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**NODAL-BOUNDARY FINITE-ELEMENT METHOD FOR THE SIGNORINI PROBLEM  
IN TWO DIMENSIONS**

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**ABSTRACT**

*The various numerical solution methods targeting the Signorini problem differ in their treatment of the complementarity boundary conditions. In the finite element method, the needed application of a Newton impact-law either produces chattering or induces energy dissipation while Nitsche's and penalty methods do not strictly enforce the impenetrability condition with the obstacle. Also, the up-wind finite volume approach does not conserve energy for multiple spatial dimensions. Hence, a new methodology called the Nodal Boundary method is proposed for treatment of Signorini conditions within the framework of Finite Element. Similarly to mass redistribution techniques, this method defines the motion of contact nodes based on internal nodes. However, this method does not require computation of a modified mass matrix. Applications of the derived scheme on different settings of the Signorini problem are presented. It is found that, while energy is not always conserved, this method allows for existence of periodic solutions together with the elimination of chattering at contact.*

**INTRODUCTION**

Periodic solutions of the Signorini problem are of interest for vibration analysis of structures in unilateral contact [1]. The Signorini conditions describe the dynamics of a deformable structure prone to frictionless unilateral contact with a rigid obstacle

via two sets of mutually exclusive boundary conditions: active contact (the structure is in contact with the obstacle) and inactive contact (the structure may move freely). Mathematically, the Signorini conditions form a complementarity problem, and exact solutions could be obtained only for a few specific cases [1, 2]. The finite element method (FEM) has been commonly used to solve the Signorini problem numerically [3]. However, applying the Signorini boundary conditions in the FEM framework leads to an ill-posed problem [4]. Hence, there exist diverse numerical schemes for treatment of the Signorini conditions in FEM. In this context, schemes using a Newtonian impact law induce questionable chattering for energy conserving schemes and theoretical sticking at contact leads to unwanted dissipation of energy [2]. Thus, schemes using Newtonian impact law do not allow for periodic solutions consisting of sticking at contact, which exist for the analytical solution [5]. In turn, penalty [2] and Nitsche's [6] methods do not enforce impenetrability of the obstacle exactly. In contrast, the up-wind finite-volume-method (FVM) applied to the one dimensional case of the Signorini problem eliminates chattering while conserving energy [5]. However, its extension to multiple spatial dimensions results in energy dissipation [7]. At last, the mass redistribution method [8], while allowing for energy conservation and elimination of chattering, requires solving a constrained optimization problem to form the reduced mass matrix. We propose here the *Nodal Boundary Treatment* of Signorini boundary conditions in FEM. Particularly, this method allows for existence of periodic solutions while eliminating chat-

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tering. Similarly to [8], the proposed method requires redistribution of masses but does not require solution of a constrained optimization problem and may apply on already existing mass and stiffness matrices.

## 1 Signorini problem with gravity

The Signorini problem in dynamics is described by a system of hyperbolic partial differential equations defining the contributions of elastic forces and inertial forces within the structure in the spatial domain  $\Omega$  and time  $t$ :

$$\rho \mathbf{u}_{tt} - \text{div}(\boldsymbol{\sigma}(\mathbf{u})) = \mathbf{f}, \quad (\mathbf{x}, t) \in \Omega \times [0, \infty) \quad (1)$$

where  $\mathbf{u}$ ,  $\rho$ , and  $\mathbf{f}$  describe the displacement field, density, and external forcing, respectively. The stress field is consistent with linear elasticity for isotropic materials. Homogeneous Neumann boundary conditions are applied on the portion  $\Gamma_N$  of the boundary not prone to contact, and Homogeneous Dirichlet conditions on the boundary  $\Gamma_D$ . The portion  $\Gamma_C$  is prone to unilateral contact and treated using the Signorini complementarity conditions

$$\begin{aligned} 0 \leq g - \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{n}, \quad \sigma_n(\mathbf{u}) \leq 0 \\ (g - \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{n})\sigma_n(\mathbf{u}) = 0 \quad \mathbf{x} \in \Gamma_C \end{aligned} \quad (2)$$

where  $g$  defines the distance between the non-deformed boundary  $\Gamma_C$  and the rigid obstacle at rest. The contact pressure is captured by  $\sigma_n(\mathbf{u}) = \mathbf{n}^\top \boldsymbol{\sigma}(\mathbf{u}) \mathbf{n}$  where  $\mathbf{n}$  is the outward normal to  $\Gamma_C$ .

## 2 Nodal Boundary Method

In FEM, enforcement of the Signorini conditions (2) results in an ill-posed problem as there exist infinitely many possible values  $\sigma_n$  satisfying (2) in the FE discretized system [2, 8]. A common work around is to explicitly add an impact law [2, 3]. In the present work, to make the problem well-posed,  $\sigma_n$  is treated as a function of the nodal displacement  $\mathbf{u}$ . This function,  $\sigma_n(\mathbf{u})$ , us the FE approximation of the stress function  $\boldsymbol{\sigma}(\mathbf{u})$  on the boundary  $\Gamma_C$ .

Here, the displacement field is approximated using shape functions and attendant nodal quantities stored in  $\mathbf{u}(t) \in \mathbb{R}^N$ . Test functions with nodal quantities stored in  $\mathbf{w}$  are used to enforce equilibrium in a weak fashion. In the nodal boundary method, we distinguish between nodal displacements  $\mathbf{u}_C$  prone to unilateral contact on  $\Gamma_C$  and the remainder of the nodal displacements  $\mathbf{u}_O$  such that  $\mathbf{u} = (\mathbf{u}_O \ \mathbf{u}_C)^\top$  (with corresponding  $N = N_O + N_C$ ). The same convention is applied to the test functions such that

$\mathbf{w} = (\mathbf{w}_O \ \mathbf{w}_C)^\top$ . The FEM formulation of (1), with the homogeneous boundary conditions on  $\Gamma_N$  applied, reads

$$\mathbf{w}^\top (\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} - \mathbf{f}) + \mathbf{w}_C^\top \mathbf{f}_C = \mathbf{0}, \quad \forall \mathbf{w}. \quad (3)$$

Here, the quantities  $\mathbf{M}$ ,  $\mathbf{K}$  and  $\mathbf{f}$  are the finite element mass matrix, stiffness matrix and external force vector, respectively. Moreover, the vector  $\mathbf{f}_C$  represents the contact-force on the boundary  $\Gamma_C$ .

In the Nodal Boundary Method, we impose that  $\mathbf{f}_C(\mathbf{u})$  is a linear function in  $\mathbf{u}$  given by the FEM approximation of the stress  $\boldsymbol{\sigma}_n$  on  $\Gamma_C$ . Next, the displacement of contact nodes  $\mathbf{u}_C$  is dictated by the internal nodes  $\mathbf{u}_O$  via Condition (2). Here, we may use the the ‘‘max’’ operator formulation of Condition (2) to express this relation [6]

$$\mathbf{F}(\mathbf{u}_C, \mathbf{u}_O) \equiv \mathbf{f}_C(\mathbf{u}_O, \mathbf{u}_C) + \mathbf{max}(\mathbf{0}, g\mathbf{1} - \mathbf{u}_C - \mathbf{f}_C(\mathbf{u}_O, \mathbf{u}_C)) = \mathbf{0} \quad (4)$$

where  $\mathbf{1}$  is a vector with unit entries. The solution to (4) is unique for given  $\mathbf{u}_O$  and  $g$  and can be found numerically [8]. In what follows, it is useful to present the solution to  $\mathbf{F}(\mathbf{u}_C, \mathbf{u}_O) = \mathbf{0}$  explicitly

$$\begin{aligned} \mathbf{u}_C &= \mathbf{A}(\mathbf{u}_O, g)\mathbf{u}_O + g\mathbf{d}(\mathbf{u}_O, g) \\ \dot{\mathbf{u}}_C &= \mathbf{A}(\mathbf{u}_O, g)\dot{\mathbf{u}}_O \\ \ddot{\mathbf{u}}_C &= \mathbf{A}(\mathbf{u}_O, g)\ddot{\mathbf{u}}_O \end{aligned} \quad (5)$$

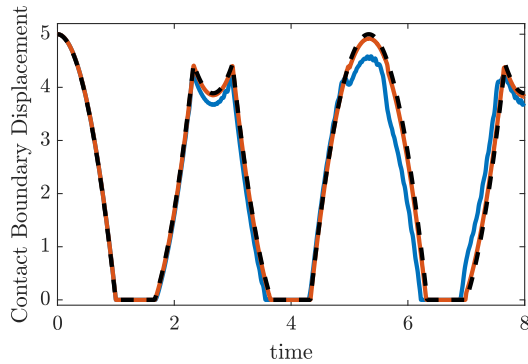
Here,  $\mathbf{A}(\mathbf{u}_O, g)$  and  $\mathbf{d}(\mathbf{u}_O, g)$  are non-smooth functions in  $\mathbf{u}_O$ . After determining  $\mathbf{u}_C(\mathbf{u}_O, g)$  using (5), the system of ODEs in (3) is over-determined since it includes ODEs in  $\mathbf{u}_C$ . To render the system of ODEs from (3) well-posed, the corresponding  $\mathbf{w}_C$  functions are substituted with a linear expression in  $\mathbf{w}_O$ . In fact, we define  $\mathbf{w}_C$  such that the total energy is conserved during active or inactive contact motion. This implies that  $\mathbf{w}_C$  behaves in a similar manner to the velocities  $\dot{\mathbf{u}}_C$ , ie  $\mathbf{w}_C = \mathbf{A}(\mathbf{u}_O, g)\mathbf{w}_O$ . Application of the latter together with with (5) to the ODE (3) for all arbitrary  $\mathbf{w}_O$  leads to a system of  $N_O$  non-smooth ODEs in  $\mathbf{u}_O$  exclusively

$$(\mathbf{I} \ \mathbf{A}^\top) \left( \mathbf{M} \begin{pmatrix} \mathbf{I} \\ \mathbf{A} \end{pmatrix} \ddot{\mathbf{u}}_O + \mathbf{K} \begin{pmatrix} \mathbf{I} \\ \mathbf{A} \end{pmatrix} \mathbf{u}_O + \mathbf{K} \begin{pmatrix} \mathbf{0} \\ g\mathbf{d} \end{pmatrix} - \mathbf{f} \right) = \mathbf{0}. \quad (6)$$

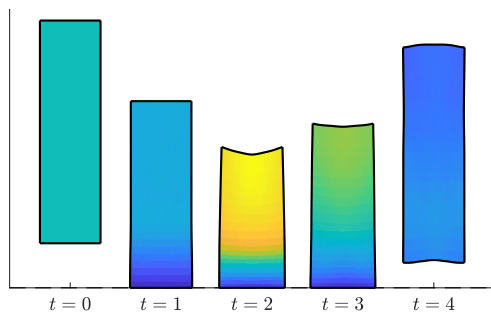
where  $\mathbf{I}$  is the identity matrix of size  $N_O \times N_O$ . The resulting non-smooth ODEs (6) can be solved using explicit or implicit schemes.

## 3 Results

The NBM has been implemented on three examples: the bouncing bar (Figure 1), the bouncing plate (Figure 2) and the



**FIGURE 1.** CONTACT NODE DISPLACEMENT FOR BOUNCING BAR PROBLEM WITH THE NBM USING 40 ELEMENTS. BLUE: QUADRATIC POLYNOMIALS. ORANGE: ORDER 4 POLYNOMIALS. BLACK: EXACT SOLUTION

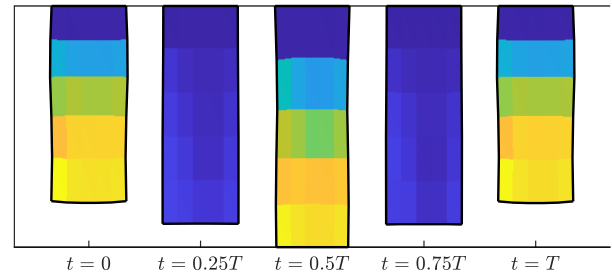


**FIGURE 2.** SOLUTION TO THE BOUNCING PLATE PROBLEM WITH THE NBM USING 400 ELEMENTS AND LINEAR SHAPE FUNCTIONS. COLOUR GRADIENT REPRESENTS  $\|\mathbf{u}\|$ .

fixed (“cantilever”) plate (Figure 3). In the case of the bouncing bar, the NBM was shown to converge to the exact solution in a similar rate to schemes utilizing the dissipative Newton impact law (zero coefficient of restitution). Next, for the case of the bouncing plate, the NBM shows a less dispersive behaviour than Nitsche’s and Newton’s impact law while converging to the same solution. At last, a periodic solution for the case of the cantilever plate was obtained using NBM and the harmonic balance method [9].

#### 4 Conclusions

The NBM for treatment of the Signorini complementarity conditions in the FE framework in multiple spatial dimensions were presented. The classical impact law was replaced by application of Signorini conditions both on the nodal displacements and the test functions. Future applications of this scheme will be in non-smooth modal analysis where families of periodic solutions of the Signorini problem must be detected.



**FIGURE 3.** PERIODIC SOLUTION OF FEM APPROXIMATION OF FIXED PLATE IN CONTACT USING NBM: 25 ELEMENTS AND LINEAR SHAPE FUNCTIONS. COLOUR GRADIENT REPRESENTS  $\|\mathbf{u}\|$ .  $T$  - PERIOD OF MOTION

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