Generalized Nash Fairness solutions for Bi-Objective Discrete Optimization: Theory and Algorithms

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Abstract

This paper deals with a particular case of Bi-Objective Optimization called *Bi-Objective Discrete Optimization* (BODO), where the feasible set is discrete, and the two objectives take only positive values. Since the feasible set of a BODO problem is discrete and usually finite, it can theoretically be enumerated to identify the Pareto set, which consists of all Pareto-optimal solutions representing different trade-offs between two objectives. However, in general, this problem is challenging due to two main issues: time complexity, as the number of Pareto-optimal solutions can be exponentially large, and lack of decisiveness. From a practical point of view, the Central Decision Maker (CDM) may be interested in a reduced Pareto set reflecting the own preference of the CDM, which a computationally tractable algorithm can obtain.

In this paper, we propose a new criterion for selecting solutions within the Pareto set of BODO. For this purpose, we focus on solutions achieving proportional fairness between two objectives, called *generalized Nash Fairness solutions* (ρ-NF solutions). The positive parameter ρ provided by the CDM reflects the relative importance of the first objective compared to the second one.

We first introduce the ρ-NF solution concept for BODO. We then show that the ρ-NF solution set is a subset of the Pareto set, and this inclusion can be strict. We also propose a recursive Newton-like algorithm for determining the ρ-NF solution set. Finally, an illustrative example of BODO is given.

Keywords: Bi-Objective Discrete Optimization, Bi-Criteria Decision Making, Pareto optimal, Proportional Fairness, Generalized Nash Fairness solution

1. Introduction

Bi-Objective Discrete Optimization (BODO) is a branch of optimization with many applications in different areas, such as applied mathematics, economics, and computer science. In practice, the feasible set of BODO is discrete and its objectives take only positive

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values, including some well-studied examples of BODO in resource allocation, such as the bi-objective knapsack problem [39], in networks, such as the bi-objective shortest path problem [40], the bi-objective spanning tree problem [41] and in capital budgeting, such as the bi-objective assignment problem. The latter problem assembles the objectives of the classic assignment problem [23] and the balanced assignment problem mentioned in [24].

In multi-objective optimization, the concept of Pareto-optimal (non-dominated) solution plays an important role as it distinguishes between efficient and non-efficient solution [5]. Based on this concept, three main approaches proposed in the literature can be applied to handle BODO: the Pareto set approach, the budget approach, and the aggregation approach. In the first approach, the BODO will be solved by determining the Pareto set, which contains all Pareto-optimal solutions. If the objective functions are linear and under some structural conditions, there exist several polynomial time algorithms to approximate the Pareto set. For example, Papadimitriou et al. constructed a ϵ-approximate Pareto set for the web access problem [6]. Notice that this theory can be applied to multi-objective optimization with more than two objectives. The budget approach reduces the multi-objective problem into a standard optimization problem. The focus here is on managing and allocating a fixed resource or 'budget' efficiently. For example, the ϵ-constraints method [11] is applied in the context of bi-objective combinatorial optimization with integer objectives [13]. Similarly, Eusebio et al. found representative non-dominated points for bi-objective integer network flow problems [12]. Moreover, several approximate algorithms have been proposed in [7], [8]. The aggregation approach usually forms a single-objective optimization problem whose optimal solutions are the Pareto-optimal solutions to the bi-objective optimization problem. The most simple aggregation approach is the weighted sum method [4]. By scalarizing two objectives into a single objective, it provides a solution that reflects preferences incorporated in selecting the weights and possibly multiple solutions when varying the weights consistently. However, its fundamental drawback is not finding a uniform spread of Pareto-optimal solutions, even if a uniform spread of weight vectors is used. More seriously, those in non-convex regions are not detected [34], [35].

Besides these three main approaches, BODO can also be solved by determining solutions that are near-optimal to all objectives. For example, Stein et al. considered a bi-objective scheduling problem and derived existence and non-existence theorems on schedules that are simultaneously near-optimal to both objectives [9]. Similarly, Angel et al. considered a generalization of the classical MAX-CUT problem and showed that under some conditions, the approximate solutions can be computed in polynomial time [10].

For a general BODO problem without any structural conditions, since its feasible set is discrete and usually finite, it can theoretically be enumerated to identify the Pareto set. However, this problem is generally known to be algorithmically unsolvable (e.g., see [30]). It is challenging due to two main issues: time complexity, as the number of Pareto-optimal solutions can be exponentially large, and lack of decisiveness. From a practical point of view, the Central Decision Maker (CDM) may be interested in a reduced Pareto set reflecting the own preference of the CDM, which a computationally tractable algorithm can obtain.

In a recent paper [27], a new criterion for selecting solutions within the Pareto set has been applied for Bi-Objective Minimization Problems, i.e., two objectives are to be simul-

taneously minimized. The authors utilized the notion of proportional fairness − which is a well-known rule in communication networks, operation research, and general social choice introduced in [1], [2], [3] − as a criterion for selecting the preferred solutions. They also presented the concept of generalized Nash Fairness (ρ-NF) solution, a Pareto-optimal solution achieving some proportional fairness between two objectives. The positive parameter ρ provided by the CDM reflects the relative importance of the first objective compared to the second one. They then focused on finding two particular ρ-NF solutions, called extreme ρ-NF solutions, having the smallest values of each objective. In this paper, we extend the findings presented in [27] by generalizing the concept of ρ-NF solution for BODO problems where each objective can be either maximized or minimized. We then show that the ρ-NF solution set is a subset of the Pareto set. We also propose a recursive Newton-like algorithm for determining the ρ-NF solution set.

Notice that we introduced a particular case of ρ-NF solution when $\rho = 1$, called NF solution, and the idea to determine the NF solution set for an example of Min-Min BODO in our conference paper [26]. Thus, this paper is an extended and generalized work of [26].

Let us introduce the notations used in this paper. Let $P(x), Q(x)$ denote two positive objective functions, and $(P, Q) = (P(x), Q(x))$ denote a feasible solution corresponding to a decision vector $x \in \mathcal{X}$ where $\mathcal{X}$ is a finite feasible set. Notice that the ρ-NF solutions will be characterized by the solutions (P, Q) and not by the corresponding decision vectors. Thus, two solutions having the same value of (P, Q) will be considered equivalent. Throughout this paper, we use the notation "$\equiv$" to denote equivalent solutions. Assume that we know the algorithms to optimize every single objective and the linear combination of two objectives P, Q over $\mathcal{X}$. These hypotheses are verified for most practical examples of BODO.

We will consider BODO in three cases: Max-Max BODO with two objectives to be maximized, such as the bi-objective knapsack problem [39], Max-Min BODO with one objective to be maximized, and another one to be minimized, such as some fractional programming problems [25] and Min-Min BODO with two objectives to be minimized, such as the bi-objective assignment problem [26].

The paper is organized as follows. Section 2 discusses the concept and the existence of ρ-NF solutions for BODO. Then, we show the Pareto efficiency of ρ-NF solutions in Section 3. Section 4 provides a recursive Newton-like algorithm for determining the ρ-NF solution set. Section 5 presents an example of BODO to illustrate this algorithm. Finally, we give some conclusions and future works in Section 6.

2. Definition and existence of ρ-NF solutions for BODO

2.1. Max-Max BODO

2.1.1. Definition of ρ-NF solution for Max-Max BODO

Max-Max BODO can be formulated as

$$\max_{x \in \mathcal{X}_1} (P(x), Q(x)),$$

where $\mathcal{X}_1$ denotes the set of all feasible decision vectors x. For Max-Max BODO, we suppose that $\mathcal{X}_1$ is finite and $P(x), Q(x) > 0, \forall x \in \mathcal{X}$.

We first recall the notion of proportional fairness introduced in [1], [2], [3]. Proportional fairness for the two-player resource allocation problem is defined using the Nash standard of comparison: a transfer of utilities between the two players is considered fair if the percentage increase in the utility of one player is larger than the percentage decrease in the utility of the other player. For the multiple-player problem, the aggregate proportional change is non-positive when comparing the fair allocation to any other feasible allocation of utilities.

The definition of ρ is related to the works of Kelly et al. [1]. They introduced the concept of proportional fairness for a general communication network problem. Then the choice of parameter ρ by users corresponds to an implicit weighting by the network of the relative utilities of two users, with weights related to the users' various marginal utilities. In the context of Max-Max BODO, we explain the parameter ρ as follows. ρ is a positive parameter supplied by the CDM to reflect the relative importance of P to Q. Notice that ρ can be defined beforehand based on the characterization of two objectives. In other words, ρ percent change of P will be comparably equivalent to a one percent change of Q. Thus, based on the proportional fairness, the ρ-NF solution for Max-Max BODO should be such that, if compared to any other feasible solutions, the sum of the factor ρ of the proportional change of P and the proportional change of Q is non-positive.

Let $\mathcal{S}_1$ be the set of all feasible solutions (P, Q) for Max-Max BODO. Since $\mathcal{X}_1$ is finite, $\mathcal{S}_1$ is also finite. If $(P^*, Q^*) \in \mathcal{S}_1$ is a ρ-NF solution for Max-Max BODO, we have

$$\rho\frac{P - P^*}{P^*} + \frac{Q - Q^*}{Q^*} \leq 0 \iff \rho\frac{P}{P^*} + \frac{Q}{Q^*} \leq \rho + 1,\ \forall (P, Q) \in \mathcal{S}_1,$$

Hence, the ρ-NF solution for Max-Max BODO can be defined as follows.

Definition 1. *$(P^*, Q^*) \in \mathcal{S}_1$ is a ρ-NF solution for Max-Max BODO if and only if*

$$\rho\frac{P}{P^*} + \frac{Q}{Q^*} \leq \rho + 1,\ \forall (P, Q) \in \mathcal{S}_1, \tag{1}$$

2.1.2. Existence of ρ-NF solution for Max-Max BODO

In this section, we show that the ρ-NF does not always exist for Max-Max BODO, and if it exists, it is the unique solution to the optimization problem

$$\mathcal{H}_1 = \max_{(P,Q)\in\mathcal{S}_1} P^\rho Q,$$

Theorem 1. *If (P^*, Q^*) is a ρ-NF solution for Max-Max BODO, then it is necessarily the unique solution of $\mathcal{H}_1$ with respect to the values of P and Q.*

Proof. Suppose that (P^*, Q^*) is a ρ-NF solution for Max-Max BODO. We have

$$\rho\frac{P}{P^*} + \frac{Q}{Q^*} \leq \rho + 1 \implies 1 \geq \frac{\rho}{\rho+1} \times \frac{P}{P^*} + \frac{1}{\rho+1} \times \frac{Q}{Q^*},\ \forall (P, Q) \in \mathcal{S}_1, \tag{2}$$

Using Young's inequality for products (see [28]), we obtain

$$\frac{\rho}{\rho+1}\times\frac{P}{P^*}+\frac{1}{\rho+1}\times\frac{Q}{Q^*}\geq(\frac{P}{P^*})^{\frac{\rho}{\rho+1}}\times(\frac{Q}{Q^*})^{\frac{1}{\rho+1}}=(\frac{P^\rho Q}{(P^*)^\rho Q^*})^{\frac{1}{\rho+1}}, \tag{3}$$

From (2) and (3), we obtain $(P^*)^\rho Q^*\geq P^\rho Q, \forall(P,Q)\in\mathcal{S}_1$ which implies (P^*,Q^*) is a solution of $\mathcal{H}_1$.

Now suppose that there exists another ρ-NF solution $(P^{**},Q^{**})\in\mathcal{S}_1$ such that $(P^{**})^\rho Q^{**}=(P^*)^\rho Q^*$. We also have

$$\rho\frac{P^{**}}{P^*}+\frac{Q^{**}}{Q^*}\leq\rho+1\implies 1\geq\frac{\rho}{\rho+1}\times\frac{P^{**}}{P^*}+\frac{1}{\rho+1}\times\frac{Q^{**}}{Q^*},$$

Using Young's inequality for products, we have

$$1\geq\frac{\rho}{\rho+1}\times\frac{P^{**}}{P^*}+\frac{1}{\rho+1}\times\frac{Q^{**}}{Q^*}\geq(\frac{P^{**}}{P^*})^{\frac{\rho}{\rho+1}}\times(\frac{Q^{**}}{Q^*})^{\frac{1}{\rho+1}}=(\frac{(P^{**})^\rho Q^{**}}{(P^*)^\rho Q^*})^{\frac{1}{\rho+1}}=1,$$

Thus, the equality in Young's inequality above must hold, which implies $P^{**}/P^*=Q^{**}/Q^*$. Moreover, since $(P^{**})^\rho Q^{**}=(P^*)^\rho Q^*$, we obtain

$$1=\frac{(P^{**})^\rho Q^{**}}{(P^*)^\rho Q^*}=(\frac{P^{**}}{P^*})^\rho(\frac{Q^{**}}{Q^*})=(\frac{P^{**}}{P^*})^{\rho+1},$$

Thus, we have $P^{**}=P^*$ and $Q^{**}=Q^*$. □

Notice that a solution of $\mathcal{H}_1$ is not necessarily the ρ-NF. Let us suppose that $\rho=1$ and the feasible set $\mathcal{S}_1$ has two solutions $(P_1,Q_1)=(13,15)$, $(P_2,Q_2)=(10,20)$. We see that (P_2,Q_2) is the solution having the maximal value of $P^\rho Q$ but it is not a ρ-NF solution because

$$\rho\frac{P_1}{P_2}+\frac{Q_1}{Q_2}=\frac{13}{10}+\frac{15}{20}=\frac{41}{20}>2=\rho+1,$$

Consequently, the ρ-NF solution for Max-Max BODO is not guaranteed to exist. If it exists, it is the unique solution of $\mathcal{H}_1$.

2.2. Max-Min BODO

2.2.1. Definition of ρ-NF solution for Max-Min BODO

Since minimizing $Q(x)$ over $\mathcal{X}_2$ is equivalent to maximizing $-Q(x)$ over $\mathcal{X}_2$, Max-Min BODO can be formulated as

$$\max_{x\in\mathcal{X}_2}(P(x),-Q(x))$$

where $\mathcal{X}_2$ denotes the set of feasible decision vectors x (i.e., we simultaneously maximize $P(x)$ and minimize $Q(x)$ over $\mathcal{X}_2$). Moreover, for Max-Min BODO, we suppose that $\mathcal{X}_2$ is finite and $P(x), Q(x) > 0, \forall x \in \mathcal{X}_2$.

In the context of Max-Min BODO, we prefer an alternative assigning a greater value for P and a smaller value for Q. Thus, the ρ-NF solution should be such that, when compared to any other feasible solutions, the difference between the factor ρ of the proportional change of P and the proportional change of Q is non-positive (i.e., from the definition (1) of ρ-NF solution for Max-Max BODO, we change the sign for the proportional change of Q as negative).

Let $\mathcal{S}_2$ be the set of all feasible solutions (P, Q) for Max-Min BODO. Since $\mathcal{X}_2$ is finite, $\mathcal{S}_2$ is also finite. If $(P^*, Q^*) \in \mathcal{S}_2$ is a ρ-NF solution for Max-Min BODO, we have

$$\rho\frac{P - P^*}{P^*} - \frac{Q - Q^*}{Q^*} \leq 0 \iff \rho\frac{P}{P^*} - \frac{Q}{Q^*} \leq \rho - 1, \ \forall (P, Q) \in \mathcal{S}_2,$$

Hence, the ρ-NF solution for Max-Min BODO can be defined as follows.

Definition 2. *$(P^*, Q^*) \in \mathcal{S}_2$ is a ρ-NF solution for Max-Min BODO if and only if*

$$\rho\frac{P}{P^*} - \frac{Q}{Q^*} \leq \rho - 1, \ \forall (P, Q) \in \mathcal{S}_2, \tag{4}$$

2.2.2. Existence of ρ-NF solutions for Max-Min BODO with $0 < \rho \leq 1$

This section discusses the existence of ρ-NF solutions for Max-Min BODO. For that, we consider the following optimization problem.

$$\mathcal{H}_2 = \max_{(P,Q)\in\mathcal{S}_2} \frac{P^\rho}{Q},$$

We will show that in case $0 < \rho \leq 1$, a ρ-NF solution is necessarily a solution of $\mathcal{H}_2$, and in case $\rho > 1$, there always exists a ρ-NF solution which is a solution of $\mathcal{H}_2$.

Theorem 2. *If (P^*, Q^*) is a ρ-NF solution for Max-Min BODO with $0 < \rho \leq 1$, then it is necessarily a solution of $\mathcal{H}_2$. Moreover, in case $0 < \rho < 1$, the ρ-NF solution (if it exists) is the unique solution of $\mathcal{H}_2$.*

Proof. Suppose that (P^*, Q^*) is a ρ-NF solution for Max-Min BODO with $0 < \rho \leq 1$. Using Young's inequality for products, we have

$$\frac{Q}{Q^*} \geq \rho \times \frac{P}{P^*} + (1 - \rho) \geq (\frac{P}{P^*})^\rho = \frac{P^\rho}{(P^*)^\rho}, \forall (P, Q) \in \mathcal{S}_2,$$

which is equivalent to

$$\frac{(P^*)^\rho}{Q^*} \geq \frac{P^\rho}{Q}, \forall (P, Q) \in \mathcal{S}_2,$$

Thus, (P^*, Q^*) is a solution of $\mathcal{H}_2$.

Now suppose that $0 < \rho < 1$ and there exists another ρ-NF solution $(P^{**}, Q^{**}) \in \mathcal{S}_2$ such that $(P^{**})^\rho/Q^{**} = (P^*)^\rho/Q^*$.

Using Young's inequality for products, we also have

$$\frac{Q^{**}}{Q^*} \geq \rho \times \frac{P^{**}}{P^*} + (1-\rho) \geq (\frac{P^{**}}{P^*})^\rho = \frac{Q^{**}}{Q^*},$$

Thus, the equality in Young's inequality above must hold, which implies $P^{**} = P^*$ and then $Q^{**} = Q^*$. □

Notice that a solution of $\mathcal{H}_2$ is not necessarily the ρ-NF solution. Let us suppose that $\rho = 1/2$ and the feasible set $\mathcal{S}_2$ has two solutions $(P_1, Q_1) = (17, 4)$, $(P_2, Q_2) = (9, 3)$. We see that (P_1, Q_1) is the solution having the maximal value of P^ρ/Q but it is not a ρ-NF solution because

$$\rho\frac{P_2}{P_1} - \frac{Q_2}{Q_1} = \frac{1}{2} \times \frac{9}{17} - \frac{3}{4} = -\frac{33}{68} > -\frac{1}{2} = \rho - 1,$$

Consequently, the ρ-NF solution for Max-Min BODO with $0 < \rho < 1$ is not guaranteed to exist. If it exists, it is the unique solution of $\mathcal{H}_2$.

2.2.3. Existence of ρ-NF solutions for Max-Min BODO with $\rho > 1$

In this case, we show that the ρ-NF solution always exists by the following theorem.

Theorem 3. *There always exists a ρ-NF solution for Max-Min BODO with $\rho > 1$.*

Proof. Since $\mathcal{S}_2$ is finite, there always exists $(P^*, Q^*) \in \mathcal{S}_2$ as a solution of $\mathcal{H}_2$.

Let us show that (P^*, Q^*) is a ρ-NF solution. We have $(P^*)^\rho/Q^* \geq P^\rho/Q$, $\forall (P, Q) \in \mathcal{S}_2$. Using Young's inequality for products, we obtain

$$\frac{1}{\rho} \times \frac{Q}{Q^*} + \frac{\rho - 1}{\rho} \geq (\frac{Q}{Q^*})^{\frac{1}{\rho}} \geq \frac{P}{P^*},$$

Thus,

$$\rho\frac{P}{P^*} - \frac{Q}{Q^*} \leq \rho - 1, \ \forall (P, Q) \in \mathcal{S}_2,$$

Consequently, (P^*, Q^*) is a ρ-NF solution. □

Let us show that there may be more than one ρ-NF solution for Max-Min BODO with $\rho > 1$. Suppose that $\rho = 2$ and the feasible set $\mathcal{S}_2$ has two solutions $(P_1, Q_1) = (10, 8)$, $(P_2, Q_2) = (8, 5)$. We see that both (P_1, Q_1) and (P_2, Q_2) are ρ-NF solutions because

$$\rho\frac{P_2}{P_1} - \frac{Q_2}{Q_1} = 2 \times \frac{10}{8} - \frac{8}{5} = \frac{9}{10} < 1 = \rho - 1,$$
$$\rho\frac{P_1}{P_2} - \frac{Q_1}{Q_2} = 2 \times \frac{8}{10} - \frac{5}{8} = \frac{39}{40} < 1 = \rho - 1,$$

2.3. Min-Min BODO

In this section, we restate the definition and the existence of ρ-NF solution for Min-Min BODO, which has been introduced in [27].

2.3.1. Definition of ρ-NF solution for Min-Min BODO

Min-Min BODO can be formulated as

$$\min_{x\in\mathcal{X}_3} (P(x), Q(x)),$$

where $\mathcal{X}_3$ denotes the set of feasible decision vectors x. Moreover, for Min-Min BODO, we suppose that $\mathcal{X}_3$ is finite and $P(x), Q(x) > 0, \forall x \in \mathcal{X}_3$.

In the context of Min-Min BODO, we prefer an alternative assigning a smaller value for both P and Q, which is opposite to Max-Max BODO. Thus, the ρ-NF solution should be such that, when compared to any other feasible solutions, the sum of the factor ρ of the proportional change of P and the proportional change of Q should be non-negative.

Let $\mathcal{S}_3$ be the set of all feasible solutions (P, Q) for Min-Min BODO. Since $\mathcal{X}_3$ is finite, $\mathcal{S}_3$ is also finite. If $(P^*, Q^*) \in \mathcal{S}_3$ is a ρ-NF solution for Min-Min BODO, we have

$$\rho\frac{P-P^*}{P^*} + \frac{Q-Q^*}{Q^*} \geq 0 \iff \rho\frac{P}{P^*} + \frac{Q}{Q^*} \geq \rho + 1,\ \forall (P, Q) \in \mathcal{S}_3,$$

Hence, the ρ-NF solution for Max-Max BODO can be defined as follows.

Definition 3. *[27] $(P^*, Q^*) \in \mathcal{S}_3$ is a ρ-NF solution for Min-Min BODO if and only if*

$$\rho\frac{P}{P^*} + \frac{Q}{Q^*} \geq \rho + 1,\ \forall (P, Q) \in \mathcal{S}_3, \tag{5}$$

2.3.2. Existence of ρ-NF solution for Min-Min BODO

For the Min-Min BODO, we consider the optimization problem:

$$\mathcal{H}_3 = \min_{(P,Q)\in\mathcal{S}_3} P^\rho Q,$$

We will show that the ρ-NF solution always exists for Min-Min BODO by the following theorem.

Theorem 4. *[27] There always exists a ρ-NF solution for Min-Min BODO.*

Proof. Since $\mathcal{S}_3$ is finite, there always exists $(P^*, Q^*) \in \mathcal{S}_3$ as a solution of $\mathcal{H}_3$.

Let us show that (P^*, Q^*) is a ρ-NF solution. We have $P^\rho Q \geq (P^*)^\rho Q^*$, $\forall (P, Q) \in \mathcal{S}_3$. Using Young's inequality for products, we obtain

$$\frac{\rho}{\rho+1} \times \frac{P}{P^*} + \frac{1}{\rho+1} \times \frac{Q}{Q^*} \geq (\frac{P}{P^*})^{\frac{\rho}{\rho+1}} \times (\frac{Q}{Q^*})^{\frac{1}{\rho+1}} = (\frac{P^\rho Q}{(P^*)^\rho Q^*})^{\frac{1}{\rho+1}} \geq 1,\ \forall (P, Q) \in \mathcal{S}_3,$$

Thus,

$$\rho\frac{P}{P^*} + \frac{Q}{Q^*} \geq \rho + 1,\ \forall (P, Q) \in \mathcal{S}_3,$$

Consequently, (P^*, Q^*) is a ρ-NF solution. □

Let us show that there may be more than one ρ-NF solution for Min-Min BODO. Suppose that $\rho = 1$ and the feasible set $\mathcal{S}_3$ has two solutions $(P_1, Q_1) = (11, 4)$, $(P_2, Q_2) = (9, 5)$. We see that both (P_1, Q_1) and (P_2, Q_2) are ρ-NF solutions because

$$\rho\frac{P_2}{P_1} + \frac{Q_2}{Q_1} = \frac{9}{11} + \frac{5}{4} = \frac{91}{44} > 2 = \rho + 1,$$
$$\rho\frac{P_1}{P_2} + \frac{Q_1}{Q_2} = \frac{11}{9} + \frac{4}{5} = \frac{91}{45} > 2 = \rho + 1,$$

3. Characterization of ρ-NF solutions

As a particular case of multi-objective optimization, the concept of Pareto efficiency is applied to describing BODO's efficient solutions (Pareto-optimal solutions). This section shows the Pareto efficiency of ρ-NF solutions. Furthermore, the ρ-NF solution set is a subset of the Pareto set, and this inclusion can be strict. Then, we show that the ρ-NF solutions can be found by optimizing a linear combination of two objectives.

Theorem 5. *The ρ-NF solution set is a subset of the Pareto set. Moreover, this inclusion can be strict.*

Proof. In fact, the Pareto efficiency of ρ-NF solutions can be directly implied from their mathematical definitions. For example, let $(P^*, Q^*) \in \mathcal{S}_1$ be a ρ-NF solution for Max-Max BODO. If it is not a Pareto-optimal solution, there exists $(P', Q') \in \mathcal{S}_1$ such that (P^*, Q^*) is strictly dominated by (P', Q') (i.e., $P^* \leq P', Q^* \leq Q'$ and $(P^*, Q^*) \not\equiv (P', Q')$). We obtain

$$\rho\frac{P'}{P^*} + \frac{Q'}{Q^*} > \rho + 1,$$

which leads to a contradiction to Definition 1.

According to the proof of Theorem 1, if $\rho = 1$ and the feasible set $\mathcal{S}_1$ has two solutions $(P_1, Q_1) = (13, 15)$ and $(P_2, Q_2) = (10, 20)$ then (P_2, Q_2) is a Pareto-optimal solution but not a ρ-NF solution. Consequently, a Pareto-optimal solution is not necessarily a ρ-NF solution for Max-Max BODO.

Similarly, the ρ-NF solutions for Max-Min BODO and Min-Min BODO are necessarily the Pareto-optimal solution, but the inverse may not be true. This concludes the proof. □

Then, we show that the ρ-NF solutions for Max-Max BODO, Max-Min BODO, and Min-Min BODO can be obtained by solving the following optimization problems, respectively.

$$\mathcal{F}_1(\alpha) = \max_{(P,Q)\in\mathcal{S}_1} f_1(\alpha, P, Q) := \rho P + \alpha Q,$$
$$\mathcal{F}_2(\alpha) = \max_{(P,Q)\in\mathcal{S}_2} f_2(\alpha, P, Q) := \rho P - \alpha Q,$$
$$\mathcal{F}_3(\alpha) = \min_{(P,Q)\in\mathcal{S}_3} f_3(\alpha, P, Q) := \rho P + \alpha Q,$$

where $\alpha > 0$ is the coefficient to be determined.

Since $f_i(\alpha, P, Q)$ are linear combinations of P and Q, we suppose we know how to solve $\mathcal{F}_i(\alpha)$, $\forall i = 1, 2, 3$.

Theorem 6. *For $i = 1, 2, 3, (P^*, Q^*) \in \mathcal{S}_i$ is a ρ-NF solution if and only if (P^*, Q^*) is a solution of $\mathcal{F}_i(\alpha^*)$ with $\alpha^* = P^*/Q^*$.*

Proof. We first proof this lemma for Max-Max BODO.

$\Longrightarrow$ Let $(P^*, Q^*) \in \mathcal{S}_1$ be a ρ-NF solution and $\alpha^* = P^*/Q^*$. We will show that (P^*, Q^*) is a solution of $\mathcal{F}_1(\alpha^*)$.

Since $P^*, Q^* > 0$, $\alpha^* > 0$. As (P^*, Q^*) is a ρ-NF solution we have

$$\rho\frac{P}{P^*} + \frac{Q}{Q^*} \leq \rho + 1, \ \forall (P, Q) \in \mathcal{S}_1, \tag{6}$$

Multiplying (6) by $P^* > 0$ and using $\alpha^* = P^*/Q^*$ give

$$\rho P + \alpha^* Q \leq \rho P^* + \alpha^* Q^*, \ \forall (P, Q) \in \mathcal{S}_1, \tag{7}$$

Hence, (P^*, Q^*) is a solution of $\mathcal{F}_1(\alpha^*)$.

$\Longleftarrow$ Suppose that (P^*, Q^*) is a solution of $\mathcal{F}_1(\alpha^*)$ with $\alpha^* = P^*/Q^*$, we have

$$\rho P + \alpha^* Q \leq \rho P^* + \alpha^* Q^*, \ \forall (P, Q) \in \mathcal{S}_1,$$

Replacing α^* by P^*/Q^* we obtain

$$\rho\frac{P}{P^*} + \frac{Q}{Q^*} \leq \rho + 1, \ \forall (P, Q) \in \mathcal{S}_1,$$

Thus, (P^*, Q^*) is a ρ-NF solution.

Since the proofs for Max-Min BODO and Min-Min BODO are similar to those for Max-Max BODO, they are omitted. □

Based on Theorem 6, we state the following lemma for verifying whether a feasible solution is a ρ-NF solution or not.

Lemma 1. *For $i = 1, 2, 3$ and a given solution $(P_0, Q_0) \in \mathcal{S}_i$, let $\alpha_1 = P_0/Q_0$ and (P_1, Q_1) be a solution of $\mathcal{F}_i(\alpha_1)$. Then (P_0, Q_0) is a ρ-NF solution if and only if $f_i(\alpha_1, P_1, Q_1) = f_i(\alpha_1, P_0, Q_0)$.*

Proof. $\Longrightarrow$ If $(P_0, Q_0) \in \mathcal{S}_i$ is a ρ-NF solution then (P_0, Q_0) is also a solution of $\mathcal{F}_i(\alpha_1)$ due to Theorem 6. Thus, $f_i(\alpha_1, P_1, Q_1) = f_i(\alpha_1, P_0, Q_0)$ since (P_0, P_1) and (P_1, Q_1) are both the solutions of $\mathcal{F}_i(\alpha_1)$.

$\Longleftarrow$ If $f_i(\alpha_1, P_1, Q_1) = f_i(\alpha_1, P_0, Q_0)$ then (P_0, Q_0) is also a solution of $\mathcal{F}_i(\alpha_1)$. Since $\alpha_1 = P_0/Q_0$, (P_0, Q_0) is a ρ-NF solution due to Theorem 6. □

In the next section, we discuss how to construct an algorithm for determining the ρ-NF solution set based on Theorem 6 and Lemma 1.

4. Determining the ρ-NF solution set for BODO

As shown in Section 2.1.2, the ρ-NF solutions for Max-Max BODO and Max-Min BODO with $\rho < 1$ are not guaranteed to exist. Thus, finding them is challenging since it is not equivalent to any known optimization problem. In this section, we focus on constructing an algorithm for determining the ρ-NF solution set for Max-Min BODO with $\rho > 1$ and Min-Min BODO.

4.1. Max-Min BODO with $\rho = 1$

As a result of Theorem 2, all the ρ-NF solutions for Max-Min BODO with $\rho = 1$ are the solutions to the following optimization problem

$$\mathcal{R}_2 = \max_{(P,Q)\in\mathcal{S}_2} \frac{P}{Q},$$

Notice that $\mathcal{R}_2$ is fractional programming, which is well-studied in the literature. There are many algorithms have been designed and analyzed for fractional programming (see, for example, surveys [15], [16], [17], [18]). Consequently, in this section, we just restate the method for solving $\mathcal{R}_2$.

Subsequently, we restate the algorithm for solving $\mathcal{R}_2$. This algorithm is inspired by the application of Newton's method to linear fractional programming that was first discussed by Isbell and Marlow [29] and then generalized to nonlinear fractional programming by Dinkelbach [19]. We first consider the following Dinkelbach's transform [19]:

$$\mathcal{D}_2(\alpha) = \max_{(P,Q)\in\mathcal{S}_2} P - \alpha Q,$$

where $\alpha \geq 0$ is a suitable efficient to be terminated.

Algorithm 1

Input: An instance of Max-Min BODO with $\rho = 1$.
Output: A ρ-NF solution.

1: $\alpha_0 \leftarrow 0$
2: Solving $\mathcal{D}_2(\alpha_0)$ to obtain a solution (P_0, Q_0)
3: $i \leftarrow 0$
4: **repeat**
5: $\quad \alpha_{i+1} \leftarrow P_i/Q_i$
6: $\quad$ Solving $\mathcal{D}_2(\alpha_{i+1})$ to obtain a solution (P_{i+1}, Q_{i+1})
7: $\quad T_i \leftarrow P_i - \alpha_i Q_i$
8: $\quad i \leftarrow i + 1$
9: **until** $T_i = 0$
10: Return (P_i, Q_i)

Notice that Algorithm 1 returns only one solution of $\mathcal{R}_2$, which corresponds to one ρ-NF solution instead of all possible ρ-NF solutions. For determining the ρ-NF solution set, we

aim to determine the solution set of $\mathcal{R}_2$. If P, Q are linear, there have been some discussions about its solution set in the literature. Most of them give a unique condition for the solution. If this condition is not satisfied, computational procedures are presented to determine the solution set [36], [37]. For example, a method for determining the solution set based on an adaptation of the convex simplex method credited to Gilmore and Gomory has been introduced in [38].

When P, Q are nonlinear, determining the solution set of $\mathcal{R}_2$ may be difficult since it is hard to describe the general structure of the solutions. Furthermore, all the solutions of $\mathcal{R}_2$ are solutions of $\mathcal{D}_2$ with a unique value of α. In the context of this paper, we will refrain from delving further into dealing with the nonlinear case.

4.2. Max-Max BODO and Max-Min BODO with $\rho < 1$

We recall that the ρ-NF solutions for Max-Max BODO and Max-Min BODO with $\rho < 1$ are not guaranteed to exist. Notice that in the simplest case when $\rho = 1$, the ρ-NF solution for Max-Max BODO (if it exists) is equivalent to the *Nash bargaining solution* − a well-known notion from cooperative game theory − since it maximizes the products of two objectives [20]. A natural idea for determining ρ-NF solution in such a case is that we first solve $\mathcal{H}_1$, then we verify whether the obtained solution is the ρ-NF solution by using Lemma 1. However, solving $\mathcal{H}_1$, even when $\rho = 1$, is computationally expensive since the objective remains nonlinear and the feasible set is non-convex. Consequently, to address such a case, a few approaches involve considering certain non-convex sets, which are convex after a logarithmic transformation [21]. An alternative approach is to introduce the concept of *local proportional fairness*, which is always achievable, and then analyze its properties [22].

Similarly, solving $\mathcal{H}_2$ with $\rho < 1$ is challenging, although it is also fractional programming. The fact is that solving the corresponding Dinkelbach's transform

$$\mathcal{D}'_2(\alpha) = \max_{(P,Q)\in\mathcal{S}_2} P^\rho - \alpha Q \text{ with } \rho < 1,$$

requires a highly non-linear optimization, even though when P, Q are linear.

In the next section, we will show the main distribution of this paper: we propose a recursive Newton-like algorithm using the weighted sum scalarization approach for determining the ρ-NF solution set for Max-Min BODO with $\rho > 1$ and Min-Min BODO.

4.3. Max-Min BODO with $\rho > 1$ and Min-Min BODO

4.3.1. General idea of algorithm for determining the ρ-NF solution set

Since many ρ-NF solutions may exist for both Min-Min BODO with $\rho > 1$ and Min-Min BODO, this section states the general idea and algorithm for obtaining their ρ-NF solution set in a unique framework.

According to Theorem 6, the ρ-NF solutions are optimal solutions of weighted sum problems. Thus, they are supported efficient solutions (i.e., the solutions that can be obtained by optimizing weighted sum problems), which are on the boundary of the convex hull of all Pareto points [32]. When dealing with bi-objective mixed integer problems, the supported

efficient solutions can be identified by the weight space decomposition method [31], [32]. Furthermore, if the objectives are linear, the indifference regions in the weight space are also computed (see, e.g., [33]). However, as the ρ-NF solution set can be a strict subset of the Pareto set, we aim to propose a novel method focusing on determining the ρ-NF solution set without computing all the supported efficient solutions. For this purpose, we construct a recursive Newton-like algorithm based on the weighted sum scalarization approach to identify all ρ-NF solutions.

Let $\mathcal{F}(\alpha)$ represent $\mathcal{F}_2(\alpha)$ and $\mathcal{F}_3(\alpha)$ in our algorithm (i.e., if we solve $\mathcal{F}(\alpha)$, that means we solve $\mathcal{F}_2(\alpha)$ or minimize $\mathcal{F}_3(\alpha)$, depending on the considered problem is Max-Min BODO with $\rho > 1$ or Min-Min BODO). Let $\mathcal{C}$ be the set containing all elements α^* such that there exists a ρ-NF solution (P^*, Q^*) as a solution of $\mathcal{F}(\alpha^*)$ with $\alpha^* = P^*/Q^*$. We can prove that each element of $\mathcal{C}$ corresponds to a unique ρ-NF solution and vice versa. Thus, determining the ρ-NF solution set is equivalent to finding all elements of $\mathcal{C}$.

Let us explain the general idea for finding all elements of C. From the given BODO problem, we first determine α^{sup} as the upper bound for the elements of $\mathcal{C}$. Then, our algorithm uses Procedure *EXPLORE*() to recursively explore all elements of $\mathcal{C}$ in the interval $[0, \alpha^{sup}]$. For each subinterval $[\alpha_i, \alpha_j] \subseteq [0, \alpha^{sup}]$, Procedure *EXPLORE*() determines a value $\alpha \in [\alpha_i, \alpha_j]$ based on the solutions of $\mathcal{F}(\alpha_i)$ and $\mathcal{F}(\alpha_j)$. Then, Procedure *VERIFY*() verifies whether $\alpha \in \mathcal{C}$ or not. If not, we use Procedure *FIND*() to find an element $\alpha^* \in \mathcal{C}$ such that there does not exist any other element of $\mathcal{C}$ in the closed interval defined by α and α^* (i.e., $[\alpha, \alpha^*]$ if $\alpha < \alpha^*$ or $[\alpha^*, \alpha]$ if $\alpha^* < \alpha$). Then, we remove such closed interval to obtain at most two remaining subintervals from $[\alpha_i, \alpha_j]$, which we will continue to explore. Although we may need two recursive calls in an iteration, the total number of recursive calls in the worst case is bounded by the number of Pareto-optimal solutions. Furthermore, our determining method of α provides some criteria for quickly asserting that no more ρ-NF solution exists in $[\alpha_i, \alpha_j]$.

In the following, we discuss how to propose Procedure *VERIFY*(α_0) for verifying $\alpha_0 \in \mathcal{C}$ from the given α_0. Notice that it may be difficult to verify if $\alpha_0 \in \mathcal{C}$ by only solving $\mathcal{F}(\alpha_0)$ and then checking if the obtained solution (P_0, Q_0) satisfies $P_0 - \alpha_0 Q_0 = 0$ (based on Theorem 6). The fact is that the problem $\mathcal{F}(\alpha_0)$ may have multiple solutions, and we only obtain one solution, which is possibly not the ρ-NF solution in case $\alpha_0 \in \mathcal{C}$ (i.e., there exists another solution (P_1, Q_1) of $\mathcal{F}(\alpha_0)$ which is a ρ-NF solution. However, (P_0, Q_0), instead of (P_1, Q_1), is returned. Thus, $\alpha_0 \in \mathcal{C}$ although $P_0 - \alpha_0 Q_0 \neq 0$).

Hence, for Procedure *VERIFY*(), we define two additional optimization problems

$$\mathcal{G}_2(\alpha) = \max_{(P,Q)\in\mathcal{S}_2} g_2(\alpha, P, Q) := \rho P - \alpha Q - |P - \alpha Q|,$$
$$\mathcal{G}_3(\alpha) = \min_{(P,Q)\in\mathcal{S}_3} g_3(\alpha, P, Q) := \rho P + \alpha Q + |P - \alpha Q|,$$

where $|.|$ denotes the absolute function.

Notice that $g_2(\alpha, P, Q)$ and $g_3(\alpha, P, Q)$ can be transformed as a linear combination of P and Q. Thus, suppose that we know how to solve $\mathcal{G}_2(\alpha)$ and $\mathcal{G}_3(\alpha)$.

Let $\mathcal{G}(\alpha)$ represent $\mathcal{G}_2(\alpha)$ and $\mathcal{G}_3(\alpha)$ in our algorithm. If we solve $\mathcal{G}(\alpha)$, that means we solve $\mathcal{G}_2(\alpha)$ or $\mathcal{G}_3(\alpha)$, depending on the considered problem is Max-Min BODO with $\rho > 1$ or Min-Min BODO . We first state Procedure *VERIFY*() as follows.

Procedure 2 Verifying if $\alpha_0 \in \mathcal{C}$

Input: $\alpha_0 \in [0, \alpha^{sup}]$.
Output: True if $\alpha_0 \in \mathcal{C}$ or False otherwise.

procedure *VERIFY*(α_0)
  Solving $\mathcal{F}(\alpha_0)$ and $\mathcal{G}(\alpha_0)$ to obtain the solutions (P_0, Q_0) and (P_1, Q_1)
  if $g(\alpha_0, P_1, Q_1) = f(\alpha_0, P_0, Q_0)$ **then**
    Return True
  else
    Return False
  end if
end procedure

For a given $\alpha_0 \in [0, \alpha^{sup}]$, if we assert that $\alpha_0 \notin \mathcal{C}$ by Procedure *VERIFY*(α_0), then we can use Procedure *FIND*(α_0) for finding an element of $\mathcal{C}$ from α_0. This procedure is also based on the application of Newton's method for linear programming. It can be stated as follows.

Procedure 3 Finding an element of $\mathcal{C}$ from $\alpha_0 \notin \mathcal{C}$

Input: $\alpha_0 \in [0, \alpha^{sup}]$ and $\alpha_0 \notin \mathcal{C}$.
Output: A unique element of $\mathcal{C}$ in the closed interval defined by itself and α_0.

procedure *FIND*(α_0)
  Solving $\mathcal{F}(\alpha_0)$ to obtain a solution (P_0, Q_0)
  $i \leftarrow 0$
  repeat
    $\alpha_{i+1} \leftarrow P_i/Q_i$
    Solving $\mathcal{F}(\alpha_{i+1})$ to obtain a solution (P_{i+1}, Q_{i+1})
    $T_i \leftarrow f(\alpha_{i+1}, P_{i+1}, Q_{i+1}) - f(\alpha_{i+1}, P_i, Q_i)$
    $i \leftarrow i + 1$
  until $T_i = 0$
  Return α_{i+1}
end procedure

Then, for each subinterval $[\alpha_i, \alpha_j] \subseteq [0, \alpha^{sup}]$, we present Procedure *EXPLORE*$([\alpha_i, \alpha_j])$ for finding all elements of $\mathcal{C}$ in such subinterval.

Procedure 4 Finding the elements of $\mathcal{C}$ in the interval $[\alpha_i, \alpha_j]$

Input: $\alpha_i < \alpha_j$ and $\alpha_i, \alpha_j \in [0, \alpha^{sup}]$, $(P_i, Q_i) \not\equiv (P_j, Q_j)$ are the solutions of $\mathcal{F}(\alpha_i)$ and $\mathcal{F}(\alpha_j)$, respectively.

Output: All elements of $\mathcal{C}$ in the interval $[\alpha_i, \alpha_j]$.

procedure *EXPLORE*$([\alpha_i, \alpha_j])$
  $\mathcal{C} = \{\}$
  if *VERIFY*(α_i) == *True* **then** $\mathcal{C} = \mathcal{C} \cup \alpha_i$
  end if
  if *VERIFY*(α_j) == *True* **then** $\mathcal{C} = \mathcal{C} \cup \alpha_j$
  end if
  $\alpha_k \leftarrow |\frac{\rho(P_i - P_j)}{Q_i - Q_j}|$
  while $\alpha_i < \alpha_k$ **and** $\alpha_k < \alpha_j$ **do**
    if *VERIFY*$(\alpha_k) == True$ **then**
      $\mathcal{C} \leftarrow \mathcal{C} \cup \alpha_k$ ▷ α_k is an element of $\mathcal{C}$
      EXPLORE$([\alpha_i, \alpha_k])$, *EXPLORE*$([\alpha_k, \alpha_j])$
    else
      Solving $\mathcal{F}(\alpha_k)$ to obtain a solution (P_k, Q_k)
      if $(P_k, Q_k) \not\equiv (P_i, Q_i)$ **and** $(P_k, Q_k) \not\equiv (P_j, Q_j)$ **then** $c_k \leftarrow$ *FIND*(α_k)
        if $\alpha_i == c_k$ **then** *EXPLORE*$([\alpha_k, \alpha_j])$
        else if $\alpha_j == c_k$ **then** *EXPLORE*$([\alpha_i, \alpha_k])$
        else if $\alpha_i < c_k$ **and** $c_k < \alpha_k$ **then** $\mathcal{C} \leftarrow \mathcal{C} \cup c_k$
          EXPLORE$([\alpha_i, c_k])$, *EXPLORE*$([\alpha_k, \alpha_j])$
        else if $\alpha_j > c_k$ **and** $c_k > \alpha_k$ **then** $\mathcal{C} \leftarrow \mathcal{C} \cup c_k$
          EXPLORE$([\alpha_i, \alpha_k])$, *EXPLORE*$([c_k, \alpha_j])$
        end if
      end if
    end if
  end while
  Return $\mathcal{C}$
end procedure

Finally, we state the main algorithm for determining the ρ-NF solution set by finding all elements of $\mathcal{C}$.

Algorithm 5 Finding the ρ-NF solution set

Input: A BODO instance (either Max-Min BODO with $\rho > 1$ or Min-Min BODO).

Output: Set $\mathcal{C}$ whose elements correspond to all ρ-NF solutions.

Compute α^{sup} from the given BODO instance.
EXPLORE$([0, \alpha^{sup}])$

In the next two sections, we show that Algorithm 5 returns all elements of $\mathcal{C}$, which correspond to all ρ-NF solutions for Max-Min BODO with $\rho > 1$ and Min-Min BODO.

Notice that the two cases' statements of lemmas and theorems are almost similar. However, their proofs are different due to their different characterization of the ρ-NF solution.

4.3.2. Proofs for Max-Min BODO with $\rho > 1$

For Max-Min BODO with $\rho > 1$, the set $\mathcal{C}$ is defined as follows: $\mathcal{C}$ contains all elements $\alpha^* > 0$ such that $\mathcal{F}_2(\alpha^*)$ has a solution (P^*, Q^*) and $\alpha^* = P^*/Q^*$.

We first assert that each element of $\mathcal{C}$ corresponds to a unique ρ-NF solution and vice versa. Consequently, determining the ρ-NF solution set is equivalent to finding all elements of $\mathcal{C}$.

Lemma 2. *For Max-Min BODO with $\rho > 1$, there is a bijection between the set $\mathcal{C}$ and the ρ-NF solution set.*

Proof. According to Theorem 6, each ρ-NF solution corresponds to a unique element of $\mathcal{C}$.

We show that each element of $\mathcal{C}$ corresponds to a unique ρ-NF solution with respect to the values of P and Q.

Suppose that $(P^*, Q^*), (P^{**}, Q^{**}) \in \mathcal{S}_2$ are two ρ-NF solutions corresponding to $\alpha^* \in \mathcal{C}$. Since both (P^*, Q^*) and (P^{**}, Q^{**}) are the solutions of $\mathcal{F}_2(\alpha^*)$, we have

$$\rho P^* - \alpha^* Q^* = \rho P^{**} - \alpha^* Q^{**}, \tag{8}$$

Furthermore, since $\alpha^* \in \mathcal{C}$, we get

$$P^* - \alpha^* Q^* = P^{**} - \alpha^* Q^{**} = 0, \tag{9}$$

From (8) and (9), we have $(\rho - 1)P^* = (\rho - 1)P^{**}$. Since $\rho - 1 > 0$, we obtain $P^* = P^{**}$ and consequently $Q^* = Q^{**}$. Thus, $(P^*, Q^*) \equiv (P^{**}, Q^{**})$. □

According to Lemma 2, although both Max-Min BODO with $\rho = 1$ and Max-Min BODO with $\rho > 1$ have possibly multiple ρ-NF solutions, they have a main difference. That is, unlike for Max-Min BODO with $\rho > 1$, all ρ-NF solutions for Max-Min BODO with $\rho = 1$ correspond to a unique value, which is the maximal value of the ratio P/Q, $\forall (P, Q) \in \mathcal{S}_2$.

We then determine α^{sup} as the upper bound for the elements of $\mathcal{C}$. Let P_{max}, Q_{min} be the optimal values of the problems maximizing P and minimizing Q over $\mathcal{X}_2$, respectively. Let $\alpha^{sup} = P_{max}/Q_{min}$. We have

$$\alpha^* = \frac{P^*}{Q^*} \leq \frac{P_{max}}{Q_{min}} = \alpha^{sup},$$

Thus, α^{sup} is an upper bound of α^*.

For a given $\alpha_0 \in [0, \alpha^{sup}]$, we show that verifying if $\alpha_0 \in \mathcal{C}$ can be done by solving $\mathcal{F}_2(\alpha)$ and $\mathcal{G}_2(\alpha)$.

Lemma 3. *For a given $\alpha_0 \in [0, \alpha^{sup}]$, let $(P_0, Q_0), (P_1, Q_1) \in \mathcal{S}_2$ be the solutions of $\mathcal{F}_2(\alpha_0)$ and $\mathcal{G}_2(\alpha_0)$, respectively. Then $\alpha_0 \in \mathcal{C}$ if and only if $g_2(\alpha_0, P_1, Q_1) = f_2(\alpha_0, P_0, Q_0)$.*

Proof. $\Longrightarrow$ Suppose that $\alpha_0 \in \mathcal{C}$. According to Theorem 6, there exists $(P^*, Q^*) \in \mathcal{S}_2$ such that (P^*, Q^*) is a solution of $\mathcal{F}_2(\alpha_0)$ and $P^* = \alpha_0 Q^*$. Since both (P_0, Q_0) and (P^*, Q^*) are the solutions of $\mathcal{F}_2(\alpha_0)$ and $P^* - \alpha_0 Q^* = 0$, we have

$$\rho P_0 - \alpha_0 Q_0 = \rho P^* - \alpha_0 Q^* - |P^* - \alpha_0 Q^*|,$$

The optimality of (P_0, Q_0) gives

$$\rho P_0 - \alpha_0 Q_0 \geq \rho P_1 - \alpha_0 Q_1,$$

Since $|P_1 - \alpha_0 Q_1| \geq 0$, we deduce $\rho P_0 - \alpha_0 Q_0 \geq \rho P_1 - \alpha_0 Q_1 - |P_1 - \alpha_0 Q_1|$. Thus,

$$\rho P^* - \alpha_0 Q^* - |P^* - \alpha_0 Q^*| \geq \rho P_1 - \alpha_0 Q_1 - |P_1 - \alpha_0 Q_1|, \tag{10}$$

Since (P_1, Q_1) is a solution of $\mathcal{G}_2(\alpha_0)$, we have

$$\rho P_1 - \alpha_0 Q_1 - |P_1 - \alpha_0 Q_1| \geq \rho P^* - \alpha_0 Q^* - |P^* - \alpha_0 Q^*|, \tag{11}$$

From (10) and (11), we get

$$\rho P_1 - \alpha_0 Q_1 - |P_1 - \alpha_0 Q_1| = \rho P^* - \alpha_0 Q^* - |P^* - \alpha_0 Q^*| = \rho P_0 - \alpha_0 Q_0,$$

which implies $g_2(\alpha_0, P_1, Q_1) = f_2(\alpha_0, P_0, Q_0)$.

$\Longleftarrow$ Suppose that $g_2(\alpha_0, P_1, Q_1) = f_2(\alpha_0, P_0, Q_0)$. We obtain $\rho P_1 - \alpha_0 Q_1 - |P_1 - \alpha_0 Q_1| = \rho P_0 - \alpha_0 Q_0$. Since $\rho P_1 - \alpha_0 Q_1 - |P_1 - \alpha_0 Q_1| \leq \rho P_1 - \alpha_0 Q_1 \leq \rho P_0 - \alpha_0 Q_0$, we must have $|P_1 - \alpha_0 Q_1| = 0$ and $\rho P_1 - \alpha_0 Q_1 = \rho P_0 - \alpha_0 Q_0$. Consequently, (P_1, Q_1) is a solution of $\mathcal{F}_2(\alpha_0)$ and $P_1 = \alpha_0 Q_1$. Thus, (P_1, Q_1) is a ρ-NF solution and $\alpha_0 \in \mathcal{C}$ due to Theorem 6. $\square$

Let $T_i = f_2(\alpha_{i+1}, P_{i+1}, Q_{i+1}) - f_2(\alpha_{i+1}, P_i, Q_i)$ where (P_i, Q_i) is the solution of $\mathcal{F}_2(\alpha_i)$ and $\{\alpha_i\}_{i \geq 0}$ (including α_0) is the sequence constructed by Procedure *FIND*(α_0). We then prove that if $\alpha_0 \notin \mathcal{C}$, Procedure *FIND*$(\alpha_0)$ returns an element of $\mathcal{C}$ such that it is a unique element of $\mathcal{C}$ in the closed interval defined by α_0 and itself.

Lemma 4. *For $\alpha_0 \in [0, \alpha^{sup}]$, during the execution of Procedure FIND(α_0), $\alpha_{i+1} > 0$, (P_{i+1}, Q_{i+1}) is a Pareto-optimal solution and $T_i \geq 0$ $\forall i \geq 0$. Furthermore, Procedure FIND(α_0) terminates after a finite number of iterations.*

Proof. Since $P_i, Q_i > 0$, we have $\alpha_{i+1} = P_i/Q_i > 0, \forall i \geq 0$. Thus, (P_{i+1}, Q_{i+1}) is a Pareto-optimal solution due to Theorem 5, $\forall i \geq 0$.

For all $i \geq 0$, the optimality of (P_{i+1}, Q_{i+1}) gives

$$\rho P_{i+1} - \alpha_{i+1} Q_{i+1} \geq \rho P_i - \alpha_{i+1} Q_i, \tag{12}$$

Thus, $T_i = f_2(\alpha_{i+1}, P_{i+1}, Q_{i+1}) - f_2(\alpha_{i+1}, P_i, Q_i) = \rho P_{i+1} - \alpha_{i+1} Q_{i+1} - (\rho P_i - \alpha_{i+1} Q_i) \geq 0$.

Moreover, replacing α_{i+1} by P_i/Q_i, (12) yields

$$\rho P_{i+1} \geq \frac{P_i}{Q_i} Q_{i+1} + (\rho - 1) P_i \implies \frac{P_{i+1}}{P_i} \geq \frac{1}{\rho} \times \frac{Q_{i+1}}{Q_i} + \frac{\rho - 1}{\rho},$$

Using Young's inequality for products, we have

$$\frac{P_{i+1}}{P_i} \geq \frac{1}{\rho} \times \frac{Q_{i+1}}{Q_i} + \frac{\rho - 1}{\rho} \geq (\frac{Q_{i+1}}{Q_i})^{\frac{1}{\rho}} \implies \frac{P_{i+1}^{\rho}}{Q_{i+1}} \geq \frac{P_i^{\rho}}{Q_i},$$

In other words, the value of P_i^{ρ}/Q_i is increasing after each iteration of Procedure *FIND*(α_0). Since $\mathcal{S}_2$ is finite, the set of values P^{ρ}/Q is also finite. Thus, we get $\frac{P_{k+1}^{\rho}}{Q_{k+1}} = \frac{P_k^{\rho}}{Q_k}$ after a finite number of iterations. Now, the equality of Young's inequality above must hold. We have then $\frac{P_{k+1}}{P_k} = \frac{Q_{k+1}}{Q_k} = 1 \implies P_{k+1} = P_k$ and $Q_{k+1} = Q_k$. Consequently, $T_k = f_2(\alpha_{k+1}, P_{k+1}, Q_{k+1}) - f_2(\alpha_{k+1}, P_k, Q_k) = \rho P_{k+1} - \alpha_{k+1} Q_{k+1} - (\rho P_k - \alpha_{k+1} Q_k) = 0$. That is to say, Procedure *FIND*(α_0) terminates after a finite number of iterations. □

Suppose that Procedure *FIND*(α_0) returns a coefficient α_{n+1} satisfying $T_n = f_2(\alpha_{n+1}, P_{n+1}, Q_{n+1}) - f_2(\alpha_{n+1}, P_n, Q_n) = 0$ where $n \geq 0$. Thus, (P_n, Q_n) is a ρ-NF solution and $\alpha_{n+1} \in \mathcal{C}$ due to Lemma 1. In addition, if $n \geq 1$ then $T_i > 0$, $\forall 0 \leq i \leq n-1$.

In the next lemma, we show the monotonic relationship between $\alpha \in [0, \alpha^{sup}]$ and the solution of $\mathcal{F}_2(\alpha)$ with respect to the values of P and Q.

Lemma 5. *Let $\alpha', \alpha'' \in [0, \alpha^{sup}]$, $\alpha' < \alpha''$ and (P', Q'), $(P'', Q'') \in \mathcal{S}_2$ be the solutions of $\mathcal{F}_2(\alpha')$ and $\mathcal{F}_2(\alpha'')$, respectively. Then $P' \geq P''$ and $Q' \geq Q''$.*

Proof. The optimality of (P', Q') and (P'', Q'') gives

$$\rho P' - \alpha' Q' \geq \rho P'' - \alpha' Q'', \text{ and} \tag{13a}$$
$$\rho P'' - \alpha'' Q'' \geq \rho P' - \alpha'' Q' \tag{13b}$$

Adding (13a) and (13b) gives $(\alpha' - \alpha'')(Q' - Q'') \leq 0$. Since $\alpha' < \alpha''$, we have $Q' \geq Q''$.

On the other hand, the inequality (13a) implies $\rho(P' - P'') \geq \alpha'(Q' - Q'') \geq 0$. Since $\rho > 1$, we get $P' \geq P''$. □

Theorem 7. *For $\alpha_0 \in [0, \alpha^{sup}]$ and $\alpha_0 \notin \mathcal{C}$, Procedure FIND$(\alpha_0)$ returns α_{n+1} as the unique element of $\mathcal{C}$ in the closed interval defined by α_0 and α_{n+1}.*

Proof. Since (P_0, Q_0) is a solution of $\mathcal{F}_2(\alpha_0)$ and $\alpha_0 \notin \mathcal{C}$, we have $P_0 - \alpha_0 Q_0 \neq 0$. Thus, we consider two cases: $P_0 - \alpha_0 Q_0 > 0$ and $P_0 - \alpha_0 Q_0 < 0$.

We first suppose that $P_0 - \alpha_0 Q_0 > 0$. We will prove $\alpha_i < \alpha_{i+1}, \forall i \geq 0$ by induction on i.

Since $P_0 - \alpha_0 Q_0 > 0$, we have $\alpha_0 < P_0/Q_0 = \alpha_1$. Thus, our hypothesis is true with $i = 0$.

Suppose our hypothesis is true until $i = k \geq 0$. We have $\alpha_i < \alpha_{i+1}$, $\forall 0 \leq i \leq k$. The optimality of (P_{k+1}, Q_{k+1}) gives

$$\rho P_{k+1} - \alpha_{k+1} Q_{k+1} \geq \rho P_k - \alpha_{k+1} Q_k,$$

which is equivalent to

$$(\rho-1)P_{k+1}+P_{k+1}-\alpha_{k+1}Q_{k+1}\geq(\rho-1)P_k+P_k-\alpha_{k+1}Q_k, \tag{14}$$

Since $\alpha_k<\alpha_{k+1}$, $P_k\geq P_{k+1}$ and $Q_k\geq Q_{k+1}$ due to Lemma 5.

If $P_k=P_{k+1}$ and $Q_k=Q_{k+1}$ then $T_k=\rho P_{k+1}-\alpha_{k+1}Q_{k+1}-(\rho P_k-\alpha_{k+1}Q_k)=0$. Consequently, Procedure *FIND*(α_0) returns the value α_{k+1} and the sequence $\{\alpha_i\}_{0\leq i\leq k+1}$ is strictly increasing.

If $P_k=P_{k+1}$ and $Q_k>Q_{k+1}$ then $\alpha_{k+1}=\frac{P_k}{Q_k}<\frac{P_{k+1}}{Q_{k+1}}=\alpha_{k+2}$.

If $P_k>P_{k+1}$ then (14) yields $P_{k+1}-\alpha_{k+1}Q_{k+1}>P_k-\alpha_{k+1}Q_k=P_k-\frac{P_k}{Q_k}Q_k=0$. Thus, $\alpha_{k+1}<\frac{P_{k+1}}{Q_{k+1}}=\alpha_{k+2}$.

Since we obtain $\alpha_{k+1}<\alpha_{k+2}$ in both two cases above, our hypothesis is also true with $i=k+1$. Hence, $\{\alpha_i\}$ is strictly increasing, $\forall i\geq 0$.

In the following, we will show that α_{n+1} is the unique element of $\mathcal{C}$ in the interval $[\alpha_0,\alpha_{n+1}]$.

Suppose that we have $\alpha^*=(\alpha_0,\alpha_{n+1})\cap\mathcal{C}$ corresponding to a ρ-NF solution (P^*,Q^*). Since the sequence $\{\alpha_i\}_{0\leq i\leq n+1}$ is strictly increasing, there exists $0\leq k\leq n$ such that $\alpha^*\in(\alpha_k,\alpha_{k+1}]$.

Since $\alpha^*>\alpha_k$, we have $P^*\leq P_k$ and $Q^*\leq Q_k$ due to Lemma 5. Furthermore, as (P^*,Q^*) is a ρ-NF solution, we get

$$\rho\frac{P_k}{P^*}-\frac{Q_k}{Q^*}\leq\rho-1\implies\frac{P_k}{P^*}-\frac{Q_k}{Q^*}\leq(\rho-1)(1-\frac{P_k}{P^*})\leq 0, \tag{15}$$

If $\frac{P_k}{P^*}=\frac{Q_k}{Q^*}$ then (15) yields $\frac{P_k}{P^*}=\frac{Q_k}{Q^*}=1$. It follows that $P_k=P^*$, $Q_k=Q^*$ and $\alpha_{k+1}=\frac{P_k}{Q_k}=\frac{P^*}{Q^*}=\alpha^*$. Since $\alpha_{k+1}=\alpha^*<\alpha_{n+1}$, we have $k\leq n-1$ and $n\geq 1$. As both (P^*,Q^*) and (P_{k+1},Q_{k+1}) are the solutions of $\mathcal{F}_2(\alpha_{k+1})$, we obtain $\rho P_{k+1}-\alpha_{k+1}Q_{k+1}-(\rho P^*-\alpha_{k+1}Q^*)=0$. Consequently, $T_k=\rho P_{k+1}-\alpha_{k+1}Q_{k+1}-(\rho P_k-\alpha_{k+1}Q_k)=0$ which leads to a contradiction due to $T_k>0$, $\forall 0\leq k\leq n-1$.

Thus, we have

$$\frac{P_k}{P^*}<\frac{Q_k}{Q^*}\implies\alpha^*=\frac{P^*}{Q^*}>\frac{P_k}{Q_k}=\alpha_{k+1},$$

which leads to a contradiction due to the fact that $\alpha^*\leq\alpha_{k+1}$.

Hence, α_{n+1} is the unique element of $\mathcal{C}$ in the interval $[\alpha_0,\alpha_{n+1}]$.

Similarly, in case $P_0-\alpha_0Q_0<0$, we obtain that the sequence $\{\alpha_i\}$ is strictly decreasing and α_{n+1} is the unique element of $\mathcal{C}$ in the interval $[\alpha_{n+1},\alpha_0]$. That concludes the proof. □

According to Theorem 7, from $\alpha_0\in[0,\alpha^{sup}]$ and $\alpha_0\notin\mathcal{C}$, we can use Procedure *FIND*(α_0) to find an element α_{n+1} of $\mathcal{C}$ such that there does not have any other element of $\mathcal{C}$ in the interval$[\alpha_0,\alpha_{n+1}]$ (if $\alpha_0<\alpha_{n+1}$) or $[\alpha_{n+1},\alpha_0]$ (if $\alpha_0>\alpha_{n+1}$).

For each interval $[\alpha_i,\alpha_j]\subseteq[0,\alpha^{sup}]$ and $(P_i,Q_i)\not\equiv(P_j,Q_j)$ are the solutions of $\mathcal{F}_2(\alpha_i)$ and $\mathcal{F}_2(\alpha_j)$ where $0\leq\alpha_i<\alpha_j\leq\alpha^{sup}$, we present some criteria to quickly verify if there does not exist any ρ-NF solution in the interval (α_i,α_j) which is different to (P_i,Q_i) and (P_j,Q_j).

Lemma 6. *Given an interval* $[\alpha_i, \alpha_j]$ *defined by* $0 \leq \alpha_i < \alpha_j \leq \alpha^{sup}$ *and let* (P_i, Q_i), (P_j, Q_j) *be the solutions of* $\mathcal{F}_2(\alpha_i)$ *and* $\mathcal{F}_2(\alpha_j)$ *such that* $(P_i, Q_i) \not\equiv (P_j, Q_j)$. *Let* $\alpha_k = |\frac{\rho(P_i - P_j)}{Q_i - Q_j}|$ *and* (P_k, Q_k) *be a solution of* $\mathcal{F}_2(\alpha_k)$. *If one of the following conditions is satisfied, then there does not exist any* ρ*-NF solution which is different to* (P_i, Q_i) *and* (P_j, Q_j) *in the interval* (α_i, α_j).

1. $\alpha_k \in \{\alpha_i, \alpha_j\}$;
2. $\alpha_k \notin \mathcal{C}$ *and* $(P_k, Q_k) \equiv (P_i, Q_i)$ *or* $(P_k, Q_k) \equiv (P_j, Q_j)$;

Proof. We first show that α_k is well defined. Since $\alpha_i < \alpha_j$, we have $P_i \geq P_j$, $Q_i \geq Q_j$ due to Lemma 5. Assume that $Q_i = Q_j$. The optimality of (P_j, Q_j) gives

$$\rho P_j - \alpha_j Q_j \geq \rho P_i - \alpha_j Q_i,$$

Since $Q_i = Q_j$, we obtain $P_j \geq P_i$. Thus, $P_i = P_j$ and then $(P_i, Q_i) \equiv (P_j, Q_j)$ which leads to a contradiction.

Hence, $Q_i > Q_j$ and consequently, α_k is well defined.

Since $P_i \geq P_j$ and $Q_i > Q_j$, we have $\alpha_k = \frac{\rho(P_i - P_j)}{Q_i - Q_j}$. Now we show that $\alpha_k \in [\alpha_i, \alpha_j]$.

The optimality of (P_i, Q_i) and (P_j, Q_j) gives

$$\rho P_i - \alpha_i Q_i \geq \rho P_j - \alpha_i Q_j,$$
$$\rho P_j - \alpha_j Q_j \geq \rho P_i - \alpha_j Q_i,$$

Thus, we obtain $\alpha_i \leq \frac{\rho(P_i - P_j)}{Q_i - Q_j} \leq \alpha_j$ which leads to $\alpha_i \leq \alpha_k \leq \alpha_j$.

If $\alpha_k = \alpha_i$ then $\rho P_i - \alpha_i Q_i = \rho P_j - \alpha_i Q_j$. Thus, (P_i, Q_i) and (P_j, Q_j) are both solutions of $\mathcal{F}_2(\alpha_i)$. Hence, for all $\alpha \in (\alpha_i, \alpha_j)$, (P_j, Q_j) is the solution of $\mathcal{F}_2(\alpha)$ as a result of Lemma 5.

Similarly, if $\alpha_k = \alpha_j$, (P_i, Q_i) is the solution of $\mathcal{F}_2(\alpha)$ for all $\alpha \in (\alpha_i, \alpha_j)$.

Consequently, in case $\alpha_k \in \{\alpha_i, \alpha_j\}$, there does not exist any ρ-NF solution which is different to (P_i, Q_i) and (P_j, Q_j) in the interval (α_i, α_j).

Now let (P_k, Q_k) be a solution of $\mathcal{F}_2(\alpha_k)$. Without loss of generality, we suppose that $\alpha_k \notin \mathcal{C}$ and $(P_k, Q_k) \equiv (P_i, Q_i)$.

Since $(P_k, Q_k) \equiv (P_i, Q_i)$, (P_i, Q_i) is the solution of $\mathcal{F}_2(\alpha_k)$. Since $\alpha_k = \frac{\rho(P_i - P_j)}{Q_i - Q_j}$, we have $\rho P_i - \alpha_k Q_i = \rho P_j - \alpha_k Q_j$. Thus, (P_j, Q_j) is also a solution of $\mathcal{F}_2(\alpha_k)$.

Consequently, if $\alpha \in (\alpha_i, \alpha_k)$ then (P_i, Q_i) is the solution of $\mathcal{F}_2(\alpha)$ and if $\alpha \in (\alpha_k, \alpha_j)$ then (P_j, Q_j) is the solution of $\mathcal{F}_2(\alpha)$. That means there does not exist any ρ-NF solution in the interval (α_i, α_j) which is different to (P_i, Q_i) and (P_j, Q_j). □

Theorem 8. *Algorithm 5 returns all the elements of* $\mathcal{C}$ *corresponding to all* ρ*-NF solutions for Max-Min BODO with* $\rho > 1$.

Proof. Let K_2 denote the finite number of Pareto-optimal solutions. Consequently, $[0, \alpha^{sup}]$ can be separated by at most K_2 consecutive subintervals $[c_i, c_j]$ where $0 \leq c_i < c_j \leq$

α^{sup} and c_i, c_j correspond to two different Pareto-optimal solutions. By using Procedure *EXPLORE*($[c_i, c_j]$), each recursive call gives us a ρ-NF solution where the corresponding coefficient in the subinterval $[c_i, c_j]$ or show that such subinterval is well explored without any new ρ-NF solution. As we use Procedure *FIND*() and Procedure *VERIFY*() in each recursive call, Procedure *EXPLORE*($[0, \alpha^{sup}]$) also terminates after a finite number of iterations. Since Algorithm 5 terminated when the interval $[0, \alpha^{sup}]$ is totally explored, it found all elements of $\mathcal{C}$ corresponding to all ρ-NF solutions. □

4.3.3. Proofs for Min-Min BODO

For Min-Min BODO, the set $\mathcal{C}$ is defined as follows: $\mathcal{C}$ contains all elements $\alpha^* > 0$ such that $\mathcal{F}_3(\alpha^*)$ has a solution (P^*, Q^*) with $\alpha^* = P^*/Q^*$.

We also assert that each element of C corresponds to a unique ρ-NF solution and vice versa.

Lemma 7. *For Min-Min BODO, there is a bijection between the set $\mathcal{C}$ and the ρ-NF solution set.*

Proof. According to Theorem 6, each ρ-NF solution corresponds to a unique element of $\mathcal{C}$.

We also show that each element of $\mathcal{C}$ corresponds to a unique ρ-NF solution with respect to the values of P and Q.

Suppose that $(P^*, Q^*), (P^{**}, Q^{**})$ are two ρ-NF solutions corresponding to $\alpha^* \in \mathcal{C}$. Since both (P^*, Q^*) and (P^{**}, Q^{**}) are the solutions of $\mathcal{F}_3(\alpha^*)$, we have

$$\rho P^* + \alpha^* Q^* = \rho P^{**} + \alpha^* Q^{**}, \tag{16}$$

Furthermore, since $\alpha^* \in \mathcal{C}$, we get

$$P^* - \alpha^* Q^* = P^{**} - \alpha^* Q^{**} = 0, \tag{17}$$

Adding (16) and (17) gives $(\rho + 1)P^* = (\rho + 1)P^{**}$. Since $\rho \geq 1$, we obtain $P^* = P^{**}$ and consequently $Q^* = Q^{**}$. Thus, $(P^*, Q^*) \equiv (P^{**}, Q^{**})$.

□

We then determine the upper bound α^{sup} for the elements of $\mathcal{C}$. Let $(P_q, Q_q) \in \mathcal{S}_3$ be a solution for minimizing Q and $\alpha^{sup} = P_q/Q_q$. We state the following lemma.

Lemma 8. *Let $\alpha^* \in \mathcal{C}$. We have $\alpha^* \leq \alpha^{sup}$.*

Proof. Let $(P^*, Q^*) \in \mathcal{S}_3$ be a ρ-NF solution for Min-Min BODO corresponding to $\alpha^* \in \mathcal{C}$. The optimality of (P^*, Q^*) gives

$$\rho P^* + \alpha^* Q^* \leq \rho P_q + \alpha^* Q_q, \tag{18}$$

Moreover, since $(P_q, Q_q) \in \mathcal{S}_3$ is a solution for minimizing Q, we have $Q^* \geq Q_q$ which implies $P^* \leq P_q$ due to (18). Thus,

$$\alpha^* = \frac{P^*}{Q^*} \leq \frac{P_q}{Q_q} = \alpha^{sup},$$

□

For a given $\alpha_0 \in [0, \alpha^{sup}]$, we show that verifying if $\alpha_0 \in \mathcal{C}$ can be done by solving $\mathcal{F}_3(\alpha)$ and $\mathcal{G}_3(\alpha)$.

Lemma 9. *For a given $\alpha_0 \in [0, \alpha^{sup}]$, let $(P_0, Q_0), (P_1, Q_1) \in \mathcal{S}_3$ be the solutions of $\mathcal{F}_3(\alpha_0)$ and $\mathcal{G}_3(\alpha_0)$, respectively. Then $\alpha_0 \in \mathcal{C}$ if and only if $g_3(\alpha_0, P_1, Q_1) = f_3(\alpha_0, P_0, Q_0)$.*

Proof. $\Longrightarrow$ Suppose that $\alpha_0 \in \mathcal{C}$. According to Theorem 6, there exists $(P^*, Q^*) \in \mathcal{S}_3$ such that (P^*, Q^*) is a solution of $\mathcal{F}_3(\alpha_0)$ and $P^* = \alpha_0 Q^*$. Since both (P_0, Q_0) and (P^*, Q^*) are the solutions of $\mathcal{F}_3(\alpha_0)$ and $P^* - \alpha_0 Q^* = 0$, we have

$$\rho P_0 + \alpha_0 Q_0 = \rho P^* + \alpha_0 Q^* + |P^* - \alpha_0 Q^*|,$$

The optimality of (P_0, Q_0) gives

$$\rho P_0 + \alpha_0 Q_0 \leq \rho P_1 + \alpha_0 Q_1,$$

Since $|P_1 - \alpha_0 Q_1| \geq 0$, we deduce $\rho P_0 + \alpha_0 Q_0 \leq \rho P_1 + \alpha_0 Q_1 + |P_1 - \alpha_0 Q_1|$. Thus,

$$\rho P^* + \alpha_0 Q^* + |P^* - \alpha_0 Q^*| \leq \rho P_1 + \alpha_0 Q_1 + |P_1 - \alpha_0 Q_1|, \tag{19}$$

Since (P_1, Q_1) is a solution of $\mathcal{G}_3(\alpha_0)$, we have

$$\rho P_1 + \alpha_0 Q_1 + |P_1 - \alpha_0 Q_1| \leq \rho P^* + \alpha_0 Q^* + |P^* - \alpha_0 Q^*|, \tag{20}$$

From (19) and (20), we get

$$\rho P_1 + \alpha_0 Q_1 + |P_1 - \alpha_0 Q_1| = \rho P^* + \alpha_0 Q^* + |P^* - \alpha_0 Q^*| = \rho P_0 + \alpha_0 Q_0,$$

which implies $g_3(\alpha_0, P_1, Q_1) = f_3(\alpha_0, P_0, Q_0)$.

$\Longleftarrow$ Suppose that $g_3(\alpha_0, P_1, Q_1) = f_3(\alpha_0, P_0, Q_0)$. We obtain $\rho P_1 + \alpha_0 Q_1 + |P_1 - \alpha_0 Q_1| = \rho P_0 + \alpha_0 Q_0$. Since $\rho P_1 + \alpha_0 Q_1 + |P_1 - \alpha_0 Q_1| \geq \rho P_1 + \alpha_0 Q_1 \geq \rho P_0 + \alpha_0 Q_0$, we must have $|P_1 - \alpha_0 Q_1| = 0$ and $\rho P_1 + \alpha_0 Q_1 = \rho P_0 + \alpha_0 Q_0$. Consequently, (P_1, Q_1) is a solution of $\mathcal{F}_3(\alpha_0)$ and $P_1 = \alpha_0 Q_1$. Thus, (P_1, Q_1) is a ρ-NF solution and $\alpha_0 \in \mathcal{C}$ due to Theorem 6. $\square$

Let $T_i = f_3(\alpha_{i+1}, P_{i+1}, Q_{i+1}) - f_3(\alpha_{i+1}, P_i, Q_i)$ where (P_i, Q_i) is the solution of $\mathcal{F}_3(\alpha_i)$ and $\{\alpha_i\}_{i \geq 0}$ (including α_0) is the sequence constructed by Procedure *FIND*(α_0). We then prove that if $\alpha_0 \notin \mathcal{C}$, Procedure *FIND*$(\alpha_0)$ returns an element of $\mathcal{C}$ such that it is a unique element of $\mathcal{C}$ in the closed interval defined by α_0 and itself.

Lemma 10. *For $\alpha_0 \in [0, \alpha^{sup}]$, during the execution of Procedure FIND(α_0), $\alpha_{i+1} > 0$, (P_{i+1}, Q_{i+1}) is a Pareto-optimal solution and $T_i \leq 0$ $\forall i \geq 0$. Furthermore, Procedure FIND(α_0) terminates after a finite number of iterations.*

Proof. Since $P_i, Q_i > 0$, we have $\alpha_{i+1} = P_i/Q_i > 0, \forall i \geq 0$. Thus, (P_{i+1}, Q_{i+1}) is a Pareto-optimal solution due to Theorem 5, $\forall i \geq 0$.

For all $i \geq 0$, the optimality of (P_{i+1}, Q_{i+1}) gives

$$\rho P_{i+1} + \alpha_{i+1} Q_{i+1} \leq \rho P_i + \alpha_{i+1} Q_i, \tag{21}$$

Thus, $T_i = f_3(\alpha_{i+1}, P_{i+1}, Q_{i+1}) - f_3(\alpha_{i+1}, P_i, Q_i) = \rho P_{i+1} + \alpha_{i+1} Q_{i+1} - (\rho P_i + \alpha_{i+1} Q_i) \leq 0$. Moreover, replacing α_{i+1} by P_i/Q_i, (21) yields

$$\rho P_{i+1} + \frac{P_i}{Q_i} Q_{i+1} \leq (\rho + 1) P_i \implies 1 \geq \frac{\rho}{\rho+1} \times \frac{P_{i+1}}{P_i} + \frac{1}{\rho+1} \times \frac{Q_{i+1}}{Q_i},$$

Using Young's inequality for products, we have

$$1 \geq \frac{\rho}{\rho+1} \times \frac{P_{i+1}}{P_i} + \frac{1}{\rho+1} \times \frac{Q_{i+1}}{Q_i} \geq (\frac{P_{i+1}}{P_i})^{\frac{\rho}{\rho+1}} (\frac{Q_{i+1}}{Q_i})^{\frac{1}{\rho+1}} \implies P_i^\rho Q_i \geq P_{i+1}^\rho Q_{i+1},$$

In other words, the value of $P_i^\rho Q_i$ is decreasing after each iteration of Procedure *FIND*(α_0). Since $\mathcal{S}_3$ is finite, the set of values $P^\rho Q$ is also finite. Thus, we get $P_{k+1}^\rho Q_{k+1} = P_k^\rho Q_k$ after a finite number of iterations. Now the equality of Young's inequality above must hold. We have then $\frac{P_{k+1}}{P_k} = \frac{Q_{k+1}}{Q_k} = 1 \implies P_{k+1} = P_k$ and $Q_{k+1} = Q_k$. Consequently, $T_k = f_3(\alpha_{k+1}, P_{k+1}, Q_{k+1}) - f_3(\alpha_{k+1}, P_k, Q_k) = \rho P_{k+1} + \alpha_{k+1} Q_{k+1} - (\rho P_k + \alpha_{k+1} Q_k) = 0$. That is to say, Procedure *FIND*(α_0) terminates after a finite number of iterations. □

Suppose that Procedure *FIND*(α_0) returns a coefficient α_{n+1} satisfying $T_n = f_3(\alpha_{n+1}, P_{n+1}, Q_{n+1}) - f_3(\alpha_{n+1}, P_n, Q_n) = 0$ where $n \geq 0$. Thus, (P_n, Q_n) is a ρ-NF solution and $\alpha_{n+1} \in \mathcal{C}$ due to Lemma 1. In addition, if $n \geq 1$ then $T_i < 0$, $\forall 0 \leq i \leq n-1$.

Like Lemma 5, we also show the monotonic relationship between $\alpha \in [0, \alpha^{sup}]$ and the solution of $\mathcal{F}_3(\alpha)$ with respect to the values of P and Q.

Lemma 11. *Let $\alpha', \alpha'' \in [0, \alpha^{sup}]$, $\alpha' < \alpha''$ and (P', Q'), $(P'', Q'') \in \mathcal{S}_3$ be the solutions of $\mathcal{F}_3(\alpha')$ and $\mathcal{F}_3(\alpha'')$, respectively. Then $P' \leq P''$ and $Q' \geq Q''$.*

Proof. The optimality of (P', Q') and (P'', Q'') gives

$$\rho P' + \alpha' Q' \leq \rho P'' + \alpha' Q'', \text{ and} \tag{22a}$$
$$\rho P'' + \alpha'' Q'' \leq \rho P' + \alpha'' Q' \tag{22b}$$

Adding (22a) and (22b) gives $(\alpha' - \alpha'')(Q' - Q'') \leq 0$. Since $\alpha' < \alpha''$, we have $Q' \geq Q''$.

On the other hand, the inequality (22a) implies $\rho(P'' - P') \geq \alpha'(Q' - Q'') \geq 0$. Since $\rho \geq 1$, we get $P' \leq P''$. □

Theorem 9. *For $\alpha_0 \in [0, \alpha^{sup}]$ and $\alpha_0 \notin \mathcal{C}$, Procedure FIND$(\alpha_0)$ returns α_{n+1} as the unique element of $\mathcal{C}$ in the closed interval defined by α_0 and α_{n+1}.*

Proof. For Min-Min BODO, we also consider two cases: $P_0 - \alpha_0 Q_0 > 0$ and $P_0 - \alpha_0 Q_0 < 0$.

We first suppose that $P_0 - \alpha_0 Q_0 > 0$. We will prove $\alpha_i < \alpha_{i+1}, \forall i \geq 0$ by induction on i.

Since $P_0 - \alpha_0 Q_0 > 0$, $\alpha_0 < P_0/Q_0 = \alpha_1$. Thus, our hypothesis is true with $i = 0$.

Suppose our hypothesis is true until $i = k \geq 0$. We have $\alpha_i < \alpha_{i+1}, \forall 0 \leq i \leq k$. Since $\alpha_k < \alpha_{k+1}$, we have $P_{k+1} \geq P_k > 0$ and $Q_k \geq Q_{k+1} > 0$ due to Lemma 11. It leads to $Q_k P_{k+1} \geq P_k Q_{k+1}$ and $Q_k P_{k+1} = P_k Q_{k+1} \iff (P_k, Q_k) \equiv (P_{k+1}, Q_{k+1})$.

If $T_k = 0$ then Procedure *FIND*(α_0) returns the value α_{k+1}. Hence, the sequence $\{\alpha_i\}$ for $i = 0, 1, ..., k+1$ is strictly increasing.

If $T_k < 0$ then we have $(P_k, Q_k) \not\equiv (P_{k+1}, Q_{k+1})$. It implies $Q_k P_{k+1} > P_k Q_{k+1}$. We get

$$\alpha_{k+1} = \frac{P_k}{Q_k} < \frac{P_{k+1}}{Q_{k+1}} = \alpha_{k+2},$$

Thus, in this case, our hypothesis is also true with $i = k+1$. Consequently, $\{\alpha_i\}$ is strictly increasing, $\forall i \geq 0$.

In the following, we will show that α_{n+1} is the unique element of $\mathcal{C}$ in the interval $[\alpha_0, \alpha_{n+1}]$.

Suppose that we have $\alpha^* = (\alpha_0, \alpha_{n+1}) \cap \mathcal{C}$ corresponding to a ρ-NF solution (P^*, Q^*). Since the sequence $\{\alpha_i\}_{0 \leq i \leq n+1}$ is strictly increasing, there exists $0 \leq k \leq n$ such that $\alpha^* \in (\alpha_k, \alpha_{k+1}]$.

Since $\alpha^* > \alpha_k$, we have $P^* \geq P_k$ and $Q^* \leq Q_k$ due to Lemma 11. Furthermore, as (P^*, Q^*) is a ρ-NF solution, we get

$$\rho \frac{P_k}{P^*} + \frac{Q_k}{Q^*} \geq \rho + 1 \implies \frac{Q_k}{Q^*} - \frac{P_k}{P^*} \geq (\rho + 1)(1 - \frac{P_k}{P^*}) \geq 0, \tag{23}$$

If $\frac{Q_k}{Q^*} = \frac{P_k}{P^*}$ then (23) yields $\frac{Q_k}{Q^*} = \frac{P_k}{P^*} = 1$. It follows that $P_k = P^*$, $Q_k = Q^*$ and $\alpha_{k+1} = \frac{P_k}{Q_k} = \frac{P^*}{Q^*} = \alpha^*$. Since $\alpha_{k+1} = \alpha^* < \alpha_{n+1}$, we have $k \leq n-1$ and $n \geq 1$. As both (P^*, Q^*) and (P_{k+1}, Q_{k+1}) are the solutions of $\mathcal{F}_2(\alpha_{k+1})$, we obtain $\rho P_{k+1} + \alpha_{k+1} Q_{k+1} - (\rho P^* + \alpha_{k+1} Q^*) = 0$. Consequently, $T_k = \rho P_{k+1} + \alpha_{k+1} Q_{k+1} - (\rho P_k + \alpha_{k+1} Q_k) = 0$ which leads to a contradiction due to $T_k < 0$, $\forall 0 \leq k \leq n-1$.

Thus, we have

$$\frac{Q_k}{Q^*} > \frac{P_k}{P^*} \implies \alpha^* = \frac{P^*}{Q^*} > \frac{P_k}{Q_k} = \alpha_{k+1},$$

which leads to a contradiction due to the fact that $\alpha^* \leq \alpha_{k+1}$.

Hence, α_{n+1} is the unique element of $\mathcal{C}$ in the interval $[\alpha_0, \alpha_{n+1}]$.

Similarly, in case $P_0 - \alpha_0 Q_0 < 0$, we obtain that the sequence $\{\alpha_i\}$ is strictly decreasing and α_{i+1} is the unique element of $\mathcal{C}$ in the interval $[\alpha_{n+1}, \alpha_0]$. That concludes the proof. □

According to Theorem 9, from $\alpha_0 \in [0, \alpha^{sup}]$ and $\alpha_0 \notin \mathcal{C}$, we can use Procedure *FIND*(α_0) to find an element α_{n+1} of $\mathcal{C}$ such that there does not exist any other element of $\mathcal{C}$ in the interval $[\alpha_0, \alpha_{n+1}]$ (if $\alpha_0 < \alpha_{n+1}$) or $[\alpha_{n+1}, \alpha_0]$ (if $\alpha_0 > \alpha_{n+1}$).

For each interval $[\alpha_i, \alpha_j]$ and $(P_i, Q_i) \not\equiv (P_j, Q_j)$ are the solutions of $\mathcal{F}_3(\alpha_i)$ and $\mathcal{F}_3(\alpha_j)$, respectively, where $0 \leq \alpha_i < \alpha_j \leq \alpha^{sup}$, we also present some criteria to quickly verify if

there does not exist any ρ-NF solution in the interval (α_i, α_j) which is different to (P_i, Q_i) and (P_j, Q_j). Then, we show that Algorithm 5 returns all the elements of $\mathcal{C}$ corresponding to all ρ-NF solutions for Min-Min BODO. Notice that the proofs of the following lemma and theorem can be omitted since they are similar to those for Max-Min BODO with $\rho > 1$.

Lemma 12. *Given an interval $[\alpha_i, \alpha_j]$ defined by $0 \leq \alpha_i < \alpha_j \leq \alpha^{sup}$ and let (P_i, Q_i), (P_j, Q_j) be the solutions of $\mathcal{F}_3(\alpha_i)$ and $\mathcal{F}_3(\alpha_j)$, respectively, such that $(P_i, Q_i) \not\equiv (P_j, Q_j)$. Let $\alpha_k = |\frac{\rho(P_i - P_j)}{Q_i - Q_j}|$ and (P_k, Q_k) be a solution of $\mathcal{F}_3(\alpha_k)$. If one of the following conditions is satisfied, then there does not exist any ρ-NF solution which is different to (P_i, Q_i) and (P_j, Q_j) in the interval (α_i, α_j).*

1. *$\alpha_k \in \{\alpha_i, \alpha_j\}$;*
2. *$\alpha_k \notin \mathcal{C}$ and $(P_k, Q_k) \equiv (P_i, Q_i)$ or $(P_k, Q_k) \equiv (P_j, Q_j)$;*

Theorem 10. *Algorithm 5 returns all the elements of $\mathcal{C}$ corresponding to all ρ-NF solutions for Min-Min BODO.*

According to Theorem 8 and 10, Algorithm 5 returns all the elements of $\mathcal{C}$, which correspond to all ρ-NF solutions, in a finite number of iterations. In general, estimating its complexity may be difficult. This difficulty comes from the fact that we do not know the BODO problem's context and the expressions for the objective functions. In some particular problems, we can show that Algorithm 5 terminates in polynomial time (for example, the Bi-Objective Assignment Problem described in Section 5).

5. Illustrative example

In this section, the recursive Newton-like algorithm for determining the ρ-NF solution set is tested on an instance of Min-Min BODO as a variant of the assignment problem called *bi-objective assignment problem* (BOAP). The BOAP can be formally defined as follows. Given a set of n workers, a set of n jobs, and a $n \times n$ cost matrix M whose positive elements m_{ij} represent the cost assignments of worker i to job j, the BOAP finds a one-to-one worker-job assignment (i.e., a perfect bipartite matching) that minimizes simultaneously the total cost and the *max-min distance*, which is the difference between the maximum cost assignment and the minimum one in the assignment solution. Notice that the problem minimizing only the max-min distance, called *balanced assignment problem*, has been introduced for finding the solution where the equitable distribution of assignments is important, which is considered in some real-life instances of the assignment problems [24].

As shown in Section 4, for finding the ρ-NF solution set, we aim to solve $\mathcal{F}_3(\alpha)$ and $\mathcal{G}_3(\alpha)$ where $\alpha \in [0, \alpha^{sup}]$. In the following, we present linear programming (LP) for solving $\mathcal{F}_3(\alpha)$.

$$\min \rho P + \alpha Q \tag{24a}$$

$$\text{s.t. } P = \sum_{i\in[n],j\in[n]} m_{i,j}x_{i,j} \tag{24b}$$

$$Q = u - l \tag{24c}$$

$$\sum_{j\in[n]} x_{j,i} = \sum_{j\in[n]} x_{i,j} = 1 \qquad \forall i \in [n] \tag{24d}$$

$$u \geq \sum_{j\in[n]} m_{i,j}x_{i,j} \qquad \forall i \in [n] \tag{24e}$$

$$l \leq \sum_{j\in[n]} m_{i,j}x_{i,j} \qquad \forall i \in [n] \tag{24f}$$

$$x_{i,j} \geq 0 \qquad \forall i, j \in [n] \tag{24g}$$

where $[n] = \{1, ..., n\}$ and $x_{i,j}$ represents the assignment between worker i and job j corresponding to the cost $m_{i,j}$. In this formulation, the value of P represents the total cost. To calculate the max-min distance Q, we determine the maximum and the minimum cost assignments u and l in the assignment solution. Constraints (24e) allow bounding u from below by the maximum cost assignment in the assignment solution. Similarly, constraints (24f) allow bounding l from above by the minimum cost assignment in the assignment solution. As $Q = u - l$ is minimized, u and l will take the maximum and minimum cost assignments, respectively.

We also present the following LP for solving $\mathcal{G}_3(\alpha)$. This LP contains all the constraints from (24b) to (24g). However, to avoid duplication, they are omitted.

$$\min \rho P + \alpha Q + t \tag{25a}$$

$$\text{s.t. } t \geq P - \alpha Q \tag{25b}$$

$$t \geq -P + \alpha Q \tag{25c}$$

Notice that using two additional constraints (25b) and (25c), the parameter t represents the absolute value of $P - \alpha Q$ in $g_3(\alpha, P, Q)$.

We will show that the solutions of LP formulations (24) and (25) are integral, which correspond to the assignment solutions (i.e., perfect bipartite matching). Moreover, for BOAP, determining the ρ-NF solution set can be done in polynomial time.

Lemma 13. *The solutions of LP formulations* (24) *and* (25) *are integral.*

Proof. The objective function of (24) assures that u and l will be equal, respectively, to the maximum and the minimum cost assignments in the optimal solution. Consequently, the

solution of LP formulation (24) is always integral since the constraints matrix of (24d) is totally unimodular (e.g., see [14]), and the constraints (24e) and (24f) are bound constraints.

Similarly, the solution of LP formulation (25) is also integral since the constraints (25b) and (25c) are bound constraints. □

Theorem 11. *For BOAP, determining the ρ-NF solution set can be done in polynomial time.*

Proof. Consider an instance of BOAP with a $n \times n$ cost matrix and suppose that there are K distinct Pareto-optimal solutions (P_i, Q_i) where $1 \leq i \leq K$. We first show that $K \leq C^2_{n^2} = \frac{n^2(n^2-1)}{2}$.

Let m_i^{max} and m_i^{min} be the maximum and the minimum cost assignment in the assignment solution corresponding to (P_i, Q_i) then $Q_i = m_i^{max} - m_i^{min}$. For two distinct Pareto-optimal solutions $(P_i, Q_i), (P_j, Q_j)$, we have $Q_i \neq Q_j$ which is equivalent to $m_i^{max} - m_i^{min} \neq m_j^{max} - m_j^{min}$. We have then $(m_i^{max}, m_i^{min}) \not\equiv (m_j^{max}, m_j^{min})$. Thus, the assignment solutions corresponding to (P_i, Q_i), (P_j, Q_j) have distinct pairs of assignments representing the maximum and the minimum cost assignment. Since we have at most n^2 distinct assignments, the number of distinct pairs of assignments is at most $C^2_{n^2}$. Thus, $K \leq C^2_{n^2}$.

According to Lemma 4, the iterations of Procedure *FIND*() return distinct Pareto-optimal solutions. Thus, it terminates in a polynomial number of iterations. Consequently, it terminates in polynomial time since LP formulation (24) for $\mathcal{F}_3(\alpha)$ can be solved in polynomial time. Notice that Procedure *VERIFY*() also terminates in polynomial time.

We know that $[0, \alpha^{sup}]$ can be separated by at most K consecutive subintervals $[c_i, c_j]$ such that $c_i < c_j$ and c_i, c_j correspond to two distinct Pareto-optimal solutions. For Procedure *EXPLORE*$([0, \alpha^{sup}])$, we use Procedure *FIND*() and Procedure *VERIFY*() in each recursive call, and we have at most K recursive calls. Hence, Procedure *EXPLORE*$([0, \alpha^{sup}])$ also terminates in polynomial time. □

We now consider an instance of BOAP with 17 workers, 17 jobs, and a cost matrix where its elements are randomly uniformly generated in the range [1,30] (see Appendix C). Let P, Q represent the total cost and the max-min distance in a feasible assignment solution. Furthermore, we first consider $\rho = 1$ (i.e., P and Q are equally important from the CDM's point of view). The experiments are conducted on a PC Intel Core i5-9500 3.00GHz with 6 cores and 6 threads. We present the computational procedures for determining the ρ-NF solution set based on Algorithm 5.

Step 1 We compute a solution (P_0, Q_0) of $\mathcal{F}_3(0)$. We obtain $(P_0, Q_0) = (68, 9)$. Then we compute a solution (P_q, Q_q) for minimizing Q. We obtain $(P_q, Q_q) = (262, 3)$. Thus, $\alpha^{sup} = P_q/Q_q = \frac{262}{3}$. We also assert that $\alpha^{sup} \notin \mathcal{C}$ by using Procedure *VERIFY*(α^{sup}). We aim to explore the interval $[0, \frac{262}{3}]$.

Step 2 Since $(131, 4)$ is a solution of $\mathcal{F}_3(\alpha^{sup})$, we take $\alpha_1 = |\frac{68-131}{9-4}| = \frac{63}{5}$ and $(80, 7)$ as a solution of $\mathcal{F}_3(\alpha_1)$. By using Procedure *VERIFY*(α_1), we obtain that $\alpha_1 \notin \mathcal{C}$. Thus, we use Procedure *FIND*(α_1) to find an element of $\mathcal{C}$. We obtain $c_1 = \frac{80}{7} \in \mathcal{C}$ corresponding to the ρ-NF solution $(80, 7)$. We have two subintervals $[0, \frac{63}{5}]$ and $[\frac{80}{7}, \frac{262}{3}]$ to be explored.

Step 3 For the subinterval $[\frac{80}{7}, \frac{262}{3}]$, we take $\alpha_2 = |\frac{80-131}{7-4}| = 17$ and $(110, 5)$ as a solution of $\mathcal{F}_3(\alpha_2)$. By using Procedure *VERIFY*(α_2), we obtain $\alpha_2 \notin \mathcal{C}$. Thus, we use Procedure *FIND*(α_2) to find another element of $\mathcal{C}$. We obtain $c_2 = \frac{131}{4} \in \mathcal{C}$ corresponding to the ρ-NF solution $(131, 4)$. According to Lemma 12, there is no ρ-NF solution in the interval $[17, \frac{262}{3}]$ except $(131, 4)$. We have two subintervals $[0, \frac{63}{5}]$ and $[\frac{80}{7}, 17]$ to be explored.

Step 4 For the subinterval $[\frac{80}{7}, 17]$, we take $\alpha_3 = |\frac{80-110}{7-5}| = 15$ and $(80, 7)$ as a solution of $\mathcal{F}_3(\alpha_3)$. We have $\alpha_3 \notin \mathcal{C}$. According to Lemma 12, there is no ρ-NF solution in this subinterval except $(80, 7)$.

Step 5 For the subinterval $[0, \frac{63}{5}]$, we take $\alpha_4 = |\frac{68-80}{9-7}| = 6$ and $(71, 8)$ as a solution of $\mathcal{F}_3(\alpha_4)$. By using Procedure *VERIFY*(α_4), we obtain $\alpha_4 \notin \mathcal{C}$. Thus, we use Procedure *FIND*(α_4) to find another element of $\mathcal{C}$. We obtain $c_3 = \frac{71}{8} \in \mathcal{C}$ corresponding to the ρ-NF solution $(71, 8)$. We have two subintervals $[0, 6]$ and $[\frac{71}{8}, \frac{63}{5}]$ to be explored.

Step 6 For the subinterval $[\frac{71}{8}, \frac{63}{5}]$, we take $\alpha_5 = |\frac{70-80}{8-7}| = 10$ and $(80, 7)$ as a solution of $\mathcal{F}_3(\alpha_5)$. We have $\alpha_5 \notin \mathcal{C}$. According to Lemma 12, there is no ρ-NF solution in this subinterval except $(71, 8)$ and $(80, 7)$.

Step 7 Finally, for the subinterval $[0, 6]$, we take $\alpha_6 = |\frac{68-71}{9-8}| = 3$ and $(68, 9)$ as a solution of $\mathcal{F}_3(\alpha_6)$. We have $\alpha_6 \notin \mathcal{C}$. According to Lemma 12, there is no ρ-NF solution in this subinterval except $(71, 8)$.

Hence, the interval $[0, \frac{262}{3}]$ is totally explored and we obtain three ρ-NF solutions: $(71, 8), (80, 7)$ and $(131, 4)$.

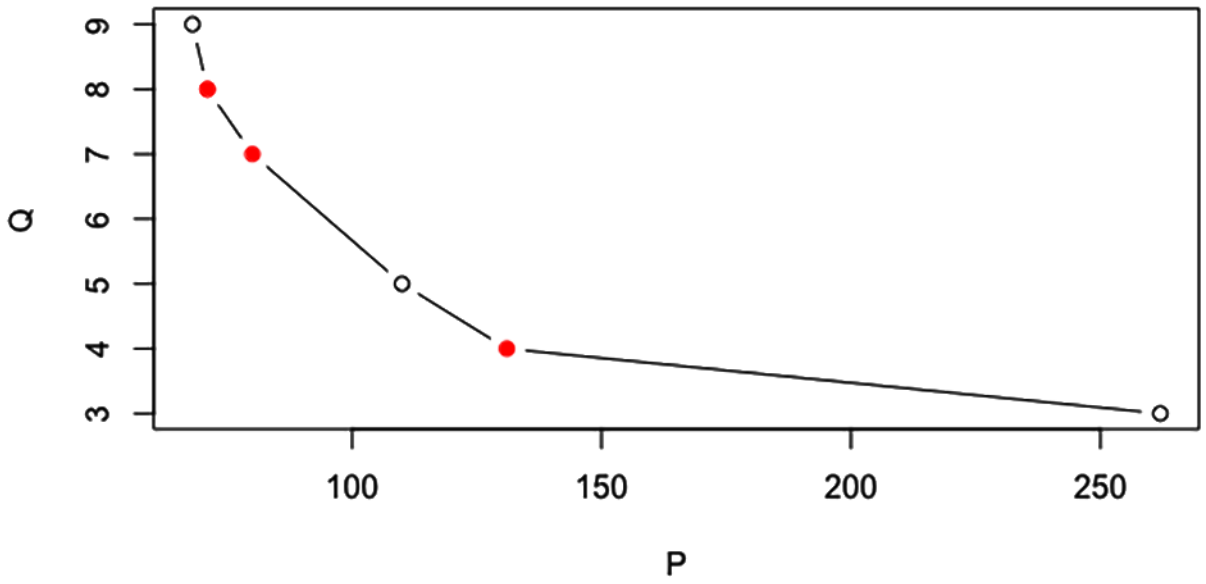

Figure 1: Pareto frontier with ρ-NF solutions in red, $\rho = 1$

We can also compute all the Pareto-optimal solutions for this instance as $(68, 9)$,$(71, 8)$, $(80, 7)$,$(110, 5)$,$(131, 4)$, and $(262, 3)$. Figure 1 shows the Pareto frontier with three ρ-NF solutions in red. Now we suppose that the CDM prefers the objective P to Q, then we can choose a bigger value of ρ, for example, $\rho = 2$ (i.e., P is relatively two times more important than Q from the CDM's point of view). Then the Pareto frontier does not change but there is only one ρ-NF solution as shown in Figure 2. In these instances, the ρ-NF solution set is a strict subset of the Pareto set. Moreover, $(131, 4)$ (resp. $(71, 8)$) is the ρ-NF solution that minimizes $P^\rho Q$ when $\rho = 1$ (resp. $\rho = 2$) among all the feasible solutions. In general, for Min-Min BODO (resp. Max-Min BODO with $\rho > 1$), we can determine the optimal solution of the problem $\mathcal{H}_3 = \min_{(P,Q)\in\mathcal{S}_3} P^\rho Q$ (resp. $\mathcal{H}_2 = \max_{(P,Q)\in\mathcal{S}_2} P^\rho/Q$ with $\rho > 1$) through determining the ρ-NF solution set, which represents all its local optimums.

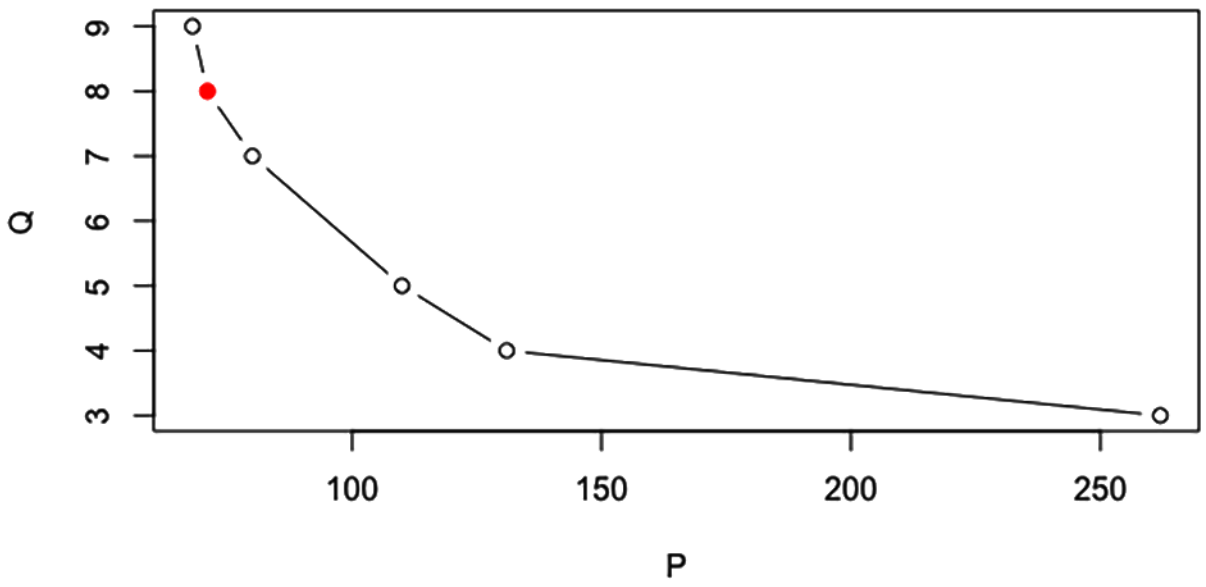

Figure 2: Pareto frontier with ρ-NF solution in red, $\rho = 2$

6. Conclusion

In this paper, we have generalized the concept of the ρ-NF solution for *Bi-Objective Discrete Optimization* (BODO), where the feasible set is discrete, and the two objectives take only positive values. We first discussed the definition and the existence of the ρ-NF solutions for BODO. Then, we showed that the ρ-NF solution set is a subset of the Pareto set, and this inclusion can be strict. We also designed a recursive Newton-like algorithm to determine the ρ-NF solution set. Finally, an illustrative example of BODO is given.

Future work should provide efficient algorithms for Max-Max BODO and Max-Min BODO with $\rho < 1$. Furthermore, we aim to clarify the quantitative link between the ρ-NF solution set and the Pareto set in more specific BODO cases (for example, in combinatorial optimization with linear objectives). The algorithm's complexity in such cases should also be studied.

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Appendix A. Cost matrix for the instance of BOAP used in Section 5

$$M = \begin{pmatrix}
1 & 29 & 14 & 22 & 12 & 27 & 1 & 11 & 9 & 10 & 4 & 1 & 3 & 28 & 5 & 9 & 5 \\
23 & 23 & 12 & 24 & 20 & 17 & 21 & 9 & 19 & 17 & 16 & 16 & 12 & 19 & 20 & 19 & 10 \\
21 & 10 & 29 & 3 & 28 & 14 & 6 & 15 & 26 & 17 & 4 & 12 & 2 & 29 & 20 & 25 & 1 \\
25 & 7 & 21 & 17 & 20 & 28 & 20 & 17 & 19 & 11 & 24 & 17 & 20 & 2 & 15 & 7 & 26 \\
19 & 3 & 14 & 10 & 18 & 20 & 7 & 9 & 17 & 11 & 16 & 19 & 8 & 20 & 13 & 22 & 23 \\
17 & 1 & 7 & 3 & 12 & 14 & 16 & 11 & 24 & 27 & 15 & 23 & 8 & 29 & 12 & 10 & 22 \\
10 & 1 & 8 & 15 & 28 & 26 & 6 & 15 & 12 & 10 & 27 & 21 & 9 & 22 & 9 & 9 & 17 \\
19 & 15 & 10 & 17 & 9 & 24 & 21 & 6 & 17 & 21 & 13 & 28 & 15 & 8 & 27 & 21 & 29 \\
12 & 3 & 20 & 4 & 24 & 13 & 13 & 18 & 23 & 23 & 18 & 23 & 9 & 10 & 15 & 28 & 28 \\
15 & 21 & 22 & 5 & 20 & 2 & 4 & 29 & 5 & 18 & 8 & 29 & 2 & 1 & 17 & 29 & 10 \\
16 & 9 & 16 & 26 & 6 & 20 & 15 & 15 & 11 & 25 & 13 & 8 & 26 & 15 & 18 & 2 & 3 \\
1 & 8 & 14 & 5 & 1 & 18 & 6 & 4 & 6 & 15 & 15 & 15 & 12 & 15 & 22 & 17 & 12 \\
7 & 9 & 2 & 25 & 14 & 1 & 25 & 13 & 28 & 6 & 16 & 3 & 4 & 5 & 28 & 28 & 1 \\
21 & 12 & 6 & 8 & 25 & 6 & 26 & 8 & 17 & 17 & 13 & 28 & 16 & 24 & 27 & 5 & 18 \\
2 & 27 & 10 & 25 & 6 & 24 & 4 & 29 & 26 & 28 & 25 & 4 & 24 & 16 & 1 & 17 & 22 \\
22 & 21 & 29 & 8 & 24 & 17 & 19 & 3 & 22 & 9 & 13 & 4 & 14 & 14 & 3 & 29 & 9 \\
27 & 27 & 6 & 18 & 23 & 11 & 4 & 16 & 19 & 15 & 5 & 15 & 3 & 8 & 22 & 3 & 28
\end{pmatrix}$$