



**HAL**  
open science

# A "Potentially Better" $\alpha$ -MaxMin Axiomatisation of Temporally-Biased Multiple Discounts

Thai Ha-Huy, Jean-Pierre Drugeon

► **To cite this version:**

Thai Ha-Huy, Jean-Pierre Drugeon. A "Potentially Better"  $\alpha$ -MaxMin Axiomatisation of Temporally-Biased Multiple Discounts. 2023. hal-04010969v1

**HAL Id: hal-04010969**

**<https://hal.science/hal-04010969v1>**

Preprint submitted on 2 Mar 2023 (v1), last revised 29 Oct 2023 (v5)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A “POTENTIALLY BETTER” $\alpha$ –MAXMIN AXIOMATISATION OF TEMPORALLY-BIASED MULTIPLE DISCOUNTS\*

Jean-Pierre Drugeon<sup>†</sup> and Thai Ha-Huy<sup>‡</sup>

March 2, 2023

---

\*The authors would like to thank Gaetano Bloise, Alain Chateauneuf, Rose-Anne Dana and Joanna Franaszek for their insightful comments. Thai Ha-Huy would like to thank the Labex MME-DII (ANR-11-LBX-0023-01) and Paris School of Economics (campus Jourdan) for supports during the writing of this article. Jean-Pierre Drugeon would like to thank PGSE and the support of the *Ambiguity in Dynamic Environments* ANR (co-funded with Queen Mary University, Bielefeld University and Kyoto University) research project. The authors would finally like to thank both referees and the associate editor for numerous suggestions and remarks that greatly helped in improving the article.

<sup>†</sup>Paris School of Economics and Centre National de la Recherche Scientifique. Email: jean-pierre.drugeon@cnrs.fr

<sup>‡</sup>Université Paris-Saclay, Univ Evry, EPEE, 91025, Evry-Courcouronnes, France. Email: thai.hahuy@univ-evry.fr

# ABSTRACT

This article introduces an axiomatic approach of utilities streams based upon three preference relations, namely the close future order, the distant future order and the main order. Assuming all these preferences to be bi-separable, the article derives an unanimous representation for weights over time periods. The analysis of two categories of a *potentially better* property allows for the establishment of *MaxMin*, *MaxMax* and  $\alpha$ -*MaxMin* representations. This is followed by the presentation of a multiple discounts rates version of  $T^*$ -temporally biased, generalising quasi-hyperbolic discounting, for the close future order. A similar analysis for the distant future is also performed where it is proved that Banach limits can be considered as the distant future counterpart of exponential discounting in the evaluation of the close future .

KEYWORDS: Axiomatisation, Myopia, Multiple Discounts,  $\alpha$ -MaxMin Criteria, Temporal Biases, Banach Limits, Infinite Dimensional Topologies.

JEL CLASSIFICATION: D11, D15, D90.

## CONTENTS

1	INTRODUCTION	1
1.1	Motivation and Results . . . . .	1
1.2	Related Literature . . . . .	5
1.3	Contents . . . . .	6
2	BASIC AXIOMS, ROBUST PRE-ORDERS, AND THE $\alpha$ -MAXMIN REPRESENTATION	7
2.1	Fundamentals, Elementary Axioms and Index Functions . . . . .	7
2.2	Robust Pre-orders . . . . .	12
2.3	Representation of the robust pre-orders . . . . .	13
2.3.1	Representation of the close future order . . . . .	13
2.3.2	Representation of the distant future order . . . . .	14
2.4	The “Potentially Better” Categories and the MaxMin, MaxMax and $\alpha$ -MaxMin Representations . . . . .	15

3	REPRESENTATIONS FOR TEMPORALLY-BIASED MULTIPLE DISCOUNTS	16
3.1	Temporal bias axiom . . . . .	17
3.2	Temporal Bias Representation of the Close future Pre-order . . . . .	18
3.3	Temporal bias representation of the distant future pre-order and Banach limits . . . . .	20

# 1. INTRODUCTION

## 1.1 MOTIVATION AND RESULTS

The introduction by [Gilboa and Schmeidler \(1989\)](#) of the *multiple priors* approach to choice under uncertainty and the relevance of the associated *worst scenario* case and Maxmin criteria has been at the very inception of the numerous contemporaneous developments of the *ambiguity* literature. It is somewhat surprising to observe that, while this *multiple priors* approach had led to a renewal of studies for many topics of the choice under uncertainty literature, it was not until [Wakai \(2007\)](#) and [Chambers and Echenique \(2018\)](#) that it got adapted to inter-temporal choice and applied to multiple discounting configurations. While the work of [Chambers and Echenique \(2018\)](#) established a worst scenario case *à la* [Gilboa and Schmeidler \(1989\)](#), [Wakai \(2007\)](#) presented a smoothing behavior on a recursive representation. Recently, [Bich et al. \(2022\)](#) extended the axiomatic system of [Chambers and Echenique \(2018\)](#) to encompass quasi-hyperbolic discounting phenomena. All of these results are however established on a *MaxMin* configuration and do not take into account the far remote future, *i.e.*, the evaluation of a utility stream is almost determined by a finite number of generations.

The central aim of this study is then to examine the scope for such an inter-temporel representation when one is concerned with discounted infinite utility streams instead of choice under uncertainty over a range of states of the world. This study develops the analysis in three regards.

First, and differing with most of the axiomatic approaches to discounting, this article takes into account the arbitrarily remote components of the utility streams. The purpose of this investigation is to provide alternative representations that would complement the ones of the near future, largely in use in most researches.

Usually, the lack of available information or the ambiguity about the system of appropriate discount rates or the difference in opinions of the experts that the economic agent consults, all concur to a situation where the decision is based upon a set of different weights systems. This study presents an approach that aggregates such

differences and extends the usual Maxmin criterion of the literature to the more general class of  $\alpha$ -MaxMin criteria, where the evaluation of an inter-temporal utility stream is defined not only from the worst evaluation but also from the best one.

While [Chambers and Echenique \(2018\)](#) analysed how regular discounting criteria could reconcile diverging opinions held by several experts, there is another part of the literature that is concerned with the anomalies and temporal biases that resulted from experimental studies. This led to an interest in the present biased *quasi-hyperbolic* discounting representation, first introduced by [Phelps and Pollack \(1968\)](#) and recently by [Laibson \(1997\)](#). This also corresponds to the third concern of this article, which is to encompass general temporal biases within a multiple discounting configuration.

Imagine indeed that a given government is to evaluate very long-run policies. One of its purposes is to balance the welfares of, on the one hand, the near, and, on the other hand, the far remote future generations. The government may rely on the advice of a group of experts in order to evaluate the close and distant future values of the projects. Such a scientific committee may gather economists, politicians, environmentalists or even, when one seeks to evaluate very long-run projects, philosophers. It is thus everything but surprising to remark that, most of the time, their opinions would significantly differ. Based upon such advices, it is assumed that the government establishes two orders  $\succeq_c$  and  $\succeq_d$  representing its preferences for the near and the far remote future.

In the evaluations under the close future order  $\succeq_c$ , the far remote future becomes negligible. Following a huge body of preferences in the literature, the value of the close future then happens to be mostly defined by the utility levels of a finite number of generations. The distant future order  $\succeq_d$ , as for itself, displays a drastically distinct behaviour: changing the utility levels of only a finite number of generations wouldn't affect such preferences. Finally, the government's total order  $\succeq$ , taking into account both the close and the distant futures, synthesizes these two classes of preferences.

Under standard conditions of axiomatic inter-temporal literature, namely translation invariance and positive homogeneity, such orders can be represented by index

functions that are constantly additive and positively homogeneous. By adding the natural Pareto condition that the options commonly preferred by  $\succeq_c$  and  $\succeq_d$  are also preferred by the total order  $\succeq$ , the index function of  $\succeq$  can be represented by a convex combination of the two other ones. Interestingly, the parameter of the convex combination is not a constant but depends on the utility stream at stake. Two configurations emerge: a first where the economic agent desires to *smooth* the difference between the close and the distant futures and a second that corresponds to the opposite behavior.

To explore further the scope for a *unanimous* comparison of experts, we present the *robust* pre-orders  $\succeq_c^*$  and  $\succeq_d^*$ . Under an order belonging to  $\{\succeq_c, \succeq_d\}$ , a given utility stream is *robustly better* than another one if and only if this comparison does not depend on the reference stream, in the sense that mixing two streams with a third one does not change the comparison between them. These robust pre-orders are generally incomplete and can be represented by an unanimity rule for multiple weights over time periods, which reflects the diversity of experts' opinions. For the close future pre-order  $\succeq_c^*$ , this set is a subset of countably additive probabilities on the set of dates, representing different discount rates systems. For the distant future pre-order  $\succeq_d^*$ , the corresponding set builds from a set of *purely finitely additive* probabilities.<sup>1</sup>

For each order  $\succeq_c$  and  $\succeq_d$ , the difference in opinions of experts may naturally lead to situations where two utility streams are *not* robustly comparable, but one has more *potential to be preferred* than the other one. This article presents two categories of the *potentially better* property. In the first category, if a utility stream is robustly better than every constant stream that is robustly dominated by the other one, then it is considered as having more *potential* to be preferred. Complementary to this, in the second category, a stream has more *potential* to be preferred if it is *not* robustly worse than a constant stream that is *not* robustly better than the other.

Under the condition ensuring that having more potential in the first category implies to be preferred, we obtain the MaxMin criterion, where the value of a utility stream is determined by the worst evaluation. Similarly, if a stream that is more potential

---

<sup>1</sup>They are also known under the name *charges*, see [Bhaskara Rao and Bhaskara Rao \(1983\)](#).

in the second category is preferred, we obtain a MaxMax criterion, where only the best evaluation is taken into account. Under a more requiring condition than the aforementioned ones, establishing that the satisfaction of having more potential in *both* categories implies to be better, we obtain an  $\alpha$ -MaxMin representation that encompasses the MaxMin and MaxMax criteria as special cases.

Facing now with the third concern of the article about the scope for present biases, we assume a weakened version of *stationarity*. The evaluation of every expert satisfies a *delayed-stationary* property in the sense that beginning after a delay of a certain period, this does not depend upon the date it was done. Therefore, in the case of a close future order, a *delayed equivalence* assumption comes into consideration. This states that, for every stream, there exists a *delayed* stream that is its *robustly improving* capacity while mixing with another delayed one. A generalization of the quasi-hyperbolic discounting representation is established. The set that characterizes the robust order is a convex hull of discount rates systems satisfying the following property: from a day in the future, the rate of trade-off between a day and its subsequence one becomes constant.

In the case of *purely finitely additive measures* that characterize the distant future robust pre-order, under the same *delay-stationarity* assumption applied to the close future, these measures actually belong to the set of *Banach limits*.<sup>2</sup> Interestingly, the evaluation of a utility stream under a Banach limit does not change if it is shifted one (or many) period(s) to the future. This property echoes the close future evaluation under *exponential discounting* for which the comparison between two streams does not depend on the period of departure. This *stability* property, or, in other words, *anonymity*, makes Banach limits, in the evaluation of the distant future, the counterpart of exponential discount rates in the evaluation of the close future.

---

<sup>2</sup>For intuition about *Banach limits*, one can have in mind the infimum limit *liminf* and the supremum limit *limsup* of utility streams. These functions satisfy every properties of *Banach limits*, minus the linearity. For a rigorous definition, see page 55 in [Becker and Boyd \(1997\)](#).



## 1.2 RELATED LITERATURE

The introduction by [Gilboa and Schmeidler \(1989\)](#) of the multiple priors approach to choice under uncertainty is the beginning of a huge literature in decision theory.<sup>3</sup> While the approach using associated worst scenario case and the Maximin criteria remains the cornerstone of most studies, other authors, such as [Ghirardato et al. \(2004\)](#), emphasized the need for a generalized criterion that distinguishes *ambiguity* from *ambiguity attitude*. They proposed the use of an alternative version which generalizes the well-known  $\alpha$ -MaxMin rule of [Hurwicz \(1951\)](#).

The  $\alpha$ -MaxMin rule of [Hurwicz \(1951\)](#) applies to settings of uncertainty where the subjective perception of ambiguity can be described through a set of probability measures and the attitude towards ambiguity. This criterion's empirical relevance in an experimental environment has been the subject of numerous studies. For a survey, see [Trautmann and van de Kuilen \(2015\)](#).

Recently, numerous topics of  $\alpha$ -MaxMin literature have been under scrutiny. To name some contributions, [Frick et al. \(2022\)](#) and [Chateauneuf et al. \(2021\)](#) have shown an interest in the non-uniqueness of a weighted representation. While the first incorporated *objective rationality* into an  $\alpha$ -MaxMin expected utility, the second studied the *falsifiability* of  $\alpha$ -Maxmin representation. In separate regards, while [Beißner et al. \(2020\)](#) was interested in the connection with time consistency, [Beißner and Werner \(2021\)](#) studied differentiability properties.

The contributions by [Kopylov \(2003\)](#) and [Ghirardato et al. \(2004\)](#) represent an axiomatization of a mix of ambiguity-averse and ambiguity-seeking tendencies. This representation is used in numerous experimental studies where the weight  $\alpha$  and the set of beliefs are often interpreted as parameters of a decision maker's ambiguity attitude and perception of ambiguity.

Due to the importance of inter-temporal analysis, efforts have been made in the axiomatic bases of discounted utilities. While pioneer studies, namely [Koopmans \(1960\)](#) and [Koopmans \(1972\)](#) are irreplaceable in this regard, [Dolmas \(1995\)](#) made an interesting clarification while [Fishburn and Rubinstein \(1982\)](#) presented an en-

---

<sup>3</sup>"A second Big Bang in theory of decision under uncertainty", see [Karni et al. \(2022\)](#).

lightening alternative. This line of research is summed up in the work of [Bleichrodt et al. \(2008\)](#). Since the prominent work of [Laibson \(1997\)](#) was published, similar efforts have been made to understand temporal biases and anomalies. Notable contributions in this direction have been made by [Chakraborty \(2017\)](#) and [Montiel Olea and Strzalecki \(2014\)](#).

Recently, two studies have been conducted on such representations generalizing the ones in [Chambers and Echenique \(2018\)](#) and [Wakai \(2007\)](#). The first, [Bich et al. \(2022\)](#), extended the [Chambers and Echenique \(2018\)](#) system of axioms to the scope for one-step present bias and quasi-hyperbolic discounting. The second, [Drugeon and Ha-Huy \(2022b\)](#), followed an approach that focuses on recursive time-dependent orders and the associated multiple time-varying discount rates. In this work, as in [Wakai \(2007\)](#), the discount rate is chosen in *each* period by a comparison between the utility value of the present and that of the future.

As a result of the technical complexity of infinite dimensional topologies, the role of arbitrarily remote components of utilities streams has been the subject of studies, such as [Brown and Lewis \(1981\)](#), [Sawyer \(1988\)](#), [Gilles \(1989\)](#), [Drugeon and Ha-Huy \(2022a\)](#) and, recently, [de Andrade et al. \(2021\)](#).

### 1.3 CONTENTS

Section 2 describes the basic axioms for a decomposition between the close and the distant future. Robust pre-orders and conditions for MaxMin, MaxMax and  $\alpha$ -MaxMin representations are also presented. Section 3 strengthens these results by incorporating them in a temporally-biased multiple discounts configuration. The mathematical preparations and the proofs are given in the Appendix.

## 2. BASIC AXIOMS, ROBUST PRE-ORDERS, AND THE $\alpha$ -MAXMIN REPRESENTATION

### 2.1 FUNDAMENTALS, ELEMENTARY AXIOMS AND INDEX FUNCTIONS

This paper adopts an axiomatic approach to the evaluation of bounded utility streams in a discrete time configuration. Letters such as  $x, y, z$  will be used for streams of utilities with values in  $\mathbb{R}$ . Denote by  $\ell_\infty$  the set of bounded real sequences.

Notation  $\mathbf{1}$  will denote the constant stream  $(1, 1, \dots)$ . Similarly,  $b\mathbf{1}$ ,  $c\mathbf{1}$  and  $c^*\mathbf{1}$  will be used for constant streams  $(b, b, \dots)$ ,  $(c, c, \dots)$  and  $(c^*, c^*, \dots)$ . The notations  $\lambda, \mu$ , and  $\chi$  will be used for constant scalars.

For every  $x \in \ell_\infty$  and  $T \geq 0$ , let  $x_{[0,T]} = (x_0, x_1, \dots, x_T)$  be its *head*  $T + 1$  first components and  $x_{[T+1,\infty)} = (x_{T+1}, x_{T+2}, \dots)$  its *tail* starting from date  $T + 1$ . Given sequences  $x$  and  $y$ , the sequence  $(y_{[0,T]}, x_{[T+1,\infty)})$  denotes  $(y_0, y_1, \dots, y_T, x_{T+1}, x_{T+2}, \dots)$ . The sequence  $(y_{[0,T]}, x)$  denotes  $(y_0, y_1, \dots, y_T, x_0, x_1, x_2, \dots)$ . By convention, if  $T = -1$ , let  $(y_{[0,T]}, x_{[T+1,\infty)})$  be the sequence  $x = (x_0, x_1, x_2, \dots)$ .

An economic agent evaluates utility streams belonging to  $\ell_\infty$ , trying to balance the welfare between the close and the distant futures. On the set of utility streams, she has two original preferences, which are represented by the *close future order*  $\succeq_c$  and the *distant future order*  $\succeq_d$ . While the close future order does not take into account the distant future, the distant future order has an opposite tendency and ignores the close future. Based on these two preferences, the economic agent establishes his or her main preference  $\succeq$ , synthesizing the comparisons made by  $\succeq_c$  and  $\succeq_d$ .

As a constructive example, imagine a situation where a government has to evaluate very long-run policies. The government may rely on the advices of a group of economists, environmentalists, politicians, or even philosophers to evaluate the values of close and distant future of projects. Each expert will give her or his opinion about an appropriate discount rates system for the close future as well as some rule

for the calculus of the distant future. The evaluation of close and distant future will be determined based upon the experts' advices.

For any expert, the rule of calculus must satisfy two following basic properties: positive homogeneity of degree one and additivity. These ensure the satisfaction of *homogeneity* and *additivity* properties. *Homogeneity* states that multiplying positive scalar to two streams does not change the expert's comparison between them. *Additivity* ensures that for the expert, the comparison between two streams is maintained after having added a third one. The preferences of an expert do not depend on the reference stream.

It goes without saying that, owing to their different backgrounds, the opinions of such experts will differ, and it is the purpose of this article to consider an axiomatic approach that aggregates the differences between them. After having consulted the experts, the preferences are described by two bi-separables orders,  $\succeq_c$  and  $\succeq_d$ . While the *additivity* property may be violated by the aggregating order  $\succeq$ , the *co-ordinality*<sup>4</sup> is preserved, ensuring that the utilities of different dates are measured by the same unit.

We now return to the basic properties concerning preferences. The intuition behind the orders in the above illustration is presented in Axiom **F1**. For any order  $\widehat{\succeq} \in \{\succeq, \succeq_c, \succeq_d\}$ , let  $x \widehat{\sim} y$  denote  $x \widehat{\succeq} y$  and  $y \widehat{\succeq} x$ . The order  $\widehat{\succeq}$  is *non-trivial* if there is  $x$  and  $y$  such that  $x \succeq y$  and  $y \not\succeq x$ . Let the notation be presented as  $x \widehat{\succ} y$ .

**AXIOM F 1.** Every order  $\widehat{\succeq}$  belonging to  $\{\succeq_c, \succeq_d, \succeq\}$  satisfies the following properties.<sup>5</sup>

- (i) *Completeness, transitivity and monotonicity* For every  $x, y \in \ell_\infty$ , either  $x \widehat{\succeq} y$  or  $y \widehat{\succeq} x$ . If  $x \widehat{\succeq} y$  and  $y \widehat{\succeq} z$ , then  $x \widehat{\succeq} z$ . If  $x_s \geq y_s$  for every  $s \in \mathbb{N}$ , then  $x \widehat{\succeq} y$ .
- (ii) *Archimedeanity* For  $x \in \ell_\infty$  and real values  $b, b'$  satisfying  $b\mathbf{1} \widehat{\succ} x \widehat{\succ} b'\mathbf{1}$ , there are  $0 < \lambda, \mu < 1$  such that  $(1 - \lambda)b\mathbf{1} + \lambda b'\mathbf{1} \widehat{\succ} x \widehat{\succ} (1 - \mu)b\mathbf{1} + \mu b'\mathbf{1}$ .

<sup>4</sup>See [Chambers and Echenique \(2018\)](#) for a detailed discussion.

<sup>5</sup>These properties are well-known in the literature, and an interested reader may find a detailed discussion about their significance in [Chambers and Echenique \(2018\)](#), [Bich et al. \(2022\)](#), or [Dugeon and Ha-Huy \(2022a\)](#).

(iii) *Constant additivity* For every stream  $x, y$  and constants  $b$ , and  $\lambda \in [0, 1)$ ,  $x \widehat{\succeq} y$  if and only if  $(1 - \lambda)x + \lambda b\mathbf{1} \widehat{\succeq} (1 - \lambda)y + \lambda b\mathbf{1}$ .

(iv) *Non-triviality of the main order* There exist  $x, y \in \ell_\infty$  such that  $x \succ y$ .

The *Completeness*, *transitivity* and *monotonicity* properties are standard in the literature. *Archimedeanity* ensures continuity with respect to the sup-norm topology in  $\ell_\infty$ . The *constant additivity* property is admittedly less immediate. It is referred to as *certainty independence* in the decision theory literature, contains the *positive homogeneity* property and ensures that constant streams are *comparison neutral*: by mixing them with another constant stream, the comparison does not change. The *positive homogeneity* property states that, for any  $x$  and  $y$ , any  $\lambda > 0$ ,  $x \succeq y$  if and only if  $\lambda x \succeq \lambda y$ . This equivalence means that every coordinate of a utilities stream is measured by the same unity.<sup>6</sup> These conditions, combined with the *non-triviality* property, have as a consequence that the order  $\widehat{\succeq}$  can be represented by an index function  $\widehat{I}$ : for every  $x$  and  $y$  in  $\ell_\infty$ ,  $x \widehat{\succeq} y$  if and only if  $\widehat{I}(x) \geq \widehat{I}(y)$ . Moreover, the function  $\widehat{I}$  satisfies constant additivity and homogeneity of degree-1 properties. More precisely, this function is defined as:

$$\widehat{I}(x) = \sup\{b \text{ such that } x \widehat{\succeq} b\mathbf{1}\}.$$

A first observation from the definition of  $\widehat{I}$  is that, for every constant  $b \in \mathbb{R}$ ,  $\widehat{I}(b\mathbf{1}) = b$ . A proof of Lemma 2.1 can be found in [Drugeon and Ha-Huy \(2022a\)](#).

**LEMMA 2.1.** *Assume axiom F1. If the order  $\widehat{\succeq}$  is non-trivial, then it can be represented by an index function  $\widehat{I}$  such that, for every  $x \in \ell_\infty$ ,  $\lambda \geq 0$  and  $b \in \mathbb{R}$ :*

(i)  $\widehat{I}(\lambda x) = \lambda \widehat{I}(x)$ ,

(ii)  $\widehat{I}(x + b\mathbf{1}) = \widehat{I}(x) + b$ .

It is easy to check that, for every constant  $b \in \mathbb{R}$ ,  $\lambda \geq 0$ ,  $\widehat{I}(\lambda x + b\mathbf{1}) = \lambda \widehat{I}(x) + b$ . By convention, if the order  $\widehat{\succeq}$  is trivial, we let  $\widehat{I}(x) = 0$  for every  $x \in \ell_\infty$ . We then have three index functions  $I_c(x)$ ,  $I_d(x)$  and  $I(x)$  corresponding to the three orders  $\succeq_c$ ,  $\succeq_d$  and  $\succeq$ .

---

<sup>6</sup>For more concrete interpretation, see [Chambers and Echenique \(2018\)](#).

The following axiom details the precise properties of these three orders. It describes the *tail insensitivity* property of the close future order and the *head insensitivity* property of the distant future order. The main order satisfies a version of the *à la Pareto* property.

**AXIOM F2.** Consider  $\{\succeq_c, \succeq_d, \succeq\}$ . The close future, distant future and main orders respectively satisfy:

- (i) *Distant future insensitivity.* Consider the close future order  $\succeq_c$ . For every  $x, y, z \in \ell_\infty$ ,  $\epsilon > 0$ , there exists  $T_0$  such that for  $T \geq T_0$ ,

$$(x_{[0,T]}, y_{[T+1,\infty)}) \succeq_c (x_{[0,T]}, z_{[T+1,\infty)}) - \epsilon \mathbf{1}.$$

- (ii) *Close future insensitivity.* Consider the distant future order  $\succeq_d$ . For every  $x, y, z \in \ell_\infty$  and  $T \geq 0$ ,

$$(y_{[0,T]}, x_{[T+1,\infty)}) \sim_d (z_{[0,T]}, x_{[T+1,\infty)}).$$

- (iii) *Consistency.* Consider the main order  $\succeq$ . For  $x, y \in \ell_\infty$ , if  $x \succeq_c y$  and  $x \succeq_d y$ , then  $x \succeq y$ .

It is easy to verify that, under axioms **F1** and **F2**, for every  $x, y \in \ell_\infty$ :

$$I_c(x) = \lim_{T \rightarrow \infty} I_c(x_{[0,T]}, y_{[T+1,\infty)}),$$

$$I_d(x) = I_d(y_{[0,T]}, x_{[T+1,\infty)}),$$

for every  $T \geq 0$ .

From now on, we always impose **F1** and **F2** on the three orders  $\succeq_c$ ,  $\succeq_d$  and  $\succeq$ .

The *consistency* condition requires that the main order never contradicts the close future and the distant future orders when these two orders are in agreement with each other. Combining with the *non-triviality* of the main order, a direct consequence of *consistency* is that at least one of the two close future and distant future orders is non-trivial.

To be more precise, if the distant future order  $\succeq_d$  is trivial, then two orders  $\succeq$  and  $\succeq_c$  are equivalent. The economic agent cares only about the close future. This is

the usual situation of literature where the remote future is negligible. Similarly, if the close future order  $\succeq_c$  is trivial, then  $\succeq$  and  $\succeq_d$  do coincide.

We now focus on the relation between the index functions. The evaluation of a utility stream is a convex combination of its close future and distant future values. Let

$$\begin{aligned}\chi_g &= \lim_{T \rightarrow \infty} I(o\mathbf{1}_{[0,T]}, \mathbf{1}_{[T+1,\infty)}), \\ \chi_\ell &= - \lim_{T \rightarrow \infty} I(o\mathbf{1}_{[0,T]}, -\mathbf{1}_{[T+1,\infty)}).\end{aligned}$$

These two values can be interpreted as the perception of the economic agent about the importance of constant gains and losses in the distant future. These two parameters both belong to the interval  $[0, 1]$ . The case  $\chi_g = \chi_\ell = 1$  corresponds to the configuration where the close future order is trivial and  $\succeq = \succeq_d$ . In opposition to this,  $\chi_g = \chi_\ell = 0$  implies that the distant future order is trivial and  $\succeq = \succeq_c$ , a well-known configuration of the literature.

Lemma 2.2 is crucial in the establishment of the formula linking the close and the distant future values of the utilities streams. The value of a stream is a convex combination of its close and distant values. If the close future value is smaller than the other one, then the parameter of this convex combination in use is  $\chi_g$ . In the opposite case, it is  $\chi_\ell$  that is selected to determine the weight in the convex combination.

**LEMMA 2.2.** *Consider a stream  $x$ .*

(i) *If  $I_c(x) \leq I_d(x)$ , then*

$$I(x) = (1 - \chi_g)I_c(x) + \chi_g I_d(x).$$

(ii) *If  $I_c(x) \geq I_d(x)$ , then*

$$I(x) = (1 - \chi_\ell)I_c(x) + \chi_\ell I_d(x).$$

From Lemma 2.2, we can establish two different behaviors. The first one corresponds to the situation where the economic agent desires to *smooth* the difference between the close future and the distant future values. The second one exhibits an opposite behavior.

**PROPOSITION 2.1.** Consider  $\chi_g$  and  $\chi_\ell$ .

(i) If  $\chi_g \leq \chi_\ell$ , then, for every stream  $x$ ,

$$I(x) = \min_{\chi_g \leq \lambda \leq \chi_\ell} [(1 - \lambda)I_c(x) + \lambda I_d(x)].$$

(ii) If  $\chi_g \geq \chi_\ell$ , then, for every stream  $x$ ,

$$I(x) = \max_{\chi_\ell \leq \lambda \leq \chi_g} [(1 - \lambda)I_c(x) + \lambda I_d(x)].$$

As an example, consider the order  $\succeq$  that is represented as follows, with  $0 < \chi_g < \chi_\ell < 1$  and  $D$  a compact subset of  $(0, 1)$ :

$$I(x) = \min_{\chi_g \leq \lambda \leq \chi_\ell} \left[ (1 - \lambda) \min_{\delta \in D} \left( (1 - \delta) \sum_{s=0}^{\infty} \delta^s x_s \right) + \lambda \liminf_{s \rightarrow \infty} x_s \right].$$

In this example, the initial order  $\succeq$  can be decomposed into two sub-orders  $\succeq_c$  and  $\succeq_d$  with two associated index functions that are available as:

$$I_c(x) = \min_{\delta \in D} \left( (1 - \delta) \sum_{s=0}^{\infty} \delta^s x_s \right),$$

$$I_d(x) = \liminf_{s \rightarrow \infty} x_s.$$

## 2.2 ROBUST PRE-ORDERS

With the understanding that the orders  $\succeq_c$  and  $\succeq_d$  are established and characterized by the set representing the divergence between the opinions, a question naturally emerges: could there be an order or a pre-order representing an *unanimous* comparison in experts' opinions about the utilities streams?

Since an expert gives her or his evaluation rule that satisfies the *homogeneity* and the *additivity* properties, it is intuitive to assume that an *unanimous* pre-order must also satisfy these two properties. We may thus consider a definition for the *pre-order* featuring the *robustness*: whatever the mixture with a common component, the comparison would not be modified.

**DEFINITION 2.1.** Consider an order  $\widehat{\succeq} \in \{\succeq_c, \succeq_d\}$ . Let the pre-order  $\widehat{\succeq}^*$  be defined by

$$x \widehat{\succeq}^* y \quad \text{iff, for every } 0 < \lambda < 1, z \in \ell_\infty, \text{ we have } (1 - \lambda)x + \lambda z \widehat{\succeq} (1 - \lambda)y + \lambda z.$$



We may interpret *robust dominance* as to be preferred independently from the reference stream. If we begin from  $z$ , it is always better mixing it with  $x$  than with  $y$ . Similarly to the [Bewley \(2002\)](#) model, this pre-order is not complete and is represented by an unanimity rule for multiple weights over time periods. These rules characterize the difference between the opinions of the experts. [Section 2.3](#) will present in detail their characterization.

## 2.3 REPRESENTATION OF THE ROBUST PRE-ORDERS

In the same vein as [Gilboa and Schmeidler \(1989\)](#), this approach leads to a characterization of the order  $\widehat{\succeq}$  by a set of continuous linear functions on  $\ell_\infty$ . These can be considered as *finitely additive* measures on the set of natural numbers  $\{0, 1, 2, \dots\}$ . Interestingly, they are *countably additive* in the case of the close future order  $\succeq_c$ , and *purely finitely additive* in the case of the distant future order  $\succeq_d$ . The definition of these notions can be found in the [Appendix A](#).

### 2.3.1 REPRESENTATION OF THE CLOSE FUTURE ORDER

In [Section 2](#), for any sequences  $x$  and  $y$ , the value  $I_c(x_{[0,T]}, y_{[T+1,\infty)})$  converges to  $I_c(x)$  when  $T$  tends to infinity. However, this convergence is not uniform: the order  $\succeq_c$  does not necessarily satisfy the usual *tail-insensitivity* condition of the literature. [Axiom A1](#) ensures the satisfaction of this property, which is the *close future* version of the well-known axiom *continuity at infinity* of [Chambers and Echenique \(2018\)](#). It assumes a strong version of myopia and imposes a limit for the disagreements between the experts in the evaluation of the close future. Though they can propose different discount rates systems, they all give a very small value for sufficiently remote dates.

**AXIOM A 1.** For every  $0 < c < 1$ , there exists  $T_0$  such that, for every  $T \geq T_0$ ,

$$(\mathbf{1}_{[0,T]}, \mathbf{0}_{[T+1,\infty)}) \succeq_c^* c\mathbf{1}.$$

Under [axiom A1](#), the robust pre-order  $\succeq_c^*$  is represented by a set of weights  $\Omega_c$  that builds from countably additive probabilities. To be more precise,  $\Omega_c$  is a set of

sequences  $\omega = (\omega_0, \omega_1, \omega_2, \dots)$  such that  $\omega_s \geq 0$  for every  $s$ , and  $\sum_{s=0}^{\infty} \omega_s = 1$ . Any probability can be considered as a possible system of discount rates that is used to evaluate the close future.

**PROPOSITION 2.2.** *Suppose that the close future order  $\succeq_c$  is non-trivial. Under the axiom **A1**, there exists a set  $\Omega_c \subset \ell_1$  that is compact with respect to the weak topology in  $\ell_1$  and satisfies the two following properties:*

(i) *For every  $\omega \in \Omega_c$ ,  $\omega_s \geq 0$ ,  $\forall s$ , and  $\sum_{s=0}^{\infty} \omega_s = 1$ .*

(ii) *For every streams  $x$  and  $y$ ,  $x \succeq_c^* y$  if and only if,*

$$\sum_{s=0}^{\infty} \omega_s x_s \geq \sum_{s=0}^{\infty} \omega_s y_s,$$

*for every  $\omega \in \Omega_c$ .*

As a remark, observe that since  $x \succeq_c^* y$  implies  $x \succeq_c y$ , for every stream  $x$ ,

$$\inf_{\omega \in \Omega_c} \sum_{s=0}^{\infty} \omega_s x_s \leq I_c(x) \leq \sup_{\omega \in \Omega_c} \sum_{s=0}^{\infty} \omega_s x_s.$$

### 2.3.2 REPRESENTATION OF THE DISTANT FUTURE ORDER

Since the distant future order  $\succeq_d$  does not take into account the close future, the robust pre-order  $\succeq_d^*$  satisfies that same property and depends only on the distant future of the utilities streams. It is characterized by a set  $\Omega_d$  that builds from *purely finitely additive* properties.

Under the evaluation of a continuous linear function  $\phi \in \Omega_d$ , the value of a stream  $x$ ,  $\phi \cdot x$  depends only on the distant behavior of  $x$  and does not change if there are only a finite number of changes in the values  $x_s$ . More precisely, for every stream  $y$ , we have

$$\phi \cdot (y_{[0,T]}, x_{[T+1,\infty)}) = \phi \cdot x, \text{ for every } T \geq 0.$$

Hence, the value  $\phi \cdot x$  can be interpreted as the evaluation of the stream  $x$  in the distant future and under  $\phi$ .

**PROPOSITION 2.3.** *Suppose that the distant future order  $\succeq_d$  is not trivial. There exists a set of purely finitely additive probabilities  $\Omega_d$  such that  $x \succeq_d^* y$  if and only if  $\phi \cdot x \geq \phi \cdot y$  for every  $\phi \in \Omega_d$ .*

Observe that since  $x \succeq_d^* y$  implies  $x \succeq_d y$ , for every stream  $x$ ,

$$\inf_{\phi \in \Omega_d} \phi \cdot x \leq I_d(x) \leq \sup_{\phi \in \Omega_d} \phi \cdot x.$$

## 2.4 THE “POTENTIALLY BETTER” CATEGORIES AND THE MAXMIN, MAXMAX AND $\alpha$ -MAXMIN REPRESENTATIONS

Once the robust pre-orders  $\succeq_c^*$  and  $\succeq_d^*$  have been established, a natural interest raises about situations where the two utility streams are *not* robustly comparable, but one has more *potential to be preferred* than the other one. For any of the two orders  $\succeq_c$  or  $\succeq_d$ , we will define two categories of being *potentially better*.

Consider two utility streams  $x$  and  $y$ . The first category of potentially better is defined as follows. If  $x$  is robustly better than every constant stream that is robustly dominated by  $y$ , then it is considered as having more *potential to be preferred*. The second category is defined as  $x$  having more *potential to be preferred* if it is *not* robustly worse than a constant stream that is *not* robustly better than  $y$ . More formally,

**DEFINITION 2.2.** Consider an order  $\widehat{\succeq} \in \{\succeq_c, \succeq_d\}$

- (i) Under the order  $\widehat{\succeq}$ ,  $x$  is potentially better than  $y$  in the first category if, for every constant  $b$ ,

$$y \widehat{\succeq}^* b\mathbf{1} \text{ implies } x \widehat{\succeq}^* b\mathbf{1}.$$

- (ii) Under the order  $\widehat{\succeq}$ ,  $x$  is potentially better than  $y$  in the second category if, for every constant  $b$ ,

$$b\mathbf{1} \not\widehat{\succeq}^* y \text{ implies } \widehat{b}\mathbf{1} \not\widehat{\succeq}^* x.$$

Under the axiom ensuring that having more potential in the first category implies to be preferred, we obtain a MaxMin criterion, where the value of a utility stream is determined by the worst evaluation. Similarly, if a stream that is potentially better in the second category is also the preferred one, we obtain a MaxMax criterion, where only the best evaluation is taken into account. Under a more requiring condition than the aforementioned ones, establishing that the satisfaction of having

more potential in both categories implies to be preferred, we obtain an  $\alpha$ -MaxMin representation that encompasses the MaxMin and the MaxMax criteria as special cases.

**PROPOSITION 2.4.** *Consider the order  $\widehat{\succeq} \in \{\succeq_c, \succeq_d\}$  and its corresponding pre-order  $\widehat{\succeq}^*$ . Assume that this order is non-trivial.*

- (i) *If, for every streams  $x$  and  $y$ ,  $x$  being potentially better than  $y$  in the first category implies  $x \widehat{\succeq} y$ , then the order  $\widehat{\succeq}$  has a MaxMin representation:*

$$\widehat{I}(x) = \inf_{P \in \widehat{\Omega}} P \cdot x$$

*for every stream  $x$ .*

- (ii) *If, for every streams  $x$  and  $y$ ,  $x$  being potentially better than  $y$  in the second category implies  $x \widehat{\succeq} y$ , then the order  $\widehat{\succeq}$  has a MaxMax representation:*

$$\widehat{I}(x) = \sup_{P \in \widehat{\Omega}} P \cdot x$$

*for every stream  $x$ .*

- (iii) *If, for every streams  $x$  and  $y$ ,  $x$  being potentially better than  $y$  in both categories implies  $x \widehat{\succeq} y$ , then the order  $\widehat{\succeq}$  has an  $\alpha$ -MaxMin representation: there exists unique  $0 \leq \alpha \leq 1$  such that for every stream  $x$ ,*

$$\widehat{I}(x) = \alpha \sup_{P \in \widehat{\Omega}} P \cdot x + (1 - \alpha) \inf_{P \in \widehat{\Omega}} P \cdot x.$$

**REMARK 2.1.** Following two distinct approaches, [Chateauneuf et al. \(2021\)](#) and [Frick et al. \(2022\)](#) are interested in the uniqueness of the  $\alpha$ -MaxMin representation. In this regard, the *potential to be preferred* categories in this article echoes in some-way *security* and *potential dominance* properties in [Frick et al. \(2022\)](#). The current representation also avoids the *falsifiability* line of criticism of the later authors. The sets of linear orders being defined as positive polar cones with respect to robust orders, they are unique as well as the parameters  $\alpha$  in Proposition 2.4.

### 3. REPRESENTATIONS FOR TEMPORALLY-BIASED MULTIPLE DISCOUNTS

### 3.1 TEMPORAL BIAS AXIOM

In order to properly characterize the sets  $\Omega_c$  and  $\Omega_d$ , consider the definition 3.1, which characterizes the impatience and stationary properties of the robust pre-orders. Fix an order  $\widehat{\succeq} \in \{\succeq_c, \succeq_d\}$  and its pre-order  $\widehat{\succeq}^*$ . Definition 3.1 represents the intuition that beginning from a sufficiently remote date in the future, the evaluation of every expert becomes *stationary*. After a certain day, her or his opinion about the comparison between two streams does not depend on the day of the start. The *temporal bias* phenomenon, one of the main causes for the violation of the stationary property, has only a *finite range of influence on the experts*. In fact, there exists a date such that their evaluations all become stationary afterwards.

**DEFINITION 3.1.** Impatience and  $T^*$ -delay stationarity Set  $T^* \geq 0$ . The order  $\widehat{\succeq}$  satisfies  $T^*$ -delay stationarity if, for every  $x \in \ell_\infty$  and a constant  $b$  such that

$$(b\mathbf{1}_{[0, T^*-1]}, x) \widehat{\succeq}^* b\mathbf{1},$$

we have

$$(b\mathbf{1}_{[0, T^*-1]}, x) \widehat{\succeq}^* (b\mathbf{1}_{[0, T^*]}, x) \widehat{\succeq}^* b\mathbf{1}.$$

To be more precise, for an order  $\widehat{\succeq}$  satisfying this definition:

- (i) Case  $T^* = 0$  corresponds to the *Stationarity* property:

$$x \widehat{\succeq}^* b\mathbf{1} \text{ implies } x \widehat{\succeq}^*(b, x) \widehat{\succeq}^* b\mathbf{1}.$$

- (ii) Case  $T^* = 1$  corresponds to the *Quasi-hyperbolic discounting* property:

$$(b, x) \widehat{\succeq}^* b\mathbf{1} \text{ implies } (b, x) \widehat{\succeq}^*(b, b, x) \widehat{\succeq}^* b\mathbf{1}.$$

- (iii) Case  $T^* \geq 1$  can be considered as a  $T^*$ -steps *quasi-hyperbolic discounting* property:

$$\underbrace{(b, b, \dots, b, x)}_{T^* \text{ times}} \widehat{\succeq}^* b\mathbf{1} \text{ implies } \underbrace{(b, b, \dots, b, x)}_{T^* \text{ times}} \widehat{\succeq}^* \underbrace{(b, b, \dots, b, x)}_{T^*+1 \text{ times}} \widehat{\succeq}^* b\mathbf{1}.$$

The choice to build the *impatience* and *stationarity* properties from the comparison of a stream with another constant one is based on the purpose of practicability. It

is indeed easier to ask an economic agent, or to observe her or his behavior whether she or he values a utility stream of at least  $b$  or not.

In definition 3.1, the first comparison characterizes impatience whereas the second one features  $T^*$ -delay stability. In other words, if a combination is robustly better than a constant sequence, it remains robustly better if it is moved forward into the future, even though the effect according to order  $\succeq^*$  becomes weaker over time.

### 3.2 TEMPORAL BIAS REPRESENTATION OF THE CLOSE FUTURE PRE-ORDER

If the close order  $\succeq_c$  satisfies definition 3.1, one can obtain a characterization for the sets of probabilities  $\Omega$ . For a stream  $x$ , define  $\mathcal{C}(x)$  as the supremum of the values  $c \in \mathbb{R}$  such that  $x \succeq_c^* c\mathbf{1}$ .

AXIOM A2. Let the close future order  $\succeq_c$  satisfy the condition in definition 3.1. For a utilities stream  $x$ , let  $c = \mathcal{C}(x)$ . There exists a utilities stream  $y$  such that

$$(i) \quad \mathcal{C}((c\mathbf{1}_{[0, T^*-1]}, y)) = c,$$

(ii) for every stream  $\hat{y}$  satisfying  $\mathcal{C}((c\mathbf{1}_{[0, T^*-1]}, \hat{y})) = c$ , one has

$$\mathcal{C}\left(\frac{1}{2}x + \frac{1}{2}(c\mathbf{1}_{[0, T^*-1]}, \hat{y})\right) > \text{ciff} \mathcal{C}\left(\frac{1}{2}(c\mathbf{1}_{[0, T^*-1]}, y) + \frac{1}{2}(c\mathbf{1}_{[0, T^*-1]}, \hat{y})\right) > c.$$

This axiom states the existence of a  $T^*$ -delay equivalence. Let us make an intuition about this property. Assume first that some *stationarity* property is imposed on the robust order  $\succeq_c^*$ . We can present this as: if  $\mathcal{C}((c\mathbf{1}_{[0, T^*-1]}, \hat{y})) = c$ , then for every  $T \geq 0$ ,

$$\mathcal{C}\left(\frac{1}{2}x + \frac{1}{2}(c\mathbf{1}_{[0, T-1]}, \hat{y})\right) > \text{ciff} \mathcal{C}\left(\frac{1}{2}(c\mathbf{1}_{[0, T-1]}, x) + \frac{1}{2}(c\mathbf{1}_{[0, T-1]}, \hat{y})\right) > c.$$

This means that, for a  $T$ -delay version of a stream  $\hat{y}$ , if *mixing* with  $x$  robustly improves the preferences, then the same is satisfied for the  $T$ -delay version of  $x$ . In other words, the robust improvement capacity does not depend on the beginning date of the stream. As a result of the temporal bias phenomenon, this stationarity property is not satisfied. Axiom A1 can be considered as a weaken version of it,

ensuring the existence of a  $T^*$ -delay equivalence  $(c\mathbf{1}_{[0, T^*-1]}, y)$  that preserves the robust improvement capacity of  $x$ .

**PROPOSITION 3.1.** *Assume axiom **A1**. Assume also that the order  $\succeq_c$  is non-trivial and satisfies the Impatience and  $T^*$ -delay stability property.*

- (i) *Stationarity* If  $T^* = 0$ , then there exists  $\mathcal{D} \subset (0, 1)$  such that  $\Omega_c$  is the convex hull of probabilities

$$\{(1 - \delta, (1 - \delta)\delta, \dots, (1 - \delta)\delta^s, \dots)\}_{\delta \in \mathcal{D}}.$$

- (ii) *Quasi-hyperbolic discounting* Consider the case  $T^* = 1$ . By adding axiom **A2**, there exists  $\mathcal{D} \in (0, 1)^2$  such that  $\Omega_c$  is the convex hull of the probabilities

$$\{(1 - \delta_o, \delta_o(1 - \delta), \delta_o\delta(1 - \delta), \delta_o\delta^2(1 - \delta), \dots, \delta_o(1 - \delta)\delta^s, \dots)\}_{(\delta_o, \delta) \in \mathcal{D}}.$$

- (iii)  *$T^*$ -steps quasi hyperbolic discounting* Consider the case  $T^* \geq 1$ . By adding axiom **A2**, there exists  $\mathcal{D} \in (0, 1)^{T^*+1}$  such that  $\Omega_c$  is the convex hull of the set of probabilities:

$$\{(1 - \delta_o, \delta_o(1 - \delta_1), \delta_o\delta_1(1 - \delta_2), \dots, \delta_o\delta_1 \cdots \delta_{T^*-1}(1 - \delta), \dots, \delta_o\delta_1 \cdots \delta_{T^*-1}\delta^s(1 - \delta), \dots)\}_{(\delta_o, \delta_1, \dots, \delta_{T^*-1}, \delta) \in \mathcal{D}}.$$

Chambers and Echenique (2018) imposed an *indifference stationarity* axiom, which supposes that for any  $x$  that is equivalent to a constant sequence  $c\mathbf{1}$ ,  $x$  is equivalent to any convex combination between  $x$  and  $(c\mathbf{1}_{[0, T]}, x)$ , for every  $T$ . In a recent work, dealing with multiple temporal biased discount rates, Bich et al. (2022), working in a similar axiomatic system configuration as Chambers and Echenique (2018), generalized the *Invariance to stationary relabelling* in Chambers and Echenique (2018) to a similar one, namely *delay-Invariance to stationary relabeling*, and obtain a multiple quasi-hyperbolic discounting representation. This condition and *Impatience and  $T^*$ -delay stationary* are similar in the sense that they both impose a weaker version of *anonymity*, usually observed in exponential discounting.

The difference between the two papers mentioned and this one essentially comes from the fact that while Chambers and Echenique (2018) and Bich et al. (2022)

worked on a complete order  $\succeq$  and established a Maxmin representation of the index function, this study works on a robust pre-order  $\succeq^*$ , corresponding to a larger family of possible orders and index functions, such as the  $\alpha$ -MaxMin representation. Unsurprisingly, the two different approaches involve two rather different systems of axioms.

### 3.3 TEMPORAL BIAS REPRESENTATION OF THE DISTANT FUTURE PRE-ORDER AND BANACH LIMITS

Under the condition that  $\succeq_d$  satisfies the *Impatience and  $T^*$ -delay stability* property, one can also obtain important properties of the set of purely finitely additive measures that characterize the pre-orders  $\succeq_d^*$ . The set  $\Omega_d$  builds from *Banach limits*. This echoes a similar property to *exponential discounting* where the comparison between two sequences does not depend on the chosen date of the departure.

**PROPOSITION 3.2.** *Suppose that  $\succeq_d$  is not trivial and satisfies Impatience and  $T^*$ -delay stability with  $T^* \geq 1$ . Then every purely finitely additive probability  $\phi \in \Omega_d$  is a Banach limit: for every  $x \in \ell_\infty$ ,*

$$\phi \cdot x = \phi \cdot (o, x).$$

From this result, we have  $\phi \cdot x \geq \phi \cdot y$  if and only if  $\phi \cdot (o, x) \geq \phi \cdot (o, y)$ . This property echoes the *anonymity* of exponential discounting. Hence, the Banach limits in the evaluation of the distant future can be considered as a counterpart of exponential discounting in the evaluation of the close future.

Axiom **A3** establishes a *prudence* property for the distant future order  $\succeq_d$ . The utilities stream dominates (or is dominated) by a constant one if and only if its utility values are all greater (or worse) in a sufficiently distant future.

**AXIOM A3.** *Prudence in the distant future* For any  $x \in \ell_\infty$  and  $b \in \mathbb{R}$ ,

- (i) If there exist  $\epsilon > 0$  and an infinite number of times  $s$  such that  $b > x_s + \epsilon$ , then  $x \not\succeq_d^* b\mathbf{1}$ .
- (ii) If there exist  $\epsilon > 0$  and an infinite number of times  $s$  such that  $x_s > b + \epsilon$ , then  $b\mathbf{1} \not\succeq_d^* x$ .



**PROPOSITION 3.3.** *Assume that the order  $\succeq_d$  satisfies condition (iii) in Proposition 2.4. Adding axiom **A3**, there exists unique  $0 \leq \alpha_d \leq 1$  such that the distant future index function can be represented as:*

$$I_d(x) = \alpha_d \limsup_{s \rightarrow \infty} x_s + (1 - \alpha_d) \liminf_{s \rightarrow \infty} x_s.$$

## A. MATHEMATICAL PREPARATIONS

### A.1 TOPOLOGIES

The set of bounded utility streams  $\ell_\infty$  is equipped the sup-norm topology. A sequence of streams  $\{x_n\}_{n=0}^\infty \subset \ell_\infty$  converges to  $x$  in this topology if

$$\lim_{n \rightarrow \infty} \left[ \sup_{s \geq 0} |x_{n,s} - x_s| \right] = 0.$$

The set of continuous linear functions defined on  $\ell_\infty$ , namely its *dual*, is denoted by  $(\ell_\infty)^*$ . This set can be decomposed as  $(\ell_\infty)^* = \ell_1 \oplus \ell_1^d$ . The set  $\ell_1$  is constituted of sequences  $\omega = (\omega_0, \omega_1, \omega_2, \dots)$  such that

$$\sum_{s=0}^{\infty} |\omega_s| < \infty.$$

A sequence of  $\{\omega^n\}_{n=0}^\infty \subset \ell_1$  converges to  $\omega$  in *weak*-topology if for every  $x \in \ell_\infty$ , we have

$$\lim_{n \rightarrow \infty} \sum_{s=0}^{\infty} \omega_s^n x_s = \sum_{s=0}^{\infty} \omega_s x_s.$$

Sometime, instead of  $\sum_{s=0}^{\infty} \omega_s x_s$ , we can write simply  $\omega \cdot x$ . If  $\omega_s \geq 0$  for every  $s$  and  $\sum_{s=0}^{\infty} \omega_s = 1$ , we call  $\omega$  a *countably additive* probability.

The set  $\ell_1^d$  is constituted by *purely finitely signed* measures.<sup>7</sup> In this article, we will focus only on a special subset of it, the set of *purely finitely* probabilities, with a characterization that will be presented in the second part of this section.

---

<sup>7</sup>See [Dunford and Schwartz \(1966\)](#).

## A.2 CHARACTERIZATION SET OF THE ROBUST PRE-ORDER AND PROBABILITY DECOMPOSITION

Fix an order  $\widehat{\succ} \in \{\succ_c, \succ_d\}$ . Suppose that this order is not trivial. Define  $\widehat{\Omega}$  as:

$$\widehat{\Omega} = \{P \in (\ell_\infty)^* \text{ such that } P \cdot x \geq 0 \text{ for every } x \widehat{\succ}^* 0 \mathbf{1} \text{ and } P \cdot \mathbf{1} = 1\}.$$

It is obvious that if a stream  $x$  satisfies  $x_s \geq 0$  for every  $x$ , then  $P \cdot x \geq 0$  for every  $P \in \widehat{\Omega}$ . This means  $P$  can be considered as a measure on the set of natural numbers  $\{0, 1, 2, \dots\}$ , in the sense that for every subset  $S \subset \{0, 1, 2, \dots\}$ , we may define  $P(S)$  as  $P \cdot x$ , where  $x_s = 1$  if  $s \in S$  and  $x_s = 0$  if  $s \notin S$ . Using Theorems 1.23 and 1.24 in [Yosida and Hewitt \(1952\)](#), each  $P$  belonging to  $\widehat{\Omega}$  is a *finitely additive* probability on  $\{0, 1, 2, \dots\}$  and can be decomposed as

$$P = (1 - \lambda)\omega + \lambda\phi,$$

where  $0 \leq \lambda \leq 1$ ,  $\omega$  is a *countably additive* probability belonging to  $\ell_1$ , and  $\phi$  is a *purely finitely additive* probability belonging to  $\ell_1^d$ .

To be precise,  $\phi$  satisfies the following property: if  $\tilde{\omega} \in \ell_1$  such that  $\tilde{\omega}_s \geq 0 \forall s$  and for every subset  $S \subset \{0, 1, 2, \dots\}$ , we have  $\sum_{s \in S} \tilde{\omega}_s \leq \phi(S)$ , then  $\tilde{\omega}_s = 0$  for every  $s$ . In other words, the evaluation of  $x \in \ell_\infty$  under  $\phi$ , the value  $\phi \cdot x$ , is not affected if we change only a finite numbers of values  $x_s$ . For every  $y \in \ell_\infty$ ,  $\phi \cdot (y_{[0,T]}, x_{[T+1,\infty)}) = \phi \cdot x, \forall T \geq 0$ .

This decomposition will be used in the establishment of finitely additive probability that characterizes the robust order  $\widehat{\succ}$ , [Propositions 2.2 and 2.3](#).

## B. PROOF OF LEMMA 2.2

Fix  $x \in \ell_\infty$ . To simplify the exposition, let  $a = I_c(x)$  and  $b = I_d(x)$ . We will prove that the evaluation of the whole stream  $I(x)$  is a convex combination of  $a$  and  $b$  and that the parameter of the convex combination depends on these two values.

According to *monotonicity*, the parameters  $\chi_g$  and  $\chi_\ell$  are limits of decreasing sequences. Hence, they are well defined.

Now, we prove the following assertion

$$I(x) = \lim_{T \rightarrow \infty} I(a\mathbf{1}_{[0,T]}, b\mathbf{1}_{[T+1,\infty)}).$$

This is obviously true if the close future order  $\succeq_c$  is trivial, with a consequence that  $\succeq = \succeq_d$ . Consider the case that  $\succeq_c$  is non-trivial. Fix any  $\epsilon > 0$ . From the *distant future insensitivity* property, we have

$$\begin{aligned} I_c(x) &> I_c((a - \epsilon)\mathbf{1}) \\ &= \lim_{T \rightarrow \infty} I_c((a - \epsilon)\mathbf{1}_{[0,T]}, (b - \epsilon)\mathbf{1}_{[T+1,\infty)}). \end{aligned}$$

Hence, for  $T$  that is sufficiently large,  $x \succeq_c ((a - \epsilon)\mathbf{1}_{[0,T]}, (b - \epsilon)\mathbf{1}_{[T+1,\infty)})$ .

From the *close future insensitivity* property,

$$\begin{aligned} I_d(x) &\geq I_d((b - \epsilon)\mathbf{1}) \\ &= I_d((a - \epsilon)\mathbf{1}_{[0,T]}, (b - \epsilon)\mathbf{1}_{[T+1,\infty)}), \end{aligned}$$

for every  $T$ . Hence,  $x \succeq_d ((a - \epsilon)\mathbf{1}_{[0,T]}, (b - \epsilon)\mathbf{1}_{[T+1,\infty)})$ .

According to the *consistency* property, for every  $T$  that is sufficiently large, we have

$$x \succeq ((a - \epsilon)\mathbf{1}_{[0,T]}, (b - \epsilon)\mathbf{1}_{[T+1,\infty)}).$$

This implies

$$I(x) \geq \limsup_{T \rightarrow \infty} I(a\mathbf{1}_{[0,T]}, b\mathbf{1}_{[T+1,\infty)}) - \epsilon.$$

Using the same arguments, we get

$$I(x) \leq \liminf_{T \rightarrow \infty} I(a\mathbf{1}_{[0,T]}, b\mathbf{1}_{[T+1,\infty)}) + \epsilon.$$

Since  $\epsilon$  is chosen arbitrarily,

$$I(x) = \lim_{T \rightarrow \infty} I(a\mathbf{1}_{[0,T]}, b\mathbf{1}_{[T+1,\infty)}).$$

The assertion proven, assume that  $I_c(x) \leq I_d(x)$ . Then

$$\begin{aligned} I(x) &= \lim_{T \rightarrow \infty} I(a\mathbf{1}_{[0,T]}, b\mathbf{1}_{[T+1,\infty)}) \\ &= \lim_{T \rightarrow \infty} I(o\mathbf{1}_{[0,T]}, (b - a)\mathbf{1}_{[T+1,\infty)}) + a \\ &= (b - a) \lim_{T \rightarrow \infty} I(o\mathbf{1}_{[0,T]}, \mathbf{1}_{[T+1,\infty)}) + a \\ &= (b - a)\chi_g + a \\ &= (1 - \chi_g)I_c(x) + \chi_g I_d(x). \end{aligned}$$

Consider the case  $I_c(x) \geq I_d(x)$ . Observe that

$$\chi_\ell = 1 - \lim_{T \rightarrow \infty} I(\mathbf{1}_{[0,T]}, 0\mathbf{1}_{[T+1,\infty)}).$$

Using the same arguments as in the first part of the proof, we have

$$I(x) = (1 - \chi_\ell)I_c(x) + \chi_\ell I_d(x).$$

## C. PROOF OF PROPOSITION 2.1

Consider the case  $\chi_g \leq \chi_\ell$ . For a stream  $x$ , if  $I_c(x) \leq I_d(x)$ , we have

$$\begin{aligned} I(x) &= (1 - \chi_g)I_c(x) + \chi_g I_d(x) \\ &= \min_{\chi_g \leq \lambda \leq \chi_\ell} [(1 - \lambda)I_c(x) + \lambda I_d(x)]. \end{aligned}$$

The last equality comes from  $I_c(x) \leq I_d(x)$  and  $\chi_g \leq \chi_\ell$ .

If  $I_c(x) \geq I_d(x)$ , we have

$$\begin{aligned} I(x) &= (1 - \chi_\ell)I_c(x) + \chi_\ell I_d(x) \\ &= \min_{\chi_g \leq \lambda \leq \chi_\ell} [(1 - \lambda)I_c(x) + \lambda I_d(x)]. \end{aligned}$$

The last equality comes from  $I_c(x) \geq I_d(x)$  and  $\chi_g \leq \chi_\ell$ .

Using the same arguments, for every stream  $x$ ,

$$I(x) = \max_{\chi_\ell \leq \lambda \leq \chi_g} [(1 - \lambda)I_c(x) + \lambda I_d(x)].$$

## D. PROOF OF PROPOSITION 2.2

Relying upon the same arguments as in Section A, there exists a set of finitely additive probabilities  $\Omega_c \subset (\ell_\infty)^*$  such that

$$x \succeq_c^* y \Leftrightarrow P \cdot x \geq P \cdot y,$$

for every  $P \in \Omega_c$ . By [Yosida and Hewitt \(1952\)](#), every  $P \in \Omega_c$  can be decomposed as  $P = (1 - \lambda)\omega + \lambda\phi$ , with  $\omega \in \ell_1$  is a countable additivity probability and  $\phi$  is a purely finitely additive probability.

Suppose that  $\lambda\phi \neq 0$ , or equivalently  $\lambda\phi \cdot \mathbf{1} > 0$ . Fix  $c$  such that  $1 - \lambda < c < 1$ .

Consider axiom **A1**, there exists a large enough  $T_0$  such that for  $T \geq T_0$ ,

$$(\mathbf{1}_{[0,T]}, 0\mathbf{1}_{[T+1,\infty)}) \succeq_c^* c\mathbf{1}.$$

Hence,

$$((1 - \lambda)\omega, \lambda\phi) \cdot (\mathbf{1}_{[0,T]}, 0\mathbf{1}_{[T+1,\infty)}) \geq c,$$

with a direct consequence that  $1 - \lambda \geq c$ , a contradiction.

To sum up, for every  $((1 - \lambda), \lambda\phi) \in \Omega_c$ ,  $\lambda\phi = 0$ . Hence,  $\Omega_c$  can be considered a subset of probabilities that is included in  $\ell_1$ . Moreover, for every  $\epsilon > 0$ , there exists  $N$  such that  $\sum_{s=0}^N \omega_s > 1 - \epsilon$  for every  $\omega \in \Omega_c$ . By Dunford-Petit criterion in [Dunford and Schwartz \(1966\)](#), this implies the weakly compactness of  $\Omega_c$  in  $\ell_1$ . This property will be used in the proof of [Proposition 3.1](#).

## E. PROOF OF PROPOSITION 2.3

Let  $\Omega_d$  be the set of finitely additive probabilities being defined as  $P \in \Omega_d$  if and only if  $P \cdot \mathbf{1} = 1$  and  $P \cdot x \geq 0$  for every  $x$  such that  $x \succeq_d^* 0\mathbf{1}$ .

As presented in [Section A](#), by [Yosida and Hewitt \(1952\)](#), a probability  $P \in \Omega_d$  can be decomposed as  $P = (1 - \lambda)\omega + \lambda\phi$ , with  $\omega \in \ell_1$  is a countably additive probability and  $\phi$  is a purely finitely additive probability belonging to  $\ell_d^1$ .

We prove that  $(1 - \lambda)\omega = 0$ . Indeed, suppose the contrary. Then  $\lambda < 1$  and there exists  $T$  such that  $\omega_T > 0$ . Take a constant  $c > 0$  such that  $(1 - \lambda)\omega_T c > \lambda$  and let  $x = (-c\mathbf{1}_{[0,T]}, \mathbf{1})$ . For every  $z \in \ell_\infty$  one has  $I_d(x + z) = I_d(\mathbf{1} + z) \geq I_d(z)$ . Hence,  $x \succeq_d^* 0\mathbf{1}$ . Then

$$(1 - \lambda)\omega \cdot x + \lambda\phi \cdot x \geq 0,$$

which implies  $-(1 - \lambda)\omega_T c + \lambda \geq 0$ , a contradiction. This contradiction implies that  $(1 - \lambda)\omega = 0$ , which also implies  $\lambda = 1$ . The weights set  $\Omega_d$  can be considered a subset of *purely finitely additive* probabilities belonging to  $\ell_d^1$ .

## F. PROOF OF PROPOSITION 2.4

Fix an order  $\widehat{\succeq}$  belonging to  $\{\succeq_c, \succeq_d\}$ . Assume that being potentially better in the

first category implies to be preferred. Consider two streams  $x$  and  $y$ , we will prove that  $x$  is potentially better than  $y$  in the first category if and only if:

$$\inf_{P \in \widehat{\Omega}} P \cdot x \geq \inf_{P \in \widehat{\Omega}} P \cdot y.$$

Assume that  $x$  is potentially better than  $y$  under  $\widehat{\succeq}$ . Stream  $y$  *robustly* dominates a constant one  $b\mathbf{1}$  if and only if  $\inf_{P \in \widehat{\Omega}} P \cdot y \geq b$ . Let  $b = \inf_{P \in \widehat{\Omega}} P \cdot y$ . By the very definition of the first category of the potentially better property, we have  $x \widehat{\succeq}^* b\mathbf{1}$ , with a direct consequence that  $\inf_{P \in \widehat{\Omega}} P \cdot x \geq \inf_{P \in \widehat{\Omega}} P \cdot y$ .

Assume that  $\inf_{P \in \widehat{\Omega}} P \cdot x \geq \inf_{P \in \widehat{\Omega}} P \cdot y$ . If  $y \widehat{\succeq}^* b\mathbf{1}$ , then  $\inf_{P \in \widehat{\Omega}} P \cdot x \geq \inf_{P \in \widehat{\Omega}} P \cdot y \geq b$ . This implies that  $x \widehat{\succeq}^* b\mathbf{1}$ .

The condition of Proposition 2.4(i) is thus equivalent to: if  $\inf_{P \in \widehat{\Omega}} P \cdot x \geq \inf_{P \in \widehat{\Omega}} P \cdot y$ , then  $x \widehat{\succeq} y$ . For every stream  $x$ , let  $b = \inf_{P \in \widehat{\Omega}} P \cdot x$  and  $y = b\mathbf{1}$ . Since  $\inf_{P \in \widehat{\Omega}} P \cdot x = \inf_{P \in \widehat{\Omega}} P \cdot y$ , we have  $x \widehat{\sim} y$ , with  $\widehat{I}(x) = b$  as a direct consequence.

As for part (ii), using the same arguments, we can prove that if having more potential in the second category implies to be preferred,  $\widehat{I}(x) = \sup_{P \in \widehat{\Omega}} P \cdot x$ , for every stream  $x$ .

Consider the most interesting part, namely (iii). Using the same arguments as in the proof of part (i), the condition in part (iii) can be rewritten as: if  $\inf_{P \in \widehat{\Omega}} P \cdot x \geq \inf_{P \in \widehat{\Omega}} P \cdot y$  and  $\sup_{P \in \widehat{\Omega}} P \cdot x \geq \sup_{P \in \widehat{\Omega}} P \cdot y$ , then  $x \widehat{\succeq} y$ .

For every  $x$ , it is obvious that  $\sup_{P \in \widehat{\Omega}} P \cdot x \widehat{\succeq} \widehat{I}(x) \widehat{\succeq} \inf_{P \in \widehat{\Omega}} P \cdot x$ . Hence, there exists  $0 \leq \alpha_x \leq 1$  such that

$$\widehat{I}(x) = \alpha_x \sup_{P \in \widehat{\Omega}} P \cdot x + (1 - \alpha_x) \inf_{P \in \widehat{\Omega}} P \cdot x.$$

If  $\inf_{P \in \widehat{\Omega}} P \cdot x < \sup_{P \in \widehat{\Omega}} P \cdot x$ , the value  $\alpha_x$  is unique.

To end the proof and establish an  $\alpha$ -MaxMin representation, we prove that for any  $x, y$  such that  $\inf_{P \in \widehat{\Omega}} P \cdot x < \sup_{P \in \widehat{\Omega}} P \cdot x$  and  $\inf_{P \in \widehat{\Omega}} P \cdot y < \sup_{P \in \widehat{\Omega}} P \cdot y$ , we obtain  $\alpha_x = \alpha_y$ .

First, observe that we can find  $\lambda > 0$  and a constant  $b$  such that

$$\begin{aligned} \lambda \sup_{P \in \widehat{\Omega}} P \cdot y + b &= \sup_{P \in \widehat{\Omega}} P \cdot x, \\ \lambda \inf_{P \in \widehat{\Omega}} P \cdot y + b &= \inf_{P \in \widehat{\Omega}} P \cdot x. \end{aligned}$$

Let  $\tilde{x} = \lambda y + b$ . We have

$$\begin{aligned}\widehat{I}(\tilde{x}) &= \lambda \widehat{I}(y) + b, \\ \sup_{P \in \widehat{\Omega}} P \cdot \tilde{x} &= \lambda \sup_{P \in \widehat{\Omega}} P \cdot y + b, \\ \inf_{P \in \widehat{\Omega}} P \cdot \tilde{x} &= \lambda \inf_{P \in \widehat{\Omega}} P \cdot y + b.\end{aligned}$$

Hence,  $\alpha_{\tilde{x}} = \alpha_y$ . Observe that

$$\begin{aligned}\sup_{P \in \widehat{\Omega}} P \cdot \tilde{x} &= \sup_{P \in \widehat{\Omega}} P \cdot x, \\ \inf_{P \in \widehat{\Omega}} P \cdot \tilde{x} &= \inf_{P \in \widehat{\Omega}} P \cdot x.\end{aligned}$$

Hence,  $\tilde{x} \widehat{\sim} x$ . This implies  $\widehat{I}(\tilde{x}) = \widehat{I}(x)$ , and  $\alpha_x = \alpha_y$ . Let the common value be  $\alpha$ . The order  $\widehat{\succeq}$  has an  $\alpha$ -MaxMin representation where, for every stream  $x$ , we have

$$\widehat{I}(x) = \alpha \sup_{P \in \widehat{\Omega}} P \cdot x + (1 - \alpha) \inf_{P \in \widehat{\Omega}} P \cdot x.$$

## G. PROOF OF PROPOSITION 3.1

The proof of this proposition begins by a preparative Lemma G.1. Under the hypothesis that the close future order  $\succeq_c$  satisfies definition 3.1, for each stream  $x$ , the value of the worst scenario corresponding to  $(c^* \mathbf{1}_{[0, T^*-1]}, x)$ , evaluated under order  $\succeq_c$ , neither change with a shift of the stream to the future nor with a convex combination with this shift. In another words, beginning from  $T^*$ , the robust order satisfies a version of *stability*.

For any  $x \in \ell_\infty$ , let  $\mathcal{C}(x)$  be the supremum value  $c$  such that  $x \succeq_c^* c \mathbf{1}$ .

**LEMMA G.1.** *Assume that axiom A1 is satisfied. Assume also that order  $\succeq_c$  is not trivial, satisfies Impatience and  $T^*$ -delay stationarity property.*

(i) *For any constant  $c$ ,  $(c \mathbf{1}_{[0, T^*-1]}, x) \succeq_c^* c \mathbf{1}$  implies:*

$$(c \mathbf{1}_{[0, T^*-1]}, x) \succeq_c^* (c \mathbf{1}_{[0, T^*]}, x) \succeq_c^* (c \mathbf{1}_{[0, T^*+1]}, x) \succeq_c^* \dots \succeq_c^* c \mathbf{1}.$$

(ii) *If  $c^* = \mathcal{C}((c^* \mathbf{1}_{[0, T^*-1]}, x))$ , then for any  $T \geq T^*$ ,*

$$\mathcal{C}(c^* \mathbf{1}_{[0, T]}, x) = c^*.$$

(iii) If  $c^* = \mathcal{C}((c^* \mathbf{1}_{[0, T^*-1]}, x))$ , then for any  $T \geq T^*$ ,

$$\mathcal{C} \left( \frac{1}{2} (c^* \mathbf{1}_{[0, T^*-1]}, x) + \frac{1}{2} (c^* \mathbf{1}_{[0, T]}, x) \right) = c^*.$$

*Proof.* The proof of part (i) is obvious, using  $T^*$ -delay stability property.

(ii) It is obvious that if  $x \succeq_c^* y$ , then  $\mathcal{C}(x) \geq \mathcal{C}(y)$ . Let  $c^* = \mathcal{C}(c^* \mathbf{1}_{[0, T^*-1]}, x)$ . From part (i), for  $T \geq T^*$ ,

$$(c^* \mathbf{1}_{[0, T^*-1]}, x) \succeq_c^* (c^* \mathbf{1}_{[0, T]}, x) \succeq_c^* c^* \mathbf{1}.$$

This implies  $c^* = \mathcal{C}(c^* \mathbf{1}_{[0, T^*-1]}, x) \geq \mathcal{C}(c^* \mathbf{1}_{[0, T]}, x) \geq c^*$ .

(iii) Since  $(c^* \mathbf{1}_{[0, T^*-1]}, x) \succeq_c^* (c^* \mathbf{1}_{[0, T]}, x)$ ,

$$\begin{aligned} c^* &= \mathcal{C}((c^* \mathbf{1}_{[0, T^*-1]}, x)) \\ &\geq \mathcal{C} \left( \frac{1}{2} (c^* \mathbf{1}_{[0, T^*-1]}, x) + \frac{1}{2} (c^* \mathbf{1}_{[0, T]}, x) \right) \\ &\geq c^*. \end{aligned}$$

QED

Now, return to the main part of the proof. For each probability  $\omega = (\omega_0, \omega_1, \dots) \in \ell_1$  and  $T \geq 0$ , let  $\omega^T$  be the probability defined as

$$\omega_s^T = \frac{\omega_{T+s}}{\sum_{s'=0}^{\infty} \omega_{T+s'}}.$$

It is worth noting that, for  $x \in \ell_\infty$  and a constant  $c$ ,  $\omega \cdot (c \mathbf{1}_{[0, T-1]}, x) = c$  if and only if  $\omega^T \cdot x = c$ .

Let  $\Omega_c^{T^*} = \{\omega^{T^*} \text{ such that } \omega \in \Omega_c\}$ . First, observe that from axiom **A1**, we have  $\Omega_c^{T^*}$  is a weakly compact subset of  $\ell_1$ . Take  $\omega \in \Omega_c$  such that  $\omega^{T^*}$  is an exposed point of  $\Omega_c^{T^*}$ . We will establish that  $\omega^{T^*} = (\omega^{T^*})^T$  for all  $T \geq 0$ .

From the definition of  $\omega$ , there exists  $x \in \ell_\infty$  such that  $\omega^{T^*} \cdot x < \tilde{\omega}^{T^*} \cdot x$  for every  $\tilde{\omega} \in \Omega_c \setminus \{\omega\}$ . Let  $c^* = \omega^{T^*} \cdot x$ . It is obvious that the following inequality is verified:

$$c^* = \omega \cdot (c^* \mathbf{1}_{[0, T^*-1]}, x) < \tilde{\omega} \cdot (c^* \mathbf{1}_{[0, T^*-1]}, x).$$

This implies that  $\mathcal{C}((c^* \mathbf{1}_{[0, T^*-1]}, x)) = c^*$ . A direct consequence is that  $(c^* \mathbf{1}_{[0, T^*-1]}, x) \succeq_c^* c^* \mathbf{1}$ . Fix  $T \geq 0$  and from Lemma **G.1**,

$$\mathcal{C} \left( \frac{1}{2} (c^* \mathbf{1}_{[0, T^*-1]}, x) + \frac{1}{2} (c^* \mathbf{1}_{[0, T^*+T]}, x) \right) = c^*.$$



This implies that there exists  $\omega'$  such that

$$c^* = \omega' \cdot \left( \frac{1}{2} (c^* \mathbf{1}_{[0, T^*-1]}, x) + \frac{1}{2} (c^* \mathbf{1}_{[0, T^*+T]}, x) \right) = \min_{\omega \in \Omega_c} \omega \cdot \left( \frac{1}{2} (c^* \mathbf{1}_{[0, T^*-1]}, x) + \frac{1}{2} (c^* \mathbf{1}_{[0, T^*+T]}, x) \right).$$

From (i),  $\omega' \cdot (c^* \mathbf{1}_{[0, T^*-1]}, x) \geq c^*$  and  $\omega' \cdot (c^* \mathbf{1}_{[0, T^*+T]}, x^*) \geq c^*$ . It follows that

$$\omega' \cdot (c^* \mathbf{1}_{[0, T^*-1]}, x) = \omega' \cdot (c^* \mathbf{1}_{[0, T^*+T]}, x^*) = c^*.$$

Hence:

$$\begin{aligned} (\omega')^{T^*} \cdot x &= c^*, \\ (\omega')^{T^*} \cdot (c^* \mathbf{1}_{[0, T^*+T]}, x) &= c^*. \end{aligned}$$

Since  $\omega^{T^*}$  is an exposed point of  $\Omega_c^{T^*}$ , the first equality implies that  $(\omega')^{T^*} = \omega^{T^*}$ .

Observe that  $\omega \cdot (c^* \mathbf{1}_{[0, T^*+T]}, x) = c^*$  is equivalent to  $(\omega^{T^*})^T \cdot x = c^*$ . Moreover,  $(\omega^{T^*})^T$  belongs to  $\Omega_c^{T^*}$ . Indeed, suppose the contrary: from the weak compactness of  $\Omega_c^{T^*}$ , there exists  $\epsilon > 0$  such that the intersection between  $\Omega_c^{T^*}$  and the open set  $\{\tilde{\omega} \text{ such that } \|\tilde{\omega} - (\omega^{T^*})^T\|_{\ell_1} < \epsilon\}$  is empty. From Hahn-Banach theorem, there exists  $x'$  and a constant  $c$  such that  $\tilde{\omega}^{T^*} \cdot x' > c > \omega^{T^*} \cdot x'$  for every  $\tilde{\omega} \in \Omega_c$ . This implies that  $(c \mathbf{1}_{[0, T^*-1]}, x') \succeq_c^* c \mathbf{1}$  and therefore that  $(c \mathbf{1}_{[0, T^*-1]}, x') \succeq_c^* (c \mathbf{1}_{[0, T^*+T]}, x') \succeq_c^* c \mathbf{1}$ , hence  $\omega \cdot (c \mathbf{1}_{[0, T^*+T]}, x') \geq c$ , which is equivalent to  $(\omega^{T^*})^T \cdot x' \geq c$ , a contradiction.

The probability  $(\omega^{T^*})^T$  belongs to  $\Omega_c^{T^*}$ , and satisfies  $(\omega^{T^*})^T \cdot x = c^*$ . From the definition of  $\omega^{T^*}$  and  $x$ ,  $\omega^{T^*} = (\omega^{T^*})^T$ , for every  $T \geq 0$ . It follows that

$$\omega_s^{T^*} = \frac{\omega_{T^*+T+s}}{\sum_{s'=0}^{\infty} \omega_{T^*+T+s'}} \text{ and } \omega_{s+1}^{T^*} = \frac{\omega_{T^*+T+s+1}}{\sum_{s'=0}^{\infty} \omega_{T^*+T+s'}}.$$

This implies that for every  $T, s$ :

$$\frac{\omega_{s+1}^{T^*}}{\omega_s^{T^*}} = \frac{\omega_{T^*+T+s+1}}{\omega_{T^*+T+s}}.$$

This is equivalent, for some  $\delta > 0$  and for every  $s \geq 0$ , to

$$\frac{\omega_{s+1}^{T^*}}{\omega_s^{T^*}} = \delta,$$

or to  $\omega_s^{T^*} = \delta^s \omega_0^{T^*}$  for every  $s \geq 0$ . Since  $\sum_{s=0}^{\infty} \omega_s^{T^*} = 1$ , it follows that  $0 < \delta < 1$  and  $\omega_s = (1 - \delta) \delta^s$  for  $s \geq 0$ .

To sum up, every exposed point of  $\Omega_c^{T^*}$  has a exponential representation. The set  $\Omega_c^{T^*}$  is weakly compact, according to Theorem 4 in [Amir and Lindentrauss \(1968\)](#),

$\Omega_c^{T^*}$  is the convex hull of its exposed points. This implies the existence of a subset  $D^* \subset (0, 1)$  such that

$$\Omega_c^{T^*} = \text{convex}\{(1 - \delta, (1 - \delta)\delta, \dots, (1 - \delta)\delta^s, \dots)\}_{\delta \in D^*}.$$

Part (i), where  $T^* = 0$  is proven.

Consider the case  $T^* \geq 1$ . Observe that if  $\omega^{T^*}$  is an exposed point of  $\Omega_c^{T^*}$ , then  $\omega$  is an exposed point of  $\Omega_c$ . Indeed, in that case, there exists  $x \in \ell_\infty$  such that  $\omega^{T^*} \cdot x < \tilde{\omega}^{T^*} \cdot x$  for every  $\tilde{\omega} \in \Omega_c \setminus \{\omega\}$ . Let  $c = \omega^{T^*} \cdot x$ . It is easy to verify that  $c = \omega \cdot (c\mathbf{1}_{[0, T^*-1]}, x)$  and  $c < \tilde{\omega} \cdot (c\mathbf{1}_{[0, T^*-1]}, x)$ . Hence,  $\omega$  is an exposed point of  $\Omega_c$ .

Consider an exposed point  $\omega$  of  $\Omega_c$ . We will prove that  $\omega^{T^*}$  is an exposed point of  $\Omega_c^{T^*}$ . In this stage of the proof, we need axiom **A2**, to prove that the  $T^*$ -delay equivalence of an exposed point of  $\Omega_c$  is an exposed point of  $\Omega_c^{T^*}$ .

By the choice of  $\omega$ , there exists  $x \in \ell_\infty$  such that  $\omega \cdot x < \tilde{\omega} \cdot x$ , for every  $\tilde{\omega} \in \Omega_c \setminus \{\omega\}$ . Let  $c = \mathcal{C}(x) = \omega \cdot x$ . Consider the utility stream  $y$ , which is a  $T^*$ -delay equivalence of  $x$ , being defined in the statement of axiom **A2**. Taking  $\hat{y} = y$ , from the obvious property  $\mathcal{C}\left(\frac{1}{2}(c\mathbf{1}_{[0, T^*-1]}, y) + \frac{1}{2}(c\mathbf{1}_{[0, T^*-1]}, y)\right) = c$ , one has

$$\mathcal{C}\left(\frac{1}{2}x + \frac{1}{2}(c\mathbf{1}_{[0, T^*-1]}, y)\right) = c.$$

Using the same arguments as those used in the proof of Lemma **G.1**, we obtain

$$\omega \cdot x = \omega \cdot (c\mathbf{1}_{[0, T^*-1]}, y) = c.$$

Since  $\mathcal{C}\left((c\mathbf{1}_{[0, T^*-1]}, y)\right) = c$ , for every  $\tilde{\omega} \in \Omega_c$ ,  $\tilde{\omega} \cdot (c\mathbf{1}_{[0, T^*-1]}, y) \geq c$ , which is equivalent to  $\tilde{\omega}^{T^*} \cdot y \geq c$ . We prove that for every exposed point  $\hat{\omega}^{T^*}$  of  $\Omega_c^{T^*}$  that differs to  $\omega^{T^*}$ ,

$$\hat{\omega}^{T^*} \cdot y > c.$$

Assume the contrary, and consider a point  $\hat{\omega}^{T^*}$ , which is an exposed point and  $\hat{\omega}^{T^*} \cdot y = c$ . There exists  $y'$  such that  $\hat{\omega}^{T^*} \cdot y' < \tilde{\omega}^{T^*} \cdot y'$ , for every  $\tilde{\omega}^{T^*} \in \Omega^{T^*} \setminus \{\hat{\omega}^{T^*}\}$ , including  $\omega^{T^*}$ . Let  $\hat{y} = y' + (c - \hat{\omega}^{T^*} \cdot y') \mathbf{1}$ . The stream  $\hat{y}$  satisfies

$$c = \hat{\omega}^{T^*} \cdot \hat{y} < \tilde{\omega}^{T^*} \cdot \hat{y},$$

for every  $\tilde{\omega}^{T^*} \in \Omega^{T^*} \setminus \{\hat{\omega}^{T^*}\}$ , including  $\omega^{T^*}$ . Moreover, for every  $\tilde{\omega}^{T^*} \in \Omega_c^{T^*}$ ,

$$\tilde{\omega}^{T^*} \cdot \left(\frac{1}{2}y + \frac{1}{2}\hat{y}\right) \geq c,$$

with the equality being obtained at  $\tilde{\omega}^{T^*} = \hat{\omega}^{T^*}$ . One has

$$\mathcal{C} \left( \frac{1}{2}(c\mathbf{1}_{[0, T^*-1]}, y) + \frac{1}{2}(c\mathbf{1}_{[0, T^*-1]}, \hat{y}) \right) = c.$$

Contrary to this, inequality  $\omega^{T^*} \cdot y' > c$  implies  $\omega \cdot (c\mathbf{1}_{[0, T^*-1]}, y') > c$ , and

$$\omega \cdot \left( \frac{1}{2}x + \frac{1}{2}(c\mathbf{1}_{[0, T^*-1]}, \hat{y}) \right) > c.$$

For any  $\tilde{\omega} \in \Omega_c \setminus \{\omega\}$ ,  $\tilde{\omega} \cdot x > c$ . Hence, the satisfaction of the *strict* inequality

$$\begin{aligned} \tilde{\omega} \left( \frac{1}{2}x + \frac{1}{2}(c\mathbf{1}_{[0, T^*-1]}, \hat{y}) \right) &= \frac{1}{2}\tilde{\omega} \cdot x + \frac{1}{2}\tilde{\omega} \cdot (c\mathbf{1}_{[0, T^*-1]}, \hat{y}) \\ &> c. \end{aligned}$$

The compactness of  $\Omega_c$  implies that  $\mathcal{C} \left( \frac{1}{2}x + \frac{1}{2}(c\mathbf{1}_{[0, T^*-1]}, \hat{y}) \right) > c$ , a contradiction. This contradiction ensures that for every  $\hat{\omega}^{T^*} \in \Omega_c^{T^*} \setminus \{\omega^{T^*}\}$ , one has  $\hat{\omega}^{T^*} \cdot y > c$ . This implies  $\omega^{T^*}$  is an exposed point of  $\Omega_c^{T^*}$ , and has an exponential representation with some discount rate  $\delta$ . It is easy to find  $\delta_0, \delta_1, \dots, \delta_{T^*-1} \in (0, 1)$  such that  $\omega_0 = 1 - \delta_0, \omega_1 = \delta_0(1 - \delta_1), \dots, \omega_{T^*-1} = \delta_0\delta_1 \dots \delta_{T^*-1}(1 - \delta)$  and  $\omega_{T+s} = \delta_0\delta_1 \dots \delta_{T^*-1} \times \delta^s(1 - \delta)$ , for  $s \geq 0$ .

The set  $\Omega_c$  being the convex hull of its exposed points, the proof is completed.

## H. PROOF OF PROPOSITION 3.2

Fix  $b \leq \inf_{s \geq 0} x_s$ . Obviously, for every  $T \geq T^*$ ,  $(b\mathbf{1}_{[0, T]}, x_{[T+1, \infty)}) \succeq_d^* b\mathbf{1}$ . It follows that

$$(b\mathbf{1}_{[0, T]}, x_{[T+1, \infty)}) \succeq^* (b\mathbf{1}_{[0, T+1]}, x_{[T+1, \infty)}).$$

From the *head-insensitivity* property of the distant future order  $\succeq_d$ ,

$$x \succeq_d^* (0, x).$$

Hence, for every purely finitely additive probability  $\phi$  belonging to  $\Omega_d$ ,

$$\phi \cdot x \geq \phi \cdot (0, x).$$

By applying the same arguments with  $-x$  in the place of  $x$ , and  $b < -\sup_{s \geq 0} x_s$ , it follows that  $\phi \cdot (-x) \geq \phi \cdot (0, -x)$ . From the linearity of  $\phi$ , we obtain

$$\phi \cdot x = \phi \cdot (0, x).$$

# I. PROOF OF PROPOSITION 3.3

First, observe that for every purely finitely additive probability  $\phi \in \Omega_d$ ,  $x \in \ell_\infty$ , one has

$$\liminf_{s \rightarrow \infty} x_s \leq \phi \cdot x \leq \limsup_{s \rightarrow \infty} x_s.$$

Axiom **A3** implies that

$$\begin{aligned} \inf_{\phi \in \Omega_d} \phi \cdot x &= \liminf_{s \rightarrow \infty} x_s, \\ \sup_{\phi \in \Omega_d} \phi \cdot x &= \limsup_{s \rightarrow \infty} x_s. \end{aligned}$$

Indeed, assume the contrary. Consider the case  $\liminf_{s \rightarrow \infty} x_s < \inf_{\phi \in \Omega_d} \phi \cdot x$ . This implies the existence of  $b \in \mathbb{R}$  and  $\epsilon > 0$  such that  $b > \liminf_{s \rightarrow \infty} x_s + \epsilon$  and  $x \not\prec_d^* b\mathbf{1}$ , a contradiction with part (i) of axiom **A3**. For the case in which  $\sup_{\phi \in \Omega_d} \phi \cdot x < \limsup_{s \rightarrow \infty} x_s$ , using part (ii), similar arguments lead us to a contradiction.

Therefore, the decomposition is a direct consequence of Proposition 2.4.

## REFERENCES

- AMIR, D. and J. LINDENTRAUSS (1968): *The Structure of Weakly Compact Sets in Banach Spaces*, Annals of Mathematics 88, 35-46.
- DE ANDRADE, M., L. BASTIANELLO and J. ORRILLO (2021): *Future Blindness*, Working Paper.
- BEISSNER, P., Q. LIN and F. RIEDEL (2020): *Dynamically Consistent  $\alpha$ -MaxMin Expected Utility*, Mathematical Finance 30, 1073-1102.
- BEISSNER, P. and J. WERNER (2021): *Optimal Allocations with  $\alpha$ -MEU, CEU and CPT*, Working Paper.
- BECKER, R.A. and J. H. BOYD (1997). *Capital Theory, Equilibrium Analysis and Recursive Utility*. Blackwell.
- BEWLEY, T.F. (2002): *Knightian Decision Theory. Part I*. Decisions in Economics and Finance 25, 79-110.

- BHASKARA RAO, K.P.S. and M. BHASKARA RAO (1983): *Theory of Charges*. Academic Press.
- BICH, P., B. DONG-XUAN and B. WIGNIOLLE (2021). *On Multiple Discount Rates and Present Bias*. Working paper.
- BLEICHRODT, H., K. ROHDE and P. WAKKER (2008): *Koopmans' constant discounting for intertemporal choice: a simplification and a generalisation*, Journal of Mathematical Psychology 52, 341-347.
- BROWN, D. and L. LEWIS (1981): *Myopic Economic Agents*, Econometrica 49, 369-268.
- CHAKRABORTY, A. (2017): *Present Bias*, Working Paper.
- CHATEAUNEUF, A., X. QU, C. VENTURA and V. VERGOPOULOS (2021): *On the Falsifiability of  $\alpha$ -MaxMin*, Working Paper.
- CHAMBERS, C. and F. ECHENIQUE (2018): *On multiple discount rates*, Econometrica 86, 1325-1346.
- DOLMAS, J. (1995): *Time-additive Representations of Preferences when Consumption grows without Bound*, Economic Letters 47, 317-325.
- DRUGEON, J.-P. and T. HA-HUY (2022a): *A not so Myopic Axiomatisation of Discounting*, Economic Theory 73, 349-376.
- DRUGEON, J.-P. and T. HA-HUY (2022b): *Time-Varying Discount Rates with Recursive Time-Dependent Orders*, Working Paper.
- DUNFORD N. and J. T. SCHWARTZ (1966): *Linear Operators*, Part I : general Theory," 3rd ed., Interscience, New York, 1966.
- FISHBURN, P. and A. RUBINSTEIN (1982): *Time Preference*, International Economic Review 23, 677-694.
- FRICK, M., R. IJIMA and Y. LE YAOUANQ (2022): *Objective Rationality Foundations for (Dynamic)  $\alpha$ -MEU*, Journal of Economic Theory <https://doi.org/10.1016/j.jet.2021.105394>.

- GHIRARDATO, P., F. MACCHERONI and M. MARINACCI (2004): *Differentiating Ambiguity and Ambiguity Attitude*, Journal of Economic Theory 118, 133-173.
- GILBOA, I. and D. SCHMEIDLER (1989): *MaxMin expected utility with non-unique prior*, Journal of Mathematical Economics 18, 141-153.
- GILLES, C. (1989): *Charges as Equilibrium prices and Asset Bubbles*, Journal of Mathematical Economics 18, 155-167.
- HURWICZ, L. (1951): *Some Specification Problems and Applications to Econometric Models*, Econometrica 19, 343-344.
- KARNI, E., F. MACCHERONI and M. MARINACCI (2022): David Schmeidler's contributions to decision theory, *Theory and Decision* **93**: 219-235.
- KOOPMANS, T. J. (1960): *Stationary Ordinal Utility and Impatience*, Econometrica 28, 287-309.
- KOOPMANS, T. J. (1972): T.J. Representation of Preference Orderings over Time. In: *Decision and Organisation*. McGuire, C. and R. Radner, eds. Amsterdam: North-Holland.
- KOPYLOV, I. (2003):  *$\alpha$ -MaxMin Expected Utility*, Working Paper.
- LAIBSON, D. (1997): *Golden Eggs and Hyperbolic Discounting*, Quarterly Journal of Economics 112, 443-478.
- MONTIEL OLEA J. L. and T. STRZALECKI (2014): *Axiomatisation and Measurement of Quasi-Hyperbolic discounting*, Quarterly Journal of Economics, 1449-1499.
- PHELPS, P. and R. POLLACK (1968): *On Second-best National Saving and Game-Equilibrium Growth*, Review of Economics Studies 35, 185-199.
- SAWYER, C. (1988): *Stationary Recursive Preferences: Myopia and Impatience Reconsidered—the Many Goods Case*, Working Paper.
- TRAUTMAN, S. T., and G. VAN DE KUILEN (2015): *Ambiguity Attitudes*, The Wiley Blackwell Handbook of Judgment and Decision Making 2, 89-116

WAKAI, T. (2007): *A Model of Utility Smoothing*, *Econometrica* 73, 157-63, 2007.

YOSIDA K., HEWITT E. (1952): *Finitely additive measures*. *Transactions of the American Mathematical Society* 72, 46-66.