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DEGENERATE COMPLEX HESSIAN EQUATIONS ON COMPACT HERMITIAN MANIFOLDS

VINCENT GUEDJ & CHINH H. LU

ABSTRACT. In this note we provide uniform a priori estimates for solutions to degenerate complex Hessian equations on compact hermitian manifolds. Our approach relies on the corresponding a priori estimates for Monge-Ampère equations; it provides an extension as well as a short alternative proof to results of Dinew-Kołodziej, Kołodziej-Nguyen and Guo-Phong-Tong.

À la mémoire de Jean-Pierre Demailly.

INTRODUCTION

Let X be a compact complex manifold equipped with a hermitian form ω_X . We fix $1 \leq k \leq n = \dim_{\mathbb{C}} X$, ω another hermitian form, and μ_X a volume form on X . The complex Hessian equation is

$$(\omega + dd^c \varphi)^k \wedge \omega_X^{n-k} = c\mu_X,$$

where $c > 0$ is a positive constant and φ (the unknown) is a smooth (ω, ω_X, k) -subharmonic function, i.e. $(\omega + dd^c \varphi)^j \wedge \omega_X^{n-j} \geq 0$ for all $1 \leq j \leq k$.

When $k = 1$ this reduces to the Laplace equation. When $k = n$ this yields the complex Monge-Ampère equation which was famously solved by Yau [Yau78] when ω is Kähler, and by Tosatti-Weinkove [TW10] in the hermitian setting.

From now on we assume that $1 < k < n$. A unique solution has been provided by Dinew-Kołodziej [DK17] when $\omega = \omega_X$ is Kähler; the hermitian setting has been resolved independently by Székelyhidi [Szek18] and Zhang [Zh17].

It is quite natural to consider such equations in degenerate settings, allowing the measure $\mu_X = f\omega_X^n$ to be merely absolutely continuous with respect to the volume form ω_X^n . Such degenerate complex Hessian equations have been intensively studied in the past fifteen years (see [Blo05, Lu13, DK14, DL15, LN15, Lu15, KN16, Char16, GN18, DDT19, BZ20, GP22] and the references therein).

In this note we go one step further by allowing the form ω to degenerate as well: we no longer assume it is positive, we merely ask that it is *big*, in analogy with the corresponding notion from algebraic geometry [BEGZ10], i.e. we assume there is an ω -psh function ρ with analytic singularities such that $\omega + dd^c \rho$ dominates a hermitian form. We let Ω denote the Zariski open set where ρ is smooth.

One can not expect any longer that (ω, ω_X, k) -subharmonic functions are smooth in X , but we show in Section 1.2 how to define the Hessian operator $H_k(f) = (\omega + dd^c \varphi)^k \wedge \omega_X^{n-k}$ for such functions φ which are *continuous* in Ω and globally bounded on X .

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In this context we obtain the following:

Theorem A. *Let ω be a semi-positive and big form and $f \in L^p(\omega_X^n)$ with $p > n/k$ and $\int_X f \omega_X^n > 0$. There exists $c > 0$ and φ a bounded (ω, ω_X, k) -subharmonic function, continuous in Ω , such that*

$$(\omega + dd^c \varphi)^k \wedge \omega_X^{n-k} = cf \omega_X^n.$$

This result extends the main result of [KN16]. It applies in particular to the case when $\pi : X \rightarrow Y$ is the desingularization of a compact hermitian variety Y , and $\omega = \pi^* \omega_Y$ is the pull-back of a hermitian form ω_Y on Y .

As is often the case in similar contexts, the main difficulty lies in establishing uniform a priori estimates. This is the content of our second main result:

Theorem B. *Let ω_0, ω be semi-positive and big forms such that $\omega_0 \leq \omega$. Let φ be a smooth (ω, ω_X, k) -subharmonic function solution to*

$$(\omega + dd^c \varphi)^k \wedge \omega_X^{n-k} = f \omega_X^n,$$

where $f \in L^p(dV_X)$, $p > n/k$. Then

$$\text{Osc}_X(\varphi) \leq C,$$

where $C > 0$ depends on ω_0, n, k, p and an upper bound for $\|f\|_{L^p}$.

The approach of Kolodziej-Nguyen relies on a pluripotential theory for degenerate complex Hessian equations on hermitian manifolds (mimicking the corresponding results obtained by these authors and Dinew for the Monge-Ampère equation [DK12], [KN15], [KN19], [Ng16]). An alternative PDE proof has been recently provided by Guo-Phong-Tong-Wang in [GPT21, GPTW21] when X is Kähler, and by Guo-Phong [GP22] when X is hermitian and ω is a hermitian form.

We provide in this note a proof which relies directly on the corresponding results for the complex Monge-Ampère operator. Given φ a solution of the k -Hessian equation with density f , we show that the ω -psh envelope of φ solves a Monge-Ampère equation whose density is bounded from above by $f^{n/k}$. A lower bound on φ thus follows from the L^∞ -a priori estimates for the Monge-Ampère equation established in [GL21].

Our method is new and also of interest when X is Kähler; it applies to more general densities (see Remark 3.2); in particular the results of Theorem A and Theorem B hold when f merely satisfies

$$\int_X f^{\frac{n}{k}} |\log f|^n (\log |\log f|)^{n+\delta} < +\infty,$$

for some $\delta > 0$. Radial examples show that this condition is almost optimal.

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1. PRELIMINARIES

In the whole article we let (X, ω_X) denote a compact complex manifold of complex dimension $n \in \mathbb{N}^*$, equipped with a fixed hermitian form ω_X . We use real differential operators $d = \partial + \bar{\partial}$, $d^c = i(\bar{\partial} - \partial)$, so that $dd^c = 2i\partial\bar{\partial}$.

We also fix a semi-positive $(1, 1)$ -form ω which is *big*:

Definition 1.1. We say that ω is big if there exists an ω -psh function ρ such that $\omega + dd^c \rho$ dominates a hermitian form.

It follows from Demailly's regularization theorem [Dem94] that one can further assume that ρ has analytic singularities, in particular $\omega + dd^c \rho$ is a hermitian form in some Zariski open set that we usually denote by Ω in what follows.

An important source of examples of big forms is provided by the following construction: if V is a compact complex space endowed with a hermitian form ω_V , and $\pi : X \rightarrow V$ is a resolution of singularities, then $\omega = \pi^* \omega_V$ is big.

1.1. (ω, ω_X, k) -subharmonic functions.

1.1.1. *Definition.* Let $U \subset X$ be an open subset and γ a hermitian form in U . A function $h \in \mathcal{C}^2(U, \mathbb{R})$ is called γ -harmonic if

$$dd^c h \wedge \gamma^{n-1} = 0 \text{ pointwise in } U.$$

We let $H_\gamma(U)$ denote the set of all γ -harmonic functions in U . A function $u : U \rightarrow \mathbb{R} \cup \{-\infty\}$ is γ -subharmonic if it is upper semicontinuous and for every $D \subset U$ and every $h \in H_\gamma(D)$ the following implication holds

$$u \leq h \text{ on } \partial D \implies u \leq h \text{ in } D.$$

As shown in [HL13, Section 9] the γ -subharmonicity can be equivalently defined using viscosity theory: u is γ -subharmonic in U if and only if the inequality $dd^c u \wedge \gamma^{n-1}$ holds in the viscosity sense in U . The latter condition means that for every $x_0 \in U$ and every \mathcal{C}^2 function χ defined in a neighborhood V of x_0 , the following implication holds:

$$[u(x) \leq \chi(x), \forall x \in V \text{ and } u(x_0) = \chi(x_0)] \implies dd^c \chi \wedge \gamma^{n-1} \geq 0 \text{ at } x_0.$$

When $u \in \mathcal{C}^2(U)$, it is γ -subharmonic if and only if $dd^c u \wedge \gamma^{n-1} \geq 0$ pointwise.

Definition 1.2. We let $SH_\gamma(U)$ denote the set of all γ -subharmonic functions in U that are locally integrable.

If $u \in SH_\gamma(U)$ then, by [HH72, Théorème 1, p.136], $dd^c u \wedge \gamma^{n-1} \geq 0$ in the sense of distributions. Conversely, by [Lit83, Theorem A], the latter condition implies that u coincides a.e. with a unique function $\hat{u} \in SH_\gamma(U)$ defined by

$$\hat{u}(x) := \operatorname{ess\,lim\,sup}_{y \rightarrow x} u(y) := \lim_{r \searrow 0} \operatorname{ess\,sup}_{B(x,r)} u,$$

where $\operatorname{ess\,sup}_{B(x,r)} u$ is the essential supremum of u in the ball $B(x, r)$.

Definition 1.3. A function $\varphi : U \rightarrow \mathbb{R} \cup \{-\infty\}$ is quasi- γ -subharmonic if locally $\varphi = u + \rho$ where u is γ -subharmonic and ρ is smooth.

We say that a function φ is (ω, γ) -subharmonic if it is quasi- γ -subharmonic and $(\omega + dd^c \varphi) \wedge \gamma^{n-1} \geq 0$ in the sense of distributions on X .

We let $SH_\gamma(U, \omega)$ denote the set of all (ω, γ) -subharmonic functions in U .

If $u \in \mathcal{C}^2(U, \mathbb{R})$ then $u \in SH_\gamma(U, \omega)$ if and only if $(\omega + dd^c u) \wedge \gamma^{n-1} \geq 0$ pointwise. In general this inequality holds in the viscosity sense:

Lemma 1.4. *If $\varphi \in SH_\gamma(U, \omega)$ and χ is a \mathcal{C}^2 function such that $\varphi - \chi$ attains a local maximum at $x_0 \in U$, then $(\omega + dd^c \chi) \wedge \gamma^{n-1} \geq 0$ at x_0 .*

Proof. Fix $a > 1$. Let ρ be a quadratic function such that $dd^c \rho = a\omega(x_0)$. By continuity of ω , we have $dd^c \rho \geq \omega$ in a small ball B around x_0 . The function $\rho + \varphi$ is then γ -subharmonic in B and the function $\rho + \varphi - (\rho + \chi)$ attains a local maximum at x_0 . It thus follows that, at x_0 ,

$$(a\omega + dd^c \chi) \wedge \gamma^{n-1} = dd^c(\rho + \chi) \wedge \gamma^{n-1} \geq 0.$$

The conclusion follows by letting $a \searrow 1$. \square

Fix an integer $1 \leq k \leq n$, and let $\Gamma_k(\omega_X)$ denote the set of all $(1, 1)$ -forms α such that

$$\alpha^j \wedge \omega_X^{n-j} > 0, \quad j = 1, \dots, k.$$

By Gårding's inequality [Gar59], if $\alpha, \alpha_1, \dots, \alpha_{k-1} \in \Gamma_k(\omega_X)$ then

$$\alpha \wedge \alpha_1 \wedge \dots \wedge \alpha_{k-1} \wedge \omega_X^{n-k} > 0.$$

In particular, $\alpha_1 \wedge \dots \wedge \alpha_{k-1} \wedge \omega_X^{n-k}$ is a strictly positive $(n-1, n-1)$ -form on X . By [Mic82], one can write $\alpha_1 \wedge \dots \wedge \alpha_{k-1} \wedge \omega_X^{n-k} = \gamma^{n-1}$, for some hermitian form γ . We let $F_k(\omega_X)$ denote the set of all such hermitian forms.

Definition 1.5. A function $\varphi : U \rightarrow \mathbb{R} \cup \{-\infty\}$ is (ω, ω_X, k) -subharmonic if it is (ω, γ) -subharmonic in U for all $\gamma \in F_k(\omega_X)$.

We let $SH_{\omega_X}(U, \omega, k)$ denote the set of all (ω, ω_X, k) -subharmonic functions on U which are locally integrable in U .

If u is of class \mathcal{C}^2 in U then, by [Gar59], u is (ω, ω_X, k) -subharmonic if and only if $\omega + dd^c u \in \Gamma_k(\omega_X)$. We refer the reader to [Blo05] and [KN16, Section 2.3] for basic properties of (ω, ω_X, k) -sh functions; they easily extend to our context.

Since ω_X is fixed once and for all in this note, we let $SH(X, \omega, k)$ denote the set of (ω, ω_X, k) -sh functions; they form a nested sequence

$$SH(X, \omega) \supset \dots \supset SH(X, \omega, k-1) \supset SH(X, \omega, k) \supset \dots \supset PSH(X, \omega),$$

where $SH(X, \omega) = SH(X, \omega, 1)$ is the set of ω -sh functions, while $PSH(X, \omega) = SH(X, \omega, n)$ is the set of ω -psh functions, for which $\omega + dd^c \varphi \geq 0$ is a positive current of bidegree $(1, 1)$ (see [GZ17, Chapter 8]).

Observe that $SH(X, \omega, k) \subset SH(X, \omega', k)$ if $\omega \leq \omega'$. Since $\omega \leq A\omega_X$ for some $A > 0$, any (ω, ω_X, k) -sh function is also $(A\omega_X, 1)$ -sh. We thus have the following compactness result (see [KN16, Lemma 3.3], [GN18, Section 9]).

Proposition 1.6. *The set of normalized (ω, ω_X, k) -subharmonic functions is compact in the L^1 -topology. In particular, there exists $C > 0$ such that for all $\varphi \in SH(X, \omega, k)$ normalized by $\sup_X \varphi = 0$,*

$$\int_X (-\varphi) \omega_X^n \leq C.$$

1.1.2. *Smooth approximation.*

Theorem 1.7. *If ω is hermitian, any (ω, k) -subharmonic function is the pointwise limit of a decreasing sequence of smooth (ω, k) -subharmonic functions.*

This result is a slight extension of [KN16, Lemma 3.20], which itself extends [LN15, Theorem 1.2] from the Kähler to the hermitian setting. We propose below an alternative and more direct proof.

Proof. Let u be a smooth function on X and let f be a smooth positive density such that

$$(\omega + dd^c u)^k \wedge \omega_X^{n-k} \leq f \omega_X^n.$$

For $j \geq 1$, we solve the complex Hessian equation

$$(\omega + dd^c \varphi_j)^k \wedge \omega_X^{n-k} = e^{j(\varphi_j - u)} f \omega_X^n.$$

As shown in [KN16], the estimates established in [Szek18] provide a smooth solution φ_j which is (ω, k) -subharmonic on X . We want to prove that φ_j converges uniformly to $P_{\omega, k}(u)$, the largest (ω, k) -subharmonic function lying below u . The theorem follows from this fact, as shown in [LN15, KN16].

The classical maximum principle ensures that $\varphi_j \leq u$, hence $\varphi_j \leq P_{\omega,k}(u)$. Fix now $\varphi \in SH(X, \omega, k)$ such that $\varphi \leq u$. Let $x_0 \in X$ be a point where the function $\varphi - (1 + 1/j)\varphi_j$ attains its maximum on X . By definition φ belongs to $SH_\gamma(X, \omega)$, where γ is the hermitian form such that

$$\gamma^{n-1} = (\omega + dd^c \varphi_j)^{k-1} \wedge \omega_X^{n-k}.$$

From now on the computations are made at x_0 . It follows from Lemma 1.4 that

$$(\omega + (1 + 1/j)dd^c \varphi_j) \wedge \gamma^{n-1} \geq 0.$$

From this and Gårding's inequality [Gar59] we obtain

$$\begin{aligned} \frac{j+1}{j}(\omega + dd^c \varphi_j)^k \wedge \omega_X^{n-k} &\geq \frac{1}{j}\omega \wedge (\omega + dd^c \varphi_j)^{k-1} \wedge \omega_X^{n-k} \\ &\geq \frac{1}{j}e^{\frac{j(k-1)}{k}(\varphi_j - u)} f^{(k-1)/k} g^{1/k} \omega_X^n, \end{aligned}$$

where $g\omega_X^n = \omega^k \wedge \omega_X^{n-k}$. Let $b \in (0, 1)$ be a positive constant such that $g \geq b^k f$ on X . The above inequality yields

$$(1 + j)e^{j(\varphi_j - u)/k} \geq b,$$

hence

$$\varphi_j - u \geq \frac{k(\log b - \log(1 + j))}{j}.$$

Thus, for some uniform constant $C \geq 1$ we have

$$\left(1 + \frac{1}{j}\right) \varphi_j - u \geq \left(1 + \frac{1}{j}\right) (\varphi_j - u) + \frac{u}{j} \geq -C \frac{\log j}{j}.$$

Since $\varphi \leq u$, we also obtain

$$\varphi - \left(1 + \frac{1}{j}\right) \varphi_j \leq C \frac{\log j}{j}.$$

This inequality holds on X , as $\varphi - (1 + 1/j)\varphi_j$ attains its global maximum at x_0 . Since this is true for all $\varphi \in SH(X, \omega, k)$ lying below u , we obtain

$$P_{\omega,k}(u) - C' \frac{\log j}{j} \leq \varphi_j \leq P_{\omega,k}(u),$$

for a uniform constant C' , proving that φ_j converges uniformly to $P_{\omega,k}(u)$. \square

1.2. The Hessian operator.

1.2.1. *Hermitian forms.* We assume in this subsection that ω is a hermitian form. For (ω, ω_X, k) -subharmonic functions φ of class \mathcal{C}^2 , the complex Hessian operator is defined by

$$H_k(\varphi) = (\omega + dd^c \varphi)^k \wedge \omega_X^{n-k}.$$

By Theorem 1.7, one can pointwise approximate from above any (ω, ω_X, k) -sh function φ by a decreasing sequence of smooth (ω, ω_X, k) -sh functions φ_j .

If φ is moreover continuous on X , then the convergence is uniform, and it was shown in [KN16] that the sequence of Hessian measures $H_k(\varphi_j)$ weakly converges to a unique positive Radon measure, independent of the approximants. One sets

$$H_k(\varphi) := \lim_{j \rightarrow +\infty} H_k(\varphi_j) := (\omega + dd^c \varphi)^k \wedge \omega_X^{n-k}.$$

It is likely that one can define these operators for (ω, ω_X, k) -sh functions that are merely bounded. This is indeed the case when ω_X is closed, as one can proceed by induction and use integration by parts following Bedford-Taylor's construction

[BT76, BT82]. The situation is however delicate in the general hermitian setting, and more subtle than for the Monge-Ampère operator.

Among the basic properties enjoyed by this operator are the following:

- $\varphi \mapsto H_k(\varphi)$ is continuous for the uniform topology;
- if $u_1, \dots, u_k \in \mathcal{SH}(X, \omega, k)$ are continuous, the mixed Hessian operators $(\omega + dd^c u_1) \wedge \dots \wedge (\omega + dd^c u_k) \wedge \omega_X^{n-k}$ are similarly defined and continuous;
- if u_1, \dots, u_k are smooth and $(\omega + dd^c u_j)^k \wedge \omega_X^{n-k} = f_j dV_X$ then

$$(1.1) \quad (\omega + dd^c u_1) \wedge \dots \wedge (\omega + dd^c u_k) \wedge \omega_X^{n-k} \geq (f_1 \dots f_k)^{1/k} dV_X.$$

These are a consequence of inequalities due to Gårding.

1.2.2. *Big forms.* We now no longer assume that ω is positive, but rather that it is semi-positive and big. Fix a quasi-plurisubharmonic function $\rho : X \rightarrow \mathbb{R} \cup \{-\infty\}$ with analytic singularities such that $\omega + dd^c \rho \geq \delta \omega_X$, for some constant $\delta > 0$. We let Ω denote the Zariski open set where ρ is smooth.

If $\varphi \in \mathcal{SH}(X, \omega, k) \cap L^\infty(X)$ is continuous in Ω , it follows from [KN16, Proposition 2.9] that for each smooth form γ , the mixed product $(\omega_X + dd^c \varphi)^j \wedge \gamma^{l-j} \wedge \omega_X^{n-l}$ is a well-defined Radon measure in Ω with finite mass.

Definition 1.8. For $\varphi \in \mathcal{SH}(X, \omega, k) \cap L^\infty(X) \cap C^0(\Omega)$ we set

$$H_k(\varphi) := \sum_{j=0}^k \binom{k}{j} (\omega_X + dd^c \varphi)^j \wedge (\omega - \omega_X)^{k-j} \wedge \omega_X^{n-k}.$$

This operator is well-defined in Ω with finite total mass, so we consider its trivial extension to X . The definition is motivated by the identity

$$\begin{aligned} (\omega + dd^c \varphi)^k \wedge \omega_X^{n-k} &= (\omega_X + dd^c \varphi + \omega - \omega_X)^k \wedge \omega_X^{n-k} \\ &= \sum_{j=0}^k \binom{k}{j} (\omega_X + dd^c \varphi)^j \wedge (\omega - \omega_X)^{k-j} \wedge \omega_X^{n-k}. \end{aligned}$$

The operator H_k is continuous along sequences that converge locally uniformly in Ω [KN16, Proposition 2.11]. The following result follows.

Proposition 1.9. *If $\varphi \in \mathcal{SH}(X, \omega, k) \cap L^\infty(X) \cap C^0(\Omega)$, then $H_k(\varphi)$ is a positive Radon measure.*

Proof. Fix $\varepsilon > 0$ and approximate φ -using Theorem 1.7- by a decreasing sequence (φ_j) of $(\omega + \varepsilon \omega_X)$ -sh functions. Since $(\omega + \varepsilon \omega_X + dd^c \varphi_j)^k \wedge \omega_X^{n-k} \geq 0$, we infer from [KN16, Proposition 2.11] that $(\omega + \varepsilon \omega_X + dd^c \varphi)^k \wedge \omega_X^{n-k} \geq 0$. The conclusion follows by letting $\varepsilon \rightarrow 0$, as we then obtain that $(\omega + dd^c \varphi)^k \wedge \omega_X^{n-k} \geq 0$. \square

Lemma 1.10. *Let φ, ψ be (ω, ω_X, k) -sh functions which are continuous in Ω and bounded on X . Then*

$$H_k(\max(\varphi, \psi)) \geq \mathbf{1}_{\{\varphi > \psi\}} H_k(\varphi) + \mathbf{1}_{\{\psi \geq \varphi\}} H_k(\psi).$$

In particular, if $\varphi \leq \psi$ then

$$\mathbf{1}_{\{\psi = \varphi\}} H_k(\varphi) \leq \mathbf{1}_{\{\psi = \varphi\}} H_k(\psi).$$

Proof. Fix $t > 0$. Since $\{\varphi < \psi + t\}$ and $\{\psi + t < \varphi\}$ are open in Ω , we have

$$\begin{aligned} H_k(\max(\varphi, \psi + t)) &\geq \mathbf{1}_{\{\varphi > \psi + t\}} (\omega + dd^c \varphi)^k \wedge \omega_X^{n-k} + \mathbf{1}_{\{\psi + t > \varphi\}} (\omega + dd^c \psi)^k \wedge \omega_X^{n-k} \\ &\geq \mathbf{1}_{\{\varphi > \psi + t\}} (\omega + dd^c \varphi)^k \wedge \omega_X^{n-k} + \mathbf{1}_{\{\psi \geq \varphi\}} (\omega + dd^c \psi)^k \wedge \omega_X^{n-k}. \end{aligned}$$

Letting $t \searrow 0$ we obtain the first statement of the lemma. The second one follows straightforwardly from the first. \square

1.2.3. *SH(X, ω, k)-envelopes.* We consider two types of envelopes in this note.

Definition 1.11. If $h : X \rightarrow \mathbb{R}$ is a continuous function, we set

$$P_\omega(h) := \sup \{u \in PSH(X, \omega), u \leq h\},$$

and $P_{\omega,k}(h) := \sup \{u \in SH(X, \omega, k), u \leq h\}$.

Here is a list of basic properties of these envelopes.

- The measure $(\omega + dd^c P_\omega(h))^n$ is concentrated on the set $\mathcal{C} = \{P_\omega(h) = h\}$.
- The complex Hessian measure $(\omega + dd^c P_{\omega,k}(h))^k \wedge \omega_X^{n-k}$ is concentrated on the contact set $\mathcal{C}_k = \{P_{\omega,k}(h) = h\}$.
- If h is $\mathcal{C}^{1,1}$ -smooth and ω is hermitian, then $P_\omega(h)$ is $\mathcal{C}^{1,1}$ -smooth and

$$(\omega + dd^c P_\omega(h))^n = \mathbf{1}_{\mathcal{C}}(\omega + dd^c h)^n.$$

- If h is $\mathcal{C}^{1,1}$ -smooth and ω is hermitian, then $P_{\omega,k}(h)$ is $\mathcal{C}^{1,1}$ -smooth and

$$(\omega + dd^c P_{\omega,k}(h))^k \wedge \omega_X^{n-k} = \mathbf{1}_{\mathcal{C}_k}(\omega + dd^c h)^k \wedge \omega_X^{n-k}.$$

We refer the reader to [Ber19, Tos18, GLZ19, CZ19, CM21] for the proof of these results, as well as further information on these envelopes.

1.3. **A priori estimates for the complex Monge-Ampère operator.** We are going to use the following a priori estimates established in [GL21].

Theorem 1.12. *Let ω, ω_0 be semi-positive and big $(1, 1)$ -forms such that $\omega \geq \omega_0$. Fix $\delta > 0$ and $g \in L^{1+\delta}(dV_X)$ a non-negative density. If $\varphi \in PSH(X, \omega) \cap L^\infty(X)$ satisfies $(\omega + dd^c \varphi)^n \leq g dV_X$, then*

$$\text{Osc}_X(\varphi) \leq C$$

for some $C \in \mathbb{R}^+$ that depends on n, δ, ω_0 and an upper bound on $\|g\|_{L^{1+\delta}}$.

Since this result is not explicitly stated in [GL21], we briefly sketch its proof.

Proof. We normalize φ so that $\sup_X \varphi = 0$. It follows from Skoda's uniform integrability theorem [GZ17, Theorem 8.11] that for $0 < \alpha$ small enough,

$$\int_X e^{-\alpha \frac{(2+\delta)(1+\delta)}{\delta} \varphi} dV_X \leq C_\alpha$$

is bounded from above independently of φ . Setting $\tilde{g} = e^{-\alpha \varphi} g$, Hölder inequality then shows that $\tilde{g} \in L^{1+\delta/2}$ with

$$\|\tilde{g}\|_{L^{1+\delta/2}} \leq C_\alpha^{1/q} \|g\|_{L^{1+\delta}}^{1/p},$$

where $p = \frac{1+\delta}{1+\delta/2}$ and $q = 2 + 2/\delta$. [GL21, Lemma 3.3] shows the existence of $c(\delta, \omega) > 0$ and $u \in PSH(X, \omega)$ such that $-1 \leq u \leq 0$ and

$$(\omega + dd^c u)^n \geq c(\delta, \omega) \frac{\tilde{g} dV_X}{\|\tilde{g}\|_{L^{1+\delta/2}}} \geq e^{\alpha u} c(\delta, \omega) \frac{\tilde{g} dV_X}{\|\tilde{g}\|_{L^{1+\delta/2}}}.$$

The function $v := u - \alpha^{-1} \log[c(\delta, \omega)/\|\tilde{g}\|_{L^{1+\delta/2}}] \in PSH(X, \omega)$ thus satisfies

$$e^{-\alpha v} (\omega + dd^c v)^n = e^{-\alpha v} (\omega + dd^c u)^n \geq \tilde{g} dV_X \geq e^{-\alpha \varphi} (\omega + dd^c \varphi)^n.$$

The comparison principle [GL21, Corollary 1.14] finally ensures that $v \leq \varphi$.

This provides a uniform lower bound on φ , which only depends on δ, ω and an upper bound on $\|g\|_{L^{1+\delta}}$. \square

2. DOMINATION PRINCIPLE

Fix ω a big form, ρ an ω -plurisubharmonic function with analytic singularities such that $\omega + dd^c\rho \geq \delta\omega_X$, $\delta > 0$, and set $\Omega := \{\rho > -\infty\}$.

We fix a constant $B_1 > 0$ such that for all $x \in \Omega$,

$$-B_1\omega_\rho^2 \leq dd^c\omega_\rho \leq B_1\omega_\rho^2 \quad \text{and} \quad -B_1\omega_\rho^3 \leq d\omega_\rho \wedge d^c\omega_\rho \leq B_1\omega_\rho^3.$$

The existence of B_1 is clear since $d\omega = d\omega_\rho$, $d^c\omega = d^c\omega_\rho$ and

$$-B\omega_X^2 \leq dd^c\omega \leq B\omega_X^2 \quad \text{and} \quad -B\omega_X^3 \leq d\omega \wedge d^c\omega \leq B\omega_X^3$$

for some $B > 0$. The following is an extension of [GL21, Theorem 1.11].

Theorem 2.1. *Assume ω is big. If u is a bounded (ω, ω_X, k) -subharmonic function which is continuous in Ω , then $\int_X (\omega + dd^c u)^k \wedge \omega_X^{n-k} > 0$.*

Proof. Fix ρ and B_1 as above. We set $m = \inf_\Omega(u - \rho)$. For $s > 0$ we set

$$\phi = \max(u, \rho + m + s) \quad \text{and} \quad U := \{u < \phi\} = \{u < \rho + m + s\}.$$

Observe that U is relatively compact in Ω , and it is non empty for all $s > 0$, by definition of m . It follows from [KN16, Corollary 2.4] that, for any $0 \leq j \leq k-1$,

$$dd^c(\omega_u^j \wedge \omega_\phi^{k-j-1} \wedge \omega_X^{n-k}) \leq C_1 \sum_{m=k-2}^{k-1} \sum_{l=0}^m \omega_u^l \wedge \omega_\phi^{m-l} \wedge \omega_X^{n-m},$$

where C_1 is a uniform constant. On the open set U since $\omega_\phi \geq \delta\omega_X$, we thus have

$$(2.1) \quad dd^c(\omega_u^j \wedge \omega_\phi^{k-j-1} \wedge \omega_X^{n-k}) \leq C_2 \sum_{l=0}^k \omega_u^l \wedge \omega_\phi^{k-l} \wedge \omega_X^{n-k}.$$

Since $u \leq \phi$, Lemma 1.10 ensures that

$$\mathbf{1}_{\{u=\phi\}} \omega_u^j \wedge \omega_\phi^{k-j} \wedge \omega_X^{n-k} \geq \mathbf{1}_{\{u=\phi\}} \omega_u^k \wedge \omega_\phi^{n-k}.$$

Noting that $X \setminus U = \{u = \phi\}$ we can write

$$\begin{aligned} \int_U (\omega_u^j \wedge \omega_\phi^{k-j} \wedge \omega_X^{n-k} - \omega_u^k \wedge \omega_X^{n-k}) &\leq \int_X (\omega_u^j \wedge \omega_\phi^{k-j} \wedge \omega_X^{n-k} - \omega_u^k \wedge \omega_X^{n-k}) \\ &= \int_X dd^c(\phi - u) \wedge \sum_{l=0}^{k-j} \omega_u^l \wedge \omega_\phi^{k-j-l} \wedge \omega_X^{n-k} \\ &= \int_X (\phi - u) dd^c \left[\sum_{l=0}^{k-j} \omega_u^l \wedge \omega_\phi^{k-j-l} \wedge \omega_X^{n-k} \right]. \end{aligned}$$

In the last line above, we have used Stokes theorem. Since $0 \leq \phi - u \leq s$, using (2.1) we can continue the above inequalities as follows:

$$\int_U (\omega_u^j \wedge \omega_\phi^{k-j} \wedge \omega_X^{n-k} - \omega_u^k \wedge \omega_X^{n-k}) \leq C_3 s \sum_{l=0}^k \int_U \omega_u^l \wedge \omega_\phi^{k-l} \wedge \omega_X^{n-k}.$$

Summing up the above inequalities for $j = 0, \dots, k$, one gets

$$(1 - C_4 s) \sum_{l=0}^k \int_U \omega_u^l \wedge \omega_\phi^{k-l} \wedge \omega_X^{n-k} \leq (k+1) \int_U \omega_u^k \wedge \omega_X^{n-k}.$$

If $\omega_u^k \wedge \omega_X^{n-k} = 0$ then, for $s > 0$ small enough, the above inequality yields $\int_U \omega_u^k \wedge \omega_X^{n-k} = 0$, hence U has Lebesgue measure 0, which is impossible. \square

Theorem 2.2. *Assume ω is hermitian, u, v are continuous (ω, ω_X, k) -subharmonic functions such that $\omega_u^k \wedge \omega_X^{n-k} = 0$ on $\{u < v\}$. Then $u \geq v$.*

Proof. By replacing v with $\max(u, v)$ and noting that $\{u < v\} = \{u < \max(u, v)\}$, we can assume that $u \leq v$. Our plan is thus to show that $u = v$.

For $b \geq 1$, we set $\phi_b := P_{\omega, k}(bu - (b-1)v)$. It is a negative (ω, k) -sh function. Since $bu - (b-1)v$ is continuous on X , so is ϕ_b . Observe also that

$$b^{-1}\phi_b + (1 - b^{-1})v \leq u$$

with equality on the contact set $\mathcal{C} := \{\phi_b = bu - (b-1)v\}$. Now by Lemma 1.10

$$\mathbf{1}_{\mathcal{C}} \left(b^{-k} \omega_{\phi_b}^k \wedge \omega_X^{n-k} + (1 - b^{-1})^k \omega_v^k \wedge \omega_X^{n-k} \right) \leq \mathbf{1}_{\mathcal{C}} \omega_u^k \wedge \omega_X^{n-k},$$

while $\int_{\mathcal{C}} H_k(\phi_b) = \int_X H_k(\phi_b) > 0$ by Theorem 2.1. Since $H_k(u) = 0$ on $\{u < v\}$, it follows that $\mathcal{C} \cap \{u = v\}$ contains at least one point, say x . We then have

$$\phi_b(x) = bu(x) - (b-1)v(x) = v(x) \geq \inf_X v;$$

in particular $\sup_X \phi_b \geq \inf_X v$.

Note that ϕ_b is decreasing in b since $u \leq v$. The limit $\phi := \lim_{b \rightarrow +\infty} \phi_b$ is a (ω, k) -sh function which is not identically $-\infty$. For any $a > 0$,

$$\phi_b \leq bu - (b-1)v \leq v - ba$$

on the open set $\{u < v - a\}$, thus $\phi = -\infty$ on this set. Since ϕ is integrable on X , the set $\{u < v - a\}$ must be empty, hence letting $a \rightarrow 0^+$ gives $u \geq v$. \square

Corollary 2.3. *Assume ω is hermitian, u and v are continuous (ω, ω_X, k) -subharmonic functions such that $\omega_u^k \wedge \omega^{n-k} \leq c\omega_v^k \wedge \omega^{n-k}$ on $\{u < v\}$ for some $c \in [0, 1)$. Then $u \geq v$.*

Proof. Take b so large that $(1 - b^{-1})^k \geq c$, and consider $\phi := P_{\omega, k}(bu - (b-1)v)$. Then $b^{-1}\phi + (1 - b^{-1})v \leq u$ with equality on the contact set

$$\mathcal{C} := \{\phi = bu - (b-1)v\}.$$

By Lemma 1.10 we have

$$\mathbf{1}_{\mathcal{C}} \left(b^{-k} \omega_{\phi}^k \wedge \omega_X^{n-k} + (1 - b^{-1})^k \omega_v^k \wedge \omega_X^{n-k} \right) \leq \mathbf{1}_{\mathcal{C}} \omega_u^k \wedge \omega_X^{n-k}.$$

Multiplying with $\mathbf{1}_{\{u < v\}}$ and using the assumption $\omega_u^k \wedge \omega^{n-k} \leq c\omega_v^k \wedge \omega^{n-k}$ on $\{u < v\}$ and $(1 - b^{-1})^k \geq c$, we then see that

$$\mathbf{1}_{\mathcal{C} \cap \{u < v\}} \omega_{\phi}^k \wedge \omega^{n-k} = 0.$$

Since $\omega_{\phi}^k \wedge \omega^{n-k}$ is supported on \mathcal{C} , we thus have

$$\int_{\{\phi < v\}} \omega_{\phi}^k \wedge \omega^{n-k} = \int_{\{bu - (b-1)v < v\}} \omega_{\phi}^k \wedge \omega^{n-k} = \int_{\{u < v\}} \omega_{\phi}^k \wedge \omega^{n-k} = 0.$$

Invoking the domination principle (Theorem 2.2) we then get $\phi \geq v$. It thus follows that $bu - (b-1)v \geq \phi \geq v$, hence $u \geq v$. \square

Corollary 2.4. *Assume ω is hermitian and u and v are continuous (ω, ω_X, k) -subharmonic functions. Then*

$$e^{-v}(\omega + dd^c v)^k \wedge \omega_X^{n-k} \geq e^{-u}(\omega + dd^c u)^k \wedge \omega_X^{n-k} \implies u \geq v.$$

Proof. Fix $a > 0$. On the set $\{u < v - a\}$ we have $H_k(u) \leq e^{-a}H_k(v)$. By Corollary 2.3, $u \geq v - a$. Since it holds for all $a > 0$, we obtain $u \geq v$. \square

3. PROOF OF THE MAIN THEOREMS

3.1. A priori estimates. In this section we prove Theorem B, which is an extension of the main result of Kołodziej-Nguyen [KN16]:

Theorem 3.1. *Let ω_0, ω be semi-positive and big forms such that $\omega_0 \leq \omega$. Let φ be a smooth (ω, ω_X, k) -subharmonic function satisfying*

$$(\omega + dd^c \varphi)^k \wedge \omega_X^{n-k} \leq f dV_X,$$

where $f \in L^p(dV_X)$, $p > n/k$. Then

$$\text{Osc}_X(\varphi) \leq C,$$

where $C > 0$ depends on ω_0, n, k, p and an upper bound for $\|f\|_{L^p}$.

Proof. We can assume that $\sup_X \varphi = 0$. Let $u := P_\omega(\varphi)$ be the largest ω -psh function lying below φ . Let \mathcal{C} denote the contact set $\mathcal{C} := \{u = \varphi\}$. Then u is (ω, ω_X, k) -sh with $u \leq \varphi$ hence Lemma 1.10 yields

$$\mathbf{1}_{\mathcal{C}}(\omega + dd^c u)^k \wedge \omega_X^{n-k} \leq \mathbf{1}_{\mathcal{C}}(\omega + dd^c \varphi)^k \wedge \omega_X^{n-k} \leq f dV_X.$$

Observe that the smooth function $h := \omega_X^n / dV_X \geq \delta_0$ is positive on X . Recall that the Monge-Ampère measure $(\omega + dd^c u)^n$ of u is concentrated on the contact set \mathcal{C} . We set $g := (\omega + dd^c u)^n / dV_X$. It follows from Gårding's inequalities (1.1) that

$$\delta_0^{1-k/n} g^{k/n} \leq h^{1-k/n} g^{k/n} \leq f,$$

hence $g \leq C_0 f^{n/k}$. Since $f \in L^p$ with $p > n/k$, we obtain $g \in L^{1+\delta}$ with $\delta > 0$. Theorem 1.12 therefore provides a uniform bound

$$\text{Osc}_X(u) \leq M.$$

This provides a uniform bound on $u \leq 0$ if we can bound $\sup_X u$ from below.

Let q' be the conjugate exponent to $p' = pk/n > 1$. We obtain a uniform lower bound on $\sup_X u$ by using [GL22, Proposition 3.4] and Hölder inequality, via

$$\begin{aligned} v_M^-(\omega)(-\sup_X u)^{1/q'} &\leq \int_X |u|^{1/q'} (\omega + dd^c u)^n \\ &\leq \left(\int_X |\varphi| \omega^n \right)^{1/q'} \left(\int_X f^{p'} \omega^n \right)^{1/p'} \\ &\leq C, \end{aligned}$$

as $\int_X |\varphi| \omega^n$ is bounded from above independently of φ by Proposition 1.6. Together with the uniform bound for $\text{Osc}_X(u)$, this yields a uniform bound for $|u|$. This implies a uniform bound for φ since $u \leq \varphi$ and $\sup_X \varphi = 0$. \square

Remark 3.2. We also have a uniform a priori bound when f satisfies the weaker integrability condition

$$\int_X f^{n/k} |\log f|^n |h \circ \log \circ \log f|^n dt < +\infty$$

where h is an increasing continuous function such that $\int^{+\infty} dt/h(t) < +\infty$. It follows indeed that g satisfies then Kołodziej's optimal integrability condition

$$\int_X g |\log g|^n |h \circ \log \circ \log g|^n dt < +\infty,$$

which ensures a uniform bound on $\text{Osc}_X(u)$ by [Kol98, Theorem 2.5.2]. We actually need here the extension of Kołodziej's result to the case of semi-positive and big forms [EGZ09], as well as to the hermitian setting [DK12, GL21].

The same proof thus applies with a minor modification: to obtain a lower bound for $\sup_X u$, we replace Hölder inequality by the additive Hölder-Young inequality. If w denotes the convex weight $w(t) \sim t(\log t)^n (h \circ \log \circ \log t)^n$, w^* is its conjugate convex weight, and $(w^*)^{-1}$ denotes the inverse of w^* , we obtain this way a uniform upper bound

$$\begin{aligned} v_M^-(\omega)(w^*)^{-1}(-\sup_X u) &\leq \int_X (w^*)^{-1}(u)(\omega + dd^c u)^n \\ &\leq \int_X w \circ f dV_X + \int_X (-u)dV_X \leq C. \end{aligned}$$

3.2. Stability estimates. We strengthen Theorem B by establishing the following stability estimate:

Theorem 3.3. *Assume ω is hermitian and $\omega \geq \omega_0$ for some semi-positive and big form ω_0 . Assume $0 \leq f, g \in L^p(dV_X)$ with $p > n/k$, φ, ψ are continuous (ω, ω_X, k) -subharmonic functions such that*

$$(\omega + dd^c \varphi)^k \wedge \omega_X^{n-k} \geq e^\varphi f \omega_X^n, \quad \text{and} \quad (\omega + dd^c \psi)^k \wedge \omega_X^{n-k} \leq e^\psi g \omega_X^n.$$

Then

$$(3.1) \quad \varphi \leq \psi + C \|f - g\|_p^{1/k}$$

for some $C > 0$ depending on ω_0, n, k and upper bounds for $\|f\|_p, \|\varphi\|_\infty, |\inf_X \psi|$.

In particular if φ, ψ are solutions of the corresponding equations, then

$$\|\varphi - \psi\|_\infty \leq C \|f - g\|_p^{1/k},$$

so that $\varphi - \psi$ is uniformly small if f is close to g in L^p .

Proof. Our proof uses a perturbation argument from [GLZ18, LPT21], which goes back to [Kol96]. We solve the complex Monge-Ampère equation

$$(\omega + dd^c u)^n = bh\omega_X^n, \quad \text{where} \quad h = \frac{|f - g|^{n/k}}{\|f - g\|_p^{n/k}} + 1,$$

$\sup_X u = 0$ and $b > 0$ is a constant. The existence of a continuous solution $u \in PSH(X, \omega)$ follows from [KN15, KN19].

We can assume without loss of generality that ω_X is a Gauduchon metric (i.e. $dd^c(\omega_X^{n-1}) = 0$) which dominates ω . Thus u is ω_X -psh with

$$(\omega_X + dd^c u)^n \geq (\omega + dd^c u)^n \geq b\omega_X^n$$

since $h \geq 1$. The mixed Monge-Ampère inequalities (see [Ng16]) yield

$$(\omega_X + dd^c u) \wedge \omega_X^{n-1} \geq b^{1/n} \omega_X^n,$$

hence $b \leq 1$ since $\int_X (\omega_X + dd^c u) \wedge \omega_X^{n-1} = \int_X \omega_X^n$ by the Gauduchon condition.

Since $h \in L^q$ with $q = pk/n > 1$ and $b \leq 1$, Theorem 3.1 ensures that $\|u\|_\infty$ is uniformly bounded. It thus follows from [GL22, Proposition 3.4] that $b \geq \delta_0 > 0$ is uniformly bounded from below.

If $\|f - g\|_p^{1/k}$ is not small, the stability inequality (3.1) is trivially obtained by adjusting the value of C . In the remainder of the proof, we can thus assume without loss of generality that

$$(3.2) \quad \varepsilon := b^{-1/n} \|f - g\|_p^{1/k} e^{\sup_X \varphi/k} \leq 1/2.$$

By the mixed Monge-Ampère inequalities (1.1) we have

$$(\omega + dd^c u)^k \wedge \omega_X^{n-k} \geq b^{k/n} \frac{|f-g|}{\|f-g\|_p} \omega_X^n.$$

We set $C_1 = 2k - \inf_X \varphi$ and

$$\psi_\varepsilon := (1 - \varepsilon)\varphi + \varepsilon u - C_1 \varepsilon,$$

The function ψ_ε belongs to $SH(X, \omega, k) \cap \mathcal{C}^0(X)$; its Hessian measure satisfies

$$\begin{aligned} (\omega + dd^c \psi_\varepsilon)^k \wedge \omega_X^{n-k} &\geq (1 - \varepsilon)^k e^\varphi f \omega_X^n + \varepsilon^k b^{k/n} \frac{|f-g|}{\|f-g\|_p} \omega_X^n \\ &= \left[(1 - \varepsilon)^k e^\varphi f + e^{\sup_X \varphi} |f-g| \right] \omega_X^n \\ &\geq e^{\varphi - 2k\varepsilon} g \omega_X^n, \end{aligned}$$

using (3.2), $|f-g| + f \geq g$ and the inequality $\log(1 - \varepsilon) \geq -2\varepsilon$ valid on $[0, 1/2]$.

Observe now that $\psi_\varepsilon \leq \varphi - \varepsilon\varphi - C_1\varepsilon \leq \varphi - 2k\varepsilon$ by the definition of C_1 and $u \leq 0$. We infer

$$(\omega + dd^c \psi_\varepsilon)^k \wedge \omega_X^{n-k} \geq e^{\psi_\varepsilon} g \omega_X^n,$$

hence $\psi_\varepsilon \leq \psi$ by Corollary 2.4. This yields

$$\varphi \leq \psi + \varepsilon(\varphi - u) + C_1\varepsilon \leq \psi + C_2 \|f-g\|_p^{1/k},$$

for some uniform constant C_2 , as desired. \square

3.3. Proof of Theorem A. We approximate f in L^p by smooth and positive densities f_j . For $j \geq 1$ we consider the hermitian forms $\omega_j = \omega + 2^{-j}\omega_X$. It follows from [Zh17, Szek18] that there exist unique constants $c_j > 0$ and smooth (ω_j, k) -sh functions φ_j such that

$$(\omega_j + dd^c \varphi_j)^k \wedge \omega_X^{n-k} = c_j f_j \omega_X^n,$$

and $\sup_X \varphi_j = 0$.

We claim that the constants c_j stay uniformly bounded, as j increases to $+\infty$. To prove the claim we use the solutions to the Monge-Ampère equations by Tosatti-Weinkove [TW10]:

$$(\omega_j + dd^c u_j)^n = b_j f_j^{n/k} \omega_X^n.$$

Set $p' = pk/n > 1$. Since the $L^{p'}$ norm of $f_j^{n/k}$ is uniformly bounded from above, by [GL21, Lemma 3.3], there exists a constant $c > 0$ independent of j and a bounded ω -psh function u such that $\omega_u^n \geq c f_j^{n/k}$. By [GL21, Corollary 1.13], $b_j \geq c$. It then follows from Gårding's inequality that

$$(\omega_j + dd^c u_j)^k \wedge \omega_X^{n-k} \geq c^{k/n} f_j \omega_X^n.$$

Corollary 2.3 thus ensures that $c_j \geq c^{k/n}$ is uniformly bounded from below by a positive constant.

We next bound c_j from above. By Gårding's inequality we have

$$(\omega_j + dd^c \varphi_j) \wedge \omega_X^{n-1} \geq c_j^{1/k} f_j^{1/k} \omega_X^n.$$

The functions φ_j belong to $SH(X, \omega_1)$ and are relatively compact in L^1 , hence integrating over X and using that $\int_X f_j^{1/k} \omega_X^n \rightarrow \int_X f^{1/k} \omega_X^n > 0$, we obtain a uniform upper bound for c_j .

Since $\omega \leq \omega_j$ and $\|c_j f_j\|_{L^p}$ is uniformly bounded from above, it follows from Theorem 3.1 that the φ_j 's are uniformly bounded:

$$-C_1 \leq \varphi_j \leq 0,$$

for a uniform constant C_1 . Extracting and using compactness in $L^1(X)$ of normalized (ω, ω_X, k) -sh functions, we can thus assume that $c_j \rightarrow c > 0$ and φ_j converges in $L^1(X)$ and almost everywhere to a bounded function φ which is (ω, ω_X, k) -sh, as $j \rightarrow +\infty$.

We next prove that the convergence $\varphi_j \rightarrow \varphi$ is locally uniform in Ω . We set $g_j := c_j f_j e^{-\varphi_j}$ and $g = c f e^{-\varphi}$. By Hölder inequality, $\|g_j - g\|_q \rightarrow 0$ for some $q > n/k$. Fix $j < l$ large enough. Since $\omega_j \geq \omega_l$, we have

$$(\omega_j + dd^c \varphi_l)^k \wedge \omega_X^{n-k} \geq e^{\varphi_l} g_l \omega_X^n.$$

By Theorem 3.3, we infer

$$(3.3) \quad \varphi_l \leq \varphi_j + C_2 \|g_j - g_l\|_q^{1/k},$$

for a uniform constant C_2 . We next consider

$$\psi_j := (1 - \varepsilon)\varphi_j + \varepsilon\rho - (2k + C_1)\varepsilon,$$

where $\varepsilon = \delta^{-1}2^{-j} < 1/2$ when j is large enough. Since $\omega + dd^c \rho \geq \delta\omega_X$, we have

$$\begin{aligned} \omega_l + dd^c \psi_j &\geq \omega + \varepsilon dd^c \rho + (1 - \varepsilon) dd^c \varphi_j \\ &\geq \varepsilon \delta \omega_X + (1 - \varepsilon)(\omega_j + dd^c \varphi_j) - 2^{-j} \omega_X \\ &= (1 - \varepsilon)(\omega_j + dd^c \varphi_j). \end{aligned}$$

It follows that $\psi_j \in SH(X, \omega_l, k)$ and

$$(\omega_l + dd^c \psi_j)^k \wedge \omega_X^{n-k} \geq e^{\varphi_j + k \log(1 - \varepsilon)} g_j \omega_X^n \geq e^{\varphi_j - 2k\varepsilon} g_j \omega_X^n \geq e^{\psi_j} g_j \omega_X^n$$

holds in Ω . The function $\psi_{j,l} := \max(\psi_j, \varphi_l)$ is (ω_l, k) -subharmonic and continuous on X . Lemma 1.10 yields

$$(\omega_l + dd^c \psi_{j,l})^k \wedge \omega_X^{n-k} \geq e^{\psi_{j,l}} \min(g_j, g_l) \omega_X^n.$$

Applying Theorem 3.3 we then get

$$\psi_{j,l} \leq \varphi_l + C_3 \|\min(g_j, g_l) - g_l\|_q^{1/k} \leq \varphi_l + C_3 \|g_j - g_l\|_q^{1/k},$$

for a uniform constant C_3 . Together with (3.3), this implies

$$\begin{aligned} \varphi_j + \delta^{-1}2^{-j}\rho - (2k + C_1)\delta^{-1}2^{-j} &\leq \varphi_l + C_3 \|g_j - g_l\|_q^{1/k} \\ &\leq \varphi_j + C_1 \|g_j - g_l\|_q^{1/k}. \end{aligned}$$

Therefore, φ_j converges locally uniformly in Ω to φ . Thus φ is continuous in Ω where it solves the complex Hessian equation

$$(\omega + dd^c \varphi)^k \wedge \omega_X^{n-k} = c f \omega_X^n.$$

Remark 3.4. When the form ω is hermitian, the solution φ that we construct is continuous on the whole of X . Continuity of φ at points where ω vanishes is a delicate issue, even in the case of complex Monge-Ampère equations. We refer the interested reader to [GGZ23] for a recent account.

The uniqueness of φ is also a subtle problem in the hermitian setting, see [KN19] for a partial result in the case of the complex Monge-Ampère equation.

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