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# Parameter estimation for a hidden linear birth and death process with immigration\*

Ibrahim Bouzalmat<sup>†</sup>      Benoîte de Saporta<sup>‡</sup>  
 Solym M. Manou-Abi<sup>§</sup>

## Abstract

In this paper, we use a linear birth and death process with immigration to model infectious disease propagation when contamination stems from both person-to-person contact and the environment. Our aim is to estimate the parameters of the process. The main originality and difficulty comes from the observation framework. Indeed, counts of infected population are hidden. The only data available are periodic cumulated new infected counts. We first derive an analytic expression of the unknown parameters as functions of well-chosen discrete time transition probabilities. Second, we extend and adapt the standard Baum-Welch algorithm in order to estimate the said discrete time transition probabilities in our hidden data framework. The performance of our estimators is illustrated both on synthetic data and real data of typhoid fever in Mayotte.

## 1 Introduction

Water borne infectious diseases [1, 2] can be life-threatening and may carry infection to a large population in a very small duration. Typhoid fever, cholera, dysentery, poliomyelitis are, among others, the most frequent water borne diseases. Such diseases are a major health issue in most parts of the world and especially in developing countries. In the French administrated island of Mayotte, typhoid fever is a notifiable endemic disease with an average of 41 cases per year between 2014 and 2021 under active surveillance of the Department of Safety and Health Emergencies (DÉSUS) of the Regional Health Agency (ARS) of Mayotte. The disease is predominantly transmitted through contact with or consumption of microbially polluted water or food, see [3, 4, 5].

Compartment models form a wide class of deterministic and random processes especially suitable to model disease propagation. A variety of such models

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<sup>†</sup>IMAG, Univ Montpellier, CNRS, Montpellier, France

<sup>‡</sup>IMAG, Univ Montpellier, CNRS, Montpellier, France

<sup>§</sup>IMAG, Univ Montpellier, CNRS, CUFR Mayotte, Montpellier, France

are available in the literature for the transmission of typhoid fever with or without vaccination, see e.g. [6, 7, 8, 9, 10]. In this paper, we focus on a simplified stochastic one-compartment model, where only counts of the infected population are taken into account. This stochastic model is a linear *birth-and-death* process with *immigration* (LBDI). Here *immigration* corresponds to exogenous contamination, or contamination from the environment; *birth* corresponds to new person-to-person contamination; and *death* corresponds to recovery. Such processes belong to the family of Markov jump processes and are used in a wide variety of quantitative disciplines, such as evolutionary biology, epidemiology and queuing theory, see for instance [11, 12, 13, 14, 15], and references therein. This LBDI model can be described with only three scalar parameters: the *immigration* (exogenous or environment contamination), *birth* (person to person contamination) and *death* (recovery) rates. Its asymptotic behavior is well-known as well as maximum likelihood estimators based on both discrete and continuous observations if available observations include the jump times and post jump locations, see for instance [16, 17, 18, 19].

However, in epidemiological practice continuous time or periodic counts of infected individuals are rarely available. For notifiable diseases, the most commonly collected data are cumulated new infected counts over given periods, typically daily or weekly intervals. Parameter inference for LBDI process in this observation framework is very challenging for two main reasons.

First, when observations are discrete, one needs to retrieve the transitions probabilities of infected counts over a given period. The immigration component invalidates the branching property, thus obtaining an analytical form for the transitions probabilities is not straightforward, and obtaining a numerical form may be liable to numerical instabilities [20]. Several researchers have proposed approaches to estimate the parameters of discretely observed general birth and death processes. In [15, 21], the authors propose to use the theory of continued fractions to find the explicit expression of the Laplace transform of transition probabilities in order to consider the inference problem with discrete observations as a missing data problem by using the expectation-maximization (EM) algorithm to find maximum likelihood estimates. In [22], the authors use the same EM algorithm method to propose a multi-type branching process approximation for birth-death shift processes. In [23], the authors propose to use two approaches to estimate the birth, death, and growth rates from a discretely observed linear birth-and-death process through an embedded Galton-Watson process and by maximizing a saddlepoint approximation to the likelihood.

Second, in all the previously cited papers, observations are discrete but correspond to infected population counts. Retrieving infected counts from cumulated new cases corresponds to a non standard hidden Markov model (HMM). Indeed, due to the underlying continuous-time dynamics, the cumulated new cases over a period of time depend on both the infected counts at the beginning and at the end of the period. Therefore the emission distribution of the HMM depends on both the current and past states of the hidden chain. Hence the standard Baum-Welch algorithm [24] cannot be directly applied and has to be adapted to this specific case.

This paper addresses both challenges. First, we establish an analytical formula for the parameters of a LBDI process in terms of some discrete time transition probabilities. Then we adapt the Baum-Welch algorithm to our special form of emission distribution to estimate the transition probabilities of the hidden chain from the cumulated new cases data. We finally obtain estimates of the original parameters by plug-in. Our results are illustrated both on synthetic and real data. The real data set concerns daily cumulated new declared confirmed cases of typhoid fever in Mayotte from 2018 to 2020. It was provided by the ARS. All codes were implemented in R and are available upon request to the corresponding author.

The paper is organized as follows. We present the stochastic LBDI model and our specific observation framework in Section 2. In Section 3, we present parameter estimation procedure and establish the main properties of the estimators. Section 4, is dedicated to the investigation of the numerical properties of the estimators, both on synthetic and real data. Finally, Section 5 gathers some concluding remarks and future work directions. All proofs are postponed to the technical appendix.

## 2 Model and observation framework

In this section, we first describe the stochastic model for counts of infected individuals. Then we define our observation framework and state our parameter estimation problem. Last, we explain our strategy to obtain estimators.

### 2.1 Linear birth and death process with immigration

For a water borne disease like typhoid fever, new infections are caused by direct or indirect contact with an infected individual or with the environment (food, water contamination). We make the following simplified assumptions. Each individual contaminates a new one with a constant infection (birth) rate  $\lambda$ , independently from other individuals. Infected individuals recover independently from one another at a constant recovery (death) rate  $\mu$ . Exogenous new cases arrive at a constant (immigration) rate  $\nu$  also independently from other contaminations and recoveries.

Let  $X_t$  be the number of infected individuals at time  $t \geq 0$ . Then the counting process  $X = (X_t)_{t \geq 0}$  is a linear birth and death process with immigration (LBDI process). It belongs to the class of Markov jump processes and its behavior is well understood, see for instance [25]. We recall the main properties here. The distribution of  $X$  is characterized by its infinitesimal generator  $Q$  as follows

$$\begin{aligned} \mathbb{P}(X_{t+h} = j | X_t = i) &= Q(i, j)h + o(h) && \text{if } j \neq i, \\ \mathbb{P}(X_{t+h} = i | X_t = i) &= 1 + Q(i, i)h + o(h), \end{aligned}$$

uniformly in  $t$ . The matrix  $Q$  has non-zero entries given by

$$\begin{aligned} Q(i, i+1) &= \lambda i + \nu && \text{for } i \geq 0, \\ Q(i, i-1) &= \mu i && \text{for } i \geq 1, \\ Q(i, i) &= -(\lambda + \mu)i - \nu && \text{for } i \geq 0. \end{aligned}$$

The transition semi-group defined by  $(P(t) = (p_{i,j}(t), i, j \in \mathbb{N}))_{t \geq 0}$ , where

$$p_{i,j}(t) = \mathbb{P}(X_t = j | X_0 = i), \quad i, j \in \mathbb{N}, t \geq 0,$$

satisfies the (forward) Kolmogorov equation

$$\frac{dP(t)}{dt} = P(t)Q, \quad (1)$$

for  $t > 0$  which can be rewritten for  $i, j \in \mathbb{N}$  with  $j \geq 1$  as

$$\begin{aligned} \frac{dp_{i,0}(t)}{dt} &= \mu p_{i,1}(t) - \nu p_{i,0}(t), \\ \frac{dp_{i,j}(t)}{dt} &= (\lambda(j-1) + \nu) p_{i,j-1}(t) + \mu(j+1) p_{i,j+1}(t) - ((\lambda + \mu)j + \nu) p_{i,j}(t), \end{aligned}$$

and with initial condition  $p_{i,i}(0) = 1$  and  $p_{i,j} = 0$  for  $j \neq i$ .

Finally, recall that a LBDI process (with  $\nu > 0$ ) has three possible regimes of long-time behavior: exponential growth (transience) if  $\lambda > \mu$ , all states are visited infinitely often with an infinite average return time (null recurrence) if  $\lambda = \mu$  and all states are visited infinitely often with a finite average return time (positive recurrence) if  $\lambda < \mu$ . In the latter case, there exists a unique invariant distribution  $\pi = (\pi_i, i \in \mathbb{N})$  given by

$$\pi_i = \binom{r+i-1}{i} \left(\frac{\lambda}{\mu}\right)^i \left(1 - \frac{\lambda}{\mu}\right)^r, \quad (2)$$

with  $r = \nu/\lambda$ .

In all the sequel, we assume that the LBDI process is in positive recurrent regime, i.e.  $\nu > 0$  and  $\lambda < \mu$ . Our aim is to estimate the triple  $(\lambda, \mu, \nu)$ .

## 2.2 Cumulated new cases

The main difficulty of this paper comes from the observation framework. We consider that  $X_t$  is not observed. Only cumulated new cases over given time lapses are available. Although this formulation is mathematically original and challenging, it is very common in public health data, especially for notifiable diseases.

Let  $Y_{(a,b]}$  denote the cumulated new infected corresponding to process  $X$  on the time interval  $(a, b]$ . Thus,  $Y_{(a,b]}$  corresponds to the number of jumps with amplitude +1 of  $X$  on the time interval  $(a, b]$ :  $Y_{(a,b]}$  is the cardinal of the set  $\{t \in (a, b]; X_t - X_{t-} = 1\}$ , where  $X_{t-} = \lim_{s \rightarrow t, s < t} X_s$ .

The joint process  $(X_t, Y_{(0,t]})$  is still a Markov jump process and its distribution is characterized by the transition probabilities

$$p_{i,(j,y)}(t) := \mathbb{P}(X_t = j, Y_{(0,t]} = y \mid X_0 = i),$$

for  $i, j, y \in \mathbb{N}$ . They also satisfy a (backward) Kolmogorov equation, see A for the proof.

**Lemma 1.** *For  $t > 0$  and  $i, j, y \in \mathbb{N}$ , one has*

$$\begin{aligned} & \frac{dp_{i,(j,y)}(t)}{dt} \\ &= -((\lambda + \mu)i + \nu)p_{i,(j,y)}(t) + (\lambda i + \nu)p_{i+1,(j,y-1)}(t) + (\mu i)p_{i-1,(j,y)}(t), \quad (3) \end{aligned}$$

for  $j \leq i + y$ , with  $p_{i,(j,y)}(t) = 0$  whenever  $j > i + y$  and with initial conditions  $p_{i,(j,y)}(0) = \delta_{i=j}\delta_{y=0}$ .

Let  $\Delta t$  be a fixed time lapse, typically one day or one week. Our aim is to estimate the triple  $(\lambda, \mu, \nu)$  with observations given by the sequence  $(Y_{(n\Delta t, (n+1)\Delta t]})_{n \in \mathbb{N}}$ .

### 2.3 Estimation strategy

Our estimation problem falls into the class of hidden information problems, as one given sequence of observations  $(Y_{(n\Delta t, (n+1)\Delta t]})_{n \in \mathbb{N}}$  may correspond to several different possible trajectories of  $X$ . However, our aim is not to reconstruct the most likely trajectory, but simply to estimate the triple  $(\lambda, \mu, \nu)$ .

Unfortunately, we were not able to obtain an analytical form of the transition probabilities  $p_{i,(j,y)}(t)$  from Equation (3). Therefore we cannot access the likelihood of the observations, and there is no hope for a direct maximization with respect to the unknown parameters. Instead, we proceed in two steps.

The first step is to study the discrete time Markov chain  $(X_{n\Delta t})_{n \in \mathbb{N}}$ . We obtain an analytical closed form for its transition probability matrix in terms of the unknown parameters  $(\lambda, \mu, \nu)$ . This formula can be inverted to obtain an expression of the parameters  $(\lambda, \mu, \nu)$  in terms of some well chosen transition probabilities. We then obtain estimators by plug-in from any estimator of the said transition probabilities, and we establish that consistence and asymptotic normality of the transition probability estimators transfer to the estimators of  $(\lambda, \mu, \nu)$ .

The second step takes into account the missing information framework by interpreting the discrete time Markov chain  $(X_{n\Delta t}, Y_{(n\Delta t, (n+1)\Delta t]})_{n \in \mathbb{N}}$  as a hidden Markov model (HMM), where  $(X_{n\Delta t})$  is the hidden chain and  $(Y_{(n\Delta t, (n+1)\Delta t]})$  the observations. However, in this framework the emission process is original as the cumulated number of new infected  $Y_{(n\Delta t, (n+1)\Delta t]}$  depends on the whole continuous time trajectory of  $X$  on the interval  $(n\Delta t, (n+1)\Delta t]$  and not only on its final value  $X_{(n+1)\Delta t}$ . Using the Markov property, one can obtain the emission probabilities of  $Y_{(n\Delta t, (n+1)\Delta t]}$  conditionally to  $X_{n\Delta t}$  and  $X_{(n+1)\Delta t}$  only. Our first idea was to run the standard HMM procedure from the R package HMM on

the two-dimensional chain  $Z_n = (X_{n-1}, X_n)$  to estimate the transition probabilities of the hidden chain  $Z$ . Unfortunately, the method was numerically unstable and marred by errors due to the many zero entries in the transition matrix of  $Z$ . Instead, we completely rewrote the forward and backward probabilities in our special context and adapted the Baum Welsh algorithm accordingly. This second steps enables us to obtain estimations of the hidden transition probabilities of the Markov chain  $(X_{n\Delta t})_{n \in \mathbb{N}}$  based on the observations  $(Y_{(n\Delta t, (n+1)\Delta t})}$  only. We then use the explicit formulas from step one to obtain estimators of the triple  $(\lambda, \mu, \nu)$ .

### 3 Parameters estimation

In this section, we detail the construction of our estimators in a discrete time framework. First, we establish an analytical formula for the parameters of a LBDI process in terms of some discrete time transition probabilities. Then we adapt the Baum-Welch algorithm to our special form of emission distribution to estimate the transition probabilities of the hidden chain from the cumulated new cases data. We finally obtain estimates of the original parameters by plug-in.

#### 3.1 Analytical expression of the parameters

The first step of our estimation procedure is to establish an analytical formula for parameters  $(\lambda, \mu, \nu)$  as functions of the discrete time transition probabilities of the Markov chain  $(X_{n\Delta t})_{n \in \mathbb{N}}$ .

We first establish an analytical solution to the Kolmogorov equation (1).

**Theorem 1.** *For  $t \geq 0$ , the transition probabilities of the continuous-time Markov chain  $X_t$  are given for  $i, j \in \mathbb{N}$  by*

$$\begin{aligned}
 p_{i,j}(t) & \tag{4} \\
 & = q(t)^r \sum_{l=0}^{\min(i,j)} \binom{i}{l} \binom{r+i+j-l-1}{j-l} \left(\frac{\mu}{\lambda}\right)^{i-l} (1-q(t))^{i+j-2l} \left(1 - \left(\frac{\mu}{\lambda} + 1\right)(1-q(t))\right)^l,
 \end{aligned}$$

where  $r = \frac{\nu}{\lambda}$  and  $q(t) = \frac{\mu - \lambda}{\mu - \lambda e^{(\lambda - \mu)t}}$ .

The proof is given in B and relies on turning the Kolmogorov equation into a standard first-order linear partial differential equation for the probability generating function of  $X_t$ . To the authors' best knowledge, the result of Theorem 1 is new. Note that a similar methodology was used and discussed in [26] yielding also closed form expressions for transition probabilities and originally comes from [27].

It is well known that the probability transition matrix of chain  $(X_{n\Delta t})_{n \in \mathbb{N}}$  is  $P(\Delta t)$ , hence we have an analytic form for the discrete time transition probabilities. We now want to inverse Equation (4) in order to obtain an expression of the parameters of interest  $(\lambda, \mu, \nu)$  as a function of the transition probabilities.

As there are only 3 unknown parameters, it should be sufficient to select 3 specific transition probabilities. In accordance with the real data set described in Section 4.2, we choose to use  $p_{0,0}(\Delta t)$ ,  $p_{0,1}(\Delta t)$  and  $p_{1,0}(\Delta t)$  as they correspond to the most frequently observed transitions and have the simplest analytical expressions. To simplify notations, in the sequel we will denote  $p_{0,0}$ ,  $p_{0,1}$  and  $p_{1,0}$  instead of  $p_{0,0}(\Delta t)$ ,  $p_{0,1}(\Delta t)$  and  $p_{1,0}(\Delta t)$ , and more generally  $p_{i,j}$  instead of  $p_{i,j}(\Delta t)$ .

**Theorem 2.** *The parameters  $\lambda$ ,  $\mu$  and  $\nu$  are given by the following relations  $(\lambda, \mu, \nu) = g(p_{0,0}, p_{0,1}, p_{1,0})$  with*

$$\begin{aligned}\lambda &= g_1(p_{0,0}, p_{0,1}, p_{1,0}) = \frac{\ln\left(\frac{u}{q}\right)(q-1)}{\Delta t(q-u)} \\ \mu &= g_2(p_{0,0}, p_{0,1}, p_{1,0}) = \frac{\ln\left(\frac{u}{q}\right)(u-1)}{\Delta t(q-u)} \\ \nu &= g_3(p_{0,0}, p_{0,1}, p_{1,0}) = \frac{\ln(p_{0,0}) \ln\left(\frac{u}{q}\right)(q-1)}{\ln(q) \Delta t(q-u)},\end{aligned}$$

where

$$\begin{aligned}q &= \frac{p_{0,1}}{p_{0,0} \ln(p_{0,0})} W\left(\frac{p_{0,0}^{\left(\frac{p_{0,0}}{p_{0,1}}+1\right)} \ln(p_{0,0})}{p_{0,1}}\right), \\ u &= 1 - \frac{p_{1,0}}{p_{0,0}},\end{aligned}$$

and  $W$  is the Lambert function, see e.g. [28, 29].

The proof is given in C and relies on inversion of the formulas in Theorem 1. In particular, if one plugs estimators of  $p_{0,0}$ ,  $p_{0,1}$  and  $p_{1,0}$  in the formulas of Theorem 2, one obtains estimators of the unknown parameters  $(\lambda, \mu, \nu)$ .

**Corollary 3.** *Let  $\hat{p}_{0,0}^n$ ,  $\hat{p}_{0,1}^n$  and  $\hat{p}_{1,0}^n$  be consistent and asymptotically normal estimators of  $p_{0,0}$ ,  $p_{0,1}$  and  $p_{1,0}$  with asymptotic variance-covariance matrix  $\Sigma'$ . Then the estimators  $(\hat{\lambda}^n, \hat{\mu}^n, \hat{\nu}^n) = g(\hat{p}_{0,0}^n, \hat{p}_{0,1}^n, \hat{p}_{1,0}^n)$  obtained by plug-in from the formulas of Theorem 2 are also consistent and asymptotically normal with asymptotic variance-covariance matrix given by  $\Sigma = Dg\Sigma'Dg^T$ , where  $Dg$  is the Jacobian matrix of application  $g$  and  $Dg^T$  is its transpose.*

The proof is straightforward as  $g$  is continuous and differentiable. The computation of the Jacobian matrix of  $Dg$  is detailed in D.

For instance, let  $\hat{p}_{i,j}^n$  be the maximum likelihood estimators of  $p_{i,j}$  for observations  $(X_{k\Delta t}, 0 \leq k \leq n)$  and for  $(i, j) \in \{(0, 0), (0, 1), (1, 0)\}$ . These estimators correspond to the ratio of the number of observed transitions from  $i$  to  $j$  over the total number of observed transitions from  $i$  in the sequence  $(X_{k\Delta t}, 0 \leq k \leq n)$ ,



see Section 4.1.1. Under the positive recurrence assumption ( $\lambda < \mu$  and  $\nu > 0$ ), they are consistent and asymptotically normal with limiting variance covariance matrix given by

$$\Sigma' = \begin{pmatrix} \frac{p_{0,0}(1-p_{0,0})}{\pi_0} & -\frac{p_{0,0}p_{0,1}}{\pi_0} & 0 \\ -\frac{p_{0,0}p_{0,1}}{\pi_0} & \frac{p_{0,1}(1-p_{0,1})}{\pi_0} & 0 \\ 0 & 0 & \frac{p_{1,0}(1-p_{1,0})}{\pi_1} \end{pmatrix}, \quad (5)$$

where  $\pi$  is the invariant distribution given in Equation (2). More specifically, one has, as  $n \rightarrow \infty$ ,

$$\sqrt{n} \begin{pmatrix} \hat{p}_{0,0}^n - p_{0,0} \\ \hat{p}_{0,1}^n - p_{0,1} \\ \hat{p}_{1,0}^n - p_{1,0} \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}_3(0, \Sigma'),$$

see e.g. [30, Theorem 2.1]. Hence they satisfy the assumptions of Corollary 3. However, they cannot be used in our context as the  $(X_{k\Delta t}, 0 \leq k \leq n)$  are hidden. Therefore we turn to another estimation procedure for  $p_{0,0}$ ,  $p_{0,1}$  and  $p_{1,0}$  and use the framework of hidden Markov models.

### 3.2 Hidden information framework

We now turn to the estimation of the transition probabilities  $p_{0,0}$ ,  $p_{0,1}$  and  $p_{1,0}$  of the hidden chain  $(X_{n\Delta t})_{n \in \mathbb{N}}$  from the available observations  $(Y_{(n\Delta t, (n+1)\Delta t]})_{n \in \mathbb{N}}$ . To simplify notations, in this section we note  $X_n$  instead of  $X_{n\Delta t}$  and  $Y_n$  instead of  $Y_{(n-1)\Delta t, n\Delta t]$ .

Our aim in this section is to adapt the standard HMM algorithms to obtain estimations of the transition probabilities of the hidden chain  $(X_n)$  from the observations  $(Y_n)$ . However, as explained in the previous section,  $Y_n$  depends on both  $X_{n-1}$  and  $X_n$ , therefore the emission framework is non standard. Note that the past  $(Y_k)_{k \leq n-1}$  and current observations  $Y_n$  are independent conditional on  $X_{n-1}$ . To circumvent the difficulty arising from the non standard emission framework, we use the two-dimensional Markov chain corresponding to consecutive pairs of hidden states.

Denote by  $(Z_n)_{n \in \mathbb{N}^*} = (X_{n-1}, X_n)_{n \in \mathbb{N}^*}$  the two-dimensional unobserved Markov chain with values in the hidden state space  $\mathbb{N}^2$ . Then it is straightforward to prove that  $(Z_n, Y_n)$  is a standard HMM with the following characteristics.

**Lemma 2.** *The process  $(Z_n, Y_n)_{n \in \mathbb{N}^*}$  is a hidden Markov model with characteristics given by the triple  $M = (Q, \psi, \rho)$ , where*

- *the transition probability matrix  $Q$  of the hidden chain  $(Z_n)$  is*

$$Q_{(i,j),(i',j')} = \mathbb{P}(Z_{n+1} = (i', j') | Z_n = (i, j)) = p_{i',j'} \delta_{i'=j},$$

for  $(i, j), (i', j') \in \mathbb{N}^2$ ;

- the emission probability  $\psi$  of process  $Y$  given process  $Z$  is

$$\psi_{(i,j)}(y) = \mathbb{P}(Y_n = y | Z_n = (i, j)) = \frac{p_{i,(j,y)}}{p_{i,j}},$$

for  $i, j, y \in \mathbb{N}$ , where  $p_{i,(j,y)} = p_{i,(j,y)}(\Delta t)$  is characterized by Equation (3);

- the initial distribution  $\rho$  of the state process  $Z$  is

$$\rho_{i,j} = \mathbb{P}(Z_1 = (i, j)) = p_{i,j} \pi_i,$$

for  $i, j \in \mathbb{N}$ , where  $\pi$  is the stationary distribution of  $X$  given in Equation (2).

The standard algorithm to estimate the most likely parameters  $M = (Q, \psi, \rho)$  of the HMM given the observations is the forward-backward or Baum-Welch algorithm a special case of the Expectation-Maximization (EM) algorithm, [24]. It is an iterative algorithm. At step  $n$ , given the current parameters  $M^n = (Q^n, \psi^n, \rho^n)$ , the likelihood of the observations  $\mathbb{P}(Y_1 = y_1, \dots, Y_T = y_T | M^n)$  is maximized and the model  $M^n$  is updated to  $M^{n+1}$  accordingly. The main asset of the Baum-Welch algorithm is that the likelihood can be maximized explicitly.

Unfortunately, running the standard HMM procedure from the R package HMM on  $(Z_n, Y_n)$  yields numerical instability and errors due to the many zero entries in the transition matrix  $Q$  of  $Z$ . Thus, we rewrote the main functions adapting the recursive formulas to the special shape of  $Q$  to obtain explicit estimators of the  $p_{i,j}$ .

**Theorem 4.** *Given the parameters  $M^n = (Q^n, \psi^n, \rho^n)$ , the maximum likelihood estimates of  $M^{n+1} = (Q^{n+1}, \psi^{n+1}, \rho^{n+1})$  given the observations  $(y_1, \dots, y_T)$  is given, for  $i, j, y$  in  $\mathbb{N}$ , by*

$$\begin{aligned} Q_{(i,j),(i',j')}^{n+1} &= \frac{\sum_{t=1}^T \xi_{(i,j),(i',j')}^n(t)}{\sum_{t=1}^T \gamma_{(i,j)}^n(t)} \delta_{i'=j}, \\ \psi_{(i,j)}^{n+1}(y) &= \frac{\sum_{t=1}^T \mathbb{1}_{y_t=y} \gamma_{(i,j)}^n(t)}{\sum_{t=1}^T \gamma_{(i,j)}^n(t)}, \\ \rho_{i,j}^{n+1} &= \gamma_{(i,j)}^n(1), \\ p_{i,j}^{n+1} &= \frac{\sum_{i' \in \mathbb{N}} Q_{(i',i),(i,j)}^{n+1}}{\sum_{i' \in \mathbb{N}} \mathbb{1}_{Q_{(i',i),(i,j)}^{n+1} \neq 0}}, \end{aligned}$$

where

$$\begin{aligned}\gamma_{(i,j)}^n(t) &= \frac{\alpha_{(i,j)}^n(t)\beta_{(i,j)}^n(t)}{\sum_{i,j \in \mathbb{N}} \alpha_{(i,j)}^n(t)\beta_{(i,j)}^n(t)}, \\ \xi_{(i,j),(i',j')}^n(t+1) &= \frac{\alpha_{(i,j)}^n(t)p_{(i',j')}^n\psi_{(i',j')}^n(y_{t+1})\beta_{(i',j')}^n(t+1)}{\sum_{i,j \in \mathbb{N}} \alpha_{(i,j)}^n(t)\beta_{(i,j)}^n(t)}\delta_{i'=j}, \\ p_{i,j}^n &= \frac{\sum_{i' \in \mathbb{N}} Q_{(i',i),(i,j)}^n}{\sum_{i' \in \mathbb{N}} \mathbb{1}_{Q_{(i',i),(i,j)}^n \neq 0}},\end{aligned}$$

and the forward and backward sequences  $\alpha^n$  and  $\beta^n$  are defined recursively for  $i, j$  in  $\mathbb{N}$  and  $2 \leq t \leq T$  by

$$\begin{cases} \alpha_{(i,j)}^n(1) = \rho_i^n \psi_{(i,j)}^n(y_1), \\ \alpha_{(i,j)}^n(t) = \psi_{(i,j)}^n(y_t) p_{i,j}^n \sum_{i' \in \mathbb{N}} \alpha_{(i',i)}^n(t-1), \\ \beta_{(i,j)}^n(T) = 1, \\ \beta_{(i,j)}^n(t-1) = \sum_{j' \in \mathbb{N}} p_{j,j'}^n \psi_{(j,j')}^n(y_t) \beta_{(j,j')}^n(t). \end{cases}$$

Elements of proof are given in Appendix E. After  $m$  iterations, we achieve the optimal parameters of the model and the estimates  $\hat{\lambda}$ ,  $\hat{\mu}$  and  $\hat{\nu}$  are given by

$$(\hat{\lambda}, \hat{\mu}, \hat{\nu}) = g(p_{0,0}^m, p_{0,1}^m, p_{1,0}^m).$$

As usual for EM algorithm, the outcome is highly dependent on the initial condition, and one must also tune the number of iterations and an appropriate way to truncate the infinite sums. All these points are discussed in the next section.

## 4 Numerical study

We now investigate the numerical performance of our estimation procedure both on synthetic and real data. All codes were developed in R on a laptop with a 1.10GHz processor. To implement the Lambert-W function, we use the `lambertW` R package [31]. For Baum-Welch algorithm we adapted functions from the package `HMM` [32].

### 4.1 Synthetic data

We first explore the performance of the estimators  $\hat{\lambda}^n$ ,  $\hat{\mu}^n$  and  $\hat{\nu}^n$  on simulated data, and investigate the sensitivity of the estimation to changes in the main hyper-parameters of the procedure. In order to quantify the loss of information due to the hidden information, we start with the setting where  $(X_n)$  is observed, then turn to the HMM framework.

### 4.1.1 Observations of infected counts

When the discrete time Markov chain  $(X_n)$  is observed, one can use the maximum likelihood estimates for the transition probabilities  $p_{i,j}$ , see e.g. [33]

$$\hat{p}_{i,j}^n = \frac{\sum_{k=0}^{n-1} \mathbb{1}_{\{X_k=i, X_{k+1}=j\}}}{\sum_{k=0}^{n-1} \mathbb{1}_{\{X_k=i\}}}.$$

We simulated a LBDI process starting from  $X_0 = 0$  with  $\lambda = 0.03$ ,  $\mu = 0.1$  and  $\nu = 0.01$ , until a (continuous) time horizon  $H$  for several values of  $\Delta t$ . To estimate the transition probabilities between discrete states by the maximum likelihood method (MLE), we use the `markovchainFit` function of the `markovchain` package for discrete-time Markov chain models [34].

Table (1) presents the estimated values of the parameters with the asymptotic standard error (asymptotic standard deviation divided by  $\sqrt{n}$ ) calculated from Corollary 3 with matrix  $\Sigma'$  from equation 5. For convenience, the numerical values of the asymptotic standard deviations are also provided in Table 2.

Table 1: Estimated values and standard error when the infected counts are observed with period  $\Delta t$  and time horizon  $H$ .

		$H = 1000$		
time lapse $\Delta t$		1	7	30
number of observations $n$		1001	143	34
$\hat{\lambda}^n$		0.027 (0.100)	0.081 (0.053)	0.049 (0.065)
$\hat{\mu}^n$		0.084 (0.035)	0.105 (0.046)	0.070 (0.128)
$\hat{\nu}^n$		0.012 (0.484)	0.007 (0.106)	0.007 (0.076)
		$H = 5000$		
time lapse $\Delta t$		1	7	30
number of observations $n$		5001	715	167
$\hat{\lambda}^n$		0.041 (0.045)	0.052 (0.024)	0.064 (0.029)
$\hat{\mu}^n$		0.088 (0.016)	0.132 (0.020)	0.117 (0.058)
$\hat{\nu}^n$		0.009 (0.216)	0.012 (0.047)	0.009 (0.034)
		$H = 50000$		
time lapse $\Delta t$		1	7	30
number of observations $n$		50001	7143	1667
$\hat{\lambda}^n$		0.030 (0.014)	0.023 (0.007)	0.020 (0.009)
$\hat{\mu}^n$		0.098 (0.004)	0.097 (0.006)	0.102 (0.018)
$\hat{\nu}^n$		0.010 (0.068)	0.010 (0.014)	0.010 (0.011)

Table (1) shows that the convergence property is satisfied, although a high number of observations is required to capture the right order of magnitude. This confirms that even in a complete observation framework, retrieving the parameters from discrete observations of the process is demanding.

Table 2 shows that the asymptotic variance of the estimators strongly depends on  $\Delta t$ , and increases as  $\Delta t$  decreases, with a different magnitude for the three coefficients, the impact being stronger on the immigration coefficient  $\mu$ . This suggests a possible optimal choice to be made between high frequency observations with high variance but a high number of observations and low frequency ones with smaller variance and smaller sample size.

Table 2: Asymptotic standard deviation of the normalized estimators

time lapse $\Delta t$	1	7	30
$\Sigma_{11}^{1/2}(\lambda)$	3.190	0.632	0.377
$\Sigma_{22}^{1/2}(\mu)$	1.112	0.552	0.749
$\Sigma_{33}^{1/2}(\nu)$	15.309	1.264	0.443

#### 4.1.2 Observations of cumulated new infected counts

We now turn to the more realistic case when only the cumulated new infected counts ( $Y_n$ ) are observed. We simulated a single trajectory of a LBDI process starting from  $X_0 \sim \pi$  with  $\lambda = 0.03$ ,  $\mu = 0.1$  and  $\nu = 0.01$ , until a time horizon  $H$ , see Fig. 1a). Then we extracted the cumulated new infected counts for a fixed value of  $\Delta t = 1$ , see Fig. 1b).

For the HMM procedure, we chose the initial parameters in the following ranges  $\lambda^{(0)} \in [0.028, 0.034]$ ,  $\mu^{(0)} \in [0.08, 0.15]$  and  $\nu^{(0)} \in [0.008, 0.015]$ , the maximum number of iterations is set to 500 and the stopping criterion (norm of the difference between the current estimate and the previous one) to  $10^{-9}$ . We executed the HMM algorithm for each value of the input triple  $\lambda^{(0)}$ ,  $\mu^{(0)}$  and  $\nu^{(0)}$ , and obtained output triples together with the likelihood of the corresponding model. Recall that the likelihood of the model is an output of the algorithm as it is given by

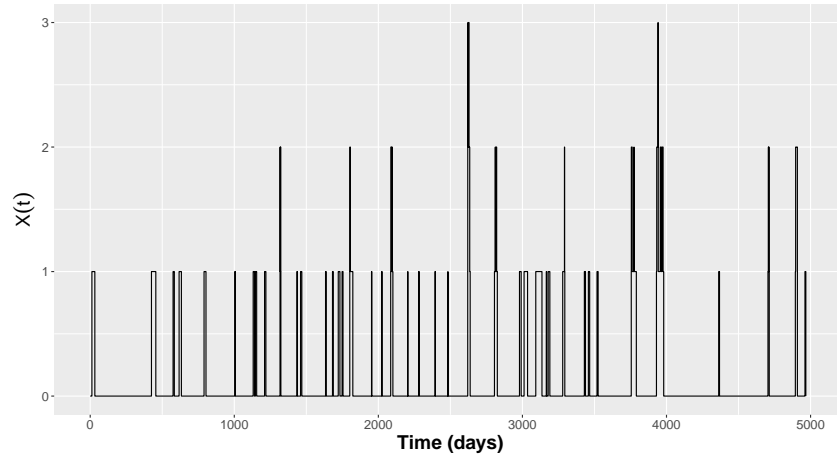
$$\sum_{i,j} \alpha_{i,j}^n(T).$$

We then selected the best output triple as the one corresponding to the model with the highest likelihood.

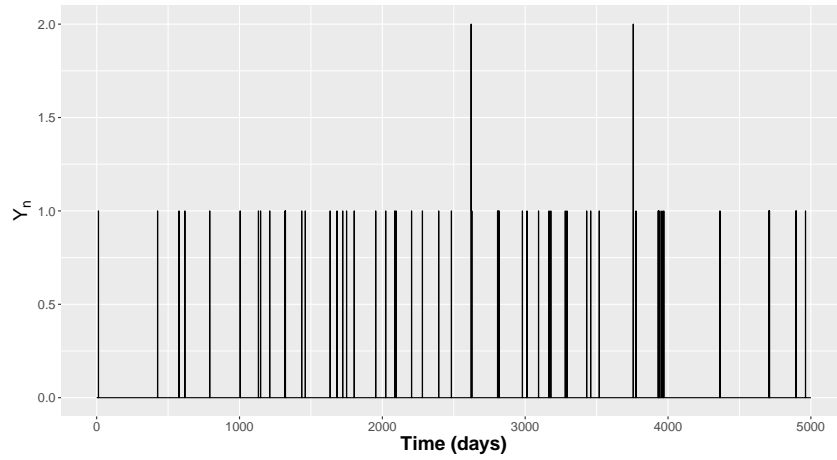
Numerically, one cannot deal with infinite state spaces. Therefore, we truncated the transition matrices on the finite state space  $\{0, 1, \dots, N\}$  as follows: we calculated the coefficients ( $p_{i,j}$ ,  $0 \leq i \leq N$ ,  $0 \leq j \leq N-1$ ) of the transition matrix by formula (4), and the last column is defined by :

$$p_{i,N} = 1 - \sum_{j=0}^{N-1} p_{i,j} \quad \text{for } i = 0, \dots, N.$$

As we are only interested in the transition probabilities from 0 to 0, 0 to 1 and 1 to 0, one can expect that this truncation procedure should not have a big impact



(a) Path of the birth-death process with immigration  $X(t)$



(b) The cumulated new infected  $Y_n$  corresponding to process  $X$ .

Figure 1: Path of the LBDI process in continuous time and the corresponding cumulated new infected counts for a time lapse  $\Delta t = 1$  until a time horizon  $H = 5000$

Table 3: Impact of the choice of the truncation parameter  $N$  on the estimation.

number of hidden states $N$	3	4	5
$\hat{\lambda}^n$	0.031	0.029	0.030
$\hat{\mu}^n$	0.110	0.098	0.105
$\hat{\nu}^n$	0.011	0.009	0.0105
$\tau$ execution time (in hours)	1.03	1.832	3.785

Table 4: The impact of the time horizon  $H$  on the estimation.

	$N = 5$			
time horizon $H$	1000	2000	3000	5000
number of observations $n$	1001	2001	3001	5001
$\hat{\lambda}^n$	0.028	0.028	0.031	0.030
$\hat{\mu}^n$	0.085	0.085	0.120	0.105
$\hat{\nu}^n$	0.008	0.008	0.012	0.011

on the estimation. The sensitivity of the output to this truncation parameter is investigated in Table (3).

In terms of complexity, increasing the size of the truncation parameter  $N$ , significantly increases the execution time of the EM algorithm, as expected as the number of parameters to estimate is proportional to  $N^2$ . In terms of accuracy, for  $N = 5$ , we get relatively good estimates compared to  $N$  equals 3 or 4. In general, the truncation of the space does not have a big impact on the accuracy of the estimation.

Finally, we set  $N = 5$  and we illustrate the impact of the time horizon  $H$  on our estimation scheme in Table (4). As expected, the quality of the estimation improves slowly with the number of observations.

## 4.2 Typhoid data in Mayotte

We now investigate a real data set from the ARS of Mayotte which contains the daily number of new confirmed cases of typhoid fever on the island between 2018 and 2020. We thus have 1086 observations, with values between 0 and 3, see Table 5 and Fig. 2. This is the data set that motivated our study.

A stochastic model for the propagation of typhoid fever in Mayotte is well adapted to the low prevalence figures. Moreover, there are long lapses without new cases followed by spontaneous surges of new cases, indicating the existence of exogenous new cases. The simplest model corresponding to this case is thus the LBDI model under positive recurrent regime. As Typhoid fever is a notifiable disease in Mayotte, we consider that all new cases are observed.

Based on literature [7, 8, 35, 36], the expert advice of typhoid specialists in

Table 5: Number of daily cumulated new declared confirmed cases of typhoid fever on Mayotte between 2018 and 2020.

observation value	0	1	2	3
number of observations	975	94	13	4

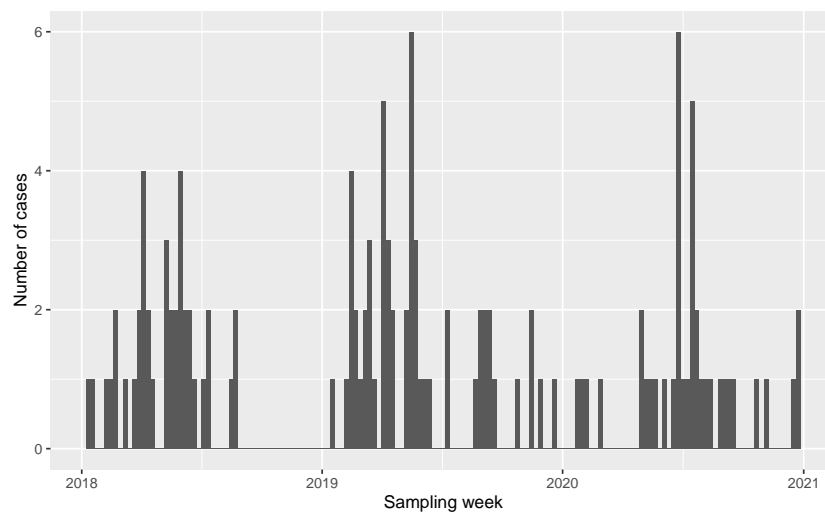


Figure 2: Number of weekly cumulated new declared confirmed cases of typhoid fever on Mayotte between 2018 and 2020.



Table 6: Estimates  $\hat{\lambda}^n$ ,  $\hat{\mu}^n$  and  $\hat{\nu}^n$  and log likelihood of the corresponding models for the different initial parameter values tested.

$(\lambda^{(0)}, \mu^{(0)}, \nu^{(0)})$	$(\hat{\lambda}^n, \hat{\mu}^n, \hat{\nu}^n)$	log-likelihood
(0.0500, 0.1100, 0.0150)	(0.0500, 0.1100, 0.0150)	-394.1107
(0.0516, 0.1174, 0.0157)	(0.0515, 0.1173, 0.0158)	-394.4322
(0.0532, 0.1247, 0.0166)	(0.0530, 0.1240, 0.0165)	-394.0771
(0.0546, 0.1321, 0.0174)	(0.0547, 0.1320, 0.0170)	<b>-393.7732</b>
(0.0563, 0.1395, 0.0182)	(0.0560, 0.1400, 0.0180)	-394.1762
(0.0579, 0.1468, 0.0189)	(0.0578, 0.1470, 0.0189)	-394.5896
(0.0595, 0.1542, 0.0197)	(0.0590, 0.1541, 0.0200)	-394.9808
(0.0611, 0.1616, 0.0205)	(0.0610, 0.1616, 0.0204)	-395.3653
(0.0626, 0.1689, 0.0213)	(0.0630, 0.1690, 0.0213)	-395.7492
(0.0642, 0.1763, 0.0221)	(0.0642, 0.1760, 0.0220)	-396.1328
(0.0658, 0.1837, 0.0229)	(0.0660, 0.1840, 0.0228)	-396.5154
(0.0674, 0.1911, 0.0237)	(0.0673, 0.1910, 0.0235)	-397.0610
(0.0689, 0.1984, 0.0245)	(0.0690, 0.1980, 0.0240)	-397.4057
(0.0705, 0.2058, 0.0253)	(0.0705, 0.2057, 0.0252)	-396.1538
(0.0721, 0.2132, 0.0261)	(0.0720, 0.2130, 0.0260)	-395.9935
(0.0737, 0.2205, 0.0268)	(0.0740, 0.2205, 0.0270)	-396.1987
(0.0753, 0.2279, 0.0276)	(0.0750, 0.2280, 0.0276)	-396.4099
(0.0768, 0.2353, 0.0284)	(0.0769, 0.2350, 0.0280)	-396.6318
(0.0784, 0.2426, 0.0292)	(0.0783, 0.2420, 0.0290)	-396.8629
(0.0800, 0.2500, 0.0300)	(0.0800, 0.2500, 0.0300)	-397.1020

Mayotte, and our synthetic data exploration, we chose initial parameters in the following ranges  $\lambda^{(0)} \in [0.05, 0.08]$ ,  $\mu^{(0)} \in [0.11, 0.25]$  and  $\nu^{(0)} \in [0.015, 0.03]$ , see Table 6. This allows to initialize  $P^0$ ,  $Q^0$ ,  $\psi^0$  and  $\rho^0$ . The stopping criterion is still set to  $10^{-9}$ , and the maximum number of iterations to 500. The output estimations and the likelihood of their corresponding model are given in Table 6. We selected the triple with the highest likelihood, namely

$$\hat{\lambda} = 0.054, \quad \hat{\mu} = 0.132 \quad \text{and} \quad \hat{\nu} = 0.017.$$

Given the comparatively low number of data, these results must be handled with care. Note that the positive recurrence condition  $\lambda < \mu$  is valid. The average recovery duration  $1/\hat{\mu} = 7.5$  days is also consistent with the average treatment duration in Mayotte.

## 5 Conclusion

We have proposed a methodology to estimate the unknown parameters of a LBDI process in the mathematically unusual but epidemiologically common case where only cumulated new infected counts over given periods are available.

Our approach relies on both analytical formulas and numerical iterations. We investigated the properties of the estimators on synthetic data and applied our procedure to a real data set of typhoid fever in Mayotte that motivated this work.

Due to the number of observed cases and hospitalizations, typhoid fever remains an endemic risk disease in the territory of Mayotte, as evidenced by the epidemic observed in 2022 in the north of the island. It is constantly monitored by the DésUS of ARS Mayotte. The strengthening of the internal crisis management strategy and the provision of innovative tools to monitor epidemics remains a priority for ARS Mayotte. This approach is a first step toward the quantitative understanding of typhoid propagation in Mayotte. We proposed a simplified model with few interpretable parameters. It would be interesting to split the data set given locally meaningful covariates such as easy access to potable water to investigate whether the coefficients guiding the propagation of the disease differ with the sanitary conditions of the population, to include a spatial propagation component, or to investigate forecasting aspects to try to characterize the risk of outbreak of an epidemic, to name just a few possible new directions with possibly high public health impact. Another promising extension would be to take into account the rainfalls. Indeed, outbreaks of the disease seem highly correlated with the rainy season. This will be the object of further work.

## A Proof of Lemma 1

We use the infinitesimal characterization of the distribution of the process. For  $h > 0$  and  $j \leq i + y$ , one has

$$\begin{aligned}
& p_{i,(j,y)}(t+h) \\
&= \mathbb{P}(X_{t+h} = j, Y_{(0,t+h]} = y \mid X_0 = i) \\
&= \mathbb{P}(X_h = i, Y_{(0,h]} = 0, X_{t+h} = j, Y_{(h,t+h]} = y \mid X_0 = i) \\
&\quad + \mathbb{P}(X_h = i+1, Y_{(0,h]} = 1, X_{t+h} = j, Y_{(h,t+h]} = y-1 \mid X_0 = i) \\
&\quad + \mathbb{P}(X_h = i-1, Y_{(0,h]} = 0, X_{t+h} = j, Y_{(h,t+h]} = y \mid X_0 = i) + o(h) \\
&= \mathbb{P}(X_h = i, Y_{(0,h]} = 0 \mid X_0 = i) \mathbb{P}(X_{t+h} = j, Y_{(h,t+h]} = y \mid X_h = i) \\
&\quad + \mathbb{P}(X_h = i+1, Y_{(0,h]} = 1 \mid X_0 = i) \mathbb{P}(X_{t+h} = j, Y_{(h,t+h]} = y-1 \mid X_h = i+1) \\
&\quad + \mathbb{P}(X_h = i-1, Y_{(0,h]} = 0 \mid X_0 = i) \mathbb{P}(X_{t+h} = j, Y_{(h,t+h]} = y \mid X_h = i-1) + o(h)
\end{aligned}$$

Since  $(X_t)$  is a homogeneous Markov process, we get :

$$\begin{aligned}
p_{i,(j,y)}(t+h) &= (1 - ((\lambda + \mu)i + \nu)h)p_{i,(j,y)}(t) \\
&\quad + (\lambda i + \nu)hp_{i+1,(j,y-1)}(t) + (\mu i)hp_{i-1,(j,y)}(t) + o(h).
\end{aligned}$$

Hence the result.

## B Proof of Theorem 1

To prove Theorem 1, we update the method of characteristics, see [37], to obtain a standard first-order linear partial differential equation for the probability generating function of  $X_t$ . Then, we derive an expression for the transition probabilities by using well-known results on series expansion. Although we made the assumption that  $\lambda < \mu$ , we give general formulas including the case  $\lambda \geq \mu$  as they are interesting per se.

First, we need some technical results.

**Lemma 3.** *Let  $i \in \mathbb{N}$  and  $G(s, t)$  be a differentiable function on  $[0, +\infty)^2$ . The solution of the following linear partial differential equation*

$$\begin{cases} \frac{\partial G(s, t)}{\partial t} &= (\mu - \lambda s)(1 - s) \frac{\partial G(s, t)}{\partial s} + \nu(s - 1)G(s, t), \\ G(s, 0) &= s^i, \end{cases} \quad (6)$$

is given by

$$G(s, t) = \begin{cases} \frac{(\mu - \lambda)^{\frac{\nu}{\lambda}} \left( s(\mu e^{(\lambda - \mu)t} - \lambda) - \mu(e^{(\lambda - \mu)t} - 1) \right)^i}{\left( \mu - \lambda e^{(\lambda - \mu)t} + \lambda s(e^{(\lambda - \mu)t} - 1) \right)^{\frac{\nu}{\lambda} + i}} & \lambda \neq \mu, \\ \left( \frac{\lambda t + s(1 - \lambda t)}{1 + \lambda t - \lambda t s} \right)^i (1 + \lambda t - \lambda t s)^{-\frac{\nu}{\lambda}} & \lambda = \mu. \end{cases}$$

*Proof.* To solve Equation (6) we make use of above mentioned method of characteristics.

*Step 1. Determining the Characteristics.* Consider the characteristic equation

$$\frac{dt}{ds} = -\frac{1}{(\mu - \lambda s)(1 - s)} = \begin{cases} \frac{1}{\lambda - \mu} \left( \frac{1}{1 - s} - \frac{\lambda}{\mu - \lambda s} \right) & \text{if } \lambda \neq \mu \\ -\frac{1}{\lambda(1 - s)^2} & \text{if } \lambda = \mu. \end{cases}$$

By integrating with respect to  $s$ , we have

$$t = k - \begin{cases} \frac{1}{\lambda - \mu} \ln \left( \frac{1 - s}{\mu - \lambda s} \right) & \text{if } \lambda \neq \mu \\ \frac{1}{\lambda(1 - s)} & \text{if } \lambda = \mu. \end{cases} \quad (7)$$

Equation (7) defines a family of characteristics parameterized by the integration constant  $k$ .

*Step 2.* Along the characteristics in (7),  $G$  satisfies

$$\frac{dG}{ds} = \frac{\nu G}{\mu - \lambda s}.$$

Integrating with respect to  $s$  yields

$$G = C(k) (\mu - \lambda s)^{-\frac{\nu}{\lambda}},$$

where  $C(k)$  is another integration constant, function of the parameter  $k$ . Using Equation (7), we obtain

$$G(s, t) = \begin{cases} C \left( t + \frac{1}{\lambda - \mu} \ln \left( \frac{1-s}{\mu - \lambda s} \right) \right) (\mu - \lambda s)^{-\frac{\nu}{\lambda}} & \text{if } \lambda \neq \mu \\ C \left( t + \frac{1}{\lambda(1-s)} \right) (\lambda(1-s))^{-\frac{\nu}{\lambda}} & \text{if } \lambda = \mu. \end{cases} \quad (8)$$

Now, we use the initial condition  $G(s, 0) = s^i$  to compute  $C(k)$ . Setting  $t = 0$  in equation (8), we get

$$s^i = \begin{cases} C \left( \frac{1}{\lambda - \mu} \ln \left( \frac{1-s}{\mu - \lambda s} \right) \right) (\mu - \lambda s)^{-\frac{\nu}{\lambda}} & \text{if } \lambda \neq \mu \\ C \left( \frac{1}{\lambda(1-s)} \right) (\lambda(1-s))^{-\frac{\nu}{\lambda}} & \text{if } \lambda = \mu. \end{cases}$$

so that

$$C(w) = \begin{cases} \left( \frac{e^{(\lambda-\mu)w} \mu - 1}{e^{(\lambda-\mu)w} \lambda - 1} \right)^i \left( \mu - \lambda \frac{e^{(\lambda-\mu)w} \mu - 1}{e^{(\lambda-\mu)w} \lambda - 1} \right)^{\frac{\nu}{\lambda}} & \text{if } \lambda \neq \mu \\ \left( 1 - \frac{1}{\lambda w} \right)^i w^{-\frac{\nu}{\lambda}} & \text{if } \lambda = \mu \end{cases}$$

where

$$w = \begin{cases} \frac{1}{\lambda - \mu} \ln \left( \frac{1-s}{\mu - \lambda s} \right) & \text{if } \lambda \neq \mu \\ \frac{1}{\lambda(1-s)} & \text{if } \lambda = \mu, \end{cases}$$

Finally, we obtain the required expression for  $\lambda \neq \mu$ ,

$$G_i(s, t) = \frac{(\mu - \lambda)^{\frac{\nu}{\lambda}} \left( s(\mu e^{(\lambda-\mu)t} - \lambda) - \mu (e^{(\lambda-\mu)t} - 1) \right)^i}{\left( \mu - \lambda e^{(\lambda-\mu)t} + \lambda s (e^{(\lambda-\mu)t} - 1) \right)^{\frac{\nu}{\lambda} + i}},$$

and for  $\lambda = \mu$

$$G_i(s, t) = \left( \frac{\lambda t + s(1 - \lambda t)}{1 + \lambda t - \lambda t s} \right)^i (1 + \lambda t - \lambda t s)^{-\frac{\nu}{\lambda}},$$

hence the result.  $\square$

We can now turn to the proof of Theorem 1.

*Proof.* of Theorem 1

Let us define the probability generating function of  $X_t$ : for any  $(s, t) \in \times [0, +\infty)^2$ , and  $i \in \mathbb{N}$  set

$$G_i(s, t) = \sum_{j=0}^{+\infty} s^j p_{i,j}(t).$$

The forward Kolmogorov equation (1) yields

$$\frac{\partial G_i(s, t)}{\partial t} = (\lambda s^2 - (\lambda + \mu)s + \mu) \sum_{j=1}^{\infty} j s^{j-1} p_{i,j}(t) + \nu(s-1) \sum_{j=0}^{\infty} s^j p_{i,j}(t),$$

so that the probability generating function satisfies the following first-order linear partial differential equation :

$$\frac{\partial G_i(s, t)}{\partial t} = (\mu - \lambda s)(1-s) \frac{\partial G_i(s, t)}{\partial s} + \nu(s-1)G_i(s, t),$$

with initial condition  $G_i(s, 0) = s^i$ . Now, using Lemma 3, we have for  $\lambda \neq \mu$ ,

$$G_i(s, t) = \frac{(\mu - \lambda)^{\frac{\nu}{\lambda}} \left( s(\mu e^{(\lambda-\mu)t} - \lambda) - \mu(e^{(\lambda-\mu)t} - 1) \right)^i}{\left( \mu - \lambda e^{(\lambda-\mu)t} + \lambda s(e^{(\lambda-\mu)t} - 1) \right)^{\frac{\nu}{\lambda} + i}}$$

and for  $\lambda = \mu$

$$G_i(s, t) = \left( \frac{\lambda t + s(1 - \lambda t)}{1 + \lambda t - \lambda t s} \right)^i (1 + \lambda t - \lambda t s)^{-\frac{\nu}{\lambda}}.$$

The case  $i = 0$  gives the following generating function of the probabilities transition  $p_{0,j}(t)$ ,  $t \geq 0$ :

$$G_0(s, t) = \begin{cases} \left( \frac{\mu - \lambda}{\mu - \lambda e^{(\lambda-\mu)t}} \right)^{\frac{\nu}{\lambda}} \left( 1 - \frac{\lambda(1 - e^{(\lambda-\mu)t})}{\mu - \lambda e^{(\lambda-\mu)t}} s \right)^{-\frac{\nu}{\lambda}} & \lambda \neq \mu, \\ \left( \frac{1}{1 + \lambda t} \right)^{\frac{\nu}{\lambda}} \left( 1 - \frac{\lambda t}{1 + \lambda t} s \right)^{-\frac{\nu}{\lambda}} & \lambda = \mu. \end{cases}$$

It is the generating function of the generalized negative binomial distribution with parameters

$$r = \frac{\nu}{\lambda} \quad \text{and} \quad q(t) = \begin{cases} \frac{\mu - \lambda}{\mu - \lambda e^{(\lambda-\mu)t}} & \lambda \neq \mu, \\ \frac{1}{1 + \lambda t} & \lambda = \mu. \end{cases}$$

Therefore, for  $j \in \mathbb{N}$  and  $t \geq 0$ , one has

$$p_{0,j}(t) = \binom{j+r-1}{r-1} q(t)^r (1-q(t))^j. \quad (9)$$

The general form of  $G_i$  for  $i \neq 0$  can be written as follows:

$$G_i(s, t) = G_0(s, t)N_i(s, t), \quad (10)$$

where

$$N_i(s, t) = \begin{cases} \left( \frac{\mu - \lambda s - \mu(1-s)e^{(\lambda-\mu)t}}{\mu - \lambda s - \lambda(1-s)e^{(\lambda-\mu)t}} \right)^i & \lambda \neq \mu, \\ \left( \frac{1 + (\lambda t - 1)(1-s)}{1 + \lambda t(1-s)} \right)^i & \lambda = \mu, \end{cases}$$

is the probability generating function of a linear birth–death process without immigration. In essence, Equation (10) reflects the independence between the process of contamination from the environment ( $G_0$ ) and the self-contamination of the population ( $N$ ). The linear birth and death process without immigration has the branching property that translates into the simple power form of its probability generating function, see for instance [38] or [39]. One can readily retrieve its transition probabilities  $p_{i,j}^{(N)}(t), t \geq 0$  from the series expansion of  $N_i(s, t)$ :

$$p_{i,j}^{(N)}(t) = \sum_{l=0}^{\min(i,j)} \binom{i}{l} \binom{i+j-l-1}{i-1} \alpha(t)^{i-l} \beta(t)^{j-l} (1 - \alpha(t) - \beta(t))^l, \quad (11)$$

where  $\alpha(t) = \frac{\mu}{\lambda}(1 - p(t))$  and  $\beta(t) = 1 - q(t)$ .

Thus, the probabilities  $p_{i,j}(t)$  are given by the following convolution from Equation (10):

$$p_{i,j}(t) = \sum_{k=0}^j p_{0,k}(t) p_{i,j-k}^{(N)}(t),$$

and Equations (9), (11), finally yield

$$\begin{aligned} p_{i,j}(t) &= q(t)^r \sum_{l=0}^{\min(i,j)} \binom{i}{l} \left( \frac{\mu}{\lambda} \right)^{i-l} (1 - q(t))^{i+j-l} \left( \frac{1 - (\frac{\mu}{\lambda} + 1)(1 - q(t))}{1 - q(t)} \right)^l \\ &\quad \times \sum_{k=0}^{j-l} \binom{r-1+k}{r-1} \binom{i-1+j-l-k}{i-1}, \end{aligned}$$

hence the result.  $\square$

One can also retrieve the well known expression for the mean of the process  $m(t) = \mathbb{E}[X_t]$ :

$$m(t) = \mathbb{E}[X_t | X_0 = i] = \begin{cases} \frac{\nu}{\lambda - \mu} (e^{(\lambda - \mu)t} - 1) + i e^{(\lambda - \mu)t} & \lambda \neq \mu, \\ \nu t + i & \lambda = \mu, \end{cases}$$

by derivating the probability generating function with respect to  $s$  and taking  $s = 1$ . The above result on the expectation is consistent with that obtained in [27] using another method which consists in deriving an equation for  $m(t + h)$ , conditioning on  $X_t$ , and then solving a differential equation on  $m(t)$  directly.

## C Proof of Theorem 2

We make use of Equation (4) for  $(i, j) \in \{(0, 0), (0, 1), (1, 0)\}$  and  $t = \Delta t$  to obtain

$$\begin{aligned} p_{0,0} &= q^r \\ p_{0,1} &= q^r r (1 - q) \\ p_{1,0} &= q^r \left(\frac{\mu}{\lambda}\right) (1 - q), \end{aligned}$$

with  $r = \frac{\nu}{\lambda}$  and  $q = q(\Delta t) = \frac{\mu - \lambda}{\mu - \lambda e^{(\lambda - \mu)\Delta t}}$ . The first 2 equations above are equivalent to the system

$$\begin{cases} \ln(p_{0,0}) &= r \ln(q) \\ \frac{p_{0,1}}{p_{0,0}} &= r (1 - q), \end{cases} \quad (12)$$

which yields

$$\frac{p_{0,0}}{p_{0,1}} \ln(p_{0,0}) q e^{\frac{p_{0,0}}{p_{0,1}} \ln(p_{0,0}) q} = \frac{p_{0,0}^{\left(\frac{p_{0,0}}{p_{0,1}} + 1\right)} \ln(p_{0,0})}{p_{0,1}},$$

so that one has

$$q = \frac{p_{0,1}}{p_{0,0}(t) \ln(p_{0,0})} W \left( \frac{p_{0,0}^{\left(\frac{p_{0,0}}{p_{0,1}} + 1\right)} \ln(p_{0,0})}{p_{0,1}} \right), \quad (13)$$

where  $W$  is the Lambert function. Now, from Equations (12) and (13), we can easily find that

$$r = \frac{\ln(p_{0,0})}{\ln(q)}.$$

Now set

$$u = q e^{(\lambda - \mu)\Delta t} = 1 - \frac{p_{1,0}}{p_{0,0}}.$$

Equation (4) for  $(i, j) \in \{(0, 0), (0, 1), (1, 0)\}$  can now be rewritten as

$$\begin{cases} r &= \frac{\nu}{\lambda} \\ q &= \frac{\mu - \lambda}{\mu - \lambda e^{(\lambda - \mu)\Delta t}} \\ u &= \frac{(\mu - \lambda) e^{(\lambda - \mu)\Delta t}}{\mu - \lambda e^{(\lambda - \mu)\Delta t}}. \end{cases} \quad (14)$$

Finally, solving System (14) give rise to the expressions of  $\lambda$ ,  $\mu$  and  $\nu$  in function of the transition probabilities  $p_{0,0}$ ,  $p_{0,1}$  and  $p_{1,0}$  given in Theorem 2. Note that these expressions are unique so that the parameters are identifiable.

## D Jacobian matrix of $Dg$

We compute the Jacobian matrix of  $Dg$ . Firstly, note that the partial derivatives of the functions  $q$ ,  $r$  and  $u$  from the proof of Theorem 2 given in C are

$$\begin{aligned} \frac{\partial q}{\partial p_{0,0}} &= -\frac{p_{0,1} (\ln(p_{0,0}) + 1)}{(p_{0,0} \ln(p_{0,0}))^2} W \left( \frac{\left(\frac{p_{0,0}}{p_{0,1}} + 1\right) \ln(p_{0,0})}{p_{0,1}} \right) \\ &\quad + \frac{p_{0,1}}{p_{0,0} \ln(p_{0,0})} (\ln(p_{0,0}) + 1) \left( \frac{1}{p_{0,1}} + \frac{1}{p_{0,0} \ln(p_{0,0})} \right) \frac{W \left( \frac{\left(\frac{p_{0,0}}{p_{0,1}} + 1\right) \ln(p_{0,0})}{p_{0,1}} \right)}{W \left( \frac{\left(\frac{p_{0,0}}{p_{0,1}} + 1\right) \ln(p_{0,0})}{p_{0,1}} \right) + 1}. \end{aligned}$$

Since

$$W \left( \frac{\left(\frac{p_{0,0}}{p_{0,1}} + 1\right) \ln(p_{0,0})}{p_{0,1}} \right) = q \frac{p_{0,0} \ln(p_{0,0})}{p_{0,1}}, \quad (15)$$

one has

$$\begin{aligned} \frac{\partial q}{\partial p_{0,0}} &= \frac{q (\ln(p_{0,0}) + 1)}{qp_{0,0} \ln(p_{0,0}) + p_{0,1}} \left( 1 + \frac{p_{0,1}}{p_{0,0} \ln(p_{0,0})} - \frac{qp_{0,0} \ln(p_{0,0}) + p_{0,1}}{p_{0,0} \ln(p_{0,0})} \right) \\ &= \frac{q (\ln(p_{0,0}) + 1)}{qp_{0,0} \ln(p_{0,0}) + p_{0,1}} (1 - q). \end{aligned}$$

Next, on has

$$\begin{aligned} \frac{\partial q}{\partial p_{0,1}} &= \frac{1}{p_{0,0} \ln(p_{0,0})} W \left( \frac{\left(\frac{p_{0,0}}{p_{0,1}} + 1\right) \ln(p_{0,0})}{p_{0,1}} \right) \\ &\quad - \frac{p_{0,1}}{p_{0,0} \ln(p_{0,0})} \left( \frac{p_{0,0} \ln(p_{0,0})}{(p_{0,1})^2} + \frac{1}{p_{0,1}} \right) \frac{W \left( \frac{\left(\frac{p_{0,0}}{p_{0,1}} + 1\right) \ln(p_{0,0})}{p_{0,1}} \right)}{W \left( \frac{\left(\frac{p_{0,0}}{p_{0,1}} + 1\right) \ln(p_{0,0})}{p_{0,1}} \right) + 1}, \end{aligned}$$

and using Equation (15) we obtain

$$\begin{aligned} \frac{\partial q}{\partial p_{0,1}} &= \frac{q}{p_{0,1}} - \frac{q}{qp_{0,0} \ln(p_{0,0}) + p_{0,1}} \left( \frac{p_{0,0} \ln(p_{0,0})}{p_{0,1}} + 1 \right) \\ &= \frac{q}{p_{0,1} \left( q + \frac{p_{0,1}}{p_{0,0} \ln(p_{0,0})} \right)} (q - 1), \end{aligned}$$



and finally one has

$$\frac{\partial q}{\partial p_{1,0}} = 0.$$

The expressions of partial differentials of  $r$  are given as follows:

$$\begin{aligned} \frac{\partial r}{\partial p_{0,0}} &= \frac{1}{p_{0,0} \ln(q)} - \frac{\ln(p_{0,0})}{(\ln(q))^2 q} \frac{\partial q}{\partial p_{0,0}} \\ &= \frac{r}{p_{0,0} \ln(p_{0,0})} - r \frac{(\ln(p_{0,0}) + 1)(1 - q)}{\ln(q)(qp_{0,0} \ln(p_{0,0}) + p_{0,1})} \\ \frac{\partial r}{\partial p_{0,1}} &= \frac{\ln(p_{0,0})}{(\ln(q))^2 q} \frac{\partial q}{\partial p_{0,1}} \\ &= r \frac{(q - 1)}{p_{0,1} \ln(q) \left( q + \frac{p_{0,1}}{p_{0,0} \ln(p_{0,0})} \right)} \\ \frac{\partial r}{\partial p_{1,0}} &= 0. \end{aligned}$$

This yields the following expressions of partial differentials of  $u$ :

$$\begin{aligned} \frac{\partial u}{\partial p_{0,0}} &= \frac{p_{1,0}}{p_{0,0}} = \frac{1 - u}{p_{0,0}} \\ \frac{\partial u}{\partial p_{0,1}} &= 0 \\ \frac{\partial u}{\partial p_{1,0}} &= -\frac{1}{p_{0,0}}. \end{aligned}$$

Defining the following functions

$$\begin{aligned} \alpha &= \frac{p(\ln(p_{0,0}) + 1)(p - 1)}{\Delta t (q p_{0,0} \ln(p_{0,0}) + p_{0,1})(q - u)} \\ \beta &= \frac{1 - p}{\Delta t p_{0,1} \left( q + \frac{p_{0,1}}{p_{0,0} \ln(p_{0,0})} \right) (p - u)} \\ \eta &= \frac{1 - u}{\Delta t p_{0,0} u (q - u)}, \end{aligned}$$

one can rewrite compactly the partial differentials of  $\lambda$ ,  $\mu$  and  $\nu$  (e.g. of  $g$ ) as follows:

$$\begin{aligned}
\frac{\partial \lambda}{\partial p_{0,0}} &= \lambda(1-u) \left( \frac{\alpha \Delta t}{1-q} + \frac{1}{p_{0,0}(q-u)} \right) - (1-q) \left( \frac{\alpha}{q} + \eta \right) \\
\frac{\partial \lambda}{\partial p_{0,1}} &= \lambda \frac{\beta \Delta t q (1-u)}{1-q} - \beta(1-q) \\
\frac{\partial \lambda}{\partial p_{1,0}} &= -\frac{\lambda}{p_{0,0}(q-u)} + \eta \frac{1-q}{1-u} \\
\frac{\partial \mu}{\partial p_{0,0}} &= \mu \left( \alpha \Delta t + \frac{1-q}{p_{0,0}(q-u)} \right) - (1-u) \left( \frac{\alpha}{q} + \eta \right) \\
\frac{\partial \mu}{\partial p_{1,0}} &= \mu \beta \Delta t q - \beta(1-u) \\
\frac{\partial \mu}{\partial p_{0,1}} &= \mu \frac{1-q}{p_{0,0}(u-1)(q-u)} + \eta \\
\frac{\partial \nu}{\partial p_{0,0}} &= \nu \left( \frac{1}{p_{0,0} \ln(p_{0,0})} + \frac{\alpha \Delta t (q-u)}{q \ln(q)} + (1-u) \left( \frac{\alpha \Delta t}{1-q} + \frac{1}{p_{0,0}(q-u)} \right) \right) \\
&\quad - r(1-q) \left( \frac{\alpha}{q} + \eta \right) \\
\frac{\partial \nu}{\partial p_{0,1}} &= \nu \beta \Delta t \left( \frac{q(1-u)}{1-q} - \frac{q-u}{\ln(q)} \right) - r\beta(1-q) \\
\frac{\partial \nu}{\partial p_{1,0}} &= -\frac{\nu}{p_{0,0}(q-u)} + r\eta \frac{1-q}{1-u}.
\end{aligned}$$

## E Proof of Theorem 4

The forward probability  $\alpha_{(i,j)}(t)$  is defined as the joint probability of observing the first  $t$  vectors and being in state  $(i, j)$  at time  $t$ . This probability can be calculated by the following recursion:

$$\begin{cases} \alpha_{(i,j)}(0) = p_{i,j} \pi_i \psi_{(i,j)}(y_0) & i, j \in \mathcal{X}, \\ \alpha_{(i,j)}(t) = \psi_{(i,j)}(y_t) p_{i,j} \sum_{i'=0}^N \alpha_{(i',i)}(t-1) & 0 < t \leq T, \quad i, j \in \mathcal{X}. \end{cases}$$

Similarly, the backward probability  $\beta_{(i,j)}(t)$  is defined as the conditional probability of observing the vectors after time  $t$ , given that the state at time  $t$  is  $(i, j)$ . The backward probability can be evaluated by the recursive formula :

$$\begin{cases} \beta_{(i,j)}(T) = 1 & i, j \in \mathcal{X}, \\ \beta_{(i,j)}(t) = \sum_{j'=0}^N p_{j,j'} \psi_{(j,j')} (y_{t+1}) \beta_{(j,j')}(t+1) & 0 \leq t < T, \quad i, j \in \mathcal{X}. \end{cases}$$

Under such consideration, the likelihood of observations rewrites as follows by summing all the forward and backward products :

$$\mathbb{P}(Y_{0:t} = y_{0:t} | M) = \sum_{i=0}^N \sum_{j=0}^N \alpha_{(i,j)}(t) \beta_{(i,j)}(t).$$

Now, let's consider the conditional probability of a transition from state  $(i, j)$  at time  $t$  to state  $(i', j')$  at time  $t + 1$ , given the observations  $Y_{0:T} = y_{0:T}$ :

$$\begin{aligned}\xi_{(i,j),(i',j')}(t) &= \mathbb{P}(Z_t = (i, j), Z_{t+1} = (i', j') | Y_{0:T} = y_{0:T}, (Q, \psi, \rho)) \\ &= \frac{\alpha_{(i,j)}(t) p_{(i',j')} \psi_{(i',j')}(y_{t+1}) \beta_{(i',j')}(t+1)}{\mathbb{P}(Y_{0:T} = y_{0:T} | (Q, \psi, \rho))} \delta_{i'=j}.\end{aligned}$$

Denote by  $\gamma_{(i,j)}(t) = \sum_{j'=0}^N \xi_{(i,j),(j',j')}(t)$  the conditional probability of being in state  $(i, j)$  at time  $t$ , given the observations  $Y_{0:T} = y_{0:T}$ :

$$\gamma_{(i,j)}(t) = \frac{\alpha_{(i,j)}(t) \beta_{(i,j)}(t)}{\mathbb{P}(Y_{0:T} = y_{0:T} | (Q, \psi, \rho))}.$$

Thus, starting from an initial characteristic  $(Q^{(0)}, \psi^{(0)}, \rho^{(0)})$ , we have the following estimations by the Baum-Welch algorithm:

$$\begin{aligned}Q_{(i,j),(i',j')}^{(n)} &= \frac{\sum_{t=0}^{T-1} \xi_{(i,j),(i',j')}(t)}{\sum_{t=0}^{T-1} \gamma_{(i,j)}(t)} \delta_{i'=j}, \\ \psi_{(i,j)}^{(n)}(y) &= \frac{\sum_{t=0}^T \mathbb{1}_{y_t=y} \gamma_{(i,j)}(t)}{\sum_{t=0}^T \gamma_{(i,j)}(t)}, \\ \rho_{i,j}^{(n)} &= \gamma_{(i,j)}(t), \\ p_{i,j}^{(n)} &= \frac{1}{N} \sum_{i'=0}^N Q_{(i',i),(i,j)}^{(n)}.\end{aligned}$$

## Declarations

The authors declare that they have no competing interests. All co-authors have seen and agree with the contents of the manuscript. We certify that the submission is original work and is not under review at any other publication.

## Availability of data and materials

The datasets used and/or analyzed during the current study available from the corresponding author on request.

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