hausdorff desingularization

We can finally conclude that there are obstructions to the desingularization of some Einstein orbifolds.

Theorem 5.8 Let (M_o, g_o) be an Einstein orbifold, and $(M_i, g_i)_i$ a sequence of Einstein manifolds converging to (M_o, g_o) in the Gromov-Hausdorff sense and assume that, at a singular point $p \in M_o$, the possible non-flat blow-up limits are integrable Ricci-flat ALE manifolds (which implies that there are no tree of singularities forming).

Then, if we denote H_2 the quadratic terms of the development of g_o in geodesic coordinates at p , and $(O_j^4)_j$ the r_b^{-4} -terms of a basis of $\mathbf{O}(g_b)$, we have:

$$(104) \quad \int_{\mathbb{S}^3} \left(3 \langle H_2, O_j^4 \rangle + O_j^4 (B_e H_2, \partial_{r_e}) \right) dv_{\mathbb{S}^3} = 0.$$

Proof Let (M_o, g_o) be an Einstein orbifold, and assume that there exists a sequence of Einstein metrics $(M_i, g_i)_i$ converging to (M_o, g_o) in the Gromov-Hausdorff sense whose blow-ups at the singular point p satisfy the assumptions of the theorem. According to Corollary 3, for i large enough, there exists a naïve desingularization of (M_o, g_o) , (M, g_i^D) , $v_i \in \tilde{\mathbf{O}}(g_i^D)$, $t_i, v_i \rightarrow 0$ and a diffeomorphism $\phi_i : M \rightarrow M$ such that $\phi_i^* g_i = \hat{g}_{t_i, v_i}$ is the Einstein modulo obstructions perturbation of $g_i^D + v_i$ of Theorem 4.6. Let us fix p a singular point of M_o and denote S the rest of the singularities of M_o . Assume that no tree of singularities forms at p and denote (N, g_b) the Ricci-flat ALE manifold limit of blow-ups at p .

According to Proposition 14, there exists an approximation g_{t_i, v_i}^A satisfying

$$\left\| \pi_{\mathbf{O}(\tilde{g}_i^D)g_i^D}(g_{t_i, v_i}^A) \right\|_{r_D^{-2}C_\beta^\alpha(g_i^D)} \leq Ct_{p,i}^{\frac{3-\beta}{4}},$$

and consequently

$$(105) \quad \|t_{p,i} \sum_j \hat{\lambda}_j(t_i, v_i) \tilde{\mathbf{o}}_{t_i, v_i}^j\|_{L^2} \leq t_{p,i}^{\frac{5}{4}}$$

where the $(\tilde{\mathbf{o}}_{t_i, v_i}^j)_j$ form an L^2 -orthonormal basis of elements of $\tilde{\mathbf{O}}_{t_p}(\tilde{g}_{b, v_{p,i}})$, and

- (1) $\|g_{t_i, v_i}^A - g_{t_i}^D\|_{C_{\beta, *}^{2, \alpha}(g_i^D)} \leq 2\epsilon$,
- (2) $g_{t_i, v_i}^A - (g_{t_i}^D + v_i)$ is $L^2(g_i^D)$ -orthogonal to $\tilde{\mathbf{O}}(g_i^D)$.

The estimate (105) implies that we have

$$(106) \quad t_{p,i} |\hat{\lambda}_j(t_i, v_i)| \leq Ct_{p,i}^{\frac{5}{4}} \ll t_{p,i}.$$

Now, we know that $t_i, v_i \rightarrow 0$, and according to Lemma 5.7, this implies that the $\hat{\lambda}_j(t_i, v_i)$ converge to $\hat{\lambda}_j$. Since the $\hat{\lambda}_j$ are only constants depending on the geometry of (M_o, g_o) and that of (N, g_b) , they must necessarily vanish to satisfy the inequality (106) for t_i arbitrarily small. By coming back to the expression of $\hat{\lambda}_j$ of (86) we find the obstruction. We can finally extend it in geodesic coordinates (for example) to obtain (104) thanks to Corollary 9. \square

6 Obstructions for known singularity models

The description of the previous section allowed us to find obstructions to the desingularization of Einstein orbifolds by smooth Einstein manifolds. We will now test them on the known examples and first show that the obstruction to the desingularization

by gluing-perturbation of an Eguchi-Hanson metric of [Biq13] also holds for any Gromov-Hausdorff desingularization of a finite number of singularities by trees of Kähler ALE Ricci-flat orbifolds in Theorem 6.7. This is conjecturally the only possible way for Einstein metrics to degenerate in a noncollapsed setting.

In dimension 4, the 2-forms decompose into self-dual and anti self-dual 2-forms which are elements of the eigenspaces of Hodge star operator $*$ (which satisfies $*^2 = \text{Id}$) respectively associated to the eigenvalues 1 and -1 . Thanks to this direct sum, the symmetric endomorphism on 2-forms, \mathbf{R} given by the Riemannian curvature decomposes into blocks,

$$\mathbf{R} =: \begin{bmatrix} \mathbf{R}^+ & \text{Ric}^0 \\ \text{Ric}^0 & \mathbf{R}^- \end{bmatrix},$$

where the Ric^0 is the traceless part of the Ricci curvature, and where \mathbf{R}^\pm are the self-dual and anti self-dual parts of the curvature.

6.1 Kähler Ricci-flat ALE metrics and obstructions

The first obstructions to the desingularization of an Einstein orbifold (M_o, g_o) by a Ricci-flat ALE manifold (N, g_b) come from the infinitesimal deformations of g_b decaying as r_b^{-4} at infinity. We will show that for any Kähler Ricci-flat ALE orbifold, there is a common obstruction to the desingularization which was already found in the case of the gluing of an Eguchi-Hanson metric for a particular gluing-perturbation procedure in [Biq13]:

$$\det \mathbf{R}^+ = 0,$$

at the singular point. We will moreover see that *generically* (see Remark 37), there are additional obstructions corresponding to

$$\mathbf{R}^+ = 0.$$

Remark 36 If we glue the Kähler Ricci-flat ALE metrics with the opposite orientation, that is with with a gluing parameter in $O(4) \setminus SO(4)$ the common obstruction becomes $\det \mathbf{R}^- = 0$ and therefore in general, since the Einstein equation implies that the curvature is block diagonal ($\text{Ric}^0 = 0$), the obstruction is

$$\det \mathbf{R} = 0.$$

The only known examples of ALE Ricci-flat orbifolds are Kähler. They have been classified and we have a satisfactory parametrization of the moduli space of these quotients of hyperkähler (hence Ricci-flat) ALE metrics.

Let us precise what the deformations of these known Ricci-flat ALE orbifolds are, in order to extend the obstruction to the desingularization by any Kähler Ricci-flat ALE space.

Proposition 15 Let Γ be a finite subgroup of $SU(2)$, (N, g_b) a hyperkähler ALE manifold asymptotic to \mathbb{R}^4/Γ and $(O_i^4)_i$ a basis of the r_b^{-4} -terms of the elements of $\mathbf{O}(g_b)$.

Then, the conditions $\int_{\mathbb{S}^3} (3\langle H_2, O_i^4 \rangle_{g_e} + O_i^4(B_e H_2, \partial_{r_e})) d\nu_{\mathbb{S}^3} = 0$ for a quadratic symmetric 2-tensor H_2 satisfying $d_e \text{Ric } H_2 = \Lambda g_e$ imply that

$$\det \mathbf{R}_+(H_2) = 0,$$

where $\mathbf{R}_+(H_2)$ is the common selfdual part of the curvature of metrics with a development $g_e + H_2 + \mathcal{O}(r_e^3)$.

Proof Denote by (x, y, z, t) the coordinates in an orthonormal basis of \mathbb{R}^4 , and define a basis of invariant 1-forms on the sphere \mathbb{S}^3 , $(\alpha_1, \alpha_2, \alpha_3)$ by,

$$\begin{aligned} \alpha_1 &:= \frac{1}{r_e^2}(xdy - ydx + zdt - tdz) \\ \alpha_2 &:= \frac{1}{r_e^2}(xdz - zdx + ydt - tdy) \\ \alpha_3 &:= \frac{1}{r_e^2}(xdt - tdx + ydz - zdy). \end{aligned}$$

Manifestly, from Theorem 5.8 these obstructions do not particularly depend on the Eguchi-Hanson metric, but on the r_b^{-4} -terms of the development of its deformations, O_i^4 , which are, by [Biq13, (27)]:

$$\begin{aligned} (1) \quad O_1^4 &:= 2 \frac{dr_e^2 + r_e^2 \alpha_1^2 - r_e^2 \alpha_2^2 - r_e^2 \alpha_3^2}{r_e^4}, \\ (2) \quad O_2^4 &:= \frac{r_e^2 \alpha_1 \cdot \alpha_2 + r_e dr_e \cdot \alpha_3}{r_e^4}, \\ (3) \quad O_3^4 &:= \frac{r_e^2 \alpha_1 \cdot \alpha_3 - r_e dr_e \cdot \alpha_2}{r_e^4}. \end{aligned}$$

Let Γ be a finite subgroup of $SU(2)$. Then, according to [Kro89a], there exists $k_\Gamma \in \mathbb{N}^*$ and D_Γ , a finite union of vector subspaces of \mathbb{R}^{3k_Γ} of codimension at least 3 containing 0 such that the set of smooth hyperkähler metrics asymptotic to \mathbb{R}^4/Γ is parametrized as $(X_\zeta, g_\zeta)_{\zeta \in \mathbb{R}^{3k_\Gamma} \setminus D_\Gamma}$. Moreover, by [Auv18, Theorem 2.1], for each $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^{3k_\Gamma} \setminus D_\Gamma$, there exists a diffeomorphism Φ_ζ from a neighborhood of

the infinity of \mathbb{R}^4/Γ to the infinity of X_ζ such that $\Phi_\zeta^*g_\zeta = g_e + h_\zeta + \mathcal{O}(r_e^{-6})$, where,

$$\begin{aligned} h_\zeta = & - \sum_{j,k,l} |\zeta_j|^2 \frac{dr_e^2 + r_e^2 \alpha_j^2 - r_e^2 \alpha_k^2 - r_e^2 \alpha_l^2}{r_e^4} \\ & - \langle \zeta_1, \zeta_2 \rangle \frac{r_e^2 \alpha_1 \cdot \alpha_2 - r_e dr_e \cdot \alpha_3}{r_e^4} \\ & - \langle \zeta_1, \zeta_3 \rangle \frac{r_e^2 \alpha_1 \cdot \alpha_3 + r_e dr_e \cdot \alpha_2}{r_e^4} \\ & - \langle \zeta_2, \zeta_3 \rangle \frac{r_e^2 \alpha_2 \cdot \alpha_3 - r_e dr_e \cdot \alpha_1}{r_e^4}, \end{aligned}$$

where the first sum is taken on the (j, k, l) satisfying $l \equiv k + 1 \equiv j + 2 \pmod{3}$.

Without loss of generality, we can assume that the first coordinates of ζ_1, ζ_2 and $\zeta_3 \in \mathbb{R}^{k_\Gamma}$ are 1, 0 and 0. Indeed, there exists $l \in \{1, \dots, k_\Gamma\}$ such that the l -th coordinate of $(\zeta_1, \zeta_2, \zeta_3), (\zeta_1^l, \zeta_2^l, \zeta_3^l)$ does not vanish. Just like for the homothetic deformations of the Eguchi-Hanson metric, thanks to an action of $SO(4)$ on the asymptotics of the metric and a homothetic transformation (which yields an action of $SO(3)$ and a common rescaling on all the ζ^l), we are able to reach another metric $g_{\tilde{\zeta}}$ with $(\tilde{\zeta}_1^l, \tilde{\zeta}_2^l, \tilde{\zeta}_3^l) = (1, 0, 0)$.

By differentiating the above expression of h_ζ , we see that the infinitesimal variations associated to the variations of $(\tilde{\zeta}_1^l, \tilde{\zeta}_2^l, \tilde{\zeta}_3^l)$ are then asymptotic to $\mathcal{O}_1^4, \mathcal{O}_2^4$ and \mathcal{O}_3^4 . The obstructions they induce by Theorem 5.8 are therefore the same as for the Eguchi-Hanson metric, and by [Biq13, Theorem 4.1] they imply the condition $\det \mathbf{R}_+ = 0$ which is independent of the above actions of $SO(4)$ and scaling. \square

Remark 37 The case of the Eguchi-Hanson metric, or when the ζ^l are parallel, is actually the least obstructed case, and by the formula [Biq13, (38)] the obstruction condition is $\mathbf{R}_+ = 0$ for ζ generic when $k_\Gamma > 1$.

We find the same obstruction for Kähler Ricci-flat ALE orbifolds which are all asymptotic to $\mathbb{R}^4/\Gamma \sim \mathbb{C}^2/\Gamma$ for a group $\Gamma = \frac{1}{dn^2}(1, dnm - 1) \subset U(2)$, that is the cyclic group generated by

$$\begin{bmatrix} e^{\frac{i2\pi}{dn^2}} & 0 \\ 0 & e^{\frac{i2\pi(dnm-1)}{dn^2}} \end{bmatrix},$$

where $d \geq 1, n \geq 2$ and, n and m are mutually prime.

Corollary 10 Let Γ be a group $\frac{1}{dn^2}(1, dnm - 1) \subset U(2)$ for $d \geq 1$, $n \geq 2$ and n and m mutually prime, and let (N, g_b) be a Kähler Ricci-flat ALE metric asymptotic to \mathbb{R}^4/Γ , and $(O_i^A)_i$ a basis of the r_b^{-4} -terms of the elements of $\mathbf{O}(g_b)$.

Then, for a quadratic symmetric 2-tensor H_2 such that $d_e \text{Ric } H_2 = \Lambda g_e$, for $i \in \{1, 2, 3\}$ we have the conditions

$$\int_{\mathbb{S}^3} (3\langle H_2, O_i^A \rangle_{g_e} + O_i^A(B_e H_2, \partial_{r_e})) dv_{\mathbb{S}^3} = 0$$

which imply that

$$\det \mathbf{R}_+(H_2) = 0.$$

Proof Let $\frac{1}{dn^2}(1, dnm - 1) \subset U(2)$ be a finite subgroup of $U(2)$, and (N, g_b) a non flat Kähler Ricci-flat ALE orbifold asymptotic to \mathbb{R}^4/Γ . According to [Suv11], (\tilde{N}, \tilde{g}_b) the universal cover of (N, g_b) is a hyperkähler orbifold asymptotic to $\tilde{\Gamma} = \frac{1}{dn}(1, -1) \subset \frac{1}{dn^2}(1, dnm - 1)$. Let H_2 be a quadratic symmetric 2-tensor on \mathbb{R}^4/Γ , we can also lift it to $\mathbb{R}^4/\tilde{\Gamma}$ as \tilde{H}_2 .

Let us come back to the origin of the obstruction in Proposition 13, and more precisely the existence of a symmetric 2-tensor h_2 asymptotic to H_2 such that $d_{g_b} \text{Ric}(h_2) = \Lambda g_b$. If such a symmetric 2-tensor exists on (N, g_b) , we can lift it as a symmetric 2-tensor \tilde{h}_2 on (\tilde{N}, \tilde{g}_b) asymptotic to \tilde{H}_2 and satisfying $d_{\tilde{g}_b} \text{Ric}(\tilde{h}_2) = \Lambda \tilde{g}_b$ which implies, again according to Proposition 13 and to Proposition 15, the condition $\det \mathbf{R}_+(\tilde{H}_2) = 0$, and finally $\det \mathbf{R}_+(H_2) = 0$. \square

Remark 38 The above proof (or any proof in this article really) does not use the Kähler nature of the studied metric. It would also apply to any Ricci-flat ALE metric whose order r_e^{-4} terms do not vanish [BH19, Proposition 2.5]. It is however not clear if these terms can vanish (in well chosen coordinates) on a non flat Ricci-flat ALE metric.

6.2 Trees of Kähler Ricci-flat ALE orbifolds and obstructions

Let us now treat the case of trees of ALE Kähler Ricci-flat orbifolds.

6.2.1 Uniform controls of gravitational instantons

Let us use the notations of the proof of Proposition 15. Let $\Gamma_1 \in SU(2)$ and $\zeta_1 \in D_{\Gamma_1}$ fixed such that the ALE orbifold $(X_{\zeta_1}, g_{\zeta_1})$ has a singularity \mathbb{R}^4/Γ for $\Gamma \subset SU(2)$. For $v \in \mathbf{O}(g_{\zeta_1})$ which we will choose small, let us then denote $\tilde{g}_{\zeta_1, v}$ the Ricci-flat (and even

hyperkähler) of g_{ζ_1} given by Definition 5.1. The goal of this section is to show that there exists a common scale $\tau > 0$ such that for any $\zeta \in \mathbb{R}^{3k_\Gamma}$ with $|\zeta| = 1$, for any $t < \tau$, the naïve gluing of g_ζ to $\bar{g}_{\zeta_1, v}$ at scale t , denoted $g_{v, t, \zeta}^B$ can be perturbed into a Ricci-flat modulo obstructions metric which we will denote $\hat{g}_{v, t, \zeta}$.

The naïve gluing $g_{v, t, \zeta}^B$ is Ricci-flat everywhere except on the annulus where the gluing takes place. We therefore have the control

$$(107) \quad \|g_{v, t, \zeta}^B(g_{v, t, \zeta}^B)\|_{r_B^{-2}C_{\beta}^{\alpha}(g_{v, t, \zeta}^B)} \leq Ct^{\frac{2-\beta}{4}}$$

where $C > 0$ only depends on ζ_1 and v because g_ζ for $|\zeta| = 1$ is controlled at infinity in the ALE coordinates of [Kro89a, Corollaire 3.14].

Going back to the proof of Theorem 4.6 by inverse function theorem, in order to perturb $g_{v, t, \zeta}^B$ to a Ricci-flat modulo obstructions metric to show the existence of $\hat{g}_{v, t, \zeta}$, it would be enough to show that the linearization of $\pi_{\tilde{\mathbf{O}}(g_{v, t, \zeta}^B)^\perp g_{v, t, \zeta}^B}$,

$$\pi_{\tilde{\mathbf{O}}(g_{v, t, \zeta}^B)^\perp} P_{g_{v, t, \zeta}^B} : \tilde{\mathbf{O}}(g_{v, t, \zeta}^B)^\perp \cap C_{\beta, *}^{2, \alpha}(g_{v, t, \zeta}^B) \rightarrow \tilde{\mathbf{O}}(g_{v, t, \zeta}^B)^\perp \cap r_B^{-2}C_{\beta}^{\alpha}(g_{v, t, \zeta}^B)$$

is uniformly bounded and has a uniformly bounded inverse for $|\zeta| = 1$ and t small enough and to control the nonlinear terms of $g_{v, t, \zeta}^B$. Fixing ζ_1 and v , It would therefore be enough thanks to Proposition 11 to show that

$$P_{g_\zeta} : \mathbf{O}(g_\zeta)^\perp \cap C_{\beta, *}^{2, \alpha}(g_\zeta) \rightarrow \mathbf{O}(g_\zeta)^\perp \cap r_\zeta^{-2}C_{\beta}^{\alpha}(g_\zeta)$$

is invertible with bounded inverse. This is however not the case when $\zeta \rightarrow D_\Gamma$, that is when g_ζ degenerates to an orbifold. In this situation, we replace the norms with respect to g_ζ by norms with respect to a naïve desingularization g_ζ^B (with additional decay in the neck regions) close to g_ζ in order to keep a uniform gluing scale.

Proposition 16 For any $\Gamma \subset SU(2)$ and $\epsilon > 0$, there exists $C = C(\Gamma, \epsilon) > 1$ such that for any $\zeta \in \mathbb{R}^{3k_\Gamma}$ with $|\zeta| = 1$, there exists a naïve desingularization (partial if $\zeta \in D_\Gamma$ and total if $\zeta \notin D_\Gamma$, see Definition 1.6) g_ζ^B of an orbifold g_{ζ_o} with $\zeta_o \in \mathbb{R}^{3k_\Gamma}$ and $|\zeta_o| = 1$ by hyperkähler orbifolds g_{ζ_j} for $|\zeta_j| = 1$ and $\zeta_j \in \mathbb{R}^{k_{\Gamma_j}}$ with $|\Gamma_j| < |\Gamma|$, for which, denoting

$$R(g_\zeta^B) := \sup_{h \in \tilde{\mathbf{O}}(g^B)^\perp \cap C_{\beta, *}^{2, \alpha}(g^B)} \frac{\|h\|_{C_{\beta, *}^{2, \alpha}(g^B)}}{\|P_{g^B} h\|_{r_B^{-2}C_{\beta}^{\alpha}(g^B)}} + \sup_{h \in C_{\beta, *}^{2, \alpha}(g^B)} \frac{\|P_{g^B} h\|_{r_B^{-2}C_{\beta}^{\alpha}(g^B)}}{\|h\|_{C_{\beta, *}^{2, \alpha}(g^B)}},$$

we have

$$(1) \quad \|g_\zeta^B - g_\zeta\|_{C_{\beta, *}^{2, \alpha}(g_\zeta^B)} < \frac{\epsilon}{R(g_\zeta^B)},$$

$$(2) \quad R(g_\zeta^B) \leq C.$$

Remark 39 The norms in the above statement are the naïve desingularizations norms (Definition 1.6) with respect to g_ζ^B . They are needed in order to obtain uniform estimates.

Remark 40 What is crucial in this statement is the fact that the constant C is independent of the metric g_ζ and in particular allows ζ to approach D_Γ with uniform controls. The constant ϵ essentially decides from which proximity we replace the metric g_ζ by a naïve desingularization. Indeed, the trivial naïve desingularization $g_\zeta^B = g_\zeta$ always satisfies the first property of the conclusion.

Proof Let $\epsilon > 0$ and let us show the result by induction on the order k of the group at infinity. It holds for \mathbb{Z}_2 of order 2 because all the metrics g_ζ for $\zeta \in \mathbb{R}^3$ are isometric for $|\zeta| = 1$, and we can take $g_\zeta^B = g_\zeta$.

Assume now that the conclusion is satisfied for any group of order less than or equal to $k - 1 \geq 2$ and consider Γ of order k .

Let us again work by induction, this time on the value of the square of the L^2 -norm of the curvature of the ALE orbifolds asymptotic to \mathbb{R}^4/Γ . This quantity is proportional to the dimension of the (orbifold) L^2 cohomology in degree 2 for these gravitational instantons.

Consider, the orbifolds g_ζ , $\zeta \in \mathbb{R}^{3k_\Gamma}$, $|\zeta| = 1$ which have the smallest energy. Assume by contradiction that there exists a sequence of orbifolds g_{ζ_i} , $|\zeta_i| = 1$ with the minimal energy such that for any naïve desingularization (Definition 1.6) g^B of a hyperkähler orbifold asymptotic to \mathbb{R}^4/Γ , we have

- (1) either, $\|g_{\zeta_i} - g^B\|_{C_{\beta, *}(g^B)}^{2, \alpha} \geq \frac{\epsilon}{R(g^B)}$,
- (2) either, $R(g^B) > i$.

In the coordinates of [Kro89a, Proposition 3.14], all of the metrics g_{ζ_i} admit coordinates of order 4 with a uniform control in $r_{\zeta_i}^{-4} C^3(g_{\zeta_i})$ in a uniform neighborhood of infinity, where r_{ζ_i} is the function of Definition 1.4. Therefore, there can only be L^2 -concentration of the curvature in a compact of diameter uniformly bounded and volume uniformly bounded below by Bishop-Gromov inequality. By minimality of the L^2 -norm of the curvature, no tree of singularities can form with $|\zeta_i| = 1$ because the limit orbifold would then have a smaller L^2 -norm of the curvature. Therefore, there exists g_{ζ_∞} such that the g_{ζ_i} converge smoothly (considering local covering at the singular

points) to g_{ζ_∞} . Since the asymptotic terms converge by [Kro89a, Proposition 3.14], we have $\|g_{\zeta_i} - g_{\zeta_\infty}\|_{C_{\beta,*}^{2,\alpha}(g_{\zeta_\infty})} \rightarrow 0$ when $i \rightarrow \infty$ where we have $\|\cdot\|_{C_{\beta,*}^{2,\alpha}(g_{\zeta_\infty})} \approx \|(1 + r_{\zeta_\infty})^\beta \cdot\|_{C_0^{2,\alpha}(g_{\zeta_\infty})}$ here since we use local coverings at the singularities instead of weights at the singularities of g_{ζ_∞} . The trivial naïve desingularization $g^B = g_{\zeta_\infty}$ therefore contradicts the assumptions. Indeed, the convergence happens in $C_{\beta,*}^{2,\alpha}(g_{\zeta_\infty})$ and since $P_{g_{\zeta_\infty}}$ is elliptic, there exists $C > 0$ such that for any $h \perp \mathbf{O}(g_{\zeta_\infty})$, we have

$$\|h\|_{C_{\beta,*}^{2,\alpha}(g_{\zeta_\infty})} \leq C \|P_{g_{\zeta_\infty}} h\|_{r_{\zeta_\infty}^{-2} C_{\beta}^\alpha(g_{\zeta_\infty})}.$$

Consider then a higher value for the L^2 -norm of the curvature $E > 0$ and assume that our property holds for the hyperkähler orbifolds asymptotic to \mathbb{R}^4/Γ whose L^2 -norm of the curvature is strictly less than E . Let us assume by contradiction that there exists a sequence $\zeta_i \in \mathbb{R}^{3k_\Gamma}$ $|\zeta_i| = 1$ satisfying $\|\text{Rm}_{g_{\zeta_i}}\|_{L^2(g_{\zeta_i})} = E$ for which for any naïve desingularization g^B of a hyperkähler orbifold asymptotic to \mathbb{R}^4/Γ , we have

- either, $\|g_{\zeta_i} - g^B\|_{C_{\beta,*}^{2,\alpha}(g^B)} \geq \frac{\epsilon}{R(g^B)}$,
- or, $R(g^B) > i$.

Like in the above argument for the minimal energy, if no singularity was forming, up to taking a subsequence, we would have a $C_{\beta,*}^{2,\alpha}(g_{\zeta_\infty})$ -convergence to a orbifold g_{ζ_∞} with $R(g_{\zeta_\infty})$ bounded. A tree of singularities is therefore forming. More precisely, up to taking a subsequence, there exists a subsequence naïve desingularizations $g_{\zeta_i}^{B'}$ of a hyperkähler orbifold asymptotic to \mathbb{R}^4/Γ , composed of orbifolds g_{ζ_o} with $|\zeta_o| = 0$ asymptotic to \mathbb{R}^4/Γ and g_{ζ_j} with $|\zeta_j| = 1$ (up to changing the scale of the gluing, we can always assume that since $g_{t\zeta}$ is isometric to tg_ζ), $\zeta_j \in \mathbb{R}^{3k_\Gamma j}$, at scales $t_{i,j}$, for which we have $\|g_{\zeta_i} - g_{\zeta_i}^{B'}\|_{C_{\beta,*}^{2,\alpha}} \rightarrow 0$ as $i \rightarrow \infty$. Moreover, all of the metrics g_{ζ_o} and g_{ζ_j} are hyperkähler according to [Ban90]. The L^2 -norm of the curvature of g_{ζ_o} is strictly inferior to E since some of the total L^2 -norm is lost in the singularities by [And89, BKN89], and we have $|\Gamma_j| < |\Gamma| = k$ by Bishop-Gromov inequality. Up to replacing g_{ζ_o} and the g_{ζ_j} by the naïve desingularizations $g_{\zeta_o}^B$ and $g_{\zeta_j}^B$ of the previous steps of our inductions, we obtain a naïve desingularization $g_{\zeta_i}^B$ for which we uniformly (depending on the constants of the previous steps of the induction only) control the operator P . Thanks to Lemma 4.3, we obtain a uniform control on the inverse of the operator $P_{g_{\zeta_i}^B}$. This contradicts the initial assumptions and proves the statement. \square

The following lemma lets us approximate the kernel of the operator P_{g_ζ} thanks to the approximate kernel $\tilde{\mathbf{O}}(g_\zeta^B)$.

Lemma 6.1 *Let P and P' be two operators between Banach spaces X and Y for which there exists $C > 0$, $\frac{1}{100C} > \epsilon > 0$, a finite dimensional space $K' \subset X$ and S' a complement of K' in X such that we have:*

(1) *for any $x \in X$,*

$$\|(P - P')x\|_Y \leq \epsilon \|x\|_X,$$

(2) *for any $x \in S'$,*

$$\|x\|_X \leq C \|P'x\|_Y,$$

(3) *for any $x \in K'$,*

$$\|P'x\|_Y \leq \epsilon \|x\|_X,$$

(4) *and $\dim(\ker P) = \dim(K')$.*

Then, for any $k \in \ker P$, there exists $k' \in K'$ such that

$$(108) \quad \|k - k'\|_X \leq \frac{2C\epsilon}{1 - C\epsilon} \|k'\|_X.$$

Proof Let $k \in \ker P$, and consider its decomposition $k = k' + s'$ in the direct sum $X = K' \oplus S'$. Thanks to the first hypothesis, we have

$$(109) \quad \|P'k\|_Y \leq \epsilon \|k\|_X,$$

thanks to the second,

$$(110) \quad \|s'\|_X \leq C \|P's'\|_Y$$

and thanks to the third one, we have

$$(111) \quad \|P'k'\|_Y \leq \epsilon \|k'\|_X.$$

Putting (109), (110) and (111) together, we find

$$(112) \quad \epsilon \|k\|_X \geq \|P's'\|_Y - \|P'k'\|_Y \geq \frac{\|s'\|_X}{C} - \epsilon \|k'\|_X,$$

hence, since $k = k' + s'$,

$$\|s'\|_X \leq C\epsilon(\|k\|_X + \|k'\|_X) \leq 2C\epsilon\|k'\|_X + C\epsilon\|s'\|_X,$$

and finally

$$\|k - k'\|_X = \|s'\|_X \leq \frac{2C\epsilon}{1 - C\epsilon} \|k'\|_X.$$

□

Remark 41 With the three first assumptions of Lemma 6.1, we still have

$$\dim \ker P \leq \dim K',$$

and any element of $\ker P$ is close to an element of K' (its projection on K' parallel to S') in the sens of (108).

We conclude from Lemma 6.1 and the estimates 1 and 2 of Proposition 16 that for any $\Gamma \subset SU(2)$ and any $\epsilon > 0$ small enough depending on Γ only, for any $\zeta \in \mathbb{R}^{3k_\Gamma}$ and for the metric g_ζ^B obtained by Proposition 16, we have the following control for $C = C(\Gamma, \epsilon) > 0$: for any $h \perp \mathbf{O}(g_\zeta)$,

$$(113) \quad \frac{1}{C} \|h\|_{C_{\beta,*}^{2,\alpha}(g_\zeta^B)} < \|P_{g_\zeta} h\|_{r_B^{-2} C_{\beta}^{\alpha}(g_\zeta^B)} < C \|h\|_{C_{\beta,*}^{2,\alpha}(g_\zeta^B)}.$$

We therefore conclude that we can define Ricci-flat modulo obstructions perturbations of our naïve desingularizations.

Corollary 11 Let $(X_{\zeta_1}, g_{\zeta_1})$ be a Ricci-flat orbifold for $\zeta_1 \in D_{\Gamma_1}$ having a singularity \mathbb{R}^4/Γ , $\Gamma \subset SU(2)$. Then, there exists $\epsilon > 0$ and $\tau > 0$ such that for any $0 < t < \tau$, $\zeta \in \mathbb{R}^{3k_\Gamma}$ with $|\zeta| = 1$ and $v \in \mathbf{O}(g_{\zeta_1})$ with $\|v\|_{C_{\beta,*}^{2,\alpha}(g_{\zeta_1})} < \tau$, if we denote

- $g_{v,t,\zeta}^{BB}$ the naïve gluing of the metric (X_ζ, g_ζ^B) , of Proposition 16 for the constant ϵ , at the singularity \mathbb{R}^4/Γ of $\bar{g}_{\zeta_1,v}$ (defined above), and
- $g_{v,t,\zeta}^B$ the naïve gluing of (X_ζ, g_ζ) at the scale t ,

there exists a unique metric $\hat{g}_{v,t,\zeta}$ satisfying

- $g_{v,t,\zeta}^B - \hat{g}_{v,t,\zeta} \perp \tilde{\mathbf{O}}(g_{v,t,\zeta}^B)$,
- $\|g_{v,t,\zeta}^B - \hat{g}_{v,t,\zeta}\|_{C_{\beta,*}^{2,\alpha}(g_{v,t,\zeta}^{BB})} \leq 2\tau$ (notice the naïve desingularization norm $g_{v,t,\zeta}^{BB}$), and
- $g_{v,t,\zeta}^B(\hat{g}_{v,t,\zeta}) \in \tilde{\mathbf{O}}(g_{v,t,\zeta}^B)$.

Moreover, the metric $\hat{g}_{v,t,\zeta}$ depends smoothly on v , t and ζ .

Proof As discussed at the beginning of this section, for the naïve gluing $g_{v,t,\zeta}^B$, we have a control

$$\|g_{v,t,\zeta}^B(g_{v,t,\zeta}^B)\|_{r_B^{-2} C_{\beta}^{\alpha}(g_{v,t,\zeta}^{BB})} \leq C t^{\frac{2-\beta}{4}}$$

where $C > 0$ only depends on ζ_1 . Now, using the control (113) to replace the controls of Lemma 4.3, we adapt the proof of Proposition 11 to show that for τ small enough depending on Γ and ζ_1 alone, the operator

$$\pi_{\tilde{\mathbf{O}}(g_{v,t,\zeta}^B)^\perp} P_{g_{v,t,\zeta}^B} : \tilde{\mathbf{O}}(g_{v,t,\zeta}^B)^\perp \cap C_{\beta,*}^{2,\alpha}(g_{v,t,\zeta}^{BB}) \rightarrow \tilde{\mathbf{O}}(g_{v,t,\zeta}^B)^\perp \cap r_B^{-2} C_{\beta}^{\alpha}(g_{v,t,\zeta}^{BB}),$$

is uniformly bounded (independently of ζ and of t and ν small enough) and invertible with a uniformly bounded inverse. Notice the norms $g_{\nu,t,\zeta}^{BB}$ again. The proof of Theorem 4.6 by inverse function theorem extends here, and there exists a unique solution $\hat{g}_{\nu,t,\zeta}$ to the equations of the statement.

Like in Corollary 4, the smooth dependence of the metric is a consequence of the implicit function theorem applied to

$$\Psi : (\nu, t, \zeta, h) \mapsto \pi_{\tilde{\mathbf{O}}(g_{\nu,t,\zeta}^B)^\perp} (g_{\nu,t,\zeta}^B + h)$$

for $h \perp \tilde{\mathbf{O}}(g_{\nu,t,\zeta}^B)$. □

6.2.2 Gluing-perturbation of gravitational instantons

Let us first make sure that all of the gluing gauges given by the isometries of \mathbb{R}^4/Γ (see Remark 3) are equivalent to gluing a gravitational instanton but with a different parameter ζ .

Lemma 6.2 *Let (M_o, g_o) be an orbifold (compact or ALE) with a singularity \mathbb{R}^4/Γ at p with $\Gamma \subset SU(2)$, and let $\zeta \in \mathbb{R}^{3k_\Gamma}$. Let moreover ψ be an isometry of \mathbb{R}^4/Γ preserving the orientation.*

Then, there exists $\zeta' \in \mathbb{R}^{3k_\Gamma}$ such that the naïve gluing of (X_ζ, g_ζ) at p with the ALE coordinates of [Kro89a, Proposition 3.14] composed with ψ when identifying with the coordinates of (M_o, g_o) (see Remark 3) is isometric to the gluing of $(X_{\zeta'}, g_{\zeta'})$ to (M_o, g_o) at p without composing with an isometry.

Proof Let $\Gamma \subset SU(2)$ be a finite subgroup. Having $\psi \in \text{Isom}(\mathbb{R}^4/\Gamma)$ implies that ψ is in the normalizer of Γ in $SO(4)$ and therefore that $\psi \in \phi(\mathbb{S}^3 \times N_\Gamma)$, where N_Γ is the normalizer of Γ in $SU(2)$ (see [McC02] for an explicit expression) and where $\phi : \mathbb{S}^3 \times \mathbb{S}^3 \rightarrow SO(4)$ is the double covering of $SO(4)$ where the first \mathbb{S}^3 is the left multiplication by a unit quaternion and the second a right multiplication. In this identification, we have $SU(2) = \phi(\{1\} \times \mathbb{S}^3)$ and we will denote $\overline{SU(2)} = \phi(\mathbb{S}^3 \times \{1\})$.

Let us now study the action of the normalizer of Γ , which is $\phi(\mathbb{S}^3 \times N_\Gamma)$, on a metric of [Kro89a] in the coordinates of [Kro89a, Proposition 3.14]. Let us come back from their construction in [Kro89a] starting with

$$P := \mathbb{C}^2 \otimes \text{End}(R),$$

where \mathbb{C}^2 is the standard representation of $SU(2)$ and where R is its regular representation $SU(2)$. Denote P^Γ the set of Γ -invariant elements of P , F the set of elements commuting with Γ seen as a subset of the unitary transformations of R , and finally denote T the subgroup of complex numbers of unit norm acting trivially. The gravitational instantons asymptotic to \mathbb{C}^2/Γ are the hyperkähler quotients of P^Γ by F/T .

By definition, the normalizer $\phi(\mathbb{S}^3 \times N_\Gamma)$ acts on P^Γ and commutes to the action of F . It consequently acts on the set of solutions of the moment map $\mu : P^\Gamma \rightarrow (\mathfrak{f}/\mathfrak{t})^* \otimes \mathbb{R}^3$ where \mathfrak{f} and \mathfrak{t} are the Lie algebras of F and T respectively. More precisely, denoting I_k , $k \in \{1, 2, 3\}$ the 3 complex structures of P^Γ given by the identification of \mathbb{C}^2 with the quaternions, for all $\xi \in \mathfrak{f}/\mathfrak{t}$, the three coordinates of μ satisfy the equations

$$\text{grad}(\mu_k \cdot \xi) = I_k(V_\xi),$$

where V_ξ is the vector field on P^Γ generated by the action of ξ . The $SU(2)$ in which Γ and N_Γ commutes to the three complex structures since it is identified to the *right* multiplication by a unit-norm quaternion. We conclude that N_Γ acts by isometry on the hyperkähler metric $\mu^{-1}(\zeta)/(F/T)$ for all $\zeta \in (\mathfrak{f}/\mathfrak{t})^* \otimes \mathbb{R}^3$. The part $\overline{SU(2)}$ of the normalizer, acts by rotation on the three complex structures. And more precisely, an element $n_- \in \overline{SU(2)}$ sends $\mu^{-1}(\zeta)/(F/T)$ to $\mu^{-1}(Ad(n_-)\zeta)/(F/T)$ where $Ad(n_-)$ is the standard action of $SU(2)/\pm \approx SO(3)$ on the factor \mathbb{R}^3 of $(\mathfrak{f}/\mathfrak{t})^* \otimes \mathbb{R}^3$.

To conclude there remains to ensure that this action of the normalizer is the standard action on the asymptotic cone \mathbb{R}^4/Γ in the coordinates of [Kro89a, Proposition 3.14]. To prove this, we use the identification $\mathbb{R}^4/\Gamma \approx \mu^{-1}(0)/(F/T)$ of [Kro89a, Corollary 3.2]. The correspondence between the infinities of $\mu^{-1}(\zeta)/(F/T)$ and $\mu^{-1}(0)/(F/T)$ for the coordinates of [Kro89a, Proposition 3.14] given by [Kro89a, (3.13)] lets us conclude that the action of the normalizer is indeed the standard action on the asymptotic cone \mathbb{R}^4/Γ .

Therefore, the gluing $(X_\zeta, g_\zeta) \approx \mu^{-1}(\zeta)/(F/T)$ for $\zeta \in \mathbb{R}^{3k_\Gamma} \approx (\mathfrak{f}/\mathfrak{t})^* \otimes \mathbb{R}^3$ composed with the isometry $\psi = \phi(\psi_-, \psi_+)$ of \mathbb{R}^4/Γ is isometric to the gluing of $(X_{\zeta'}, g_{\zeta'})$ with $\zeta' = Ad(\psi_-)\zeta$. \square

Let us then remark that any ALE hyperkähler orbifold can be desingularized by a sequence of ALE hyperkähler manifolds.

Lemma 6.3 ([Ban90]) *Let $(X_{\zeta_1}, g_{\zeta_1})$ be an ALE hyperkähler orbifold with a singularity \mathbb{R}^4/Γ . Then, for any $t > 0$, there exists $\zeta_0^t \in \mathbb{R}^{3k_\Gamma} \setminus D_\Gamma$ and $v_t \in \mathbf{O}(g_{\zeta_1})$ satisfying $\lim_{t \rightarrow 0} v_t = 0$ and $\lim_{t \rightarrow 0} X_{\zeta_0^t} = X_{\zeta_0} \in \mathbb{R}^{3k_\Gamma} \setminus D_\Gamma$, such that the Einstein modulo obstructions desingularization $(X_{\zeta_1} \# X_{\zeta_0}, \hat{g}_{v_t, t, \zeta_0^t})$ of Corollary 11 is hyperkähler.*

Proof The result of [Ban90, Theorem 4] shows that for t small enough, there exists ζ_0 for which there is a hyperkähler metric g_t satisfying

$$\|g_t - g_{0,t,\zeta_0}^B\|_{(1+r_B)^{-\beta} C_{0,*}^{2,\alpha}(g_{0,t,\zeta_0}^B)} \leq \gamma(t),$$

where $\lim_{t \rightarrow 0} \gamma(t) = 0$. We can then apply our construction of coordinates of [Ozu19a] to $(X_{\zeta_1} \# X_{\zeta_0}, g_t)$ in order to obtain a diffeomorphism $\phi : X_{\zeta_1} \# X_{\zeta_0} \rightarrow X_{\zeta_1} \# X_{\zeta_0}$ thanks to which we have the better control

$$\|\phi^* g_t - g_{0,t,\zeta_0}^B\|_{C_{\beta,*}^{2,\alpha}(g_{0,t,\zeta_0}^B)} \leq C\gamma(t).$$

Thanks to Proposition 10, there exists a small diffeomorphism $\psi : X_{\zeta_1} \# X_{\zeta_0} \rightarrow X_{\zeta_1} \# X_{\zeta_0}$ such that $\tilde{\delta}_{g_{0,t,\zeta_0}^B} \psi^* \phi^* g_t = 0$. The uniqueness of Theorem 4.6 and the fact that the infinitesimal deformations of $(X_{\zeta_0}, g_{\zeta_0})$ integrate to curves $t \mapsto (X_{\zeta_0(t)}, g_{\zeta_0(t)})$ by the classification of [Kro89a, Kro89b] ensures that there exists $v_t \in \tilde{\mathbf{O}}(g_{0,t,\zeta_0}^B)$ and ζ_0^t such that we have

$$\psi^*(\phi^* g_t) = \hat{g}_{v_t, t, \zeta_0^t},$$

with $\hat{g}_{v_t, t, \zeta_0^t}$ one of the metrics of Theorem 4.6. \square

Remark 42 Note that [HV20, Theorem 6.2] provides such gluings-perturbations in all directions $\zeta \in \mathbb{R}^{3k_\Gamma} \setminus D_\Gamma$. The admissible scales are however not independent on $|\zeta| = 1$, and arbitrary trees of singularities are not allowed.

Proposition 17 There exists $\tau > 0$ such that for all

$$(v, t, \zeta) \in (B_{C_{\beta,*}^{2,\alpha}(g_{\zeta_1})}(0, \tau) \cap \mathbf{O}(g_{\zeta_1})) \times (0, \tau) \times (\mathbb{S}^{3k_\Gamma-1} \setminus D_\Gamma),$$

the metric $\hat{g}_{v,t,\zeta}$ of Proposition 11 is isometric to a hyperkähler metric of [Kro89a].

Proof Let us start by noting that the set

$$(B_{C_{\beta,*}^{2,\alpha}(g_{\zeta_1})}(0, \tau) \cap \mathbf{O}(g_{\zeta_1})) \times (0, \tau) \times (\mathbb{S}^{3k_\Gamma-1} \setminus D_\Gamma)$$

is connected since D_Γ is a finite union of spaces of codimension at least 2 in $\mathbb{S}^{3k_\Gamma-1}$. We therefore just need to prove that the set

$$E = \{(v, t, \zeta) \in (B_{C_{\beta,*}^{2,\alpha}(g_{\zeta_1})}(0, \tau) \cap \mathbf{O}(g_{\zeta_1})) \times (0, \tau) \times (\mathbb{S}^{3k_\Gamma-1} \setminus D_\Gamma), \text{Ric}(\hat{g}_{v,t,\zeta}) = 0\},$$

which is non empty by Lemma 6.3 is open and closed. It is therefore isometric to $g_{\xi(v,t,\zeta)}$ for $\xi(v, t, \zeta) \in \mathbb{R}^{3k_\Gamma} \setminus D_{\Gamma_1}$.

The set E is closed by the continuity of

$$(v, t, \zeta) \in (B_{C_{\beta,*}^{2,\alpha}(g_{\zeta_1})}(0, \tau) \cap \mathbf{O}(g_{\zeta_1})) \times (0, \tau) \times (\mathbb{S}^{3k_\Gamma-1} \setminus D_\Gamma) \mapsto \text{Ric}(\hat{g}_{v,t,\zeta})$$

proven in Corollary 11.

For the openness, let us assume that for (v, t, ζ) fixed, we have $\text{Ric}(\hat{g}_{v,t,\zeta}) = 0$. The metric $\hat{g}_{v,t,\zeta}$ is therefore one of the metrics of [Kro89a]. There exists a space of the same dimension as $\tilde{\mathbf{O}}(g_{v,t,\zeta}^B)$ of hyperkähler deformations in the neighborhood of $g_{\xi(v,t,\zeta)}$ and therefore the Ricci-flat modulo obstructions deformations of $\hat{g}_{v,t,\zeta}$ are hyperkähler. Indeed, since the metrics $\bar{g}_{\zeta_1,v}$, g_{ζ_2} and $g_{\xi(v,t,\zeta)}$ are hyperkähler, they have a hyperkähler deformation space of dimension three times that of their L^2 -cohomology in degree 2, and the dimension of this cohomology is additive for our gluings.

According to Lemma 6.1 applied to $P = P_{\hat{g}_{v,t,\zeta}}$, $P' = P_{g_{v,t,\zeta}^B}$, $K' = \tilde{\mathbf{O}}(g_{v,t,\zeta}^B)$ and $S' = \tilde{\mathbf{O}}(g_{v,t,\zeta}^B)^\perp$, the Ricci-flat deformations of $\hat{g}_{v,t,\zeta}$ are arbitrarily close to elements of $\tilde{\mathbf{O}}(g_{v,t,\zeta}^B)$. By Corollary 7, the metrics $g_{v',t',\zeta'}^B$ approximate the small Ricci-flat deformations $\hat{g}_{v',t',\zeta'}$ of $\hat{g}_{v,t,\zeta}$ staying in $B_{C_{\beta,*}^{2,\alpha}(g_{v,t,\zeta}^B, 2\epsilon)}$, for $\epsilon > 0$ the constant of Theorem 4.6. We therefore reach metrics isometric to all the $\hat{g}_{v',t',\zeta'}$ for (v', t', ζ') in a neighborhood of (v, t, ζ) in $(B_{C_{\beta,*}^{2,\alpha}(g_{\zeta_1})}(0, \tau) \cap \mathbf{O}(g_{\zeta_1})) \times (0, \tau) \times (\mathbb{S}^{3k_\Gamma-1} \setminus D_\Gamma)$. The set E is therefore open. \square

6.2.3 Obstruction for trees of Kähler Ricci-flat ALE orbifolds

Let us use the notations of the proof of Proposition 15, and parametrize the set of Kähler Ricci-flat manifolds asymptotic to \mathbb{R}^4/Γ as $(X_\zeta, g_\zeta)_{\zeta \in \mathbb{R}^{3k_\Gamma} \setminus D_\Gamma}$ in the following statement.

Lemma 6.4 *Let $(X_{\zeta_0}, g_{\zeta_0})$ be a Kähler Ricci-flat orbifold asymptotic to \mathbb{R}^4/Γ , and let $\zeta \in \mathbb{R}^{d_\Gamma} \setminus D_\Gamma$ be close to ζ_0 . Then, there exists a naïve desingularization g_t^B of $(X_{\zeta_0}, g_{\zeta_0})$ by Kähler Ricci-flat ALE orbifolds glued in the same orientation and $v \in \tilde{\mathbf{O}}(g_t^B)$ such that $(X_\zeta, g_\zeta) = (N, \bar{g}_{b,t,v})$ is the (iterated) perturbation of Lemma 17 of $(N, g_t^B + v)$.*

Moreover, there exists $\epsilon > 0$ such that for ζ close enough to ζ_0 , there exists a diffeomorphism Φ_ζ between neighborhoods of the infinities of (X_ζ, g_ζ) and of \mathbb{R}^4/Γ such that there exists $\mathbf{o}_1(\zeta)$, $\mathbf{o}_2(\zeta)$ and $\mathbf{o}_3(\zeta)$ elements of $\mathbf{O}(g_\zeta)$ satisfying for all $i \in \{1, 2, 3\}$,

$$\Phi_\zeta^* \mathbf{o}_i(\zeta) = O_i^4 + \mathcal{O}(r_B^{4+\beta}),$$

where $O_i^4 = \mathcal{O}(r_B^{-4})$ is the homogeneous symmetric 2-tensor used in the proof of Proposition 15, and with $\|\mathbf{o}_i(\zeta)\|_{L^2(g_\zeta)} \geq \epsilon$.

Proof According to Corollary 7, $(X_\zeta, g_\zeta) = (N, \bar{g}_{b,t})$ is a Kähler Ricci-flat perturbation of a naïve desingularization (N, g_t^B) for some small t depending on ζ . Moreover, the

r_e^{-4} terms of (X_ζ, g_ζ) converge to those of $(X_{\zeta_0}, g_{\zeta_0})$ as $\zeta \rightarrow \zeta_0$ by Corollary 7. By assumption, there exists $l \in \{1, \dots, k_\Gamma\}$ such that $\zeta_0^l \neq 0$, hence, for ζ close enough to ζ_0 , $\zeta^l \neq 0$ by continuity. Just like in the proof of Proposition 15, this implies that there exists a diffeomorphism Φ_ζ between neighborhoods of the infinities of (X_ζ, g_ζ) and of \mathbb{R}^4/Γ and infinitesimal deformations of g_ζ , $\mathbf{o}_1(\zeta)$, $\mathbf{o}_2(\zeta)$ and $\mathbf{o}_3(\zeta)$ such that there exists $C > 0$ independent of ζ for which we have for all $i \in \{1, 2, 3\}$,

$$|\Phi_\zeta^* \mathbf{o}_i(\zeta) - O_i^4| \leq Cr_B^{-4-\beta}$$

by Corollary 7. In particular, since $O_i^4 \neq 0$, there exists $\epsilon > 0$ depending on C and β , but independent of ζ such that we have $\|\mathbf{o}_i(\zeta)\|_{L^2(g_\zeta)} \geq \epsilon$. \square

Lemma 6.5 *Let g_t^B be a naïve gluing of Kähler Ricci-flat ALE orbifolds, and $\bar{g}_{b,t}$ its Kähler Ricci-flat perturbation of Lemma 17.*

Then, for any symmetric 2-tensor $w \in r_B^{-2}C_\beta^\alpha(g_t^B)$, there exists a unique symmetric 2-tensor $u \in \mathbf{O}(\bar{g}_{b,t})^{\perp \bar{g}_{b,t}} \cap C_{\beta,}^{2,\alpha}(g_t^B)$, such that*

$$(114) \quad \bar{P}_{\bar{g}_{b,t}} u = \pi_{\mathbf{O}(\bar{g}_{b,t})^\perp} w.$$

We moreover have the following control for $C = C(g_t^B) > 0$,

$$\|u\|_{C_{\beta,*}^{2,\alpha}(g_t^B)} \leq C \|\pi_{\mathbf{O}(\bar{g}_{b,t})^\perp} w\|_{r_B^{-2}C_\beta^\alpha(g_t^B)}.$$

Remark 43 The crucial part of this lemma is the fact that the solution is controlled in the tree of singularities norm $C_{\beta,*}^{2,\alpha}(g_t^B)$ which behaves well as $t \rightarrow 0$.

Proof According to Theorem 4.6, we have

$$(115) \quad \|\bar{g}_{b,t} - g_t^B\|_{C_{\beta,*}^{2,\alpha}(g_t^B)} \leq Ct_{\max}^{\frac{2-\beta}{4}},$$

which, combined with the proof of Theorem 4.6 implies that, for t_{\max} small enough, the operator $\bar{P}_{\bar{g}_{b,t}}$ is injective on $\bar{\mathbf{O}}(g_t^B)^\perp \cap C_{\beta,*}^{2,\alpha}(g_t^B)$.

Moreover, for $0 < \beta < 1$, its cokernel on $r_D^{-2}C_\beta^\alpha(g_t^B)$ is equal to its kernel on $r_D^{-2}C_{-\beta}^\alpha(g_t^B)$ which is equal to $\mathbf{O}(\bar{g}_{b,t})$. Indeed, for any g_b a Ricci-flat ALE metric, the kernel and the cokernel of $\bar{P}_{g_b} : C_{\beta,*}^{2,\alpha}(g_b) \rightarrow r_b^{-2}C_\beta^\alpha(g_b)$ are equal to $\mathbf{O}(g_b)$ because taking the divergence of $\bar{P}_{g_b}(h) = 0$ for $h \in C_{\beta,*}^{2,\alpha}(g_b)$, yields $\delta_{g_b} \delta_{g_b}^*(\delta_{g_b} h) = 0$, and finally $\delta_{g_b} h = 0$ by Proposition 9. By taking the trace of the remaining of the equation, we find that $\nabla_{g_b}^* \nabla_{g_b}(\text{tr}_{g_b} h) = 0$, and since h decays at infinity, $\text{tr}_{g_b} h = 0$. Finally $\bar{P}_{g_b}(h) = 0$, and we conclude that the kernel of $\bar{P}_{g_b} : C_{\beta,*}^{2,\alpha} \rightarrow r_b^{-2}C_\beta^\alpha$ is $\mathbf{O}(g_b)$, and similarly, its cokernel is also $\mathbf{O}(g_b)$. \square

Corollary 12 Let $t = (t_1 = 1, \dots, t_k) > 0$, and let

- $(N_k, g_{b_k})_k$ be a tree of ALE Kähler Ricci-flat orbifolds desingularizing \mathbb{R}^4/Γ ,
- (N, g_t^B) the naïve gluing of the (N_k, g_{b_k}) at the relative scales t_k to (N_1, g_{b_1}) , small enough for $k \neq 1$, and
- $(N, \bar{g}_{b,t})$ be the Kähler Ricci-flat ALE perturbation of (N, g_t^B) of Lemma 17.

Let us assume that (N_1, g_{b_1}) is asymptotic to \mathbb{R}^4/Γ , consider \hat{H}_S a quadratic symmetric 2-tensor on \mathbb{R}^4/Γ such that $\bar{P}_e \hat{H}_S + \lambda g_e = \mathbf{O}_S$ for a constant symmetric 2-tensor \mathbf{O}_S .

Then, there exists $C > 0$ independent of the t_k and χ , a cut-off function supported in a neighborhood of infinity of $(N, \bar{g}_{b,t})$ independent of the t_k , and there exists \hat{h}_2 a symmetric 2-tensor on N such that we have

$$\bar{P}_{\bar{g}_{b,t}} \hat{h}_2 + \lambda \bar{g}_{b,t} - \chi \mathbf{O}_S = \sum_i \hat{\lambda}_i \mathbf{o}_i \in \mathbf{O}(\bar{g}_{b,t}),$$

and

$$\|\hat{h}_S - \chi \hat{H}_S\|_{C_{\beta,*}^{2,\alpha}(g_t^B)} \leq C \|\hat{H}_S\|_{r_e^2 C^0(g_e)}.$$

Proof Let us consider \hat{H}_S a quadratic symmetric 2-tensor on \mathbb{R}^4/Γ such that $\bar{P}_e \hat{H}_S + \lambda g_e = \mathbf{O}_S$, and let χ be a cut-off function on N_1 supported in a neighborhood of infinity where (N_1, g_{b_1}) has ALE coordinates we will also denote χ on N the cut-off function extended by 0 on the deeper ALE orbifolds.

We then have

$$\|\bar{P}_{\bar{g}_{b,t}}(\chi \hat{H}_S) + \lambda \bar{g}_{b,t} - \chi \mathbf{O}_S\|_{r_D^{-2} C_{\beta}^{\alpha}(g_t^B)} \leq C \|\hat{H}_S\|_{r_e^2 C_0^0}.$$

Indeed, in a neighborhood of infinity where $\chi \equiv 1$, since $\bar{g}_{b,t} - g_e = \mathcal{O}(r_B^{-4})$, we have $\bar{P}_{\bar{g}_{b,t}}(\chi \hat{H}_S) + \lambda \bar{g}_{b,t} = \mathbf{O}_S + \mathcal{O}(r_B^{-4})$, and on the rest of the manifold, we have the expected control by definition of the norm $r_D^{-2} C_{\beta}^{\alpha}(g_t^B)$. According to Lemma 6.5 applied to $g = \bar{g}_{b,t}$, there exists a unique symmetric 2-tensor $h' \in C_{\beta,*}^{2,\alpha}(g_t^B) \cap \mathbf{O}(\bar{g}_{b,t})^{\perp}$, such that we have

$$\bar{P}_{\bar{g}_{b,t}}(\chi \hat{H}_S + h') + \lambda \bar{g}_{b,t} - \chi \mathbf{O}_S \in \mathbf{O}(\bar{g}_{b,t}).$$

Moreover, according to Proposition 12, the element of $\mathbf{O}(\bar{g}_{b,t})$ is explicit. More precisely, consider $(\mathbf{o}_i)_i$ an orthonormal basis of $\mathbf{O}(\bar{g}_{b,t})$, and thanks to the diffeomorphism Φ_t of Lemma 6.4, let us assume that the three first elements are asymptotic to $c_i \Phi_{t,*} O_i^4$ for $c_i > \frac{1}{\epsilon}$. We have

$$\bar{P}_{\bar{g}_{b,t}}(\chi \hat{H}_S + h') + \lambda \bar{g}_{b,t} - \chi \mathbf{O}_S = \sum_i \hat{\lambda}_i \mathbf{o}_i \in \mathbf{O}(\bar{g}_{b,t}),$$

where, for $i = 1, 2, 3$,

$$\hat{\lambda}_i := - \int_{\mathbb{S}^3/\Gamma} (3\langle \hat{H}_S, O_i^A \rangle_{g_e} + O_i^A(\nabla_e \text{tr}_e \hat{H}_S, \partial_{r_e})) dv_{\mathbb{S}^3/\Gamma} + \int_N \chi \langle \mathbf{O}_S, \mathbf{o}_i \rangle_{g_b} dv_b.$$

□

Let (M_o, g_o) be an Einstein orbifold and p one of its singular points of singularity \mathbb{R}^4/Γ , S the set of singularities of $M_o \setminus \{p\}$, and let $(N_k, g_{b_k})_k$ be a tree of ALE Kähler Ricci-flat orbifolds desingularizing \mathbb{R}^4/Γ . Let moreover \hat{g}_S be a naïve desingularization modulo obstructions of (M_o, g_o, S) and \hat{H}_S the quadratic terms of a development in divergence-free gauge at $p \in M_o$, $t_1 > 0$, $\bar{g}_{b,t}$ a Kähler Ricci-flat gluing of the $(N_k, g_{b_k})_k$ at relative scales $t = (t_k)_k > 0$ produced by Lemma 17, and a symmetric 2-tensor \hat{h}_2 on N and the real numbers $\hat{\lambda}_i$ of Lemma 12.

Definition 6.6 (Metric \hat{g}^A) Let us define the approximate metric \hat{g}^A as the naïve gluing (Definition 1.6) of \hat{g}_S and $t_1(\bar{g}_{b,t} + t_1\hat{h}_S)$.

We have the following control whose proof is the same as Theorem 5.8.

Corollary 13 Let $\beta > 0$, and let us use the above notations. For $t_{\max} > 0$ small enough we have the following controls: for all $k \in \mathbb{N}$ there exists $C_k > 0$,

$$(116) \quad \|\pi_{\tilde{\mathbf{O}}(g^D) \perp g^D}(\hat{g}^A)\|_{r_D^{-2}C_{\beta}^k(g^D)} \leq C_k t_1^{\frac{3-\beta}{4}},$$

and for all $\tilde{\mathbf{o}} \in \tilde{\mathbf{O}}_{t_1}(\bar{g}_{b,t})$, and denoting $(\tilde{\mathbf{o}}_i)_i$ and orthonormal basis of $\tilde{\mathbf{O}}_{t_1}(\bar{g}_{b,t})$

$$(117) \quad \left\langle g^D(\hat{g}^A) - t_1 \sum_i \hat{\lambda}_i \tilde{\mathbf{o}}_i, \tilde{\mathbf{o}} \right\rangle_{L^2(g^D)} \leq C_0 \|\mathbf{o}\|_{L^2(g_b)} t_1^{\frac{5}{4}},$$

while satisfying,

- (1) $\|\hat{g}^A - g^D\|_{C_{\beta,*}^{2,\alpha}(g^D)} \leq 2\epsilon$,
- (2) $\hat{g}^A - g^D$ is $L^2(g^D)$ -orthogonal to $\tilde{\mathbf{O}}(g^D)$,

Remark 44 The crucial part here is that, by considering the right weighted spaces, $C_{\beta,*}^{2,\alpha}(g_t^B)$, and Kähler Ricci-flat perturbations of our tree of singularities, we obtain a control by powers of t_1 only.

We then conclude, exactly like in Theorem 5.8 that the obstruction is satisfied in the limit at every singular point of (M_o, g_o) where the trees of singularities appearing are composed of Kähler Ricci-flat orbifolds ALE.

Theorem 6.7 *Let (M_o, g_o) an Einstein orbifold, and assume that there exists $(M_i, g_i)_i$ a sequence of Einstein manifolds such that*

$$(M_i, g_i) \xrightarrow{GH} (M_o, g_o).$$

Then, (M_o, g_o) satisfies $\det \mathbf{R}(g_o) = 0$ at every singular point where the trees of singularities forming in the Gromov-Hausdorff sense according to Corollary 2 are composed of ALE Kähler Ricci-flat orbifolds glued in the same orientation.

Remark 45 The result is optimal in the sense that it is the only local obstruction to the desingularization of a $\mathbb{R}^4/\mathbb{Z}_2$ singularity. Indeed, together with the existence of Einstein desingularizations of [Biq13], proven in the case of rigid asymptotically hyperbolic Einstein metrics with a singularity $\mathbb{R}^4/\mathbb{Z}_2$ singularity, we see that there exists a desingularization in the Gromov-Hausdorff sense by Eguchi-Hanson metrics *if and only if* the condition $\det \mathbf{R}(g_o) = 0$ is satisfied.

Remark 46 For now, we cannot prove any obstruction result if trees of non Kähler Ricci-flat ALE orbifolds were to appear. The reason is that it might not be possible to glue and perturb them into a single Ricci-flat ALE manifold. The obstructions to such a gluing could possibly compensate the ones coming from the gluing to the orbifold.

Example 2 Like in Example 1, let us consider the sphere \mathbb{S}^4 as $\mathbb{S}^4 \subset \mathbb{R}^5 = \mathbb{R} \times \mathbb{R}^4$. We define \mathbb{S}^4/Γ , the orbifold obtained as the quotient of \mathbb{S}^4 by the action of Γ for the first 4 coordinates of \mathbb{R}^5 . \mathbb{S}^4/Γ has its sectional curvatures constant equal to 1, and two singularities modeled on \mathbb{R}^4/Γ . The condition $\det \mathbf{R} = 0$ is therefore not satisfied for this orbifold.

7 Obstructions under topological assumptions

Let us now give topological conditions which will ensure that the Ricci-flat ALE orbifolds appearing as blow ups in our degenerations are Kähler and glued in the same orientation, and therefore that the obstruction $\det \mathbf{R} = 0$ holds.

All of these topological conditions come from the topological characterization of [Nak90], see also [LV16] for a generalization. Basically, if a desingularization has the topology of a minimal resolution of a $SU(2)$ -singularity (or a quotient for the $U(2)$ singularities) in a neighborhood of a singularity, then, all of the bubbles are Kähler and glued in the same orientation and we can apply Theorem 6.7. We will state more global

topological conditions based on Hitchin-Thorpe inequality in Theorem 7.2, and a spin condition in Theorem 7.4. We will then finally comment on the desingularization of Einstein orbifolds with various pinching conditions and bound on the Ricci curvature in Corollary 15 and Remark 51.

7.1 Hitchin-Thorpe inequality and desingularization of Einstein orbifolds

Let us first notice that desingularizing an Einstein orbifold by smooth Einstein manifolds necessarily damages the Hitchin-Thorpe inequality satisfied by the orbifold, see Theorem 7.2. The equality case is exactly when all the Ricci-flat ALE orbifolds are Kähler and glued in a common orientation.

For an Einstein manifold of dimension 4, Chern-Gauss-Bonnet formula implies,

$$(118) \quad \chi(M) = \frac{1}{8\pi^2} \int_M |\text{Rm}|^2 dv = \frac{1}{8\pi^2} \int_M \left(\frac{R^2}{24} + |W^+|^2 + |W^-|^2 \right) dv,$$

and Hirzebruch's signature formula gives,

$$(119) \quad \tau(M) = \frac{1}{12\pi^2} \int_M (|W^+|^2 - |W^-|^2) dv.$$

Simply because $\int_M |W^\pm|^2 dv \geq 0$ and $\int_M R^2 dv \geq 0$, thanks to (118) and (119), we have the following Hitchin-Thorpe inequality for Einstein 4-manifolds,

$$2\chi(M) \geq 3|\tau(M)|.$$

with equality if and only if (M, g) is a quotient of the flat torus or of a hyperkähler metric on the $K3$ surface.

In the case of orbifolds and ALE metrics, to be consistent with Chern-Gauss-Bonnet and Hirzebruch formulas, (118) and (119) for compact Einstein manifolds of dimension 4, we have to modify the Euler characteristic and the signature thanks to a boundary term. The integral quantities (118) and (119) above are topological invariants for Einstein orbifolds and Ricci-flat ALE orbifolds. We will denote them $\tilde{\chi}$ and $\tilde{\tau}$.

For Ricci-flat ALE manifolds, Nakajima obtained an Hitchin-Thorpe inequality.

Lemma 7.1 ([Nak90, Theorem 4.2]) *Let (N, g_b) be a Ricci-flat ALE manifold of dimension 4. Then, we have the following inequality between the modified Euler characteristic and the modified signature of Ricci-flat ALE orbifolds,*

$$2\tilde{\chi}(N) \geq 3|\tilde{\tau}(N)|,$$

with equality if and only if (N, g_b) is a Kähler Ricci-flat ALE orbifold.

Remark 47 In particular, the only Ricci-flat ALE manifolds diffeomorphic to a minimal resolution of a singularity \mathbb{C}^2/Γ for $\Gamma \subset SU(2)$ or one of its quotients are Kähler.

The topological invariants $\tilde{\tau}$ and $\tilde{\chi}$ are additive by gluing ALE spaces to orbifold singularities like in Definition 1.6. If $M = M_o \#_j N_j$, we then have

$$\tau(M) = \tilde{\tau}(M_o) + \sum_j \tilde{\tau}(N_j),$$

and

$$\chi(M) = \tilde{\chi}(M_o) + \sum_j \tilde{\chi}(N_j).$$

This directly implies:

$$\begin{aligned} 2\chi(M) - 3|\tau(M)| &= 2\left(\tilde{\chi}(M_o) + \sum_j \tilde{\chi}(N_j)\right) - 3\left|\tilde{\tau}(M_o) + \sum_j \tilde{\tau}(N_j)\right| \\ &\geq 2\tilde{\chi}(M_o) - 3|\tilde{\tau}(M_o)| + \sum_j (2\tilde{\chi}(N_j) - 3|\tilde{\tau}(N_j)|) \\ &\geq 2\tilde{\chi}(M_o) - 3|\tilde{\tau}(M_o)|. \end{aligned}$$

Since every term is nonnegative by Hitchin-Thorpe inequality and Lemma 7.1, we see that there is equality if and only if for all j we have $2\tilde{\chi}(N_j) - 3|\tilde{\tau}(N_j)| = 0$ and that the gluings are done in the same orientation for which $\tilde{\tau}(M_o)$ and all the $\tilde{\tau}(N_j)$ have the same sign.

Example 3 If (M_o, g_o) is a hyperkähler orbifold, then the only Gromov-Hausdorff desingularizations preserving the inequality are hyperkähler and correspond to gluing hyperkähler ALE in the same orientation.

Example 4 For $\Gamma \subset SU(2)$, an Einstein desingularization of \mathbb{S}^4/Γ preserving Hitchin-Thorpe inequality is diffeomorphic to $M = \mathbb{S}^4/\Gamma \# X_\Gamma \# X_\Gamma$ for X_Γ a minimal resolution of the singularity \mathbb{C}^2/Γ .

By studying the equality case in the previous inequalities, we get a quite restrictive situation.

Theorem 7.2 Let (M_o, g_o) be an Einstein orbifold oriented so that $\tilde{\tau}(M_o) \geq 0$, and assume that $(M, g_i)_i$ is a sequence of Einstein metrics converging in the Gromov-Hausdorff sense to (M_o, g_o) .

We then have the following inequality,

$$2\chi(M) - 3|\tau(M)| \geq 2\tilde{\chi}(M_o) - 3\tilde{\tau}(M_o).$$

Moreover, we have equality, if and only if M is a desingularization of M_o by gluing trees of Kähler Ricci-flat ALE orbifolds in the same orientation (that is with gluing parameters in $SO(4)$), and in this equality case we have the condition

$$\det \mathbf{R}_+(g_o) = 0$$

at all of the singular points of M_o .

Remark 48 The equality condition limits the possible group actions of the singularities.

This for example implies the following.

Corollary 14 Let $\Gamma \subset SU(2)$, $(M_i, g_i)_i$ be a sequence of Einstein manifolds converging in the Gromov-Hausdorff sense to the spherical orbifold \mathbb{S}^4/Γ . Then, for i large enough, we have

$$2\chi(M_i) - 3|\tau(M_i)| > 2\tilde{\chi}(M_o) - 3|\tilde{\tau}(M_o)|.$$

7.2 Spin manifolds

Another way to ensure that the Ricci-flat ALE orbifolds appearing are Kähler and glued in the same orientation is to impose that the sequence of differentiable manifolds is spin. Our result is essentially an application of the following Lemma of Nakajima.

Lemma 7.3 ([Nak90, Corollary 3.3]) *Let (N, g_b) be a Ricci-flat ALE metric on a spin manifold which is asymptotic to \mathbb{R}^4/Γ for Γ a finite subgroup of $SU(2)$, then, (N, g_b) is a hyperkähler metric.*

As a consequence, there is also an obstruction to the desingularizations of Einstein orbifolds by smooth Einstein metrics on spin manifolds. The proof of Theorem 1.1 of [KL16] whose main tool is Lemma 7.3 implies that the limit orbifold and the Ricci-flat ALE metrics are spin and glued in the same orientation for a degeneration of Einstein metrics on spin manifolds. If the group at infinity of the ALE spaces, which are also the groups of the singularities of the orbifold are in $SU(2)$, we use Lemma 7.3 to get the following obstruction.

Theorem 7.4 *Let $(M_i, g_i)_i$ be a sequence of Einstein spin manifolds of dimension 4 converging to an Einstein orbifold (M_o, g_o) . Then, (M_o, g_o) is spin and at its singular points whose groups are in $SU(2)$, we have*

$$\det \mathbf{R} = 0.$$

Remark 49 There is no restriction on the group singularities in [KL16]. This comes from their additional assumption on the kernel of the Dirac operator of the sequence which actually implies that all singularities have their group in $SU(2)$.

7.3 Pinched Ricci curvature and the Einstein condition

Our result shows that there is a fundamental difference between the Einstein condition and some pinching conditions on the Ricci curvature once we authorize the formation of singularities. From Theorems 7.2 and 7.4, we deduce that there exists an obstruction to the desingularization of Einstein orbifolds by smooth Einstein metrics which is not there if we consider pinching conditions on the Ricci curvature. Let us illustrate this with the simple example of a spherical orbifold, even though a similar result obviously holds for general orbifolds with singularity groups in $SU(2)$.

Corollary 15 Let Γ be a finite subgroup of $SU(2)$, and $M = \mathbb{S}^4/\Gamma \# X_\Gamma \# X_\Gamma$ ($\#$ means gluing at both orbifold singularities in an orientation), where X_Γ is the minimal resolution of the singularity \mathbb{C}^2/Γ . Then, for all $1 \leq p < +\infty$,

(1) there exists a sequence of metrics $(M, g_i)_i$ such that we have

- $\|\text{Ric}(g_i) - 3g_i\|_{L^p(g_i)} \leq \frac{1}{i}$, and
- $(M, g_i) \xrightarrow{GH} (\mathbb{S}^4/\Gamma, g_{\mathbb{S}^4/\Gamma})$,

but,

(2) there does *not* exist any sequence of Einstein metrics $(M, g_i^\mathcal{E})$ such that

- $\text{Ric}(g_i^\mathcal{E}) = 3g_i^\mathcal{E}$, and
- $(M, g_i^\mathcal{E}) \xrightarrow{GH} (\mathbb{S}^4/\Gamma, g_{\mathbb{S}^4/\Gamma})$.

Proof The second part is a consequence of Theorem 5.8 because the curvature of the sphere never satisfies the condition $\det \mathbf{R} = 0$ since $\mathbf{R} = \text{Id}$ for such a metric.

For the first part, we can just remark that our approximation metric g_t^A with fixed Kähler Ricci-flat ALE metrics satisfies $\|\text{Ric}(g_t^A) - \Lambda g_t^A\|_{L^\infty(g_t^A)} = \mathcal{O}(1)$ and that $\text{Ric}(g_t^A) - \Lambda g_t^A$ is supported in regions with a volume of order t , therefore, if we choose t small enough, we have the control in L^p -norm for $p < +\infty$. \square

Question: Can we desingularize $\mathbb{S}^4/\mathbb{Z}_2$ thanks to the Eguchi-Hanson metric by metrics with Ricci curvature converging to 3 in the L^∞ -sense?

Remark 50 By being more precise in the expression of the obstructions to the desingularization of $\mathbb{S}^4/\mathbb{Z}_2$ by two Eguchi-Hanson metrics, for t_{\max} small enough,

$$(120) \quad |\text{Ric}(g_t^A) - 3g_t^A|_{g_t^A} \leq 1 + \eta(t_{\max})$$

where $\eta(t_{\max}) \rightarrow 0$ when $t_{\max} \rightarrow 0$.

Remark 51 It is possible to desingularize a spherical orbifold \mathbb{S}^4/Γ for $\Gamma \subset SU(2)$ by metrics with $\text{Ric} \geq 3$ (or $\text{Ric} \leq 3$) while Ric is pinched in L^p .

Consider for $\epsilon > 0$ and $b > 1$, choose a cut-off function, $\chi_{b,\epsilon}$, supported on $[0, b\epsilon]$ and equal to 1 on $[0, \epsilon]$ whose k -th derivatives are $\mathcal{O}\left(\frac{1}{\log(b)}\epsilon^{-k}\right)$, and define the metric

$$g_{b,\epsilon} := dr^2 + \sin((1 + \chi_{b,\epsilon})r)g_{\mathbb{S}^3/\Gamma}.$$

Assume that $\epsilon \rightarrow 0$, $b \rightarrow +\infty$ and $b\epsilon \rightarrow 0$, the orbifold metric therefore becomes arbitrarily close in the Gromov-Hausdorff sense to \mathbb{S}^4/Γ . Moreover, the sectional curvatures of $g_{b,\epsilon}$ are bounded below by $1 - \frac{C}{\log b} \rightarrow 1$ for some uniform $C > 0$. Let us finally glue $t(g_{EH} + 2h_2)$, where h_2 is asymptotic to $-\frac{1}{3}r_e^2 g_{\mathbb{S}^3/\Gamma}$ at the singular points for $t_{\max} \ll \epsilon^2$, so that the gluing happens in $0 \leq r < \epsilon$ where the metric $g_{b,\epsilon}$ equals $dr^2 + \sin^2(2r)g_{\mathbb{S}^3/\Gamma}$ just like on the sphere of radius $\frac{1}{2}$ whose sectional curvatures are constant equal to 4. For $r < \epsilon$, the controls are the same as on $\frac{g_t^A}{4}$, and therefore the metric satisfies $\text{Ric} \geq 3$ by (120) since $4(3 - (1 - \eta(t_{\max}))) > 3$. Since the metric satisfies $\text{Ric} \geq 3 - \frac{C}{\log(b)}$ for larger r , we can simply rescale it a little to ensure that we have $\text{Ric} > 3$ everywhere.

8 A general obstruction for spherical and hyperbolic orbifolds

Let us finally exhibit an obstruction to the desingularization of spherical and hyperbolic orbifolds by general Ricci-flat orbifolds (not necessarily Kähler) in Theorem 8.1. We will deduce from it that there does not exist any smooth desingularization of spherical or hyperbolic orbifolds whose blow ups are integrable Ricci-flat ALE spaces in Corollary 16.

8.1 A general infinitesimal deformation for Ricci-flat ALE spaces

On $(\mathbb{R}^4/\Gamma, g_e)$, the vector field $2r_e\partial_{r_e}$ is a conformal Killing vector field. It is moreover the gradient of the function $u := r_e^2$ which is a solution to $-\nabla_e^*\nabla_e u = 8$, and we have $\mathcal{L}_{\nabla_e u}g_e = \text{Hess}_{g_e}u = 4g_e$. On a Ricci-flat ALE we can approximate this by an infinitesimal deformation.

Proposition 18 Let (N, g_b) be a Ricci-flat ALE orbifold asymptotic to \mathbb{R}^4/Γ , and consider a diffeomorphism Φ between neighborhoods of the infinities of N and \mathbb{R}^4/Γ .

Then, there exists a unique vector field X on (N, g_b) such that $\Phi^*X = 2r_b\partial_{r_b} + o(r_b)$, and $\nabla^*\nabla X = 0$. We actually have $X = \nabla u$, where u is the unique solution of $-\nabla^*\nabla u = 8$, such that $u = r_b^2 + o(1)$.

Moreover, $(\mathcal{L}_X g_b)^\circ = \mathcal{L}_X g_b - 4g_b$, the traceless part of $\mathcal{L}_X g_b$ is an infinitesimal Ricci-flat deformation of g_b which is trace-free and divergence-free.

Proof The proof of the existence and the uniqueness of the function u can be found in the proof of Theorem B of [BH19]. The symmetric 2-tensor $(\mathcal{L}_{\nabla u} g_b)^\circ$ is indeed an infinitesimal deformation of g_b because the equation $\text{Ric} = 0$ is invariant by scaling and pull-back by diffeomorphisms, and the divergence and the trace of $(\mathcal{L}_{\nabla u} g_b)^\circ = 2\text{Hess}_{g_b}u - 4g_b$ vanish because $-\nabla^*\nabla u = 8$.

Moreover, $(\mathcal{L}_{\nabla u} g_b)^\circ$ vanishes exactly for flat cones. Indeed, if it vanishes, then ∇u is a conformal Killing vector field and therefore generates a family of conformal diffeomorphisms. By considering the maximum of the pointwise norm of the curvature of (N, g_b) which is preserved by this family of diffeomorphism, we see that it has to vanish. \square

Remark 52 This deformation is integrable because it simply comes from a rescaling and a change of coordinates.

8.2 Obstructions to the desingularization of spherical and hyperbolic orbifolds

Let us now use the above deformation $\mathbf{o}_1 := (2\text{Hess}_{g_b}u - 4g_b)$ in order to deduce some general obstructions to the desingularization of spherical and hyperbolic orbifolds.

Theorem 8.1 *Let (N, g_b) be a Ricci-flat ALE orbifold and H_2 be the quadratic terms of a spherical or hyperbolic metric in geodesic coordinates, and O_1^4 terms of order r_b^{-4} of the deformation $\mathbf{o}_1 = (2\text{Hess}_{g_b}u - 4g_b)$.*

Then, we have

$$\int_{\mathbb{S}^3} (3\langle H_2, O_1^4 \rangle_{g_e} + O_1^4(B_e H_2, \partial_{r_e})) dv \neq 0,$$

and therefore the perturbation of g_t^D to an Einstein metric orthogonally to $\tilde{\mathbf{O}}(g_t^D)$ is always obstructed.

Proof Let (N, g_b) be a Ricci-flat ALE orbifold asymptotic to a flat cone \mathbb{R}^4/Γ , and let $\mathbf{o}_1 = (2\text{Hess}_{g_b}u - 4g_b) = O_1^4 + \mathcal{O}(r_b^{-5})$ be the infinitesimal deformation of Proposition 18. Let us start by proving that $O_1^4(\partial_{r_e}, \partial_{r_e})$ does not vanish. There exists a compact $K \subset M$ such that $M \setminus K$ is foliated by hypersurfaces Σ_ρ whose mean curvature is constant equal to $\frac{3}{\rho}$. If we denote Ω_ρ the interior of Σ_ρ , then, by [BH19, Theorem A] the following limit exists and is finite:

$$(121) \quad \mathcal{V} := \lim_{\rho \rightarrow \infty} [\text{Vol}_{g_b}(\Omega_\rho) - \text{Vol}_{g_e}(B(0, \rho)/\Gamma)],$$

and we actually have $\mathcal{V} \leq 0$, with equality if and only if $(N, g_b) = (\mathbb{R}^4/\Gamma, g_e)$.

Moreover, let u be the unique solution of $-\nabla^* \nabla u = 8$ with $u = r_b^2 + o(1)$, then, we actually have

$$u = r_b^2 + \frac{b}{r_b^2} + \mathcal{O}(r_b^{-3}),$$

and by the proof of [BH19, Theorem B], we have the explicit value

$$b = -4 \frac{\mathcal{V}}{|\partial B(0, 1)/\Gamma|} \geq 0.$$

We also deduce the following development of $\mathbf{o}_1 = (2\text{Hess}_{g_b}u - 4g_b)$,

$$(122) \quad \mathbf{o}_1(\partial_{r_b}, \partial_{r_b}) = \frac{8b}{r_b^4} + \mathcal{O}(r_b^{-5})$$

which is strictly positive if g_b is not flat.

Now, for a hyperbolic metric, we have $H_2 = \frac{r_e^4}{3}(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)$ in geodesic coordinates, and for a spherical metric, $H_2 = -\frac{r_e^4}{3}(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)$. Notice moreover that, since $g_e = dr_e^2 + r_e^2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)$, we have $0 = \text{tr}_{g_e} O_1^4 = O_1^4(\partial_{r_e}, \partial_{r_e}) + \langle r_e^2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2), O_1^4 \rangle$ and therefore

$$(123) \quad \langle r_e^2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2), O_1^4 \rangle = \text{tr}_{g_e} O_1^4 - O_1^4(\partial_{r_e}, \partial_{r_e})$$

$$(124) \quad = -O_1^4(\partial_{r_e}, \partial_{r_e}).$$

For the other part of the obstruction, we have $B_e(r_e^4(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)) = 6r_e\partial_{r_e}$. Indeed, $r_e^4(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) = r_e^2g_e - r_e^2dr_e^2$, and we have

$$\begin{aligned} B_e(r_e^2g_e) &= \delta_e(r_e^2g_e) + \frac{1}{2}d\text{tr}_e(r_e^2g_e) \\ &= -2r_e g_e(\partial_{r_e}, \cdot) + 4r_e dr_e \\ &= 2r_e dr_e, \end{aligned}$$

and

$$\begin{aligned} B_e(r_e^2dr_e^2) &= \delta_e(r_e^2dr_e^2) + \frac{1}{2}d\text{tr}_e(r_e^2dr_e^2) \\ &= \delta_e\left(\sum_{ij} x^i x^j dx^i dx^j\right) + r_e dr_e \\ &= -\sum_{i \neq j} x^j dx^j - 2\sum_j x^j dx^j + r_e dr_e \\ &= -4r_e dr_e. \end{aligned}$$

Finally, for $r_e = 1$,

$$(125) \quad O_1^4(B_e(r_e^4(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)), \partial_{r_e}) = 6O_1^4(\partial_{r_e}, \partial_{r_e}).$$

The obstruction generated by \mathbf{o}_1 , that is $\int_{S^3} (3\langle H_2, O_1^4 \rangle_{g_e} + O_1^4(B_e H_2, \partial_{r_e})) dv$ therefore never vanishes by (124) and (125). \square

Remark 53 It is also possible to extend the deformations given by the Killing vector fields at infinity to generate more obstructions, but it is not clear if a Ricci-flat ALE space can have vanishing terms of order r_b^{-4} . Indeed, the quantity \mathcal{V} is global and does not tell anything on the asymptotics of the metrics, but as we just saw, it tells something about their derivatives along the deformation $(\mathcal{L}_{\nabla_{bu}g_b})^\circ$.

We deduce that we get a general obstruction to a Gromov-Hausdorff desingularization if we assume that the Ricci-flat ALE spaces are integrable.

Corollary 16 Let (M_o, g_o) be a compact spherical or hyperbolic orbifold. Then, there does not exist any sequence of Einstein manifolds (M_i, g_i) such that

$$(M_i, g_i) \xrightarrow{GH} (M_o, g_o),$$

while the non-flat limits of $(M_i, \frac{g_i}{t_i}, p_i)$ for $t_i \rightarrow 0$, $t_i > 0$ and $p_i \in M_i$ converge to *smooth* and *integrable* Ricci-flat ALE manifolds (which means that there are no trees of singularities forming).

Proof According to Theorem 5.8, if the quadratic terms of the development of g_o are H_2 , the obstruction induced by the deformation \mathbf{o}_1 is

$$\int_{\mathbb{S}^3} (3\langle H_2, O_1^4 \rangle_{g_e} + O_1^4(B_e H_2, \partial_{r_e})) dv = 0,$$

which is never satisfied according to Theorem 8.1.

The obstruction of Theorem 5.8 is therefore never satisfied for spherical and hyperbolic metrics, and it is impossible to desingularize it by Ricci-flat ALE manifolds which are integrable. \square

The obstruction to the desingularization of spherical and hyperbolic manifolds is therefore identified, but we need the technical integrability assumption to deduce a Gromov-Hausdorff obstruction thanks to it. We believe that this is only a technicality and conjecture the following statement.

Conjecture 8.2 *Singular spherical and hyperbolic orbifolds cannot be Gromov-Hausdorff limits of smooth Einstein manifolds.*

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