# Determining the generalized Nash Fairness solution set for Bi-Objective Discrete Optimization

Minh Hieu Nguyen, Mourad Baiou, Viet Hung Nguyen

# Determining the generalized Nash Fairness solution set for Bi-Objective Discrete Optimization

Minh Hieu Nguyen[a], Mourad Baiou[a], Viet Hung Nguyen[a,*]

[a]*INP Clermont Auvergne, Univ Clermont Auvergne, Mines Saint-Etienne, CNRS, UMR 6158 LIMOS 1 Rue de la Chebarde, Aubiere Cedex, France*

**Abstract**

This paper deals with a particular case of Bi-Objective Optimization called *Bi-Objective Discrete Optimization* (BODO), where the feasible set is discrete, and the two objectives take only positive values. Since the feasible set of a BODO problem is discrete and usually finite, it can theoretically be enumerated to identify the Pareto set, which consists of all Pareto-optimal solutions representing different trade-offs between two objectives. However, in general, this problem is algorithmically unsolvable. From a practical point of view, the Central Decision Maker (CDM) may be interested in a reduced Pareto set reflecting the own preference of the CDM, which a computationally tractable algorithm can obtain.

In a recent paper [14], we introduced the concept of generalized Nash Fairness ($\rho$-$NF$) solution for Bi-Objective Minimization Problems, where their two positive objectives are to be simultaneously minimized. The $\rho$-$NF$ solutions are the Pareto-optimal solutions achieving some proportional fairness between two objectives. The positive parameter $\rho$ provided by the CDM reflects the relative importance of the first objective compared to the second one. We then focused on finding two particular $\rho$-$NF$ solutions, called extreme $\rho$-$NF$ solutions, having the smallest values of each objective.

In this paper, our main contribution is a stronger result than in [14]: we generalize the concept of $\rho$-$NF$ solution for BODO problems where each objective can be either maximized or minimized. We then show that the $\rho$-$NF$ solution set is a subset of the Pareto set, and this inclusion can be strict. We also propose several algorithms for determining the $\rho$-$NF$ solution set, including a recursive Newton-like algorithm. Finally, an illustrative example of BODO is given.

*Keywords:* Bi-Objective Discrete Optimization, Bi-Criteria Decision Making, Pareto optimal, Proportional Fairness, Generalized Nash Fairness solution

## 1. Introduction

Bi-Objective Discrete Optimization (BODO) is a branch of optimization with many applications in different areas, such as applied mathematics, economics, and computer science.

*Corresponding author: viet_hung.nguyen@uca.fr

*March 2, 2023*

In practice, the objectives of BODO take only positive values, including some well-studied examples of BODO in resource allocation, such as the bi-objective knapsack problem [28], in networks, such as the bi-objective shortest path problem [29], the bi-objective spanning tree problem [30] and in capital budgeting, such as the bi-objective assignment problem. The latter problem assembles the objectives of the classic assignment problem [9] and the balanced assignment problem mentioned in [11]. Therefore, this paper considers the BODO problems with positive objectives.

Since the feasible set of a BODO problem is discrete and usually finite, it can theoretically be enumerated to identify the Pareto set, which consists of all Pareto-optimal solutions. However, this problem is generally known to be algorithmically unsolvable (e.g., see [18]). From a practical point of view, the Central Decision Maker (CDM) may be interested in a reduced Pareto set reflecting the own preference of the CDM, which a computationally tractable algorithm can obtain.

In a recent paper [14], we considered the Bi-Objective Minimization Problems, where their two positive objectives are to be simultaneously minimized. We utilized the notion of proportional fairness - which is a well-known rule in communication networks, operation research, and general social choice introduced in [1], [2], [3] - as a criterion for selecting the preferred solutions in the Pareto set. In [14], we introduced the concept of generalized Nash Fairness ($\rho$-$NF$) solution, a Pareto-optimal solution achieving some proportional fairness between two objectives. The positive parameter $\rho$ provided by the CDM reflects the relative importance of the first objective compared to the second one. We then focused on finding two particular $\rho$-$NF$ solutions, called extreme $\rho$-$NF$ solutions, having the smallest values of each objective. In this paper, our main contribution is a stronger result than in [14]: we generalize the concept of $\rho$-$NF$ solution for BODO problems where each objective can be either maximized or minimized. We then show that the $\rho$-$NF$ solution set is a subset of the Pareto set. We also propose several algorithms for determining the $\rho$-$NF$ solution set, including a recursive Newton-like algorithm.

Note that we also introduced a particular case of $\rho$-$NF$ solution when $\rho = 1$, called $NF$ solution, and the idea to determine the NF solution set for an example of Min-Min BODO in our conference paper [13]. This paper is an extended and generalized work of [13].

Let us introduce the notations used in this paper. Let $P(x), Q(x)$ denote two positive objective functions, and $(P, Q) = (P(x), Q(x))$ denote a feasible solution corresponding to a decision vector $x \in \mathcal{X}$ where $\mathcal{X}$ is a finite feasible set. Note that the $\rho$-$NF$ solutions will be characterized by the solutions $(P, Q)$ and not by the corresponding decision vectors. Thus, two solutions having the same value of $(P, Q)$ will be considered the same. Assume that each objective is a convex or concave function, depending on whether it is minimized or maximized, and we know the algorithms to minimize a convex function (or equivalently maximize a concave function) over $\mathcal{X}$. These hypotheses are verified for most practical examples of BODO.

We will consider BODO in three cases: Max-Max BODO with two objectives to be maximized, such as the bi-objective knapsack problem [28], Max-Min BODO with one objective to be maximized, and another one to be minimized, such as some fractional programming problems [12] and Min-Min BODO with two objectives to be minimized, such as the bi-

objective assignment problem [13].

The paper is organized as follows. Section 2 discusses the concept and the existence of $\rho$-$NF$ solutions for BODO. Then, we show the Pareto efficiency of $\rho$-$NF$ solutions in Section 3. Section 4 provides the algorithms for determining the $\rho$-$NF$ solution set. Section 5 presents an example of BODO to illustrate these algorithms. Finally, we give some conclusions and future works in Section 6.

## 2. Definition and existence of $\rho$-$NF$ solutions for BODO

### *2.1. Max-Max BODO*

#### *2.1.1. Definition of $\rho$-$NF$ solution for Max-Max BODO*

Max-Max BODO can be formulated as

$$\max_{x\in\mathcal{X}_1} (P(x), Q(x)),$$

where $\mathcal{X}_1$ is a finite feasible set. Moreover, for Max-Max BODO, we suppose that $P(x), Q(x)$ are positive concave functions.

We first recall the notion of proportional fairness introduced in [1], [2], [3]. Proportional fairness for the two-player resource allocation problem is defined using the Nash standard of comparison: a transfer of utilities between the two players is considered fair if the percentage increase in the utility of one player is larger than the percentage decrease in the utility of the other player. For the multiple-player problem, when comparing the fair allocation to any other feasible allocation of utilities, the aggregate proportional change is non-positive.

We explain the parameter $\rho$ as follows. $\rho$ is a positive parameter supplied by the CDM to reflect the relative importance of $P$ to $Q$. In other words, $\rho$ percent change of $P$ will be comparably equivalent to a one percent change of $Q$. Note that the contexts of Max-Max BODO and the two-player resource allocation problem introduced in [1] are similar since their objective functions are to be both maximized. Thus, based on the proportional fairness, the $\rho$-$NF$ solution for Max-Max BODO should be such that, if compared to any other feasible solutions, the sum of the factor $\rho$ of the proportional change of $P$ and the proportional change of $Q$ is non-positive.

Let $\mathcal{S}_1$ be the set of all feasible solutions $(P, Q)$ for Max-Max BODO. Since $\mathcal{X}_1$ is finite, $\mathcal{S}_1$ is also finite. If $(P^*, Q^*) \in \mathcal{S}_1$ is a $\rho$-$NF$ solution for Max-Max BODO, we have

$$\rho\frac{P-P^*}{P^*} + \frac{Q-Q^*}{Q^*} \leq 0 \iff \rho\frac{P}{P^*} + \frac{Q}{Q^*} \leq \rho + 1,\ \forall (P,Q) \in \mathcal{S}_1,$$

Hence, the $\rho$-$NF$ solution for Max-Max BODO can be defined as follows.

**Definition 1.** *$(P^*, Q^*) \in \mathcal{S}_1$ is a $\rho$-$NF$ solution for Max-Max BODO if and only if*

$$\rho\frac{P}{P^*} + \frac{Q}{Q^*} \leq \rho + 1,\ \forall (P,Q) \in \mathcal{S}_1, \tag{1}$$

*2.1.2. Existence of $\rho$-NF solution for Max-Max BODO*

In this section, we show that the $\rho$-$NF$ does not always exist for Max-Max BODO, and if it exists, it is the unique solution to the optimization problem

$$\mathcal{H}_1 = \max_{(P,Q)\in\mathcal{S}_1} P^\rho Q,$$

**Theorem 1.** *If $(P^*, Q^*)$ is a $\rho$-$NF$ solution for Max-Max BODO, then it is necessarily the unique solution of $\mathcal{H}_1$ with respect to the values of $P$ and $Q$.*

*Proof.* Suppose that $(P^*, Q^*)$ is a $\rho$-$NF$ solution for Max-Max BODO. We have

$$\rho\frac{P}{P^*} + \frac{Q}{Q^*} \leq \rho + 1 \implies 1 \geq \frac{\rho}{\rho+1} \times \frac{P}{P^*} + \frac{1}{\rho+1} \times \frac{Q}{Q^*},\ \forall (P,Q) \in \mathcal{S}_1, \tag{2}$$

Using Young's inequality for products (see [15]), we obtain

$$\frac{\rho}{\rho+1} \times \frac{P}{P^*} + \frac{1}{\rho+1} \times \frac{Q}{Q^*} \geq (\frac{P}{P^*})^{\frac{\rho}{\rho+1}} \times (\frac{Q}{Q^*})^{\frac{1}{\rho+1}} = (\frac{P^\rho Q}{P^{*\rho} Q^*})^{\frac{1}{\rho+1}}, \tag{3}$$

From (2) and (3), we obtain $P^{*\rho}Q^* \geq P^\rho Q, \forall (P,Q) \in \mathcal{S}_1$ which implies that $(P^*, Q^*)$ is a solution of $\mathcal{H}_1$.

Now suppose that there exists another $\rho$-$NF$ solution $(P^{**}, Q^{**}) \in \mathcal{S}_1$ such that $P^{**\rho}Q^{**} = P^{*\rho}Q^*$. We also have

$$\rho\frac{P^{**}}{P^*} + \frac{Q^{**}}{Q^*} \leq \rho + 1 \implies 1 \geq \frac{\rho}{\rho+1} \times \frac{P^{**}}{P^*} + \frac{1}{\rho+1} \times \frac{Q^{**}}{Q^*},$$

Using Young's inequality for products, we have

$$1 \geq \frac{\rho}{\rho+1} \times \frac{P^{**}}{P^*} + \frac{1}{\rho+1} \times \frac{Q^{**}}{Q^*} \geq (\frac{P^{**}}{P^*})^{\frac{\rho}{\rho+1}} \times (\frac{Q^{**}}{Q^*})^{\frac{1}{\rho+1}} = (\frac{P^{**\rho}Q^{**}}{P^{*\rho}Q^*})^{\frac{1}{\rho+1}} = 1,$$

Thus, the equality in Young's inequality above must hold, which implies $\frac{P^{**}}{P^*} = \frac{Q^{**}}{Q^*}$. Moreover, since $P^{**\rho}Q^{**} = P^{*\rho}Q^*$, we obtain

$$1 = \frac{P^{**\rho}Q^{**}}{P^{*\rho}Q^*} = (\frac{P^{**}}{P^*})^\rho(\frac{Q^{**}}{Q^*}) = (\frac{P^{**}}{P^*})^{\rho+1},$$

Thus, we have $P^{**} = P^*$ and $Q^{**} = Q^*$. □

Note that the unique solution of $\mathcal{H}_1$ is not necessarily the $\rho$-$NF$. Let us suppose that $\rho = 1$ and the feasible set $\mathcal{S}_1$ has two solutions $(P_1, Q_1) = (13, 15)$, $(P_2, Q_2) = (10, 20)$. We see that $(P_2, Q_2)$ is the solution having the maximal value of $P^\rho Q$ but it is not a $\rho$-$NF$ solution because

$$\rho\frac{P_1}{P_2} + \frac{Q_1}{Q_2} = \frac{13}{10} + \frac{15}{20} = \frac{41}{20} > 2 = \rho + 1,$$

Consequently, the $\rho$-$NF$ solution for Max-Max BODO is not guaranteed to exist. If it exists, it is the unique solution of $\mathcal{H}_1$.

*2.2. Max-Min BODO*

*2.2.1. Definition of $\rho$-NF solution for Max-Min BODO*

Since minimizing $Q(x)$ over $\mathcal{X}_2$ is equivalent to maximizing $-Q(x)$ over $\mathcal{X}_2$, Max-Min BODO can be formulated as

$$\max_{x\in\mathcal{X}_2} (P(x), -Q(x))$$

where $\mathcal{X}_2$ is a finite feasible set (i.e., we simultaneously maximize $P(x)$ and minimize $Q(x)$ over $\mathcal{X}_2$). Moreover, for Max-Min BODO, we suppose that $P(x)$ is a positive concave function and $Q(x)$ is a positive convex function.

In the context of Max-Min BODO, we prefer an alternative assigning a greater value for $P$ and a smaller value for $Q$. Thus, the $\rho$-$NF$ solution should be such that, when compared to any other feasible solutions, the difference between the factor $\rho$ of the proportional change of $P$ and the proportional change of $Q$ is non-positive (i.e., from the definition (1) of $\rho$-$NF$ solution for Max-Max BODO, we change the sign for the proportional change of $Q$ as negative).

Let $\mathcal{S}_2$ be the set of all feasible solutions $(P, Q)$ for Max-Min BODO. Since $\mathcal{X}_2$ is finite, $\mathcal{S}_2$ is also finite. If $(P^*, Q^*) \in \mathcal{S}_2$ is a $\rho$-$NF$ solution for Max-Min BODO, we have

$$\rho\frac{P-P^*}{P^*} - \frac{Q-Q^*}{Q^*} \leq 0 \iff \rho\frac{P}{P^*} - \frac{Q}{Q^*} \leq \rho - 1,\ \forall (P,Q) \in \mathcal{S}_2,$$

Hence, the $\rho$-$NF$ solution for Max-Min BODO can be defined as follows.

**Definition 2.** *$(P^*, Q^*) \in \mathcal{S}_2$ is a $\rho$-NF solution for Max-Min BODO if and only if*

$$\rho\frac{P}{P^*} - \frac{Q}{Q^*} \leq \rho - 1,\ \forall (P,Q) \in \mathcal{S}_2, \tag{4}$$

*2.2.2. Existence of $\rho$-NF solutions for Max-Min BODO with $0 < \rho \leq 1$*

This section discusses the existence of $\rho$-$NF$ solutions for Max-Min BODO. For that, we consider the following optimization problem.

$$\mathcal{H}_2 = \max_{(P,Q)\in\mathcal{S}_2} \frac{P^\rho}{Q},$$

We will show that in case $0 < \rho \leq 1$, a $\rho$-$NF$ solution is necessarily a solution of $\mathcal{H}_2$, and in case $\rho > 1$, there always exists a $\rho$-$NF$ solution which is a solution of $\mathcal{H}_2$.

**Theorem 2.** *If $(P^*, Q^*)$ is a $\rho$-NF solution for Max-Min BODO with $0 < \rho \leq 1$, then it is necessarily a solution of $\mathcal{H}_2$. Moreover, in case $0 < \rho < 1$, the $\rho$-NF solution (if it exists) is the unique solution of $\mathcal{H}_2$.*

*Proof.* Suppose that $(P^*, Q^*)$ is a $\rho$-$NF$ solution for Max-Min BODO with $0 < \rho \leq 1$. Using Young's inequality for products, we have

$$\frac{Q}{Q^*} \geq \rho \times \frac{P}{P^*} + (1-\rho) \geq (\frac{P}{P^*})^\rho = \frac{P^\rho}{P^{*\rho}}, \forall (P,Q) \in \mathcal{S}_2,$$

which is equivalent to

$$\frac{P^{*\rho}}{Q^*} \geq \frac{P^\rho}{Q}, \forall (P,Q) \in \mathcal{S}_2,$$

Thus, $(P^*, Q^*)$ is a solution of $\mathcal{H}_2$.

Now suppose that $0 < \rho < 1$ and there exists another $\rho$-$NF$ solution $(P^{**}, Q^{**}) \in \mathcal{S}_2$ such that $\frac{P^{**\rho}}{Q^{**}} = \frac{P^{*\rho}}{Q^*}$.

Using Young's inequality for products, we also have

$$\frac{Q^{**}}{Q^*} \geq \rho \times \frac{P^{**}}{P^*} + (1-\rho) \geq (\frac{P^{**}}{P^*})^\rho = \frac{Q^{**}}{Q^*},$$

Thus, the equality in Young's inequality above must hold, which implies $\frac{P^{**}}{P^*} = 1$. Consequently, we have $P^{**} = P^*$ and $Q^{**} = Q^*$. □

Note that the unique solution of $\mathcal{H}_2$ is not necessarily the $\rho$-$NF$ solution. Let us suppose that $\rho = 1/2$ and the feasible set $\mathcal{S}_2$ has two solutions $(P_1, Q_1) = (17, 4)$, $(P_2, Q_2) = (9, 3)$. We see that $(P_1, Q_1)$ is the solution having the maximal value of $P^\rho/Q$ but it is not a $\rho$-$NF$ solution because

$$\rho\frac{P_2}{P_1} - \frac{Q_2}{Q_1} = \frac{1}{2} \times \frac{9}{17} - \frac{3}{4} = -\frac{33}{68} > -\frac{1}{2} = \rho - 1,$$

Consequently, the $\rho$-$NF$ solution for Max-Min BODO with $0 < \rho < 1$ is not guaranteed to exist. If it exists, it is the unique solution of $\mathcal{H}_2$.

*2.2.3. Existence of $\rho$-$NF$ solutions for Max-Min BODO with $\rho > 1$*

In this case, we show that the $\rho$-$NF$ solution always exists by the following theorem.

**Theorem 3.** *There always exists a $\rho$-$NF$ solution for Max-Min BODO with $\rho > 1$.*

*Proof.* Since $\mathcal{S}_2$ is finite, there always exists $(P^*, Q^*) \in \mathcal{S}_2$ as a solution of $\mathcal{H}_2$.

Let us show that $(P^*, Q^*)$ is a $\rho$-$NF$ solution. We have $\frac{P^{*\rho}}{Q^*} \geq \frac{P^\rho}{Q}$, $\forall (P,Q) \in \mathcal{S}_2$. Using Young's inequality for products, we obtain

$$\frac{1}{\rho} \times \frac{Q}{Q^*} + \frac{\rho - 1}{\rho} \geq (\frac{Q}{Q^*})^{\frac{1}{\rho}} \geq \frac{P}{P^*},$$

Thus,

$$\rho\frac{P}{P^*} - \frac{Q}{Q^*} \leq \rho - 1, \ \forall (P,Q) \in \mathcal{S}_2,$$

Consequently, $(P^*, Q^*)$ is a $\rho$-$NF$ solution. □

Let us show that there may be more than one $\rho$-$NF$ solution for Max-Min BODO with $\rho > 1$. Suppose that $\rho = 2$ and the feasible set $\mathcal{S}_2$ has two solutions $(P_1, Q_1) = (10, 8)$, $(P_2, Q_2) = (8, 5)$. We see that both $(P_1, Q_1)$ and $(P_2, Q_2)$ are $\rho$-$NF$ solutions because

$$\rho\frac{P_2}{P_1} - \frac{Q_2}{Q_1} = 2 \times \frac{10}{8} - \frac{8}{5} = \frac{9}{10} < 1 = \rho - 1,$$
$$\rho\frac{P_1}{P_2} - \frac{Q_1}{Q_2} = 2 \times \frac{8}{10} - \frac{5}{8} = \frac{39}{40} < 1 = \rho - 1,$$

*2.3. Min-Min BODO*

In this section, we restate the definition and the existence of $\rho$-$NF$ solution for Min-Min BODO, which has been introduced in our recent paper [14]. Note that if $\rho > 0$ reflects the relative importance of the first objective to the second one, then $1/\rho$ reflects the relative importance of the second objective to the first one. Furthermore, the two objectives play the same role in Min-Min BODO since they are both to be minimized. Thus, without loss of generality, we only consider $\rho \geq 1$, more precisely, which reflects the relative importance of $P$ to $Q$.

*2.3.1. Definition of $\rho$-$NF$ solution for Min-Min BODO*

Min-Min BODO can be formulated as

$$\min_{x \in \mathcal{X}_3} (P(x), Q(x)),$$

where $\mathcal{X}_3$ is a finite feasible set. Moreover, for Min-Min BODO, we suppose that $P(x), Q(x)$ are positive convex functions.

In the context of Min-Min BODO, we prefer an alternative assigning a smaller value for both $P$ and $Q$, which is opposite to Max-Max BODO. Thus, the $\rho$-$NF$ solution should be such that, when compared to any other feasible solutions, the sum of the factor $\rho$ of the proportional change of $P$ and the proportional change of $Q$ should be non-negative.

Let $\mathcal{S}_3$ be the set of all feasible solutions $(P, Q)$ for Min-Min BODO. Since $\mathcal{X}_3$ is finite, $\mathcal{S}_3$ is also finite. If $(P^*, Q^*) \in \mathcal{S}_3$ is a $\rho$-$NF$ solution for Min-Min BODO, we have

$$\rho\frac{P - P^*}{P^*} + \frac{Q - Q^*}{Q^*} \geq 0 \iff \rho\frac{P}{P^*} + \frac{Q}{Q^*} \geq \rho + 1, \ \forall (P, Q) \in \mathcal{S}_3,$$

Hence, the $\rho$-$NF$ solution for Max-Max BODO can be defined as follows.

**Definition 3.** *[14] $(P^*, Q^*) \in \mathcal{S}_3$ is a $\rho$-$NF$ solution for Min-Min BODO if and only if*

$$\rho\frac{P}{P^*} + \frac{Q}{Q^*} \geq \rho + 1, \ \forall (P, Q) \in \mathcal{S}_3, \tag{5}$$

*2.3.2. Existence of $\rho$-$NF$ solution for Min-Min BODO*

For the Min-Min BODO, we consider the optimization problem:

$$\mathcal{H}_3 = \min_{(P,Q)\in\mathcal{S}_3} P^\rho Q,$$

We will show that the $\rho$-$NF$ solution always exists for Min-Min BODO by the following theorem.

**Theorem 4.** *[14] There always exists a $\rho$-$NF$ solution for Min-Min BODO.*

*Proof.* Since $\mathcal{S}_3$ is finite, there always exists $(P^*, Q^*) \in \mathcal{S}_3$ as a solution of $\mathcal{H}_3$.

Let us show that $(P^*, Q^*)$ is a $\rho$-$NF$ solution. We have $P^\rho Q \geq P^{*\rho} Q^*$, $\forall (P, Q) \in \mathcal{S}_3$. Using Young's inequality for products, we obtain

$$\frac{\rho}{\rho+1} \times \frac{P}{P^*} + \frac{1}{\rho+1} \times \frac{Q}{Q^*} \geq (\frac{P}{P^*})^{\frac{\rho}{\rho+1}} \times (\frac{Q}{Q^*})^{\frac{1}{\rho+1}} = (\frac{P^\rho Q}{P^{*\rho} Q^*})^{\frac{1}{\rho+1}} \geq 1,\ \forall (P, Q) \in \mathcal{S}_3,$$

Thus,

$$\rho\frac{P}{P^*} + \frac{Q}{Q^*} \geq \rho + 1,\ \forall (P, Q) \in \mathcal{S}_3,$$

Consequently, $(P^*, Q^*)$ is a $\rho$-$NF$ solution. □

Let us show that there may be more than one $\rho$-$NF$ solution for Min-Min BODO. Suppose that $\rho = 1$ and the feasible set $\mathcal{S}_3$ has two solutions $(P_1, Q_1) = (11, 4)$, $(P_2, Q_2) = (9, 5)$. We see that both $(P_1, Q_1)$ and $(P_2, Q_2)$ are $\rho$-$NF$ solutions because

$$\rho\frac{P_2}{P_1} + \frac{Q_2}{Q_1} = \frac{9}{11} + \frac{5}{4} = \frac{91}{44} > 2 = \rho + 1,$$
$$\rho\frac{P_1}{P_2} + \frac{Q_1}{Q_2} = \frac{11}{9} + \frac{4}{5} = \frac{91}{45} > 2 = \rho + 1,$$

## 3. Pareto efficiency of $\rho$-$NF$ solutions

As a particular case of multi-objective optimization, the concept of Pareto efficiency [17] is applied to describing BODO's efficient solutions (Pareto-optimal solutions). This section shows the Pareto efficiency of $\rho$-$NF$ solutions. More precisely, the $\rho$-$NF$ solution set is a subset of the Pareto set.

We first show that the $\rho$-$NF$ solutions for Max-Max BODO, Max-Min BODO, and Min-Min BODO can be respectively obtained by solving the optimization problems.

$$\max_{(P,Q)\in\mathcal{S}_1} F_1(\alpha, P, Q) = \rho P + \alpha Q,$$
$$\max_{(P,Q)\in\mathcal{S}_2} F_2(\alpha, P, Q) = \rho P - \alpha Q,$$
$$\min_{(P,Q)\in\mathcal{S}_3} F_3(\alpha, P, Q) = \rho P + \alpha Q,$$

where $\alpha > 0$ is the coefficient to be determined.

According to the convexity and the concavity of $P, Q$ in each case, $F_1(\alpha, P, Q), F_2(\alpha, P, Q)$ are concave, and $F_3(\alpha, P, Q)$ is convex. Consequently, we suppose we know how to solve the three problems above. Note that if we optimize $F_i(\alpha, P, Q)$ for $i = 1, 2, 3$, that means we maximize $F_1(\alpha, P, Q), F_2(\alpha, P, Q)$ and minimize $F_3(\alpha, P, Q)$.

**Lemma 1.** *For $i = 1, 2, 3, (P^*, Q^*) \in \mathcal{S}_i$ is a $\rho$-NF solution if and only if $(P^*, Q^*)$ is a solution for optimizing $F_i(\alpha^*, P, Q)$ with $\alpha^* = P^*/Q^*$.*

*Proof.* We first proof this lemma for Max-Max BODO.

$\Longrightarrow$ Let $(P^*, Q^*) \in \mathcal{S}_1$ be a $\rho$-$NF$ solution and $\alpha^* = P^*/Q^*$. We will show that $(P^*, Q^*)$ is a solution for maximizing $F_1(\alpha^*, P, Q)$.

Since $P^*, Q^* > 0$, $\alpha^* > 0$. As $(P^*, Q^*)$ is a $\rho$-$NF$ solution we have

$$\rho\frac{P}{P^*} + \frac{Q}{Q^*} \leq \rho + 1, \ \forall (P, Q) \in \mathcal{S}_1, \tag{6}$$

Multiplying (6) by $P^* > 0$ gives

$$\rho P + \frac{P^*}{Q^*}Q \leq \rho P^* + P^*, \ \forall (P, Q) \in \mathcal{S}_1, \tag{7}$$

Since $\alpha^* = P^*/Q^*$, we deduce from (7)

$$\rho P + \alpha^* Q \leq \rho P^* + \alpha^* Q^*, \ \forall (P, Q) \in \mathcal{S}_1,$$

Hence, $(P^*, Q^*)$ is a solution for maximizing $F_1(\alpha^*, P, Q)$.

$\Longleftarrow$ Suppose that $(P^*, Q^*)$ is a solution for maximizing $F_1(\alpha^*, P, Q)$ with $\alpha^* = P^*/Q^*$, we show that $(P^*, Q^*)$ is a $\rho$-$NF$ solution.

As $(P^*, Q^*)$ is a solution for maximizing $F_1(\alpha^*, P, Q)$ we have

$$\rho P + \alpha^* Q \leq \rho P^* + \alpha^* Q^*, \ \forall (P, Q) \in \mathcal{S}_1,$$

Replacing $\alpha^*$ by $P^*/Q^*$ we obtain

$$\rho\frac{P}{P^*} + \frac{Q}{Q^*} \leq \rho + 1, \ \forall (P, Q) \in \mathcal{S}_1,$$

That means $(P^*, Q^*)$ is a $\rho$-$NF$ solution.

Since the proofs for Max-Min BODO and Min-Min BODO are similar to those for Max-Max BODO, they are omitted.

□

**Theorem 5.** *The $\rho$-NF solution set is a subset of the Pareto set. Moreover, this inclusion can be strict.*

*Proof.* We first show that for $\alpha_0 > 0$, a solution for maximizing $F_1(\alpha_0, P, Q)$ is a Pareto-optimal solution.

Let $(P_0, Q_0) \in \mathcal{S}_1$ be a solution for maximizing $F_1(\alpha_0, P, Q)$. If $(P_0, Q_0)$ is not a Pareto-optimal solution, then there exists a feasible solution $(P_1, Q_1) \in \mathcal{S}_1$ such that $(P_0, Q_0)$ is dominated by $(P_1, Q_1)$. Without loss of generality, we suppose that $P_1 > P_0$ and $Q_1 \geq Q_0$. Since $\rho, \alpha_0 > 0$, we have

$$\rho P_0 + \alpha_0 Q_0 < \rho P_1 + \alpha_0 Q_1,$$

which is contradict to the optimality of $(P_0, Q_0)$.

Thus, $(P_0, Q_0)$ is a Pareto-optimal solution, and then a $\rho$-$NF$ solution for Max-Max BODO is necessarily a Pareto-optimal solution due to Lemma 1.

Using the proof of Theorem 1, if $\rho = 1$ and the feasible set $\mathcal{S}_1$ has two solutions $(P_1, Q_1) = (13, 15)$ and $(P_2, Q_2) = (10, 20)$ then $(P_2, Q_2)$ is a Pareto-optimal solution but not a $\rho$-$NF$ solution. Consequently, a Pareto-optimal solution is not necessarily a $\rho$-$NF$ solution for Max-Max BODO.

Similarly, the $\rho$-$NF$ solutions for Max-Min BODO and Min-Min BODO are necessarily the Pareto-optimal solution, but the inverse may not be true. This concludes the proof. □

Note that using the definitions of $\rho$-$NF$ solutions, we can also obtain their Pareto efficiency. However, we state Lemma 1 because it is necessary for determining the $\rho$-$NF$ solution set in the next section.

## 4. Determining the $\rho$-$NF$ solution set for BODO

As shown in Section 2.1.2, the $\rho$-$NF$ solution for Max-Max BODO is not guaranteed to exist. Thus, after obtaining the solution of $\mathcal{H}_1$, we verify if it is exactly the $\rho$-$NF$ solution.

In general, verifying if a feasible solution is the $\rho$-$NF$ solution can be done by the following lemma.

**Lemma 2.** *For $i = 1, 2, 3$ and a given solution $(P_0, Q_0) \in \mathcal{S}_i$, let $\alpha_1 = P_0/Q_0$ and $(P_1, Q_1)$ be a solution for optimizing $F_i(\alpha_1, P, Q)$. Then $(P_0, Q_0)$ is a $\rho$-$NF$ solution if and only if $F_i(\alpha_1, P_1, Q_1) = F_i(\alpha_1, P_0, Q_0)$.*

*Proof.* $\Longrightarrow$ If $(P_0, Q_0) \in \mathcal{S}_i$ is a $\rho$-$NF$ solution then $(P_0, Q_0)$ is also a solution for optimizing $F_i(\alpha_1, P, Q)$ due to Lemma 1. Thus, $F_i(\alpha_1, P_1, Q_1) = F_i(\alpha_1, P_0, Q_0)$ since $(P_0, P_1)$ and $(P_1, Q_1)$ are both the solutions for optimizing $F_i(\alpha_1, P, Q)$.

$\Longleftarrow$ If $F_i(\alpha_1, P_1, Q_1) = F_i(\alpha_1, P_0, Q_0)$ then $(P_0, Q_0)$ is also a solution for optimizing $F_i(\alpha_1, P, Q)$. Since $\alpha_1 = P_0/Q_0$, $(P_0, Q_0)$ is a $\rho$-$NF$ solution due to Lemma 1. □

### *4.1. Max-Max BODO*

We recall that the $\rho$-$NF$ solution for Max-Max BODO is not guaranteed to exist. Thus, we first solve $\mathcal{H}_1$, then verify if the obtained solution is the $\rho$-$NF$ solution by using Lemma 2. Note that solving $\mathcal{H}_1$ is equivalent to solving the following problem

$$\mathcal{H}^*{}_1 = \max_{(P,Q)\in\mathcal{S}_1} \rho \log P + \log Q,$$

where *log* denotes the natural logarithmic function.

According to the concavity of $P, Q$, $\rho \log P + \log Q$ is concave, $\forall \rho > 0$. Thus, suppose that we know how to solve $\mathcal{H}^*{}_1$ over $\mathcal{X}_1$. The algorithm for finding the $\rho$-$NF$ solution for Max-Max BODO can be stated as follows.

**Algorithm 1**

**Input:** An instance of Max-Max BODO with $\rho > 0$.
**Output:** The $\rho$-$NF$ solution (if it exists) or $-1$ (otherwise).

Solving $\mathcal{H}^*{}_1$ to obtain a solution $(P_0, Q_0)$
$\alpha_1 \leftarrow P_0/Q_0$
Maximizing $F_1(\alpha_1, P, Q)$ to obtain a solution $(P_1, Q_1)$
**if** $F_1(\alpha_1, P_0, Q_0) = F_1(\alpha_1, P_1, Q_1)$ **then**
    Return $(P_0, Q_0)$
**else**
    Return -1
**end if**

*4.2. Max-Min BODO with* $0 < \rho \leq 1$

As a result of Theorem 2, the $\rho$-$NF$ solution for Max-Min BODO with $0 < \rho \leq 1$ is necessarily the solution of $\mathcal{H}_2$. Since $P^\rho$ is concave and $Q$ is convex, $\mathcal{H}_2$ can be considered as a concave-convex fractional maximization problem, which is well-known in literature [6], [7]. Note that the objective function of $\mathcal{H}_2$ is generally not concave, although it is composed of a concave and a convex function. For solving $\mathcal{H}_2$, we consider the following Dinkelbach's transform [8]

$$\mathcal{D}_2(\alpha) = \max_{(P,Q)\in\mathcal{S}_2} P^\rho - \alpha Q,$$

Since $P^\rho - \alpha Q$ is a concave function, suppose that we know how to solve $\mathcal{D}_2(\alpha)$ over $\mathcal{X}_2$. Thus, the algorithm for finding $\rho$-$NF$ solution can be stated as follows. This algorithm is inspired by the application of Newton's method to linear fractional programming that was first discussed by Isbell and Marlow [16] and then generalized to nonlinear fractional programming by Dinkelbach [8].

**Algorithm 2**

**Input:** An instance of Max-Min BODO with $0 < \rho \leq 1$.
**Output:** The $\rho$-$NF$ solution (if it exists) or $-1$ (otherwise).

$\alpha_0 \leftarrow 0$
Solving $\mathcal{D}_2(\alpha_0)$ to obtain a solution $(P_0, Q_0)$
$i \leftarrow 0$
**repeat**
    $\alpha_{i+1} \leftarrow P_i^\rho / Q_i$
    Solving $\mathcal{D}_2(\alpha_{i+1})$ to obtain a solution $(P_{i+1}, Q_{i+1})$
    $T_i \leftarrow P_i^\rho - \alpha_i Q_i$
    $i \leftarrow i + 1$
**until** $T_i = 0$
**if** $\rho = 1$ **then**
    Return $(P_i, Q_i)$
**else** ▷ $0 < \rho < 1$
    $\alpha' \leftarrow P_i / Q_i$
    Maximizing $F_2(\alpha', P, Q)$ to obtain a solution $(P', Q')$
    **if** $F_2(\alpha', P_i, Q_i) = F_2(\alpha', P', Q')$ **then**
        Return $(P_i, Q_i)$
    **else**
        Return $-1$
    **end if**
**end if**

Like for Max-Max BODO, the $\rho$-$NF$ solution for Max-Min BODO with $0 < \rho < 1$ is not guaranteed to exist. Thus, after obtaining the solution $(P_i, Q_i)$ for which $T_i = 0$, we verify if it is the $\rho$-$NF$ solution based on Lemma 2.

Let us consider the special case when $\rho = 1$. In this case, all $\rho$-$NF$ solutions are the solutions to the following concave-convex fractional maximization problem (see [7])

$$\mathcal{R}_2 = \max_{(P,Q)\in\mathcal{S}_2} \frac{P}{Q},$$

Consequently, determining the $\rho$-$NF$ solution set is equivalent to determining the solution set of $\mathcal{R}_2$. If $P, Q$ are linear, there have been some discussions about its solution set in the literature. Most of them give a unique condition for the solution. If this condition is not satisfied, computational procedures are presented to determine the solution set [26], [27]. For example, a method for determining the solution set based on an adaptation of the convex simplex method credited to Gilmore and Gomory has been introduced in [25].

However, when $P, Q$ are nonlinear, determining the solution set of $\mathcal{R}_2$ may be difficult since it is hard to describe the general structure of the solutions. Furthermore, they are the optimal solutions of Dinkelbach's transform $\mathcal{D}_2(\alpha)$ with a unique value of $\alpha$ [6].

In the next section, we will show the main distribution of this paper: we propose a recursive Newton-like algorithm using the weighted sum scalarization approach for determining

the $\rho$-$NF$ solution set for Max-Min BODO with $\rho > 1$ and Min-Min BODO.

### *4.3. Max-Min BODO with $\rho > 1$ and Min-Min BODO*

#### *4.3.1. General idea of algorithm for determining the $\rho$-$NF$ solution set*

Since many $\rho$-$NF$ solutions may exist for both Min-Min BODO with $\rho > 1$ and Min-Min BODO, this section states the general idea and algorithm for obtaining their $\rho$-$NF$ solution set in a unique framework.

As the $\rho$-$NF$ solution set is a subset of the Pareto set, one may theoretically find the latter and check for every Pareto-optimal solution if it is also a $\rho$-$NF$ solution. However, most of the methods related to finding the Pareto set in the literature, such as the Normal Boundary Intersection (NBI) [20] and the Non-dominated Sorting Genetic Algorithm (NSGA-II) [21], provide an approximate Pareto set instead of the exact one. Moreover, they are also expensive to implement [22].

Another BODO's popular solving method is the weighted sum scalarization approach [4] due to its computational efficiency. By scalarizing two objectives into a single objective, it provides a solution that reflects preferences incorporated in selecting the weights and possibly multiple solutions when varying the weights consistently. However, its fundamental drawback is not finding a uniform spread of Pareto-optimal solutions, even if a uniform spread of weight vectors is used. More seriously, Pareto-optimal solutions in non-convex regions are not detected [23], [24]. Thus, it may not provide the Pareto set completely.

Hence, a novel method is required for determining the $\rho$-$NF$ solution set. For this purpose, we propose a recursive Newton-like algorithm using the modified weighted sum scalarization approach to identify all $\rho$-$NF$ solutions.

Let $F(\alpha, P, Q)$ represent $F_2(\alpha, P, Q)$ and $F_3(\alpha, P, Q)$ in our algorithm (i.e., if we optimize $F(\alpha, P, Q)$, that means we maximize $F_2(\alpha, P, Q)$ over $\mathcal{X}_2$ or minimize $F_3(\alpha, P, Q)$ over $\mathcal{X}_3$, depending on the considered problem is Max-Min BODO with $\rho > 1$ or Min-Min BODO). According to Lemma 1, let $\mathcal{C}$ be the set containing all elements $\alpha^*$ such that there exists a $\rho$-$NF$ solution $(P^*, Q^*)$ as a solution for optimizing $F(\alpha^*, P, Q)$ with $\alpha^* = P^*/Q^*$. We can prove that each element of $\mathcal{C}$ corresponds to a unique $\rho$-$NF$ solution and vice versa. Thus, determining the $\rho$-$NF$ solution set is equivalent to finding all elements of $\mathcal{C}$.

Let us explain the general idea for finding all elements of $C$. From the given BODO problem, we first determine $\alpha^{sup}$ as the upper bound for the elements of $\mathcal{C}$. Then, our algorithm uses Procedure *EXPLORE*() to recursively explore all elements of $\mathcal{C}$ in the interval $[0, \alpha^{sup}]$. For each subinterval $[\alpha_i, \alpha_j] \subseteq [0, \alpha^{sup}]$, Procedure *EXPLORE*() determines a value $\alpha \in [\alpha_i, \alpha_j]$ based on the solutions for optimizing $F(\alpha_i, P, Q)$ and $F(\alpha_j, P, Q)$. Then, Procedure *VERIFY*() verifies whether $\alpha \in \mathcal{C}$ or not. If not, we use Procedure *FIND*() to find an element $\alpha^* \in \mathcal{C}$ such that there does not exist any other element of $\mathcal{C}$ in the closed interval defined by $\alpha$ and $\alpha^*$ (i.e., $[\alpha, \alpha^*]$ if $\alpha < \alpha^*$ or $[\alpha^*, \alpha]$ if $\alpha^* < \alpha$). Then, we remove such closed interval to obtain at most two remaining subintervals from $[\alpha_i, \alpha_j]$, which we will continue to explore. Although we may need two recursive calls in an iteration, the total number of recursive calls in the worst case is bounded by the number of Pareto-optimal solutions. Furthermore, our determining method of $\alpha$ provides some criteria for quickly asserting that no more $\rho$-$NF$ solution exists in $[\alpha_i, \alpha_j]$.

In summary, we divide recursively the interval $[0, \alpha^{sup}]$ by the consecutive subintervals. Then, either our algorithm shows that there is no element of $\mathcal{C}$ in each subinterval or it provides an element of $\mathcal{C}$ and continues for the other subintervals until the interval $[0, \alpha^{sup}]$ is totally explored. Notice that our algorithm may not provide the Pareto set since some Pareto-optimal solutions are not guaranteed to be found by the weighted sum scalarization approach. However, it provides completely the $\rho$-$NF$ solution set.

In the following, we discuss how to propose Procedure $VERIFY(\alpha_0)$ for verifying $\alpha_0 \in \mathcal{C}$ from the given $\alpha_0$. Notice that it may be difficult to verify if $\alpha_0 \in \mathcal{C}$ by only optimizing $F(\alpha_0, P, Q)$ and then checking if the obtained solution $(P_0, Q_0)$ satisfies $P_0 - \alpha_0 Q_0 = 0$ (based on Lemma 1). The fact is that the problem optimizing $F(\alpha_0, P, Q)$ may have multiple solutions, and we only obtain one solution, which is possibly not the $\rho$-$NF$ solution in case $\alpha_0 \in \mathcal{C}$ (i.e., there exists another solution $(P_1, Q_1)$ for optimizing $F(\alpha_0, P, Q)$ which is a $\rho$-$NF$ solution. However, $(P_0, Q_0)$, instead of $(P_1, Q_1)$, is returned. Thus, $\alpha_0 \in \mathcal{C}$ although $P_0 - \alpha_0 Q_0 \neq 0$).

Hence, for Procedure $VERIFY()$, we define two additional optimization problems

$$\max_{(P,Q)\in\mathcal{S}_2} G_2(\alpha, P, Q) = \rho P - \alpha Q - |P - \alpha Q|,$$
$$\min_{(P,Q)\in\mathcal{S}_3} G_3(\alpha, P, Q) = \rho P + \alpha Q + |P - \alpha Q|,$$

where $|.|$ denotes the absolute function.

Note that $G_2(\alpha, P, Q)$ can be transformed as a linear combination of $P$ and $Q$. More precisely, if $P \geq \alpha Q$ then $G_2(\alpha, P, Q) = (\rho+1)P - 2\alpha Q$, and if $P \leq \alpha Q$ then $G_2(\alpha, P, Q) = (\rho - 1)P$. In both cases, $G_2(\alpha, P, Q)$ is concave since $\rho > 1$, $P$ is concave and $Q$ is convex. Thus, suppose that we know how to maximize $G_2(\alpha, P, Q)$ over $\mathcal{X}_2$.

Similarly, if $P \geq \alpha Q$ then $G_3(\alpha, P, Q) = (\rho + 1)P$, and if $P \leq \alpha Q$ then $G_3(\alpha, P, Q) = (\rho-1)P+2\alpha Q$. In both cases, $G_3(\alpha, P, Q)$ is convex since $\rho \geq 1$ and $P, Q$ are convex. Thus, suppose that we also know how to minimize $G_3(\alpha, P, Q)$ over $\mathcal{X}_3$.

Let $G(\alpha, P, Q)$ represent $G_2(\alpha, P, Q)$ and $G_3(\alpha, P, Q)$ in our algorithm. Note that if we optimize $G(\alpha, P, Q)$, that means we maximize $G_2(\alpha, P, Q)$ over $\mathcal{X}_2$ or minimize $G_3(\alpha, P, Q)$ over $\mathcal{X}_3$, depending on the considered problem is Max-Min BODO with $\rho > 1$ or Min-Min BODO . We first state Procedure $VERIFY()$ as follows.

**Algorithm 3** Verifying if $\alpha_0 \in \mathcal{C}$

**Input:** $\alpha_0 \in [0, \alpha^{sup}]$.
**Output:** True if $\alpha_0 \in \mathcal{C}$ and False otherwise.

**procedure** *VERIFY*$(\alpha_0)$
    Optimizing $F(\alpha_0, P, Q)$ and $G(\alpha_0, P, Q)$ to obtain the solutions $(P_0, Q_0)$ and $(P_1, Q_1)$
    **if** $G(\alpha_0, P_1, Q_1) = F(\alpha_0, P_0, Q_0)$ **then**
        Return True
    **else**
        Return False
    **end if**
**end procedure**

For a given $\alpha_0 \in [0, \alpha^{sup}]$, if we assert that $\alpha_0 \notin \mathcal{C}$ by Procedure *VERIFY*$(\alpha_0)$, then we can use Procedure *FIND*$(\alpha_0)$ for finding an element of $\mathcal{C}$ from $\alpha_0$. This procedure is also based on the application of Newton's method for linear programming. It can be stated as follows.

**Algorithm 4** Finding an element of $\mathcal{C}$ from $\alpha_0 \notin \mathcal{C}$

**Input:** $\alpha_0 \in [0, \alpha^{sup}]$ and $\alpha_0 \notin \mathcal{C}$.
**Output:** A unique element of $\mathcal{C}$ in the closed interval defined by itself and $\alpha_0$.

**procedure** *FIND*$(\alpha_0)$
    Optimizing $F(\alpha_0, P, Q)$ to obtain a solution $(P_0, Q_0)$
    $i \leftarrow 0$
    **repeat**
        $\alpha_{i+1} \leftarrow P_i/Q_i$
        Optimizing $F(\alpha_{i+1}, P, Q)$ to obtain a solution $(P_{i+1}, Q_{i+1})$
        $T_i \leftarrow F(\alpha_{i+1}, P_{i+1}, Q_{i+1}) - F(\alpha_{i+1}, P_i, Q_i)$
        $i \leftarrow i + 1$
    **until** $T_i = 0$
    Return $\alpha_{i+1}$
**end procedure**

Then, for each subinterval $[\alpha_i, \alpha_j] \subseteq [0, \alpha^{sup}]$, we present Procedure *EXPLORE*$([\alpha_i, \alpha_j])$ for finding all elements of $\mathcal{C}$ in such subinterval.

**Algorithm 5** Finding the elements of $\mathcal{C}$ in the interval $[\alpha_i, \alpha_j]$

**Input:** $\alpha_i < \alpha_j$ and $\alpha_i, \alpha_j \in [0, \alpha^{sup}]$, $(P_i, Q_i) \not\equiv (P_j, Q_j)$ are respectively the solutions for optimizing $F(\alpha_i, P, Q)$ and $F(\alpha_j, P, Q)$.

**Output:** All elements of $\mathcal{C}$ in the interval $[\alpha_i, \alpha_j]$.

**procedure** *EXPLORE*$([\alpha_i, \alpha_j])$
  $\mathcal{C} = \{\}$
  **if** *VERIFY*$(\alpha_i)$ == *True* **then** $\mathcal{C} = \mathcal{C} \cup \alpha_i$
  **end if**
  **if** *VERIFY*$(\alpha_j)$ == *True* **then** $\mathcal{C} = \mathcal{C} \cup \alpha_j$
  **end if**
  $\alpha_k \leftarrow |\frac{\rho(P_i - P_j)}{Q_i - Q_j}|$
  **while** $\alpha_i < \alpha_k$ **and** $\alpha_k < \alpha_j$ **do**
    **if** *VERIFY*$(\alpha_k) == True$ **then**
      $\mathcal{C} \leftarrow \mathcal{C} \cup \alpha_k$ ▷ $\alpha_k$ is an element of $\mathcal{C}$
      *EXPLORE*$([\alpha_i, \alpha_k])$, *EXPLORE*$([\alpha_k, \alpha_j])$
    **else**
      Optimizing $F(\alpha_k, P, Q)$ to obtain a solution $(P_k, Q_k)$
      **if** $(P_k, Q_k) \not\equiv (P_i, Q_i)$ **and** $(P_k, Q_k) \not\equiv (P_j, Q_j)$ **then** $c_k \leftarrow$ *FIND*$(\alpha_k)$
        **if** $\alpha_i == c_k$ **then** *EXPLORE*$([\alpha_k, \alpha_j])$
        **else if** $\alpha_j == c_k$ **then** *EXPLORE*$([\alpha_i, \alpha_k])$
        **else if** $\alpha_i < c_k$ **and** $c_k < \alpha_k$ **then** $\mathcal{C} \leftarrow \mathcal{C} \cup c_k$
          *EXPLORE*$([\alpha_i, c_k])$, *EXPLORE*$([\alpha_k, \alpha_j])$
        **else if** $\alpha_j > c_k$ **and** $c_k > \alpha_k$ **then** $\mathcal{C} \leftarrow \mathcal{C} \cup c_k$
          *EXPLORE*$([\alpha_i, \alpha_k])$, *EXPLORE*$([c_k, \alpha_j])$
        **end if**
      **end if**
    **end if**
  **end while**
  Return $\mathcal{C}$
**end procedure**

Finally, we state the main algorithm for determining the $\rho$-$NF$ solution set by finding all elements of $\mathcal{C}$.

**Algorithm 6** Finding the $\rho$-$NF$ solution set

**Input:** A BODO instance (either Max-Min BODO with $\rho > 1$ or Min-Min BODO).

**Output:** Set $\mathcal{C}$ whose elements correspond to all $\rho$-$NF$ solutions.

Compute $\alpha^{sup}$ from the given BODO instance.
*EXPLORE*$([0, \alpha^{sup}])$

In the next two sections, we show that Algorithm 6 returns all elements of $\mathcal{C}$, which correspond to all $\rho$-$NF$ solutions for Max-Min BODO with $\rho > 1$ and Min-Min BODO.

Note that the two cases' statements of lemmas and theorems are almost similar. However, their proofs are different due to their different characterization of the $\rho$-$NF$ solution.

*4.3.2. Proofs for Max-Min BODO with $\rho > 1$*

For Max-Min BODO with $\rho > 1$, the set $\mathcal{C}$ is defined as follows: $\mathcal{C}$ contains all elements $\alpha^* > 0$ such that there exists a $\rho$-$NF$ solution $(P^*, Q^*) \in \mathcal{S}_2$ for maximizing $F_2(\alpha^*, P, Q)$ with $\alpha^* = P^*/Q^*$. Moreover, $F(\alpha, P, Q)$, $G(\alpha, P, Q)$ become respectively $F_2(\alpha, P, Q)$ and $G_2(\alpha, P, Q)$ in our algorithm.

We first assert that each element of $\mathcal{C}$ corresponds to a unique $\rho$-$NF$ solution and vice versa. Consequently, determining the $\rho$-$NF$ solution set is equivalent to finding all elements of $\mathcal{C}$.

**Lemma 3.** *For Max-Min BODO with $\rho > 1$, there is a bijection between the set $\mathcal{C}$ and the $\rho$-$NF$ solution set.*

*Proof.* According to Lemma 1, each $\rho$-$NF$ solution corresponds to a unique element of $\mathcal{C}$.

We show that each element of $\mathcal{C}$ corresponds to a unique $\rho$-$NF$ solution with respect to the values of $P$ and $Q$.

Suppose that $(P^*, Q^*), (P^{**}, Q^{**}) \in \mathcal{S}_2$ are two $\rho$-$NF$ solutions corresponding to $\alpha^* \in \mathcal{C}$. Since both $(P^*, Q^*)$ and $(P^{**}, Q^{**})$ are the solutions for maximizing $F_2(\alpha^*, P, Q)$, we have

$$\rho P^* - \alpha^* Q^* = \rho P^{**} - \alpha^* Q^{**}, \tag{8}$$

Furthermore, since $\alpha^* \in \mathcal{C}$, we get

$$P^* - \alpha^* Q^* = P^{**} - \alpha^* Q^{**} = 0, \tag{9}$$

From (8) and (9), we have $(\rho - 1)P^* = (\rho - 1)P^{**}$. Since $\rho - 1 > 0$, we obtain $P^* = P^{**}$ and consequently $Q^* = Q^{**}$. Thus, $(P^*, Q^*) \equiv (P^{**}, Q^{**})$. □

According to Lemma 3, although both Max-Min BODO with $\rho = 1$ and Max-Min BODO with $\rho > 1$ have possibly multiple $\rho$-$NF$ solutions, they have a main difference. That is, unlike for Max-Min BODO with $\rho > 1$, all $\rho$-$NF$ solutions for Max-Min BODO with $\rho = 1$ correspond to a unique value, which is the maximal value of the ratio $P/Q$, $\forall (P, Q) \in \mathcal{S}_2$.

We then determine $\alpha^{sup}$ as the upper bound for the elements of $\mathcal{C}$. Let $\alpha^{sup}$ be exactly the maximal value of the ratio $P/Q$, $\forall (P, Q) \in \mathcal{S}_2$. We recall that for Max-Min BODO with $\rho = 1$, each $\rho$-$NF$ solution $(P_f, Q_f)$ is a solution of the optimization problem

$$\max_{(P,Q)\in\mathcal{S}_2} P - \alpha^{sup} Q,$$

satisfying $P_f - \alpha^{sup} Q_f = 0$ (i.e., $\alpha^{sup} = P_f/Q_f$), see [6]. By the following lemma, we prove $\alpha^{sup}$ is the upper bound for the elements of $\mathcal{C}$.

**Lemma 4.** *Let $\alpha^* \in \mathcal{C}$. We have $\alpha^* \leq \alpha^{sup}$.*

*Proof.* Let $(P^*, Q^*) \in \mathcal{S}_2$ be a $\rho$-$NF$ solution for Max-Min BODO with $\rho > 1$ corresponding to $\alpha^* \in \mathcal{C}$ and $(P_f, Q_f) \in \mathcal{S}_2$ be a $\rho$-$NF$ solution for Max-Min BODO with $\rho = 1$ corresponding to $\alpha_f$. From $P^* = \alpha^* Q^*$ and the optimality of $(P_f, Q_f)$, we have

$$P^* - \alpha^* Q^* = 0 = P_f - \alpha^{sup} Q_f \geq P^* - \alpha^{sup} Q^*,$$

Thus, $\alpha^* \leq \alpha^{sup}$. □

For a given $\alpha_0 \in [0, \alpha^{sup}]$, we show that verifying if $\alpha_0 \in \mathcal{C}$ can be done by solving $F_2(\alpha, P, Q)$ and $G_2(\alpha, P, Q)$.

**Lemma 5.** *For a given $\alpha_0 \in [0, \alpha^{sup}]$, let $(P_0, Q_0), (P_1, Q_1) \in \mathcal{S}_2$ be respectively the solutions for maximizing $F_2(\alpha_0, P, Q)$ and $G_2(\alpha_0, P, Q)$. Then $\alpha_0 \in \mathcal{C}$ if and only if $G_2(\alpha_0, P_1, Q_1) = F_2(\alpha_0, P_0, Q_0)$.*

*Proof.* $\Longrightarrow$ Suppose that $\alpha_0 \in \mathcal{C}$. According to Lemma 1, there exists a feasible solution $(P^*, Q^*) \in \mathcal{S}_2$ such that $(P^*, Q^*)$ is a solution for maximizing $F_2(\alpha_0, P, Q)$ and $P^* = \alpha_0 Q^*$. Since both $(P_0, Q_0)$ and $(P^*, Q^*)$ are the solutions for maximizing $F_2(\alpha_0, P, Q)$ and $P^* - \alpha_0 Q^* = 0$, we have

$$\rho P_0 - \alpha_0 Q_0 = \rho P^* - \alpha_0 Q^* - |P^* - \alpha_0 Q^*|,$$

The optimality of $(P_0, Q_0)$ gives

$$\rho P_0 - \alpha_0 Q_0 \geq \rho P_1 - \alpha_0 Q_1,$$

Since $|P_1 - \alpha_0 Q_1| \geq 0$, we deduce $\rho P_0 - \alpha_0 Q_0 \geq \rho P_1 - \alpha_0 Q_1 - |P_1 - \alpha_0 Q_1|$. Thus,

$$\rho P^* - \alpha_0 Q^* - |P^* - \alpha_0 Q^*| \geq \rho P_1 - \alpha_0 Q_1 - |P_1 - \alpha_0 Q_1|, \tag{10}$$

Since $(P_1, Q_1)$ is a solution for maximizing $G_2(\alpha_0, P, Q)$, we have

$$\rho P_1 - \alpha_0 Q_1 - |P_1 - \alpha_0 Q_1| \geq \rho P^* - \alpha_0 Q^* - |P^* - \alpha_0 Q^*|, \tag{11}$$

From (10) and (11), we get

$$\rho P_1 - \alpha_0 Q_1 - |P_1 - \alpha_0 Q_1| = \rho P^* - \alpha_0 Q^* - |P^* - \alpha_0 Q^*| = \rho P_0 - \alpha_0 Q_0,$$

which implies $G_2(\alpha_0, P_1, Q_1) = F_2(\alpha_0, P_0, Q_0)$.

$\Longleftarrow$ Suppose that $G_2(\alpha_0, P_1, Q_1) = F_2(\alpha_0, P_0, Q_0)$. We obtain $\rho P_1 - \alpha_0 Q_1 - |P_1 - \alpha_0 Q_1| = \rho P_0 - \alpha_0 Q_0$. Since $\rho P_1 - \alpha_0 Q_1 - |P_1 - \alpha_0 Q_1| \leq \rho P_1 - \alpha_0 Q_1 \leq \rho P_0 - \alpha_0 Q_0$, we must have $|P_1 - \alpha_0 Q_1| = 0$ and $\rho P_1 - \alpha_0 Q_1 = \rho P_0 - \alpha_0 Q_0$. Consequently, $(P_1, Q_1)$ is a solution for maximizing $F_2(\alpha_0, P, Q)$ and $P_1 = \alpha_0 Q_1$. Thus, $(P_1, Q_1)$ is a $\rho$-NF solution and $\alpha_0 \in \mathcal{C}$ due to Lemma 1. □

Let $T_i = F_2(\alpha_{i+1}, P_{i+1}, Q_{i+1}) - F_2(\alpha_{i+1}, P_i, Q_i)$ where $(P_i, Q_i)$ is the solution for maximizing $F_2(\alpha_i, P, Q)$ and $\{\alpha_i\}_{i\geq 0}$ (including $\alpha_0$) is the sequence constructed by Procedure *FIND*($\alpha_0$). We then prove that if $\alpha_0 \notin \mathcal{C}$, Procedure *FIND*($\alpha_0$) returns an element of $\mathcal{C}$ such that it is a unique element of $\mathcal{C}$ in the closed interval defined by $\alpha_0$ and itself.

**Lemma 6.** *For $\alpha_0 \in [0, \alpha^{sup}]$, during the execution of Procedure FIND($\alpha_0$), $\alpha_{i+1} > 0$, $(P_{i+1}, Q_{i+1})$ is a Pareto-optimal solution and $T_i \geq 0$ $\forall i \geq 0$. Furthermore, Procedure FIND($\alpha_0$) terminates after a finite number of iterations.*

*Proof.* Since $P_i, Q_i > 0$, we have $\alpha_{i+1} = P_i/Q_i > 0, \forall i \geq 0$. Thus, $(P_{i+1}, Q_{i+1})$ is a Pareto-optimal solution due to Theorem 5, $\forall i \geq 0$.

For all $i \geq 0$, the optimality of $(P_{i+1}, Q_{i+1})$ gives

$$\rho P_{i+1} - \alpha_{i+1} Q_{i+1} \geq \rho P_i - \alpha_{i+1} Q_i, \tag{12}$$

Thus, $T_i = F_2(\alpha_{i+1}, P_{i+1}, Q_{i+1}) - F_2(\alpha_{i+1}, P_i, Q_i) = \rho P_{i+1} - \alpha_{i+1} Q_{i+1} - (\rho P_i - \alpha_{i+1} Q_i) \geq 0$. Moreover, replacing $\alpha_{i+1}$ by $P_i/Q_i$, (12) yields

$$\rho P_{i+1} \geq \frac{P_i}{Q_i} Q_{i+1} + (\rho - 1) P_i \implies \frac{P_{i+1}}{P_i} \geq \frac{1}{\rho} \times \frac{Q_{i+1}}{Q_i} + \frac{\rho - 1}{\rho},$$

Using Young's inequality for products, we have

$$\frac{P_{i+1}}{P_i} \geq \frac{1}{\rho} \times \frac{Q_{i+1}}{Q_i} + \frac{\rho - 1}{\rho} \geq \left(\frac{Q_{i+1}}{Q_i}\right)^{\frac{1}{\rho}} \implies \frac{P_{i+1}^{\rho}}{Q_{i+1}} \geq \frac{P_i^{\rho}}{Q_i},$$

In other words, the value of $P_i^{\rho}/Q_i$ is increasing after each iteration of Procedure *FIND*($\alpha_0$). Since $\mathcal{S}_2$ is finite, the set of values $P^{\rho}/Q$ is also finite. Thus, we get $\frac{P_{k+1}^{\rho}}{Q_{k+1}} = \frac{P_k^{\rho}}{Q_k}$ after a finite number of iterations. Now the equality of Young's inequality above must hold. We have then $\frac{P_{k+1}}{P_k} = \frac{Q_{k+1}}{Q_k} = 1 \implies P_{k+1} = P_k$ and $Q_{k+1} = Q_k$. Consequently, $T_k = F_2(\alpha_{k+1}, P_{k+1}, Q_{k+1}) - F_2(\alpha_{k+1}, P_k, Q_k) = \rho P_{k+1} - \alpha_{k+1} Q_{k+1} - (\rho P_k - \alpha_{k+1} Q_k) = 0$. That is to say, Procedure *FIND*($\alpha_0$) terminates after a finite number of iterations. □

Suppose that Procedure *FIND*($\alpha_0$) returns a coefficient $\alpha_{n+1}$ satisfying $T_n = F_2(\alpha_{n+1}, P_{n+1}, Q_{n+1}) - F_2(\alpha_{n+1}, P_n, Q_n) = 0$ where $n \geq 0$. Thus, $(P_n, Q_n)$ is a $\rho$-*NF* solution and $\alpha_{n+1} \in \mathcal{C}$ due to Lemma 2. In addition, if $n \geq 1$ then $T_i > 0$, $\forall 0 \leq i \leq n-1$.

In the next lemma, we show the monotonic relationship between $\alpha \in [0, \alpha^{sup}]$ and the solution for maximizing $F_2(\alpha, P, Q)$ with respect to the values of $P$ and $Q$.

**Lemma 7.** *Let $\alpha', \alpha'' \in [0, \alpha^{sup}]$, $\alpha' < \alpha''$ and $(P', Q')$, $(P'', Q'') \in \mathcal{S}_2$ be respectively the solutions for maximizing $F_2(\alpha', P, Q)$ and $F_2(\alpha'', P, Q)$. Then $P' \geq P''$ and $Q' \geq Q''$.*

*Proof.* The optimality of $(P', Q')$ and $(P'', Q'')$ gives

$$\rho P' - \alpha' Q' \geq \rho P'' - \alpha' Q'', \text{ and} \tag{13a}$$

$$\rho P'' - \alpha'' Q'' \geq \rho P' - \alpha'' Q' \tag{13b}$$

Adding (13a) and (13b) gives $(\alpha' - \alpha'')(Q' - Q'') \leq 0$. Since $\alpha' < \alpha''$, we have $Q' \geq Q''$.

On the other hand, the inequality (13a) implies $\rho(P' - P'') \geq \alpha'(Q' - Q'') \geq 0$. Since $\rho > 1$, we get $P' \geq P''$. □

**Theorem 6.** *For $\alpha_0 \in [0, \alpha^{sup}]$ and $\alpha_0 \notin \mathcal{C}$, Procedure FIND($\alpha_0$) returns $\alpha_{n+1}$ as the unique element of $\mathcal{C}$ in the closed interval defined by $\alpha_0$ and $\alpha_{n+1}$.*

*Proof.* Since $(P_0, Q_0)$ is a solution for maximizing $F_2(\alpha_0, P, Q)$ and $\alpha_0 \notin \mathcal{C}$, we have $P_0 - \alpha_0 Q_0 \neq 0$. Thus, we consider two cases: $P_0 - \alpha_0 Q_0 > 0$ and $P_0 - \alpha_0 Q_0 < 0$.

We first suppose that $P_0 - \alpha_0 Q_0 > 0$. We will prove $\alpha_i < \alpha_{i+1}, \forall i \geq 0$ by induction on $i$.

Since $P_0 - \alpha_0 Q_0 > 0$, we have $\alpha_0 < P_0/Q_0 = \alpha_1$. Thus, our hypothesis is true with $i = 0$.

Suppose our hypothesis is true until $i = k \geq 0$. We have $\alpha_i < \alpha_{i+1},\ \forall 0 \leq i \leq k$. The optimality of $(P_{k+1}, Q_{k+1})$ gives

$$\rho P_{k+1} - \alpha_{k+1} Q_{k+1} \geq \rho P_k - \alpha_{k+1} Q_k,$$

which is equivalent to

$$(\rho - 1) P_{k+1} + P_{k+1} - \alpha_{k+1} Q_{k+1} \geq (\rho - 1) P_k + P_k - \alpha_{k+1} Q_k, \tag{14}$$

Since $\alpha_k < \alpha_{k+1}$, $P_k \geq P_{k+1}$ and $Q_k \geq Q_{k+1}$ due to Lemma 7.

If $P_k = P_{k+1}$ and $Q_k = Q_{k+1}$ then $T_k = \rho P_{k+1} - \alpha_{k+1} Q_{k+1} - (\rho P_k - \alpha_{k+1} Q_k) = 0$. Consequently, Procedure *FIND*($\alpha_0$) returns the value $\alpha_{k+1}$ and the sequence $\{\alpha_i\}_{0 \leq i \leq k+1}$ is strictly increasing.

If $P_k = P_{k+1}$ and $Q_k > Q_{k+1}$ then $\alpha_{k+1} = \frac{P_k}{Q_k} < \frac{P_{k+1}}{Q_{k+1}} = \alpha_{k+2}$.

If $P_k > P_{k+1}$ then (14) yields $P_{k+1} - \alpha_{k+1} Q_{k+1} > P_k - \alpha_{k+1} Q_k = P_k - \frac{P_k}{Q_k} Q_k = 0$. Thus, $\alpha_{k+1} < \frac{P_{k+1}}{Q_{k+1}} = \alpha_{k+2}$.

Since we obtain $\alpha_{k+1} < \alpha_{k+2}$ in both two cases above, our hypothesis is also true with $i = k + 1$. Hence, $\{\alpha_i\}$ is strictly increasing, $\forall i \geq 0$.

In the following, we will show that $\alpha_{n+1}$ is the unique element of $\mathcal{C}$ in the interval $[\alpha_0, \alpha_{n+1}]$.

Suppose that we have $\alpha^* = (\alpha_0, \alpha_{n+1}) \cap \mathcal{C}$ corresponding to a $\rho$-$NF$ solution $(P^*, Q^*)$. Since the sequence $\{\alpha_i\}_{0 \leq i \leq n+1}$ is strictly increasing, there exists $0 \leq k \leq n$ such that $\alpha^* \in (\alpha_k, \alpha_{k+1}]$.

Since $\alpha^* > \alpha_k$, we have $P^* \leq P_k$ and $Q^* \leq Q_k$ due to Lemma 7. Furthermore, as $(P^*, Q^*)$ is a $\rho$-$NF$ solution, we get

$$\rho \frac{P_k}{P^*} - \frac{Q_k}{Q^*} \leq \rho - 1 \implies \frac{P_k}{P^*} - \frac{Q_k}{Q^*} \leq (\rho - 1)(1 - \frac{P_k}{P^*}) \leq 0, \tag{15}$$

If $\frac{P_k}{P^*} = \frac{Q_k}{Q^*}$ then (15) yields $\frac{P_k}{P^*} = \frac{Q_k}{Q^*} = 1$. It follows that $P_k = P^*$, $Q_k = Q^*$ and $\alpha_{k+1} = \frac{P_k}{Q_k} = \frac{P^*}{Q^*} = \alpha^*$. Since $\alpha_{k+1} = \alpha^* < \alpha_{n+1}$, we have $k \leq n - 1$ and $n \geq 1$. As both $(P^*, Q^*)$ and $(P_{k+1}, Q_{k+1})$ are the solutions for maximizing $F_2(\alpha_{k+1}, P, Q)$, we obtain $\rho P_{k+1} - \alpha_{k+1} Q_{k+1} - (\rho P^* - \alpha_{k+1} Q^*) = 0$. Consequently, $T_k = \rho P_{k+1} - \alpha_{k+1} Q_{k+1} - (\rho P_k - \alpha_{k+1} Q_k) = 0$ which leads to a contradiction due to $T_k > 0,\ \forall 0 \leq k \leq n - 1$.

Thus, we have

$$\frac{P_k}{P^*} < \frac{Q_k}{Q^*} \implies \alpha^* = \frac{P^*}{Q^*} > \frac{P_k}{Q_k} = \alpha_{k+1},$$

which leads to a contradiction due to the fact that $\alpha^* \leq \alpha_{k+1}$.

Hence, $\alpha_{n+1}$ is the unique element of $\mathcal{C}$ in the interval $[\alpha_0, \alpha_{n+1}]$.

Similarly, in case $P_0 - \alpha_0 Q_0 < 0$, we obtain that the sequence $\{\alpha_i\}$ is strictly decreasing and $\alpha_{n+1}$ is the unique element of $\mathcal{C}$ in the interval $[\alpha_{n+1}, \alpha_0]$. That concludes the proof. □

According to Theorem 6, from $\alpha_0 \in [0, \alpha^{sup}]$ and $\alpha_0 \notin \mathcal{C}$, we can use Procedure *FIND*($\alpha_0$) to find an element $\alpha_{n+1}$ of $\mathcal{C}$ such that there does not have any other element of $\mathcal{C}$ in the half-open interval$[\alpha_0, \alpha_{n+1})$ (if $\alpha_0 < \alpha_{n+1}$) or $(\alpha_{n+1}, \alpha_0]$ (if $\alpha_0 > \alpha_{n+1}$). Note that the $\rho$-$NF$ solution $(P_n, Q_n)$ obtained by Procedure *FIND*($\alpha_0$) is also a local optimum of $\mathcal{H}_2$ with $\rho > 1$ in the such interval (see Appendix A).

For each interval $[\alpha_i, \alpha_j] \subseteq [0, \alpha^{sup}]$ and $(P_i, Q_i) \not\equiv (P_j, Q_j)$ are respectively the solutions of $F_2(\alpha_i, P, Q)$ and $F_2(\alpha_j, P, Q)$ where $0 \leq \alpha_i < \alpha_j \leq \alpha^{sup}$, we present some criteria to quickly verify if there does not exist any $\rho$-$NF$ solution in the interval $(\alpha_i, \alpha_j)$ which is different to $(P_i, Q_i)$ and $(P_j, Q_j)$.

**Lemma 8.** *Given an interval* $[\alpha_i, \alpha_j]$ *defined by* $0 \leq \alpha_i < \alpha_j \leq \alpha^{sup}$ *and let* $(P_i, Q_i)$, $(P_j, Q_j)$ *be respectively the solutions for maximizing* $F_2(\alpha_i, P, Q)$ *and* $F_2(\alpha_j, P, Q)$ *such that* $(P_i, Q_i) \not\equiv (P_j, Q_j)$. *Let* $\alpha_k = |\frac{\rho(P_i - P_j)}{Q_i - Q_j}|$ *and* $(P_k, Q_k)$ *be a solution for maximizing* $F_2(\alpha_k, P, Q)$. *If one of the following conditions is satisfied, then there does not exist any* $\rho$*-NF solution which is different to* $(P_i, Q_i)$ *and* $(P_j, Q_j)$ *in the interval* $(\alpha_i, \alpha_j)$.

1. $\alpha_k \in \{\alpha_i, \alpha_j\}$.
2. $\alpha_k \notin \mathcal{C}$ *and* $(P_k, Q_k) \equiv (P_i, Q_i)$ *or* $(P_k, Q_k) \equiv (P_j, Q_j)$.

*Proof.* We first show that $\alpha_k$ is well defined. Since $\alpha_i < \alpha_j$, we have $P_i \geq P_j$, $Q_i \geq Q_j$ due to Lemma 7. Assume that $Q_i = Q_j$. The optimality of $(P_j, Q_j)$ gives

$$\rho P_j - \alpha_j Q_j \geq \rho P_i - \alpha_j Q_i,$$

Since $Q_i = Q_j$, we obtain $P_j \geq P_i$. Thus, $P_i = P_j$ and then $(P_i, Q_i) \equiv (P_j, Q_j)$ which leads to a contradiction.

Hence, $Q_i > Q_j$ and consequently, $\alpha_k$ is well defined.

Since $P_i \geq P_j$ and $Q_i > Q_j$, we have $\alpha_k = \frac{\rho(P_i - P_j)}{Q_i - Q_j}$. Now we show that $\alpha_k \in [\alpha_i, \alpha_j]$.

The optimality of $(P_i, Q_i)$ and $(P_j, Q_j)$ gives

$$\rho P_i - \alpha_i Q_i \geq \rho P_j - \alpha_i Q_j,$$
$$\rho P_j - \alpha_j Q_j \geq \rho P_i - \alpha_j Q_i,$$

Thus, we obtain $\alpha_i \leq \frac{\rho(P_i - P_j)}{Q_i - Q_j} \leq \alpha_j$ which leads to $\alpha_i \leq \alpha_k \leq \alpha_j$.

If $\alpha_k = \alpha_i$ then $\rho P_i - \alpha_i Q_i = \rho P_j - \alpha_i Q_j$. Thus, $(P_i, Q_i)$ and $(P_j, Q_j)$ are both solutions for maximizing $F_2(\alpha_i, P, Q)$. Hence, for all $\alpha \in (\alpha_i, \alpha_j)$, $(P_j, Q_j)$ is the solution for maximizing $F_2(\alpha, P, Q)$ as a result of Lemma 7.

Similarly, if $\alpha_k = \alpha_j$, $(P_i, Q_i)$ is the solution for maximizing $F_2(\alpha, P, Q)$ for all $\alpha \in (\alpha_i, \alpha_j)$.

As a result, in case $\alpha_k \in \{\alpha_i, \alpha_j\}$, there does not exist any $\rho$-$NF$ solution which is different to $(P_i, Q_i)$ and $(P_j, Q_j)$ in the interval $(\alpha_i, \alpha_j)$.

Now let $(P_k, Q_k)$ be a solution for maximizing $F_2(\alpha_k, P, Q)$. Without loss of generality, we suppose that $\alpha_k \notin \mathcal{C}$ and $(P_k, Q_k) \equiv (P_i, Q_i)$.

Since $(P_k, Q_k) \equiv (P_i, Q_i)$, $(P_i, Q_i)$ is the solution for maximizing $F_2(\alpha_k, P, Q)$. Since $\alpha_k = \frac{\rho(P_i - P_j)}{Q_i - Q_j}$, we have $\rho P_i - \alpha_k Q_i = \rho P_j - \alpha_k Q_j$. Thus, $(P_j, Q_j)$ is also a solution for maximizing $F_2(\alpha_k, P, Q)$.

Consequently, if $\alpha \in (\alpha_i, \alpha_k)$ then $(P_i, Q_i)$ is the solution for maximizing $F_2(\alpha, P, Q)$ and if $\alpha \in (\alpha_k, \alpha_j)$ then $(P_j, Q_j)$ is the solution for maximizing $F_2(\alpha, P, Q)$. That means there does not exist any $\rho$-$NF$ solution in the interval $(\alpha_i, \alpha_j)$ which is different to $(P_i, Q_i)$ and $(P_j, Q_j)$. □

**Theorem 7.** *Algorithm 6 returns all the elements of $\mathcal{C}$ corresponding to all $\rho$-$NF$ solutions for Max-Min BODO with $\rho > 1$.*

*Proof.* According to Lemma 4, the interval $[0, \alpha^{sup}]$ contains all the elements of $\mathcal{C}$.

Let $K_2$ denote the finite number of Pareto-optimal solutions. Consequently, $[0, \alpha^{sup}]$ can be separated by at most $K_2$ consecutive subintervals $[c_i, c_j]$ where $0 \leq c_i < c_j \leq \alpha^{sup}$ and $c_i, c_j$ correspond to two different Pareto-optimal solutions. By using Procedure *EXPLORE*$([c_i, c_j])$, each recursive call gives us a $\rho$-$NF$ solution where the corresponding coefficient in the subinterval $[c_i, c_j]$ or show that such subinterval is well explored without any new $\rho$-$NF$ solution. As we use Procedure *FIND*() and Procedure *VERIFY*() in each recursive call, Procedure *EXPLORE*$([0, \alpha^{sup}])$ also terminates after a finite number of iterations. Since Algorithm 6 terminated when the interval $[0, \alpha^{sup}]$ is totally explored, it found all elements of $\mathcal{C}$ corresponding to all $\rho$-$NF$ solutions. □

*4.3.3. Proofs for Min-Min BODO*

For Min-Min BODO, the set $\mathcal{C}$ is defined as follows: $\mathcal{C}$ contains all elements $\alpha^* > 0$ such that there exists a $\rho$-$NF$ solution $(P^*, Q^*) \in \mathcal{S}_3$ for minimizing $F_3(\alpha^*, P, Q)$ with $\alpha^* = P^*/Q^*$. Moreover, $F(\alpha, P, Q)$, $G(\alpha, P, Q)$ become respectively $F_3(\alpha, P, Q)$ and $G_3(\alpha, P, Q)$ in our algorithm.

We also assert that each element of $C$ corresponds to a unique $\rho$-$NF$ solution and vice versa.

**Lemma 9.** *For Min-Min BODO, there is a bijection between the set $\mathcal{C}$ and the $\rho$-$NF$ solution set.*

*Proof.* According to Lemma 1, each $\rho$-$NF$ solution corresponds to a unique element of $\mathcal{C}$.

We also show that each element of $\mathcal{C}$ corresponds to a unique $\rho$-$NF$ solution with respect to the values of $P$ and $Q$.

Suppose that $(P^*, Q^*), (P^{**}, Q^{**})$ are two $\rho$-$NF$ solutions corresponding to $\alpha^* \in \mathcal{C}$. Since both $(P^*, Q^*)$ and $(P^{**}, Q^{**})$ are the solutions for minimizing $F_3(\alpha^*, P, Q)$, we have

$$\rho P^* + \alpha^* Q^* = \rho P^{**} + \alpha^* Q^{**}, \tag{16}$$

Furthermore, since $\alpha^* \in \mathcal{C}$, we get

$$P^* - \alpha^* Q^* = P^{**} - \alpha^* Q^{**} = 0, \tag{17}$$

Adding (16) and (17) gives $(\rho+1)P^* = (\rho+1)P^{**}$. Since $\rho \geq 1$, we obtain $P^* = P^{**}$ and consequently $Q^* = Q^{**}$. Thus, $(P^*, Q^*) \equiv (P^{**}, Q^{**})$.

□

We then determine the upper bound $\alpha^{sup}$ for the elements of $\mathcal{C}$. Let $(P_q, Q_q) \in \mathcal{S}_3$ be a solution for minimizing $Q$ and $\alpha^{sup} = P_q/Q_q$. We state the following lemma.

**Lemma 10.** *Let $\alpha^* \in \mathcal{C}$. We have $\alpha^* \leq \alpha^{sup}$.*

*Proof.* Let $(P^*, Q^*) \in \mathcal{S}_3$ be a *$\rho$-NF* solution for Min-Min BODO corresponding to $\alpha^* \in \mathcal{C}$. The optimality of $(P^*, Q^*)$ gives

$$\rho P^* + \alpha^* Q^* \leq \rho P_q + \alpha^* Q_q, \tag{18}$$

Moreover, since $(P_q, Q_q) \in \mathcal{S}_3$ is a solution for minimizing $Q$, we have $Q^* \geq Q_q$ which implies $P^* \leq P_q$ due to (18). Thus,

$$\alpha^* = \frac{P^*}{Q^*} \leq \frac{P_q}{Q_q} = \alpha^{sup},$$

□

For a given $\alpha_0 \in [0, \alpha^{sup}]$, we show that verifying if $\alpha_0 \in \mathcal{C}$ can be done by solving $F_3(\alpha, P, Q)$ and $G_3(\alpha, P, Q)$.

**Lemma 11.** *For a given $\alpha_0 \in [0, \alpha^{sup}]$, let $(P_0, Q_0), (P_1, Q_1) \in \mathcal{S}_3$ be respectively the solutions for minimizing $F_3(\alpha_0, P, Q)$ and $G_3(\alpha_0, P, Q)$. Then $\alpha_0 \in \mathcal{C}$ if and only if $G_3(\alpha_0, P_1, Q_1) = F_3(\alpha_0, P_0, Q_0)$.*

*Proof.* $\implies$ Suppose that $\alpha_0 \in \mathcal{C}$. According to Lemma 1, there exists a feasible solution $(P^*, Q^*) \in \mathcal{S}_3$ such that $(P^*, Q^*)$ is a solution for minimizing $F_3(\alpha_0, P, Q)$ and $P^* = \alpha_0 Q^*$. Since both $(P_0, Q_0)$ and $(P^*, Q^*)$ are the solutions for minimizing $F_3(\alpha_0, P, Q)$ and $P^* - \alpha_0 Q^* = 0$, we have

$$\rho P_0 + \alpha_0 Q_0 = \rho P^* + \alpha_0 Q^* + |P^* - \alpha_0 Q^*|,$$

The optimality of $(P_0, Q_0)$ gives

$$\rho P_0 + \alpha_0 Q_0 \leq \rho P_1 + \alpha_0 Q_1,$$

Since $|P_1 - \alpha_0 Q_1| \geq 0$, we deduce $\rho P_0 + \alpha_0 Q_0 \leq \rho P_1 + \alpha_0 Q_1 + |P_1 - \alpha_0 Q_1|$. Thus,

$$\rho P^* + \alpha_0 Q^* + |P^* - \alpha_0 Q^*| \leq \rho P_1 + \alpha_0 Q_1 + |P_1 - \alpha_0 Q_1|, \tag{19}$$

Since $(P_1, Q_1)$ is a solution for minimizing $G_3(\alpha_0, P, Q)$, we have

$$\rho P_1 + \alpha_0 Q_1 + |P_1 - \alpha_0 Q_1| \leq \rho P^* + \alpha_0 Q^* + |P^* - \alpha_0 Q^*|, \tag{20}$$

From (19) and (20), we get

$$\rho P_1 + \alpha_0 Q_1 + |P_1 - \alpha_0 Q_1| = \rho P^* + \alpha_0 Q^* + |P^* - \alpha_0 Q^*| = \rho P_0 + \alpha_0 Q_0,$$

which implies $G_3(\alpha_0, P_1, Q_1) = F_3(\alpha_0, P_0, Q_0)$.

$\Longleftarrow$ Suppose that $G_3(\alpha_0, P_1, Q_1) = F_3(\alpha_0, P_0, Q_0)$. We obtain $\rho P_1 + \alpha_0 Q_1 + |P_1 - \alpha_0 Q_1| = \rho P_0 + \alpha_0 Q_0$. Since $\rho P_1 + \alpha_0 Q_1 + |P_1 - \alpha_0 Q_1| \geq \rho P_1 + \alpha_0 Q_1 \geq \rho P_0 + \alpha_0 Q_0$, we must have $|P_1 - \alpha_0 Q_1| = 0$ and $\rho P_1 + \alpha_0 Q_1 = \rho P_0 + \alpha_0 Q_0$. Consequently, $(P_1, Q_1)$ is a solution for minimizing $F_3(\alpha_0, P, Q)$ and $P_1 = \alpha_0 Q_1$. Thus, $(P_1, Q_1)$ is a $\rho$-NF solution and $\alpha_0 \in \mathcal{C}$ due to Lemma 1. □

Let $T_i = F_3(\alpha_{i+1}, P_{i+1}, Q_{i+1}) - F_3(\alpha_{i+1}, P_i, Q_i)$ where $(P_i, Q_i)$ is the solution for minimizing $F_3(\alpha_i, P, Q)$ and $\{\alpha_i\}_{i\geq 0}$ (including $\alpha_0$) is the sequence constructed by Procedure *FIND*$(\alpha_0)$. We then prove that if $\alpha_0 \notin \mathcal{C}$, Procedure *FIND*$(\alpha_0)$ returns an element of $\mathcal{C}$ such that it is a unique element of $\mathcal{C}$ in the closed interval defined by $\alpha_0$ and itself.

**Lemma 12.** *For $\alpha_0 \in [0, \alpha^{sup}]$, during the execution of Procedure FIND($\alpha_0$), $\alpha_{i+1} > 0$, $(P_{i+1}, Q_{i+1})$ is a Pareto-optimal solution and $T_i \leq 0$ $\forall i \geq 0$. Furthermore, Procedure FIND($\alpha_0$) terminates after a finite number of iterations.*

*Proof.* Since $P_i, Q_i > 0$, we have $\alpha_{i+1} = P_i/Q_i > 0, \forall i \geq 0$. Thus, $(P_{i+1}, Q_{i+1})$ is a Pareto-optimal solution due to Theorem 5, $\forall i \geq 0$.

For all $i \geq 0$, the optimality of $(P_{i+1}, Q_{i+1})$ gives

$$\rho P_{i+1} + \alpha_{i+1} Q_{i+1} \leq \rho P_i + \alpha_{i+1} Q_i, \tag{21}$$

Thus, $T_i = F_3(\alpha_{i+1}, P_{i+1}, Q_{i+1}) - F_3(\alpha_{i+1}, P_i, Q_i) = \rho P_{i+1} + \alpha_{i+1} Q_{i+1} - (\rho P_i + \alpha_{i+1} Q_i) \leq 0$.

Moreover, replacing $\alpha_{i+1}$ by $P_i/Q_i$, (21) yields

$$\rho P_{i+1} + \frac{P_i}{Q_i} Q_{i+1} \leq (\rho + 1) P_i \implies 1 \geq \frac{\rho}{\rho + 1} \times \frac{P_{i+1}}{P_i} + \frac{1}{\rho + 1} \times \frac{Q_{i+1}}{Q_i},$$

Using Young's inequality for products, we have

$$1 \geq \frac{\rho}{\rho + 1} \times \frac{P_{i+1}}{P_i} + \frac{1}{\rho + 1} \times \frac{Q_{i+1}}{Q_i} \geq (\frac{P_{i+1}}{P_i})^{\frac{\rho}{\rho+1}} (\frac{Q_{i+1}}{Q_i})^{\frac{1}{\rho+1}} \implies P_i^\rho Q_i \geq P_{i+1}^\rho Q_{i+1},$$

In other words, the value of $P_i^\rho Q_i$ is decreasing after each iteration of Procedure *FIND*$(\alpha_0)$. Since $\mathcal{S}_3$ is finite, the set of values $P^\rho Q$ is also finite. Thus, we get $P_{k+1}^\rho Q_{k+1} = P_k^\rho Q_k$ after a finite number of iterations. Now the equality of Young's inequality above must hold. We have then $\frac{P_{k+1}}{P_k} = \frac{Q_{k+1}}{Q_k} = 1 \implies P_{k+1} = P_k$ and $Q_{k+1} = Q_k$. Consequently, $T_k = F_3(\alpha_{k+1}, P_{k+1}, Q_{k+1}) - F_3(\alpha_{k+1}, P_k, Q_k) = \rho P_{k+1} + \alpha_{k+1} Q_{k+1} - (\rho P_k + \alpha_{k+1} Q_k) = 0$. That is to say, Procedure *FIND*$(\alpha_0)$ terminates after a finite number of iterations. □

Suppose that Procedure *FIND*($\alpha_0$) returns a coefficient $\alpha_{n+1}$ satisfying $T_n = F_3(\alpha_{n+1}, P_{n+1}, Q_{n+1}) - F_3(\alpha_{n+1}, P_n, Q_n) = 0$ where $n \geq 0$. Thus, $(P_n, Q_n)$ is a $\rho$-$NF$ solution and $\alpha_{n+1} \in \mathcal{C}$ due to Lemma 2. In addition, if $n \geq 1$ then $T_i < 0,\ \forall 0 \leq i \leq n-1$.

Like Lemma 7, we also show the monotonic relationship between $\alpha \in [0, \alpha^{sup}]$ and the solution for minimizing $F_3(\alpha, P, Q)$ with respect to the values of $P$ and $Q$.

**Lemma 13.** *Let $\alpha', \alpha'' \in [0, \alpha^{sup}]$, $\alpha' < \alpha''$ and $(P', Q')$, $(P'', Q'') \in \mathcal{S}_3$ be respectively the solutions for minimizing $F_3(\alpha', P, Q)$ and $F_3(\alpha'', P, Q)$. Then $P' \leq P''$ and $Q' \geq Q''$.*

*Proof.* The optimality of $(P', Q')$ and $(P'', Q'')$ gives

$$\rho P' + \alpha' Q' \leq \rho P'' + \alpha' Q'', \text{ and} \tag{22a}$$
$$\rho P'' + \alpha'' Q'' \leq \rho P' + \alpha'' Q' \tag{22b}$$

Adding (22a) and (22b) gives $(\alpha' - \alpha'')(Q' - Q'') \leq 0$. Since $\alpha' < \alpha''$, we have $Q' \geq Q''$.

On the other hand, the inequality (22a) implies $\rho(P'' - P') \geq \alpha'(Q' - Q'') \geq 0$. Since $\rho \geq 1$, we get $P' \leq P''$. □

**Theorem 8.** *For $\alpha_0 \in [0, \alpha^{sup}]$ and $\alpha_0 \notin \mathcal{C}$, Procedure FIND($\alpha_0$) returns $\alpha_{n+1}$ as the unique element of $\mathcal{C}$ in the closed interval defined by $\alpha_0$ and $\alpha_{n+1}$.*

*Proof.* For Min-Min BODO, we also consider two cases: $P_0 - \alpha_0 Q_0 > 0$ and $P_0 - \alpha_0 Q_0 < 0$.

We first suppose that $P_0 - \alpha_0 Q_0 > 0$. We will prove $\alpha_i < \alpha_{i+1}, \forall i \geq 0$ by induction on $i$.

Since $P_0 - \alpha_0 Q_0 > 0$, $\alpha_0 < P_0/Q_0 = \alpha_1$. Thus, our hypothesis is true with $i = 0$.

Suppose our hypothesis is true until $i = k \geq 0$. We have $\alpha_i < \alpha_{i+1}, \forall 0 \leq i \leq k$. Since $\alpha_k < \alpha_{k+1}$, we have $P_{k+1} \geq P_k > 0$ and $Q_k \geq Q_{k+1} > 0$ due to Lemma 13. It leads to $Q_k P_{k+1} \geq P_k Q_{k+1}$ and $Q_k P_{k+1} = P_k Q_{k+1} \iff (P_k, Q_k) \equiv (P_{k+1}, Q_{k+1})$.

If $T_k = 0$ then Procedure *FIND*($\alpha_0$) returns the value $\alpha_{k+1}$. Hence, the sequence $\{\alpha_i\}$ for $i = 0, 1, ..., k+1$ is strictly increasing.

If $T_k < 0$ then we have $(P_k, Q_k) \not\equiv (P_{k+1}, Q_{k+1})$. It implies $Q_k P_{k+1} > P_k Q_{k+1}$. We get

$$\alpha_{k+1} = \frac{P_k}{Q_k} < \frac{P_{k+1}}{Q_{k+1}} = \alpha_{k+2},$$

Thus, in this case, our hypothesis is also true with $i = k+1$. Consequently, $\{\alpha_i\}$ is strictly increasing, $\forall i \geq 0$.

In the following, we will show that $\alpha_{n+1}$ is the unique element of $\mathcal{C}$ in the interval $[\alpha_0, \alpha_{n+1}]$.

Suppose that we have $\alpha^* = (\alpha_0, \alpha_{n+1}) \cap \mathcal{C}$ corresponding to a $\rho$-$NF$ solution $(P^*, Q^*)$. Since the sequence $\{\alpha_i\}_{0 \leq i \leq n+1}$ is strictly increasing, there exists $0 \leq k \leq n$ such that $\alpha^* \in (\alpha_k, \alpha_{k+1}]$.

Since $\alpha^* > \alpha_k$, we have $P^* \geq P_k$ and $Q^* \leq Q_k$ due to Lemma 13. Furthermore, as $(P^*, Q^*)$ is a $\rho$-$NF$ solution, we get

$$\rho \frac{P_k}{P^*} + \frac{Q_k}{Q^*} \geq \rho + 1 \implies \frac{Q_k}{Q^*} - \frac{P_k}{P^*} \geq (\rho + 1)(1 - \frac{P_k}{P^*}) \geq 0, \tag{23}$$

If $\frac{Q_k}{Q^*} = \frac{P_k}{P^*}$ then (23) yields $\frac{Q_k}{Q^*} = \frac{P_k}{P^*} = 1$. It follows that $P_k = P^*$, $Q_k = Q^*$ and $\alpha_{k+1} = \frac{P_k}{Q_k} = \frac{P^*}{Q^*} = \alpha^*$. Since $\alpha_{k+1} = \alpha^* < \alpha_{n+1}$, we have $k \leq n-1$ and $n \geq 1$. As both $(P^*, Q^*)$ and $(P_{k+1}, Q_{k+1})$ are the solutions for minimizing $F_2(\alpha_{k+1}, P, Q)$, we obtain $\rho P_{k+1} + \alpha_{k+1} Q_{k+1} - (\rho P^* + \alpha_{k+1} Q^*) = 0$. Consequently, $T_k = \rho P_{k+1} + \alpha_{k+1} Q_{k+1} - (\rho P_k + \alpha_{k+1} Q_k) = 0$ which leads to a contradiction due to $T_k < 0$, $\forall 0 \leq k \leq n-1$.

Thus, we have

$$\frac{Q_k}{Q^*} > \frac{P_k}{P^*} \implies \alpha^* = \frac{P^*}{Q^*} > \frac{P_k}{Q_k} = \alpha_{k+1},$$

which leads to a contradiction due to the fact that $\alpha^* \leq \alpha_{k+1}$.

Hence, $\alpha_{n+1}$ is the unique element of $\mathcal{C}$ in the interval $[\alpha_0, \alpha_{n+1}]$.

Similarly, in case $P_0 - \alpha_0 Q_0 < 0$, we obtain that the sequence $\{\alpha_i\}$ is strictly decreasing and $\alpha_{i+1}$ is the unique element of $\mathcal{C}$ in the interval $[\alpha_{n+1}, \alpha_0]$. That concludes the proof. □

According to Theorem 8, from $\alpha_0 \in [0, \alpha^{sup}]$ and $\alpha_0 \notin \mathcal{C}$, we can use Procedure *FIND*($\alpha_0$) to find an element $\alpha_{n+1}$ of $\mathcal{C}$ such that there does not exist any other element of $\mathcal{C}$ in the half-open interval $[\alpha_0, \alpha_{n+1})$ (if $\alpha_0 < \alpha_{n+1}$) or $(\alpha_{n+1}, \alpha_0]$ (if $\alpha_0 > \alpha_{n+1}$). Note that the $\rho$-$NF$ solution $(P_n, Q_n)$ obtained by Procedure *FIND*($\alpha_0$) is also a local optimum of $\mathcal{H}_3$ in the such interval (see Appendix B).

For each interval $[\alpha_i, \alpha_j]$ and $(P_i, Q_i) \not\equiv (P_j, Q_j)$ are respectively the solutions for minimizing $F_3(\alpha_i, P, Q)$ and $F_3(\alpha_j, P, Q)$ where $0 \leq \alpha_i < \alpha_j \leq \alpha^{sup}$, we also present some criteria to quickly verify if there does not exist any $\rho$-$NF$ solution in the interval $(\alpha_i, \alpha_j)$ which is different to $(P_i, Q_i)$ and $(P_j, Q_j)$.

**Lemma 14.** *Given an interval $[\alpha_i, \alpha_j]$ defined by $0 \leq \alpha_i < \alpha_j \leq \alpha^{sup}$ and let $(P_i, Q_i)$, $(P_j, Q_j)$ be respectively the solutions for minimizing $F_3(\alpha_i, P, Q)$ and $F_3(\alpha_j, P, Q)$ such that $(P_i, Q_i) \not\equiv (P_j, Q_j)$. Let $\alpha_k = |\frac{\rho(P_i - P_j)}{Q_i - Q_j}|$ and $(P_k, Q_k)$ be a solution for maximizing $F_2(\alpha_k, P, Q)$. If one of the following conditions is satisfied, then there does not exist any $\rho$-NF solution which is different to $(P_i, Q_i)$ and $(P_j, Q_j)$ in the interval $(\alpha_i, \alpha_j)$.*

1. $\alpha_k \in \{\alpha_i, \alpha_j\}$.
2. $\alpha_k \notin \mathcal{C}$ *and* $(P_k, Q_k) \equiv (P_i, Q_i)$ *or* $(P_k, Q_k) \equiv (P_j, Q_j)$.

*Proof.* We first show that $\alpha_k$ is well defined. Since $\alpha_i < \alpha_j$, we have $P_i \leq P_j$, $Q_i \geq Q_j$ due to Lemma 13. Assume that $Q_i = Q_j$. The optimality of $(P_j, Q_j)$ gives

$$\rho P_j + \alpha_j Q_j \leq \rho P_i + \alpha_j Q_i,$$

Since $Q_i = Q_j$, we obtain $P_j \leq P_i$. Thus, $P_i = P_j$ and then $(P_i, Q_i) \equiv (P_j, Q_j)$ which leads to a contradiction.

Hence, $Q_i > Q_j$ and consequently, $\alpha_k$ is well defined.

Since $P_i \leq P_j$ and $Q_i > Q_j$, we have $\alpha_k = \frac{\rho(P_j - P_i)}{Q_i - Q_j}$. Now we show that $\alpha_k \in [\alpha_i, \alpha_j]$.

The optimality of $(P_i, Q_i)$ and $(P_j, Q_j)$ gives

$$\rho P_i + \alpha_i Q_i \leq \rho P_j + \alpha_i Q_j,$$
$$\rho P_j + \alpha_j Q_j \leq \rho P_i + \alpha_j Q_i,$$

Thus, we obtain $\alpha_i \leq \frac{\rho(P_j - P_i)}{Q_i - Q_j} \leq \alpha_j$ which leads to $\alpha_i \leq \alpha_k \leq \alpha_j$.

If $\alpha_k = \alpha_i$ then $\rho P_i + \alpha_i Q_i = \rho P_j + \alpha_i Q_j$. Thus, $(P_i, Q_i$ and $(P_j, Q_j)$ are both solutions for minimizing $F_3(\alpha_i, P, Q)$. Hence, for all $\alpha \in (\alpha_i, \alpha_j)$, $(P_j, Q_j)$ is the solution for minimizing $F_3(\alpha, P, Q)$ as a result of Lemma 13.

Similarly, if $\alpha_k = \alpha_j$, $(P_i, Q_i)$ is the solution for minimizing $F_3(\alpha, P, Q)$ for all $\alpha \in (\alpha_i, \alpha_j)$.

As a result, in case $\alpha_k \in \{\alpha_i, \alpha_j\}$, there does not exist any $\rho$-$NF$ solution which is different to $(P_i, Q_i)$ and $(P_j, Q_j)$ in the interval $(\alpha_i, \alpha_j)$.

Now let $(P_k, Q_k)$ be a solution for minimizing $F_3(\alpha_k, P, Q)$. Without loss of generality, we suppose that $\alpha_k \notin \mathcal{C}$ and $(P_k, Q_k) \equiv (P_i, Q_i)$.

Since $(P_k, Q_k) \equiv (P_i, Q_i)$, $(P_i, Q_i)$ is a solution for minimizing $F_3(\alpha_k, P, Q)$. Since $\alpha_k = \frac{\rho(P_j - P_i)}{Q_i - Q_j}$, we have $\rho P_i + \alpha_k Q_i = \rho P_j + \alpha_k Q_j$. Thus, $(P_j, Q_j)$ is also a solution for minimizing $F_3(\alpha_k, P, Q)$.

Consequently, if $\alpha \in (\alpha_i, \alpha_k)$ then $(P_i, Q_i)$ is the solution for minimizing $F_3(\alpha, P, Q)$ and if $\alpha \in (\alpha_k, \alpha_j)$ then $(P_j, Q_j)$ is the solution for minimizing $F_3(\alpha, P, Q)$. That means there does not exist any $\rho$-$NF$ solution in the interval $(\alpha_i, \alpha_j)$ which is different to $(P_i, Q_i)$ and $(P_j, Q_j)$. □

**Theorem 9.** *Algorithm 6 returns all the elements of $\mathcal{C}$ corresponding to all $\rho$-$NF$ solutions for Min-Min BODO.*

*Proof.* The proof of this theorem is similar to the proof of Theorem 7.

According to Lemma 10, the interval $[0, \alpha^{sup}]$ contains all the elements of $\mathcal{C}$.

Like Theorem 7, let $K_3$ denote the finite number of Pareto-optimal solutions for Min-Min BODO. Consequently, $[0, \alpha^{sup}]$ can be separated by at most $K_3$ subintervals $[c_i, c_j]$ where $0 \leq c_i < c_j \leq \alpha^{sup}$ and $c_i, c_j$ correspond to two different Pareto-optimal solutions. Using Procedure *EXPLORE*$([c_i, c_j])$, each recursive call gives us a $\rho$-$NF$ solution where the corresponding coefficient in the subinterval $[c_i, c_j]$ or show that such subinterval is well explored without any new $\rho$-$NF$ solution. As we use Procedure *FIND*() and Procedure *VERIFY*() in each recursive call, Procedure *EXPLORE*$([0, \alpha^{sup}])$ also terminates after a finite number of iterations. Since Algorithm 6 terminated when the interval $[0, \alpha^{sup}]$ is totally explored, it found all elements of $\mathcal{C}$ corresponding to all $\rho$-$NF$ solutions. □

According to Theorem 7 and 9, Algorithm 6 returns all the elements of $\mathcal{C}$, which correspond to all $\rho$-$NF$ solutions, in a finite number of iterations. In general, estimating its complexity may be difficult. This difficulty comes from the fact that we do not know the BODO problem's context and the expressions for the objective functions. In some particular problems, we can show that Algorithm 6 terminates in polynomial time (for example, the bi-objective assignment problem described in Section 5).

## 5. Illustrative example

In this section, the recursive Newton-like algorithm for determining the $\rho$-$NF$ solution set is tested on an instance of Min-Min BODO as a variant of the assignment problem, called *bi-objective assignment problem* (BOAP). The BOAP can be formally defined as follows. Given

a set of $n$ workers, a set of $n$ jobs, and a $n \times n$ cost matrix $M$ whose positive elements $m_{ij}$ represent the cost assignments of worker $i$ to job $j$, the BOAP finds a one-to-one worker-job assignment (i.e., a perfect bipartite matching) that minimizes simultaneously the total cost and the *max-min distance*, which is the difference between the maximum cost assignment and the minimum one in the assignment solution. Note that the problem minimizing only the max-min distance, called *balanced assignment problem*, has been introduced for finding the solution where the equitable distribution of assignments is important, which is considered in some real-life instances of the assignment problems [11].

As shown in Section 4, for finding the $\rho$-$NF$ solution set, we aim to minimize $F_3(\alpha, P, Q)$ and $G_3(\alpha, P, Q)$ where $\alpha \in [0, \alpha^{sup}]$. In the following, we present linear programming (LP) for minimizing $F_3(\alpha, P, Q)$.

$$\min \rho P + \alpha Q \tag{24a}$$

$$\text{s.t. } P = \sum_{i \in [n], j \in [n]} m_{i,j} x_{i,j} \tag{24b}$$

$$Q = u - l \tag{24c}$$

$$\sum_{j \in [n]} x_{j,i} = \sum_{j \in [n]} x_{i,j} = 1 \qquad \forall i \in [n] \tag{24d}$$

$$u \geq \sum_{j \in [n]} m_{i,j} x_{i,j} \qquad \forall i \in [n] \tag{24e}$$

$$l \leq \sum_{j \in [n]} m_{i,j} x_{i,j} \qquad \forall i \in [n] \tag{24f}$$

$$x_{i,j} \geq 0 \qquad \forall i, j \in [n] \tag{24g}$$

where $[n] = \{1, ..., n\}$ and $x_{i,j}$ represents the assignment between worker $i$ and job $j$ corresponding to the cost $m_{i,j}$. In this formulation, the value of $P$ represents the total cost. To calculate the max-min distance $Q$, we determine the maximum and the minimum cost assignments $u$ and $l$ in the assignment solution. Constraints (24e) allow bounding $u$ from below by the maximum cost assignment in the assignment solution. Similarly, constraints (24f) allow bounding $l$ from above by the minimum cost assignment in the assignment solution. As $Q = u - l$ is minimized, $u$ and $l$ will take the maximum and minimum cost assignments, respectively.

We also present the following LP for minimizing $G_3(\alpha, P, Q)$. This LP contains all the constraints from (24b) to (24g). However, to avoid duplication, they are omitted.

$$\min \rho P + \alpha Q + t \tag{25a}$$

$$\text{s.t. } t \geq P - \alpha Q \tag{25b}$$

$$t \geq -P + \alpha Q \tag{25c}$$

Note that using two additional constraints (25b) and (25c), the parameter $t$ represents the absolute value of $P - \alpha Q$ in $G_3(\alpha, P, Q)$.

We will show that the solutions of LP formulations (24) and (25) are integral, which correspond to the assignment solutions (i.e., perfect bipartite matching). Moreover, for BOAP, determining the $\rho$-$NF$ solution set can be done in polynomial time.

**Lemma 15.** *The solutions of LP formulations* (24) *and* (25) *are integral.*

*Proof.* The objective function of (24) assures that $u$ and $l$ will be equal, respectively, to the maximum and the minimum cost assignments in the optimal solution. Consequently, the solution of LP formulation (24) is always integral since the constraints matrix of (24d) is totally unimodular (e.g., see [5]), and the constraints (24e) and (24f) are bound constraints.

Similarly, the solution of LP formulation (25) is also integral since the constraints (25b) and (25c) are bound constraints. □

**Theorem 10.** *For BOAP, determining the $\rho$-$NF$ solution set can be done in polynomial time.*

*Proof.* Consider an instance of BOAP with a $n \times n$ cost matrix and suppose that there are $K$ distinct Pareto-optimal solutions $(P_i, Q_i)$ where $1 \leq i \leq K$. We first show that $K \leq C^2_{n^2} = \frac{n^2(n^2-1)}{2}$.

Let $m_i^{max}$ and $m_i^{min}$ be the maximum and the minimum cost assignment in the assignment solution corresponding to $(P_i, Q_i)$ then $Q_i = m_i^{max} - m_i^{min}$. For two distinct Pareto-optimal solutions $(P_i, Q_i), (P_j, Q_j)$, we have $Q_i \neq Q_j$ which is equivalent to $m_i^{max} - m_i^{min} \neq m_j^{max} - m_j^{min}$. We have then $(m_i^{max}, m_i^{min}) \not\equiv (m_j^{max}, m_j^{min})$. Thus, the assignment solutions corresponding to $(P_i, Q_i)$, $(P_j, Q_j)$ have distinct pairs of assignments representing the maximum and the minimum cost assignment. Since we have at most $n^2$ distinct assignments, the number of distinct pairs of assignments is at most $C^2_{n^2}$. Thus, $K \leq C^2_{n^2}$.

According to Lemma 6, the iterations of Procedure *FIND*() return distinct Pareto-optimal solutions. Thus, it terminates in a polynomial number of iterations. Consequently, it terminates in polynomial time since LP formulation (24) for minimizing $F_3(\alpha, P, Q)$ can be solved in polynomial time. Note that Procedure *VERIFY*() also terminates in polynomial time.

We know that $[0, \alpha^{sup}]$ can be separated by at most $K$ consecutive subintervals $[c_i, c_j]$ such that $c_i < c_j$ and $c_i, c_j$ correspond to two distinct Pareto-optimal solutions. For Procedure $EXPLORE([0, \alpha^{sup}])$, we use Procedure *FIND*() and Procedure *VERIFY*() in each recursive call, and we have at most $K$ recursive calls. Hence, Procedure $EXPLORE([0, \alpha^{sup}])$ also terminates in polynomial time. □

We now consider an instance of BOAP with 17 workers, 17 jobs, and a cost matrix where its elements are randomly uniformly generated in the range [1,30] (see Appendix C). Let $P, Q$ represent the total cost and the max-min distance in a feasible assignment solution. Note that we have $Q > 0$ in this instance. Furthermore, we consider $\rho = 1$ (i.e., $P$ and

$Q$ are equally important from the CDM's point of view). We present the computational procedures for determining the $\rho$-$NF$ solution set based on Algorithm 6.

**Step 1** We compute a solution $(P_0, Q_0)$ for minimizing $F_3(0, P, Q)$. We obtain $(P_0, Q_0) = (68, 9)$. Then we compute a solution $(P_q, Q_q)$ for minimizing $Q$. We obtain $(P_q, Q_q) = (262, 3)$. Thus, $\alpha^{sup} = P_q/Q_q = \frac{262}{3}$. We also assert that $\alpha^{sup} \notin \mathcal{C}$ by using Procedure *VERIFY*$(\alpha^{sup})$. We aim to explore the interval $[0, \frac{262}{3}]$.

**Step 2** Since $(131, 4)$ is a solution for minimizing $F_3(\alpha^{sup}, P, Q)$, we take $\alpha_1 = |\frac{68-131}{9-4}| = \frac{63}{5}$ and $(80, 7)$ as a solution for minimizing $F_3(\alpha_1, P, Q)$. By using Procedure *VERIFY*$(\alpha_1)$, we obtain that $\alpha_1 \notin \mathcal{C}$. Thus, we use Procedure *FIND*$(\alpha_1)$ to find an element of $\mathcal{C}$. We obtain $c_1 = \frac{80}{7} \in \mathcal{C}$ corresponding to the $\rho$-$NF$ solution $(80, 7)$. We have two subintervals $[0, \frac{63}{5}]$ and $[\frac{80}{7}, \frac{262}{3}]$ to be explored.

**Step 3** For the subinterval $[\frac{80}{7}, \frac{262}{3}]$, we take $\alpha_2 = |\frac{80-131}{7-4}| = 17$ and $(110, 5)$ as a solution for minimizing $F_3(\alpha_2, P, Q)$. By using Procedure *VERIFY*$(\alpha_2)$, we obtain $\alpha_2 \notin \mathcal{C}$. Thus, we use Procedure *FIND*$(\alpha_2)$ to find another element of $\mathcal{C}$. We obtain $c_2 = \frac{131}{4} \in \mathcal{C}$ corresponding to the $\rho$-$NF$ solution $(131, 4)$. According to Lemma 14, there is no $\rho$-$NF$ solution in the interval $[17, \frac{262}{3}]$ except $(131, 4)$. We have two subintervals $[0, \frac{63}{5}]$ and $[\frac{80}{7}, 17]$ to be explored.

**Step 4** For the subinterval $[\frac{80}{7}, 17]$, we take $\alpha_3 = |\frac{80-110}{7-5}| = 15$ and $(80, 7)$ as a solution for minimizing $F_3(\alpha_3, P, Q)$. We have $\alpha_3 \notin \mathcal{C}$. According to Lemma 14, there is no $\rho$-$NF$ solution in this subinterval except $(80, 7)$.

**Step 5** For the subinterval $[0, \frac{63}{5}]$, we take $\alpha_4 = |\frac{68-80}{9-7}| = 6$ and $(71, 8)$ as a solution for minimizing $F_3(\alpha_4, P, Q)$. By using Procedure *VERIFY*$(\alpha_4)$, we obtain $\alpha_4 \notin \mathcal{C}$. Thus, we use Procedure *FIND*$(\alpha_4)$ to find another element of $\mathcal{C}$. We obtain $c_3 = \frac{71}{8} \in \mathcal{C}$ corresponding to the $\rho$-$NF$ solution $(71, 8)$. We have two subintervals $[0, 6]$ and $[\frac{71}{8}, \frac{63}{5}]$ to be explored.

**Step 6** For the subinterval $[\frac{71}{8}, \frac{63}{5}]$, we take $\alpha_5 = |\frac{70-80}{8-7}| = 10$ and $(80, 7)$ as a solution for minimizing $F_3(\alpha_5, P, Q)$. We have $\alpha_5 \notin \mathcal{C}$. According to Lemma 14, there is no $\rho$-$NF$ solution in this subinterval except $(71, 8)$ and $(80, 7)$.

**Step 7** Finally, for the subinterval $[0, 6]$, we take $\alpha_6 = |\frac{68-71}{9-8}| = 3$ and $(68, 9)$ as a solution for minimizing $F_3(\alpha_6, P, Q)$. We have $\alpha_6 \notin \mathcal{C}$. According to Lemma 14, there is no $\rho$-$NF$ solution in this subinterval except $(71, 8)$.

Hence, the interval $[0, \frac{262}{3}]$ is totally explored and we obtain three $\rho$-$NF$ solutions: $(71, 8), (80, 7)$ and $(131, 4)$.

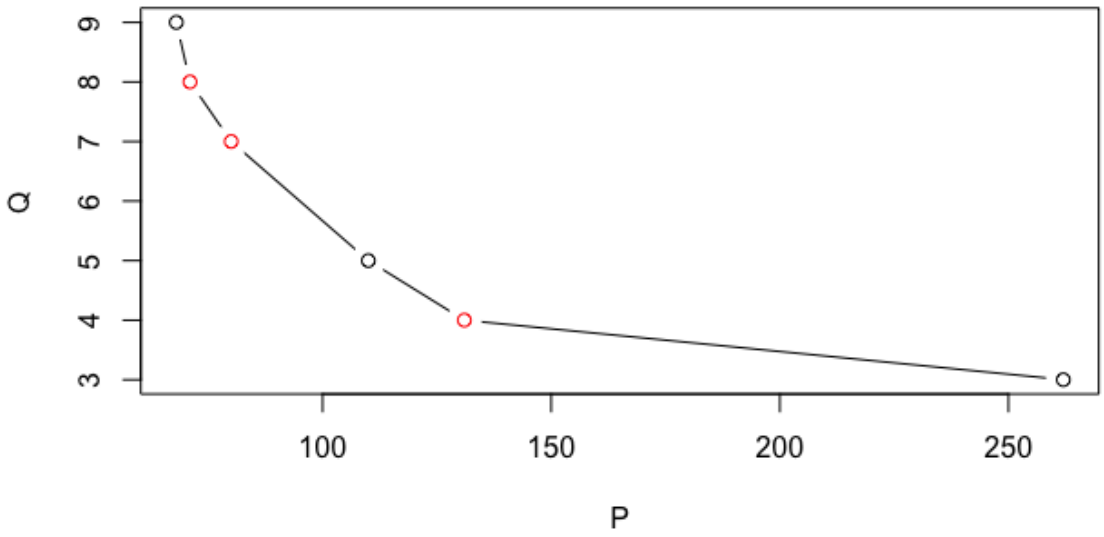


Figure 1: Pareto frontier with $\rho$-$NF$ solutions in red

We can also compute all the Pareto-optimal solutions for this instance as $(68, 9)$,$(71, 8)$, $(80, 7)$,$(110, 5)$,$(131, 4)$, and $(262, 3)$. Figure 1 shows the Pareto frontier with three $\rho$-$NF$ solutions in red. In this instance, we can easily see that the $\rho$-$NF$ solution set is a strict subset of the Pareto set. Moreover, $(131, 4)$ is the $\rho$-$NF$ solution that minimizes the product of $P$ and $Q$ among all the feasible solutions. In general, for Min-Min BODO (resp. Max-Min BODO with $\rho > 1$), we can determine the global optimums of the optimization problem $\mathcal{H}_3 = \max_{(P,Q)\in\mathcal{S}_3} P^\rho Q$ (resp. $\mathcal{H}_2 = \max_{(P,Q)\in\mathcal{S}_2} P^\rho/Q$ with $\rho > 1$) through determining the $\rho$-$NF$ solution set, which represents all its local optimums.

## 6. Conclusion

In this paper, we have generalized the concept of the $\rho$-$NF$ solution for *Bi-Objective Discrete Optimization* (BODO), where the feasible set is discrete, and the two objectives take only positive values. We first discussed the definition and the existence of the $\rho$-$NF$ solutions for BODO. Then, we showed that the $\rho$-$NF$ solution set is a subset of the Pareto set, and this inclusion can be strict. We also designed several algorithms to determine the $\rho$-$NF$ solution set, including a recursive Newton-like algorithm. Finally, an illustrative example of BODO is given.

Future work should clarify the quantitative link between the $\rho$-$NF$ solution set and the Pareto set in more specific BODO cases (for example, the bi-objective combinatorial optimization problems with linear objectives). We are also interested in determining our algorithm's complexity in such cases.

*March 2, 2023*

## References


[1] D. Bertsimas, V.F. Farias, N. Trichakis, The Price of Fairness, *Operations Research*, January-February 2011, 59(1), pp 17-31.

[2] W. Ogryczak, H. Luss, M. Pioro, D. Nace, A. Tomaszewski, Fair Optimization and Networks: A survey, *Journal of Applied Mathematics*, *Hindawi Publishing Coporation* (2014). pp 1-26.

[3] FP Kelly, AK Maullo, DKH Tan, Rate control for communication networks: shadow prices, proportional fairness, and stability, *Journal of the Operational Research Society* 49(3), November 1997.

[4] R. Timothy Marler, J. Singh Arora, The weighted sum method for multi-objective optimization: New insights, *Structural and Multidisciplinary Optimization* 41(6), June 2010. pp 853-862.

[5] I. Heller, C.B. Tompkins, An Extension of a Theorem of Dantzig's, *Annals of Mathematics Studies*, *Princeton University Press* (1956). 247-254(38).

[6] S. Schaible, Fractional programming: applications and algorithms, *European Journal of Operational Research*, 7 (1981), no. 2, pp 111-120.

[7] S. Schaible, T. Ibaraki, Fractional programming, *European Journal of Operational Research*, 12 (1983), no. 2, pp 219-229.

[8] W. Dinkelbach, On nonlinear fractional programming, *Management Sci*, (1967) 492-498(13).

[9] H.W. Kuhn, The Hungarian method for assignment problem, *Naval Research Logistics Quarterly* 2 (1–2): 83–97, March 1955.

[10] James Munkres, Algorithms for the Assignment and Transportation Problems, *Journal of the Society for Industrial and Applied Mathematics* Vol. 5, No. 1, pp. 32-38, March 1957.

[11] S. Martello, W.R. Pulleyblank, P. Toth, D. de Werra, Balanced optimization problems, *Operations Research Letters* Vol. 3, No. 5, August 1984.

[12] Y. Almogy, and O. Levin, A class of Fractional Programming Problems, *Operations Research*, Vol 19, No. 1, (1971).

[13] M.H. Nguyen, M. Baiou, and V.H. Nguyen, Nash balanced assignment problem, *Proceeding of International Symposium on Combinatorial Optimization, May 2022 (ISCO2022).*

[14] M.H. Nguyen, M. Baiou, V.H. Nguyen and T.Q.T. Vo, Generalized Nash Fairness solutions for Bi-Objective Minimization Problems, *Submitted*. https://hal.archives-ouvertes.fr/hal-03890300v2

[15] H. Yarchow, Locally convex spaces, *Stuttgart: B.G. Teubner*, (1981), pp 47-55.

[16] J.R. Isbell, W.H. Marlow, Attrition games. *Naval Res. Logist. Quart 3*, (1956), pp 71-94.

[17] F. Mornati, Pareto optimality in the work of Pareto. *European Journal of Social Sciences*, 51-2 (2013), pp 65-82.

[18] S. Ruzika, and M.M. Wiecek, Approximation methods in multiobjective programming. *Journal of Optimization Theory and Applications*, 126, (2005), pp 473-501.

[19] V. Kreinovich, A. Lakeyev, J. Rohn, and P. Kahl, Computational Complexity and Feasibility of Data Processing and Interval Computations. *Reliable Computing* Vol 4, (1998), pp 405–409. https://doi.org/10.1023/A:1024484203503

[20] I. Das, and J.E. Dennis, Normal-boundary intersection: A new method for generating the Pareto surface in nonlinear multicriteria optimization problems. *SIAM Journal on Optimization*, Vol 8-3, (1998).

[21] K. Ded, A. Pratap, S. Agarwal, and T. Meyarivan, A fast and elitist multiobjective genetic algorithm: NSGA-II. *IEEE Transactions on Evolutionary Computation*, Vol 6-2, (2002).

[22] C, Coello, Multi-Objective Evolutionary Algorithms in Real-World Applications: Some Recent Results and Current Challenges. *Springer: Cham, Switzerland*, (2015), pp 3-18.

[23] J. Koski, Defectiveness of weighting method in multicriterion optimization of structures. *Communications in Applied Numerical Methods*, Vol 1, (1985), pp 333-337.

[24] I. Das, and J.E. Dennis, A closer look at drawbacks of minimizing weighted sums of objectives for Pareto set generation in multicriteria optimization problems. *Structural Optimization*, Vol 14, (1997), pp 63-69.

[25] S. Xue, Determining the Optimal Solution Set for Linear Fractional Programming. *Journal of Systems Engineering and Electronics*, Vol 13, No 3 (2002), pp 40-45.

[26] Z. Zheng, Structure of the Optimal Solution Set and Sufficient Condition for the Unique Optimal

Solution in Linear Programming Problems. *Communication on Applied Mathematics and Computation*, 1992(1), pp 42-46.

[27] X. Zuo, S. Xue, L. Luo, A Way to Find all Optimal Solutions in Linear Programming. *Journal of Systems Engineering and Electronics*, 2002(2), pp 11-16.

[28] C. Bazgan, H. Hugot, and D. Vanderpooten, Solving efficiently the 0–1 multi-objective knapsack problem. *Computers & Operations Research*, Vol 36(1), 2009, pp 260-279.

[29] A.J.V. Skriver, A classification of bicriterion shortest path (bsp) algorithms. *Asia-Pacific J. Oper. Res.*, Vol.17, 2000, pp 199-212.

[30] G.S. Ernando, C.S. Andrea, and J.A. Dario, An exact method for solving the bi-objective minimum diameter-cost spanning tree problem. *RAIRO-Operations Res.*, Vol.49, 2015, pp 143-160.

## Appendix A.

**Proposition 1.** *For Max-Min BODO with $\rho > 1$, the $\rho$-NF solution $(P_n, Q_n)$ obtained by Procedure FIND($\alpha_0$) is a local optimum of $\mathcal{H}_2$ with $\rho > 1$ in the half-open interval $[\alpha_0, \alpha_{n+1})$ (if $\alpha_0 < \alpha_{n+1}$) or $(\alpha_{n+1}, \alpha_0]$ (if $\alpha_0 > \alpha_{n+1}$). More precisely, if $(P', Q') \neq (P_n, Q_n)$ be a solution for maximizing $F_2(\alpha', P, Q)$ where $\alpha'$ in the such interval. Then*

$$\frac{P'^{\rho}}{Q'} < \frac{P_n^{\rho}}{Q_n},$$

*Proof.* Without loss of generality, we suppose that the sequence $\{\alpha_i\}_{0 \leq i \leq n+1}$ is strictly increasing.

If $\alpha' = \alpha_k$ for any $0 \leq k \leq n$ then we have $\frac{P'^{\rho}}{Q'} < \frac{P_n^{\rho}}{Q_n}$ due to Lemma 6 and the fact that $(P', Q') \neq (P_n, Q_n)$.

If $\alpha' \neq \alpha_k$ for $0 \leq k \leq n$, there exists $0 \leq i \leq n$ such that $\alpha' \in (\alpha_i, \alpha_{i+1})$. Thus, $P_i \geq P' \geq P_{i+1}$ and $Q_i \geq Q' \geq Q_{i+1}$ due to Lemma 7.

The optimality of $(P', Q')$ gives

$$\rho P' - \alpha' Q' \geq \rho P_i - \alpha' Q_i \implies \alpha'(Q_i - Q') \geq \rho(P_i - P'), \tag{A.1}$$

Since $\alpha' < \alpha_{i+1} = P_i/Q_i$, $Q_i - Q' \geq 0$ and $P_i - P' \geq 0$, (A.1) yields

$$\frac{P_i}{Q_i}(Q_i - Q') \geq \rho(P_i - P') \implies \frac{Q_i - Q'}{Q_i} \geq \frac{\rho(P_i - P')}{P_i} \implies \frac{P'}{P_i} - \frac{Q'}{Q_i} \geq (\rho - 1)(1 - \frac{P'}{P_i}) \geq 0,$$

Thus, $\frac{P'}{P_i} \geq \frac{Q'}{Q_i} \implies \alpha_{i+1} = \frac{P_i}{Q_i} \leq \frac{P'}{Q'}$. Moreover, the optimality of $(P_{i+1}, Q_{i+1})$ gives

$$\rho P_{i+1} - \alpha_{i+1} Q_{i+1} \geq \rho P' - \alpha_{i+1} Q' \implies \alpha_{i+1}(Q' - Q_{i+1}) \geq \rho(P' - P_{i+1}),$$

Consequently, we obtain

$$\frac{P'}{Q'}(Q' - Q_{i+1}) \geq \rho(P' - P_{i+1}) \implies \frac{Q' - Q_{i+1}}{Q'} \geq \frac{\rho(P' - P_{i+1})}{P'} \implies \rho\frac{P_{i+1}}{P'} \geq \frac{Q_{i+1}}{Q'} + \rho - 1,$$

Using Young's inequality for products, we have

$$\frac{P_{i+1}}{P'} \geq \frac{1}{\rho} \times \frac{Q_{i+1}}{Q'} + \frac{\rho - 1}{\rho} \geq (\frac{Q_{i+1}}{Q'})^{\frac{1}{\rho}} \implies \frac{P_{i+1}^{\rho}}{Q_{i+1}} \geq \frac{P'^{\rho}}{Q'},$$

Since the value of $P_i^{\rho}/Q_i$ is increasing after each iteration of Procedure *FIND*($\alpha_0$) due to Lemma 6 and $(P', Q') \neq (P_n, Q_n)$, we get

$$\frac{P'^{\rho}}{Q'} < \frac{P_n^{\rho}}{Q_n},$$

□

*March 2, 2023*

## Appendix B.

**Proposition 2.** *For Min-Min BODO, the $\rho$-NF solution $(P_n, Q_n)$ obtained by Procedure FIND($\alpha_0$) is a local optimum of $\mathcal{H}_3$ in the half-open interval $[\alpha_0, \alpha_{n+1})$ (if $\alpha_0 < \alpha_{n+1}$) or $(\alpha_{n+1}, \alpha_0]$ (if $\alpha_0 > \alpha_{n+1}$). More precisely, if $(P', Q') \neq (P_n, Q_n)$ be a solution for minimizing $F_3(\alpha', P, Q)$ where $\alpha'$ in the such interval. Then*

$$P'^{\rho}Q' > P_n^{\rho}Q_n,$$

*Proof.* Without loss of generality, we suppose that the sequence $\{\alpha_i\}_{0\leq i\leq n+1}$ is strictly increasing.

If $\alpha' = \alpha_k$ for any $0 \leq k \leq n$ then we have $P'^{\rho}Q' > P_n^{\rho}Q_n$ due to Lemma 12 and the fact that $(P', Q') \neq (P_n, Q_n)$.

If $\alpha' \neq \alpha_k$ for $0 \leq k \leq n$, there exists $0 \leq i \leq n$ such that $\alpha' \in (\alpha_i, \alpha_{i+1})$. Thus, $P_i \leq P' \leq P_{i+1}$ and $Q_i \geq Q' \geq Q_{i+1}$ due to Lemma 13.

The optimality of $(P_{i+1}, Q_{i+1})$ gives

$$\rho P_{i+1} + \alpha_{i+1}Q_{i+1} \leq \rho P' + \alpha_{i+1}Q' \implies \rho(P_{i+1} - P') \leq \alpha_{i+1}(Q' - Q_{i+1}),$$

In addition, $\alpha_{i+1} = \frac{P_i}{Q_i} \leq \frac{P'}{Q'}$ due to $P_i \leq P'$ and $Q_i \geq Q'$. Thus, we obtain

$$\rho(P_{i+1} - P') \leq \frac{P'}{Q'}(Q' - Q_{i+1}) \implies \frac{\rho(P_{i+1} - P')}{P'} \leq \frac{Q' - Q_{i+1}}{Q'} \implies \rho\frac{P_{i+1}}{P'} + \frac{Q_{i+1}}{Q'} \leq \rho + 1,$$

Using Young's inequality for products, we have

$$1 \geq \frac{\rho}{\rho+1} \times \frac{P_{i+1}}{P'} + \frac{1}{\rho} \times \frac{Q_{i+1}}{Q'} \geq (\frac{P_{i+1}}{P'})^{\frac{\rho}{\rho+1}}(\frac{Q_{i+1}}{Q'})^{\frac{1}{\rho+1}} \implies P'^{\rho}Q' \geq P_{i+1}^{\rho}Q_{i+1},$$

Since the value of $P_i^{\rho}/Q_i$ is decreasing after each iteration of Procedure *FIND*($\alpha_0$) due to Lemma 12 and $(P', Q') \neq (P_n, Q_n)$, we get $P'^{\rho}Q' > P_n^{\rho}Q_n$. □

## Appendix C. Cost matrix for the instance of BOAP used in Section 5

$$M = \begin{pmatrix}
1 & 29 & 14 & 22 & 12 & 27 & 1 & 11 & 9 & 10 & 4 & 1 & 3 & 28 & 5 & 9 & 5 \\
23 & 23 & 12 & 24 & 20 & 17 & 21 & 9 & 19 & 17 & 16 & 16 & 12 & 19 & 20 & 19 & 10 \\
21 & 10 & 29 & 3 & 28 & 14 & 6 & 15 & 26 & 17 & 4 & 12 & 2 & 29 & 20 & 25 & 1 \\
25 & 7 & 21 & 17 & 20 & 28 & 20 & 17 & 19 & 11 & 24 & 17 & 20 & 2 & 15 & 7 & 26 \\
19 & 3 & 14 & 10 & 18 & 20 & 7 & 9 & 17 & 11 & 16 & 19 & 8 & 20 & 13 & 22 & 23 \\
17 & 1 & 7 & 3 & 12 & 14 & 16 & 11 & 24 & 27 & 15 & 23 & 8 & 29 & 12 & 10 & 22 \\
10 & 1 & 8 & 15 & 28 & 26 & 6 & 15 & 12 & 10 & 27 & 21 & 9 & 22 & 9 & 9 & 17 \\
19 & 15 & 10 & 17 & 9 & 24 & 21 & 6 & 17 & 21 & 13 & 28 & 15 & 8 & 27 & 21 & 29 \\
12 & 3 & 20 & 4 & 24 & 13 & 13 & 18 & 23 & 23 & 18 & 23 & 9 & 10 & 15 & 28 & 28 \\
15 & 21 & 22 & 5 & 20 & 2 & 4 & 29 & 5 & 18 & 8 & 29 & 2 & 1 & 17 & 29 & 10 \\
16 & 9 & 16 & 26 & 6 & 20 & 15 & 15 & 11 & 25 & 13 & 8 & 26 & 15 & 18 & 2 & 3 \\
1 & 8 & 14 & 5 & 1 & 18 & 6 & 4 & 6 & 15 & 15 & 15 & 12 & 15 & 22 & 17 & 12 \\
7 & 9 & 2 & 25 & 14 & 1 & 25 & 13 & 28 & 6 & 16 & 3 & 4 & 5 & 28 & 28 & 1 \\
21 & 12 & 6 & 8 & 25 & 6 & 26 & 8 & 17 & 17 & 13 & 28 & 16 & 24 & 27 & 5 & 18 \\
2 & 27 & 10 & 25 & 6 & 24 & 4 & 29 & 26 & 28 & 25 & 4 & 24 & 16 & 1 & 17 & 22 \\
22 & 21 & 29 & 8 & 24 & 17 & 19 & 3 & 22 & 9 & 13 & 4 & 14 & 14 & 3 & 29 & 9 \\
27 & 27 & 6 & 18 & 23 & 11 & 4 & 16 & 19 & 15 & 5 & 15 & 3 & 8 & 22 & 3 & 28
\end{pmatrix}$$