



Degenerate complex Hessian equations on compact Hermitian manifolds

Vincent Guedj, Chinh H. Lu

► To cite this version:

Vincent Guedj, Chinh H. Lu. Degenerate complex Hessian equations on compact Hermitian manifolds. *Pure and Applied Mathematics Quarterly*, 2023, 21 (3), pp.1171-1194. <10.4310/pamq.250116014513>. <hal-04010270>

HAL Id: hal-04010270

<https://hal.science/hal-04010270v1>

Submitted on 1 Mar 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

DEGENERATE COMPLEX HESSIAN EQUATIONS ON COMPACT HERMITIAN MANIFOLDS

VINCENT GUEDJ & CHINH H. LU

ABSTRACT. In this note we provide uniform a priori estimates for solutions to degenerate complex Hessian equations on compact hermitian manifolds. Our approach relies on the corresponding a priori estimates for Monge-Ampère equations; it provides an extension as well as a short alternative proof to results of Dinew-Kołodziej, Kołodziej-Nguyen and Guo-Phong-Tong.

À la mémoire de Jean-Pierre Demailly.

INTRODUCTION

Let X be a compact complex manifold equipped with a hermitian form ω_X . We fix $1 \leq k \leq n = \dim_{\mathbb{C}} X$, ω another hermitian form, and μ_X a volume form on X . The complex Hessian equation is

$$(\omega + dd^c \varphi)^k \wedge \omega_X^{n-k} = c \mu_X,$$

where $c > 0$ is a positive constant and φ (the unknown) is a smooth (ω, ω_X, k) -subharmonic function, i.e. $(\omega + dd^c \varphi)^j \wedge \omega_X^{n-j} \geq 0$ for all $1 \leq j \leq k$.

When $k = 1$ this reduces to the Laplace equation. When $k = n$ this yields the complex Monge-Ampère equation which was famously solved by Yau [Yau78] when ω is Kähler, and by Tosatti-Weinkove [TW10] in the hermitian setting.

From now on we assume that $1 < k < n$. A unique solution has been provided by Dinew-Kołodziej [DK17] when $\omega = \omega_X$ is Kähler; the hermitian setting has been resolved independently by Székelyhidi [Szek18] and Zhang [Zh17].

It is quite natural to consider such equations in degenerate settings, allowing the measure $\mu_X = f \omega_X^n$ to be merely absolutely continuous with respect to the volume form ω_X^n . Such degenerate complex Hessian equations have been intensively studied in the past fifteen years (see [Blo05, Lu13, DK14, DL15, LN15, Lu15, KN16, Char16, GN18, DDT19, BZ20, GP22] and the references therein).

In this note we go one step further by allowing the form ω to degenerate as well: we no longer assume it is positive, we merely ask that it is *big*, in analogy with the corresponding notion from algebraic geometry [BEGZ10], i.e. we assume there is an ω -psh function ρ with analytic singularities such that $\omega + dd^c \rho$ dominates a hermitian form. We let Ω denote the Zariski open set where ρ is smooth.

One can not expect any longer that (ω, ω_X, k) -subharmonic functions are smooth in X , but we show in Section 1.2 how to define the Hessian operator $H_k(f) = (\omega + dd^c \varphi)^k \wedge \omega_X^{n-k}$ for such functions φ which are *continuous* in Ω and globally bounded on X .

Date: March 1, 2023.

2010 Mathematics Subject Classification. 32W20, 32U05, 32Q15, 35A23.

Key words and phrases. Complex Hessian and Monge-Ampère equations, a priori estimates.

In this context we obtain the following:

Theorem A. *Let ω be a semi-positive and big form and $f \in L^p(\omega_X^n)$ with $p > n/k$ and $\int_X f \omega_X^n > 0$. There exists $c > 0$ and φ a bounded (ω, ω_X, k) -subharmonic function, continuous in Ω , such that*

$$(\omega + dd^c \varphi)^k \wedge \omega_X^{n-k} = cf \omega_X^n.$$

This result extends the main result of [KN16]. It applies in particular to the case when $\pi : X \rightarrow Y$ is the desingularization of a compact hermitian variety Y , and $\omega = \pi^* \omega_Y$ is the pull-back of a hermitian form ω_Y on Y .

As is often the case in similar contexts, the main difficulty lies in establishing uniform a priori estimates. This is the content of our second main result:

Theorem B. *Let ω_0, ω be semi-positive and big forms such that $\omega_0 \leq \omega$. Let φ be a smooth (ω, ω_X, k) -subharmonic function solution to*

$$(\omega + dd^c \varphi)^k \wedge \omega_X^{n-k} = f \omega_X^n,$$

where $f \in L^p(dV_X)$, $p > n/k$. Then

$$\text{Osc}_X(\varphi) \leq C,$$

where $C > 0$ depends on ω_0, n, k, p and an upper bound for $\|f\|_{L^p}$.

The approach of Kolodziej-Nguyen relies on a pluripotential theory for degenerate complex Hessian equations on hermitian manifolds (mimicking the corresponding results obtained by these authors and Dinew for the Monge-Ampère equation [DK12], [KN15], [KN19], [Ng16]). An alternative PDE proof has been recently provided by Guo-Phong-Tong-Wang in [GPT21, GPTW21] when X is Kähler, and by Guo-Phong [GP22] when X is hermitian and ω is a hermitian form.

We provide in this note a proof which relies directly on the corresponding results for the complex Monge-Ampère operator. Given φ a solution of the k -Hessian equation with density f , we show that the ω -psh envelope of φ solves a Monge-Ampère equation whose density is bounded from above by $f^{n/k}$. A lower bound on φ thus follows from the L^∞ -a priori estimates for the Monge-Ampère equation established in [GL21].

Our method is new and also of interest when X is Kähler; it applies to more general densities (see Remark 3.2); in particular the results of Theorem A and Theorem B hold when f merely satisfies

$$\int_X f^{\frac{n}{k}} |\log f|^n (\log |\log f|)^{n+\delta} < +\infty,$$

for some $\delta > 0$. Radial examples show that this condition is almost optimal.

Acknowledgements. This work is partially supported by the research project HERMETIC (ANR-11-LABX-0040), as well as by the ANR project PARAPLUI.

1. PRELIMINARIES

In the whole article we let (X, ω_X) denote a compact complex manifold of complex dimension $n \in \mathbb{N}^*$, equipped with a fixed hermitian form ω_X . We use real differential operators $d = \partial + \bar{\partial}$, $d^c = i(\bar{\partial} - \partial)$, so that $dd^c = 2i\partial\bar{\partial}$.

We also fix a semi-positive $(1, 1)$ -form ω which is *big*:

Definition 1.1. We say that ω is big if there exists an ω -psh function ρ such that $\omega + dd^c \rho$ dominates a hermitian form.

It follows from Demailly's regularization theorem [Dem94] that one can further assume that ρ has analytic singularities, in particular $\omega + dd^c \rho$ is a hermitian form in some Zariski open set that we usually denote by Ω in what follows.

An important source of examples of big forms is provided by the following construction: if V is a compact complex space endowed with a hermitian form ω_V , and $\pi : X \rightarrow V$ is a resolution of singularities, then $\omega = \pi^* \omega_V$ is big.

1.1. (ω, ω_X, k) -subharmonic functions.

1.1.1. *Definition.* Let $U \subset X$ be an open subset and γ a hermitian form in U . A function $h \in \mathcal{C}^2(U, \mathbb{R})$ is called γ -harmonic if

$$dd^c h \wedge \gamma^{n-1} = 0 \text{ pointwise in } U.$$

We let $H_\gamma(U)$ denote the set of all γ -harmonic functions in U . A function $u : U \rightarrow \mathbb{R} \cup \{-\infty\}$ is γ -subharmonic if it is upper semicontinuous and for every $D \subset U$ and every $h \in H_\gamma(D)$ the following implication holds

$$u \leq h \text{ on } \partial D \implies u \leq h \text{ in } D.$$

As shown in [HL13, Section 9] the γ -subharmonicity can be equivalently defined using viscosity theory: u is γ -subharmonic in U if and only if the inequality $dd^c u \wedge \gamma^{n-1}$ holds in the viscosity sense in U . The latter condition means that for every $x_0 \in U$ and every \mathcal{C}^2 function χ defined in a neighborhood V of x_0 , the following implication holds:

$$[u(x) \leq \chi(x), \forall x \in V \text{ and } u(x_0) = \chi(x_0)] \implies dd^c \chi \wedge \gamma^{n-1} \geq 0 \text{ at } x_0.$$

When $u \in \mathcal{C}^2(U)$, it is γ -subharmonic if and only if $dd^c u \wedge \gamma^{n-1} \geq 0$ pointwise.

Definition 1.2. We let $SH_\gamma(U)$ denote the set of all γ -subharmonic functions in U that are locally integrable.

If $u \in SH_\gamma(U)$ then, by [HH72, Théorème 1, p.136], $dd^c u \wedge \gamma^{n-1} \geq 0$ in the sense of distributions. Conversely, by [Lit83, Theorem A], the latter condition implies that u coincides a.e. with a unique function $\hat{u} \in SH_\gamma(U)$ defined by

$$\hat{u}(x) := \operatorname{ess\,lim\,sup}_{y \rightarrow x} u(y) := \lim_{r \searrow 0} \operatorname{ess\,sup}_{B(x,r)} u,$$

where $\operatorname{ess\,sup}_{B(x,r)} u$ is the essential supremum of u in the ball $B(x, r)$.

Definition 1.3. A function $\varphi : U \rightarrow \mathbb{R} \cup \{-\infty\}$ is quasi- γ -subharmonic if locally $\varphi = u + \rho$ where u is γ -subharmonic and ρ is smooth.

We say that a function φ is (ω, γ) -subharmonic if it is quasi- γ -subharmonic and $(\omega + dd^c \varphi) \wedge \gamma^{n-1} \geq 0$ in the sense of distributions on X .

We let $SH_\gamma(U, \omega)$ denote the set of all (ω, γ) -subharmonic functions in U .

If $u \in \mathcal{C}^2(U, \mathbb{R})$ then $u \in SH_\gamma(U, \omega)$ if and only if $(\omega + dd^c u) \wedge \gamma^{n-1} \geq 0$ pointwise. In general this inequality holds in the viscosity sense:

Lemma 1.4. *If $\varphi \in SH_\gamma(U, \omega)$ and χ is a \mathcal{C}^2 function such that $\varphi - \chi$ attains a local maximum at $x_0 \in U$, then $(\omega + dd^c \chi) \wedge \gamma^{n-1} \geq 0$ at x_0 .*

Proof. Fix $a > 1$. Let ρ be a quadratic function such that $dd^c \rho = a\omega(x_0)$. By continuity of ω , we have $dd^c \rho \geq \omega$ in a small ball B around x_0 . The function $\rho + \varphi$ is then γ -subharmonic in B and the function $\rho + \varphi - (\rho + \chi)$ attains a local maximum at x_0 . It thus follows that, at x_0 ,

$$(a\omega + dd^c \chi) \wedge \gamma^{n-1} = dd^c(\rho + \chi) \wedge \gamma^{n-1} \geq 0.$$

The conclusion follows by letting $a \searrow 1$. □

Fix an integer $1 \leq k \leq n$, and let $\Gamma_k(\omega_X)$ denote the set of all $(1, 1)$ -forms α such that

$$\alpha^j \wedge \omega_X^{n-j} > 0, \quad j = 1, \dots, k.$$

By Gårding's inequality [Gar59], if $\alpha, \alpha_1, \dots, \alpha_{k-1} \in \Gamma_k(\omega_X)$ then

$$\alpha \wedge \alpha_1 \wedge \dots \wedge \alpha_{k-1} \wedge \omega_X^{n-k} > 0.$$

In particular, $\alpha_1 \wedge \dots \wedge \alpha_{k-1} \wedge \omega_X^{n-k}$ is a strictly positive $(n-1, n-1)$ -form on X . By [Mic82], one can write $\alpha_1 \wedge \dots \wedge \alpha_{k-1} \wedge \omega_X^{n-k} = \gamma^{n-1}$, for some hermitian form γ . We let $F_k(\omega_X)$ denote the set of all such hermitian forms.

Definition 1.5. A function $\varphi : U \rightarrow \mathbb{R} \cup \{-\infty\}$ is (ω, ω_X, k) -subharmonic if it is (ω, γ) -subharmonic in U for all $\gamma \in F_k(\omega_X)$.

We let $SH_{\omega_X}(U, \omega, k)$ denote the set of all (ω, ω_X, k) -subharmonic functions on U which are locally integrable in U .

If u is of class \mathcal{C}^2 in U then, by [Gar59], u is (ω, ω_X, k) -subharmonic if and only if $\omega + dd^c u \in \Gamma_k(\omega_X)$. We refer the reader to [Blo05] and [KN16, Section 2.3] for basic properties of (ω, ω_X, k) -sh functions; they easily extend to our context.

Since ω_X is fixed once and for all in this note, we let $SH(X, \omega, k)$ denote the set of (ω, ω_X, k) -sh functions; they form a nested sequence

$$SH(X, \omega) \supset \dots \supset SH(X, \omega, k-1) \supset SH(X, \omega, k) \supset \dots \supset PSH(X, \omega),$$

where $SH(X, \omega) = SH(X, \omega, 1)$ is the set of ω -sh functions, while $PSH(X, \omega) = SH(X, \omega, n)$ is the set of ω -psh functions, for which $\omega + dd^c \varphi \geq 0$ is a positive current of bidegree $(1, 1)$ (see [GZ17, Chapter 8]).

Observe that $SH(X, \omega, k) \subset SH(X, \omega', k)$ if $\omega \leq \omega'$. Since $\omega \leq A\omega_X$ for some $A > 0$, any (ω, ω_X, k) -sh function is also $(A\omega_X, 1)$ -sh. We thus have the following compactness result (see [KN16, Lemma 3.3], [GN18, Section 9]).

Proposition 1.6. *The set of normalized (ω, ω_X, k) -subharmonic functions is compact in the L^1 -topology. In particular, there exists $C > 0$ such that for all $\varphi \in SH(X, \omega, k)$ normalized by $\sup_X \varphi = 0$,*

$$\int_X (-\varphi) \omega_X^n \leq C.$$

1.1.2. *Smooth approximation.*

Theorem 1.7. *If ω is hermitian, any (ω, k) -subharmonic function is the point-wise limit of a decreasing sequence of smooth (ω, k) -subharmonic functions.*

This result is a slight extension of [KN16, Lemma 3.20], which itself extends [LN15, Theorem 1.2] from the Kähler to the hermitian setting. We propose below an alternative and more direct proof.

Proof. Let u be a smooth function on X and let f be a smooth positive density such that

$$(\omega + dd^c u)^k \wedge \omega_X^{n-k} \leq f \omega_X^n.$$

For $j \geq 1$, we solve the complex Hessian equation

$$(\omega + dd^c \varphi_j)^k \wedge \omega_X^{n-k} = e^{j(\varphi_j - u)} f \omega_X^n.$$

As shown in [KN16], the estimates established in [Szek18] provide a smooth solution φ_j which is (ω, k) -subharmonic on X . We want to prove that φ_j converges uniformly to $P_{\omega, k}(u)$, the largest (ω, k) -subharmonic function lying below u . The theorem follows from this fact, as shown in [LN15, KN16].

The classical maximum principle ensures that $\varphi_j \leq u$, hence $\varphi_j \leq P_{\omega,k}(u)$. Fix now $\varphi \in SH(X, \omega, k)$ such that $\varphi \leq u$. Let $x_0 \in X$ be a point where the function $\varphi - (1 + 1/j)\varphi_j$ attains its maximum on X . By definition φ belongs to $SH_\gamma(X, \omega)$, where γ is the hermitian form such that

$$\gamma^{n-1} = (\omega + dd^c \varphi_j)^{k-1} \wedge \omega_X^{n-k}.$$

From now on the computations are made at x_0 . It follows from Lemma 1.4 that

$$(\omega + (1 + 1/j)dd^c \varphi_j) \wedge \gamma^{n-1} \geq 0.$$

From this and Gårding's inequality [Gar59] we obtain

$$\begin{aligned} \frac{j+1}{j}(\omega + dd^c \varphi_j)^k \wedge \omega_X^{n-k} &\geq \frac{1}{j}\omega \wedge (\omega + dd^c \varphi_j)^{k-1} \wedge \omega_X^{n-k} \\ &\geq \frac{1}{j}e^{\frac{j(k-1)}{k}(\varphi_j - u)} f^{(k-1)/k} g^{1/k} \omega_X^n, \end{aligned}$$

where $g\omega_X^n = \omega^k \wedge \omega_X^{n-k}$. Let $b \in (0, 1)$ be a positive constant such that $g \geq b^k f$ on X . The above inequality yields

$$(1 + j)e^{j(\varphi_j - u)/k} \geq b,$$

hence

$$\varphi_j - u \geq \frac{k(\log b - \log(1 + j))}{j}.$$

Thus, for some uniform constant $C \geq 1$ we have

$$\left(1 + \frac{1}{j}\right) \varphi_j - u \geq \left(1 + \frac{1}{j}\right) (\varphi_j - u) + \frac{u}{j} \geq -C \frac{\log j}{j}.$$

Since $\varphi \leq u$, we also obtain

$$\varphi - \left(1 + \frac{1}{j}\right) \varphi_j \leq C \frac{\log j}{j}.$$

This inequality holds on X , as $\varphi - (1 + 1/j)\varphi_j$ attains its global maximum at x_0 . Since this is true for all $\varphi \in SH(X, \omega, k)$ lying below u , we obtain

$$P_{\omega,k}(u) - C' \frac{\log j}{j} \leq \varphi_j \leq P_{\omega,k}(u),$$

for a uniform constant C' , proving that φ_j converges uniformly to $P_{\omega,k}(u)$. \square

1.2. The Hessian operator.

1.2.1. Hermitian forms. We assume in this subsection that ω is a hermitian form. For (ω, ω_X, k) -subharmonic functions φ of class \mathcal{C}^2 , the complex Hessian operator is defined by

$$H_k(\varphi) = (\omega + dd^c \varphi)^k \wedge \omega_X^{n-k}.$$

By Theorem 1.7, one can pointwise approximate from above any (ω, ω_X, k) -sh function φ by a decreasing sequence of smooth (ω, ω_X, k) -sh functions φ_j .

If φ is moreover continuous on X , then the convergence is uniform, and it was shown in [KN16] that the sequence of Hessian measures $H_k(\varphi_j)$ weakly converges to a unique positive Radon measure, independent of the approximants. One sets

$$H_k(\varphi) := \lim_{j \rightarrow +\infty} H_k(\varphi_j) := (\omega + dd^c \varphi)^k \wedge \omega_X^{n-k}.$$

It is likely that one can define these operators for (ω, ω_X, k) -sh functions that are merely bounded. This is indeed the case when ω_X is closed, as one can proceed by induction and use integration by parts following Bedford-Taylor's construction

[BT76, BT82]. The situation is however delicate in the general hermitian setting, and more subtle than for the Monge-Ampère operator.

Among the basic properties enjoyed by this operator are the following:

- $\varphi \mapsto H_k(\varphi)$ is continuous for the uniform topology;
- if $u_1, \dots, u_k \in \mathcal{SH}(X, \omega, k)$ are continuous, the mixed Hessian operators $(\omega + dd^c u_1) \wedge \dots \wedge (\omega + dd^c u_k) \wedge \omega_X^{n-k}$ are similarly defined and continuous;
- if u_1, \dots, u_k are smooth and $(\omega + dd^c u_j)^k \wedge \omega_X^{n-k} = f_j dV_X$ then

$$(1.1) \quad (\omega + dd^c u_1) \wedge \dots \wedge (\omega + dd^c u_k) \wedge \omega_X^{n-k} \geq (f_1 \dots f_k)^{1/k} dV_X.$$

These are a consequence of inequalities due to Gårding.

1.2.2. Big forms. We now no longer assume that ω is positive, but rather that it is semi-positive and big. Fix a quasi-plurisubharmonic function $\rho : X \rightarrow \mathbb{R} \cup \{-\infty\}$ with analytic singularities such that $\omega + dd^c \rho \geq \delta \omega_X$, for some constant $\delta > 0$. We let Ω denote the Zariski open set where ρ is smooth.

If $\varphi \in \mathcal{SH}(X, \omega, k) \cap L^\infty(X)$ is continuous in Ω , it follows from [KN16, Proposition 2.9] that for each smooth form γ , the mixed product $(\omega_X + dd^c \varphi)^j \wedge \gamma^{l-j} \wedge \omega_X^{n-l}$ is a well-defined Radon measure in Ω with finite mass.

Definition 1.8. For $\varphi \in \mathcal{SH}(X, \omega, k) \cap L^\infty(X) \cap \mathcal{C}^0(\Omega)$ we set

$$H_k(\varphi) := \sum_{j=0}^k \binom{k}{j} (\omega_X + dd^c \varphi)^j \wedge (\omega - \omega_X)^{k-j} \wedge \omega_X^{n-k}.$$

This operator is well-defined in Ω with finite total mass, so we consider its trivial extension to X . The definition is motivated by the identity

$$\begin{aligned} (\omega + dd^c \varphi)^k \wedge \omega_X^{n-k} &= (\omega_X + dd^c \varphi + \omega - \omega_X)^k \wedge \omega_X^{n-k} \\ &= \sum_{j=0}^k \binom{k}{j} (\omega_X + dd^c \varphi)^j \wedge (\omega - \omega_X)^{k-j} \wedge \omega_X^{n-k}. \end{aligned}$$

The operator H_k is continuous along sequences that converge locally uniformly in Ω [KN16, Proposition 2.11]. The following result follows.

Proposition 1.9. *If $\varphi \in \mathcal{SH}(X, \omega, k) \cap L^\infty(X) \cap \mathcal{C}^0(\Omega)$, then $H_k(\varphi)$ is a positive Radon measure.*

Proof. Fix $\varepsilon > 0$ and approximate φ -using Theorem 1.7- by a decreasing sequence (φ_j) of $(\omega + \varepsilon \omega_X)$ -sh functions. Since $(\omega + \varepsilon \omega_X + dd^c \varphi_j)^k \wedge \omega_X^{n-k} \geq 0$, we infer from [KN16, Proposition 2.11] that $(\omega + \varepsilon \omega_X + dd^c \varphi)^k \wedge \omega_X^{n-k} \geq 0$. The conclusion follows by letting $\varepsilon \rightarrow 0$, as we then obtain that $(\omega + dd^c \varphi)^k \wedge \omega_X^{n-k} \geq 0$. \square

Lemma 1.10. *Let φ, ψ be (ω, ω_X, k) -sh functions which are continuous in Ω and bounded on X . Then*

$$H_k(\max(\varphi, \psi)) \geq \mathbf{1}_{\{\varphi > \psi\}} H_k(\varphi) + \mathbf{1}_{\{\psi \geq \varphi\}} H_k(\psi).$$

In particular, if $\varphi \leq \psi$ then

$$\mathbf{1}_{\{\psi = \varphi\}} H_k(\varphi) \leq \mathbf{1}_{\{\psi = \varphi\}} H_k(\psi).$$

Proof. Fix $t > 0$. Since $\{\varphi < \psi + t\}$ and $\{\psi + t < \varphi\}$ are open in Ω , we have

$$\begin{aligned} H_k(\max(\varphi, \psi + t)) &\geq \mathbf{1}_{\{\varphi > \psi + t\}} (\omega + dd^c \varphi)^k \wedge \omega_X^{n-k} + \mathbf{1}_{\{\psi + t > \varphi\}} (\omega + dd^c \psi)^k \wedge \omega_X^{n-k} \\ &\geq \mathbf{1}_{\{\varphi > \psi + t\}} (\omega + dd^c \varphi)^k \wedge \omega_X^{n-k} + \mathbf{1}_{\{\psi \geq \varphi\}} (\omega + dd^c \psi)^k \wedge \omega_X^{n-k}. \end{aligned}$$

Letting $t \searrow 0$ we obtain the first statement of the lemma. The second one follows straightforwardly from the first. \square

1.2.3. *SH(X, ω, k)-envelopes.* We consider two types of envelopes in this note.

Definition 1.11. If $h : X \rightarrow \mathbb{R}$ is a continuous function, we set

$$P_\omega(h) := \sup \{u \in PSH(X, \omega), u \leq h\},$$

and $P_{\omega,k}(h) := \sup \{u \in SH(X, \omega, k), u \leq h\}.$

Here is a list of basic properties of these envelopes.

- The measure $(\omega + dd^c P_\omega(h))^n$ is concentrated on the set $\mathcal{C} = \{P_\omega(h) = h\}.$
- The complex Hessian measure $(\omega + dd^c P_{\omega,k}(h))^k \wedge \omega_X^{n-k}$ is concentrated on the contact set $\mathcal{C}_k = \{P_{\omega,k}(h) = h\}.$
- If h is $\mathcal{C}^{1,1}$ -smooth and ω is hermitian, then $P_\omega(h)$ is $\mathcal{C}^{1,1}$ -smooth and

$$(\omega + dd^c P_\omega(h))^n = \mathbf{1}_{\mathcal{C}}(\omega + dd^c h)^n.$$

- If h is $\mathcal{C}^{1,1}$ -smooth and ω is hermitian, then $P_{\omega,k}(h)$ is $\mathcal{C}^{1,1}$ -smooth and

$$(\omega + dd^c P_{\omega,k}(h))^k \wedge \omega_X^{n-k} = \mathbf{1}_{\mathcal{C}_k}(\omega + dd^c h)^k \wedge \omega_X^{n-k}.$$

We refer the reader to [Ber19, Tos18, GLZ19, CZ19, CM21] for the proof of these results, as well as further information on these envelopes.

1.3. **A priori estimates for the complex Monge-Ampère operator.** We are going to use the following a priori estimates established in [GL21].

Theorem 1.12. *Let ω, ω_0 be semi-positive and big $(1, 1)$ -forms such that $\omega \geq \omega_0$. Fix $\delta > 0$ and $g \in L^{1+\delta}(dV_X)$ a non-negative density. If $\varphi \in PSH(X, \omega) \cap L^\infty(X)$ satisfies $(\omega + dd^c \varphi)^n \leq g dV_X$, then*

$$\text{Osc}_X(\varphi) \leq C$$

for some $C \in \mathbb{R}^+$ that depends on n, δ, ω_0 and an upper bound on $\|g\|_{L^{1+\delta}}.$

Since this result is not explicitly stated in [GL21], we briefly sketch its proof.

Proof. We normalize φ so that $\sup_X \varphi = 0$. It follows from Skoda's uniform integrability theorem [GZ17, Theorem 8.11] that for $0 < \alpha$ small enough,

$$\int_X e^{-\alpha \frac{(2+\delta)(1+\delta)}{\delta} \varphi} dV_X \leq C_\alpha$$

is bounded from above independently of φ . Setting $\tilde{g} = e^{-\alpha \varphi} g$, Hölder inequality then shows that $\tilde{g} \in L^{1+\delta/2}$ with

$$\|\tilde{g}\|_{L^{1+\delta/2}} \leq C_\alpha^{1/q} \|g\|_{L^{1+\delta}}^{1/p},$$

where $p = \frac{1+\delta}{1+\delta/2}$ and $q = 2 + 2/\delta$. [GL21, Lemma 3.3] shows the existence of $c(\delta, \omega) > 0$ and $u \in PSH(X, \omega)$ such that $-1 \leq u \leq 0$ and

$$(\omega + dd^c u)^n \geq c(\delta, \omega) \frac{\tilde{g} dV_X}{\|\tilde{g}\|_{L^{1+\delta/2}}} \geq e^{\alpha u} c(\delta, \omega) \frac{\tilde{g} dV_X}{\|\tilde{g}\|_{L^{1+\delta/2}}}.$$

The function $v := u - \alpha^{-1} \log[c(\delta, \omega)/\|\tilde{g}\|_{L^{1+\delta/2}}] \in PSH(X, \omega)$ thus satisfies

$$e^{-\alpha v} (\omega + dd^c v)^n = e^{-\alpha v} (\omega + dd^c u)^n \geq \tilde{g} dV_X \geq e^{-\alpha \varphi} (\omega + dd^c \varphi)^n.$$

The comparison principle [GL21, Corollary 1.14] finally ensures that $v \leq \varphi$.

This provides a uniform lower bound on φ , which only depends on δ, ω and an upper bound on $\|g\|_{L^{1+\delta}}.$ \square

2. DOMINATION PRINCIPLE

Fix ω a big form, ρ an ω -plurisubharmonic function with analytic singularities such that $\omega + dd^c \rho \geq \delta \omega_X$, $\delta > 0$, and set $\Omega := \{\rho > -\infty\}$.

We fix a constant $B_1 > 0$ such that for all $x \in \Omega$,

$$-B_1 \omega_\rho^2 \leq dd^c \omega_\rho \leq B_1 \omega_\rho^2 \quad \text{and} \quad -B_1 \omega_\rho^3 \leq d\omega_\rho \wedge d^c \omega_\rho \leq B_1 \omega_\rho^3.$$

The existence of B_1 is clear since $d\omega = d\omega_\rho$, $d^c \omega = d^c \omega_\rho$ and

$$-B\omega_X^2 \leq dd^c \omega \leq B\omega_X^2 \quad \text{and} \quad -B\omega_X^3 \leq d\omega \wedge d^c \omega \leq B\omega_X^3$$

for some $B > 0$. The following is an extension of [GL21, Theorem 1.11].

Theorem 2.1. *Assume ω is big. If u is a bounded (ω, ω_X, k) -subharmonic function which is continuous in Ω , then $\int_X (\omega + dd^c u)^k \wedge \omega_X^{n-k} > 0$.*

Proof. Fix ρ and B_1 as above. We set $m = \inf_\Omega (u - \rho)$. For $s > 0$ we set

$$\phi = \max(u, \rho + m + s) \quad \text{and} \quad U := \{u < \phi\} = \{u < \rho + m + s\}.$$

Observe that U is relatively compact in Ω , and it is non empty for all $s > 0$, by definition of m . It follows from [KN16, Corollary 2.4] that, for any $0 \leq j \leq k-1$,

$$dd^c(\omega_u^j \wedge \omega_\phi^{k-j-1} \wedge \omega_X^{n-k}) \leq C_1 \sum_{m=k-2}^{k-1} \sum_{l=0}^m \omega_u^l \wedge \omega_\phi^{m-l} \wedge \omega_X^{n-m},$$

where C_1 is a uniform constant. On the open set U since $\omega_\phi \geq \delta \omega_X$, we thus have

$$(2.1) \quad dd^c(\omega_u^j \wedge \omega_\phi^{k-j-1} \wedge \omega_X^{n-k}) \leq C_2 \sum_{l=0}^k \omega_u^l \wedge \omega_\phi^{k-l} \wedge \omega_X^{n-k}.$$

Since $u \leq \phi$, Lemma 1.10 ensures that

$$\mathbf{1}_{\{u=\phi\}} \omega_u^j \wedge \omega_\phi^{k-j} \wedge \omega_X^{n-k} \geq \mathbf{1}_{\{u=\phi\}} \omega_u^k \wedge \omega_\phi^{n-k}.$$

Noting that $X \setminus U = \{u = \phi\}$ we can write

$$\begin{aligned} \int_U (\omega_u^j \wedge \omega_\phi^{k-j} \wedge \omega_X^{n-k} - \omega_u^k \wedge \omega_X^{n-k}) &\leq \int_X (\omega_u^j \wedge \omega_\phi^{k-j} \wedge \omega_X^{n-k} - \omega_u^k \wedge \omega_X^{n-k}) \\ &= \int_X dd^c(\phi - u) \wedge \sum_{l=0}^{k-j} \omega_u^l \wedge \omega_\phi^{k-j-l} \wedge \omega_X^{n-k} \\ &= \int_X (\phi - u) dd^c \left[\sum_{l=0}^{k-j} \omega_u^l \wedge \omega_\phi^{k-j-l} \wedge \omega_X^{n-k} \right]. \end{aligned}$$

In the last line above, we have used Stokes theorem. Since $0 \leq \phi - u \leq s$, using (2.1) we can continue the above inequalities as follows:

$$\int_U (\omega_u^j \wedge \omega_\phi^{k-j} \wedge \omega_X^{n-k} - \omega_u^k \wedge \omega_X^{n-k}) \leq C_3 s \sum_{l=0}^k \int_U \omega_u^l \wedge \omega_\phi^{k-l} \wedge \omega_X^{n-k}.$$

Summing up the above inequalities for $j = 0, \dots, k$, one gets

$$(1 - C_4 s) \sum_{l=0}^k \int_U \omega_u^l \wedge \omega_\phi^{k-l} \wedge \omega_X^{n-k} \leq (k+1) \int_U \omega_u^k \wedge \omega_X^{n-k}.$$

If $\omega_u^k \wedge \omega_X^{n-k} = 0$ then, for $s > 0$ small enough, the above inequality yields $\int_U \omega_\phi^k \wedge \omega_X^{n-k} = 0$, hence U has Lebesgue measure 0, which is impossible. \square

Theorem 2.2. *Assume ω is hermitian, u, v are continuous (ω, ω_X, k) -subharmonic functions such that $\omega_u^k \wedge \omega_X^{n-k} = 0$ on $\{u < v\}$. Then $u \geq v$.*

Proof. By replacing v with $\max(u, v)$ and noting that $\{u < v\} = \{u < \max(u, v)\}$, we can assume that $u \leq v$. Our plan is thus to show that $u = v$.

For $b \geq 1$, we set $\phi_b := P_{\omega, k}(bu - (b-1)v)$. It is a negative (ω, k) -sh function. Since $bu - (b-1)v$ is continuous on X , so is ϕ_b . Observe also that

$$b^{-1}\phi_b + (1 - b^{-1})v \leq u$$

with equality on the contact set $\mathcal{C} := \{\phi_b = bu - (b-1)v\}$. Now by Lemma 1.10

$$\mathbf{1}_{\mathcal{C}} \left(b^{-k} \omega_{\phi_b}^k \wedge \omega_X^{n-k} + (1 - b^{-1})^k \omega_v^k \wedge \omega_X^{n-k} \right) \leq \mathbf{1}_{\mathcal{C}} \omega_u^k \wedge \omega_X^{n-k},$$

while $\int_{\mathcal{C}} H_k(\phi_b) = \int_X H_k(\phi_b) > 0$ by Theorem 2.1. Since $H_k(u) = 0$ on $\{u < v\}$, it follows that $\mathcal{C} \cap \{u = v\}$ contains at least one point, say x . We then have

$$\phi_b(x) = bu(x) - (b-1)v(x) = v(x) \geq \inf_X v;$$

in particular $\sup_X \phi_b \geq \inf_X v$.

Note that ϕ_b is decreasing in b since $u \leq v$. The limit $\phi := \lim_{b \rightarrow +\infty} \phi_b$ is a (ω, k) -sh function which is not identically $-\infty$. For any $a > 0$,

$$\phi_b \leq bu - (b-1)v \leq v - ba$$

on the open set $\{u < v - a\}$, thus $\phi = -\infty$ on this set. Since ϕ is integrable on X , the set $\{u < v - a\}$ must be empty, hence letting $a \rightarrow 0^+$ gives $u \geq v$. \square

Corollary 2.3. *Assume ω is hermitian, u and v are continuous (ω, ω_X, k) -subharmonic functions such that $\omega_u^k \wedge \omega^{n-k} \leq c\omega_v^k \wedge \omega^{n-k}$ on $\{u < v\}$ for some $c \in [0, 1)$. Then $u \geq v$.*

Proof. Take b so large that $(1 - b^{-1})^k \geq c$, and consider $\phi := P_{\omega, k}(bu - (b-1)v)$. Then $b^{-1}\phi + (1 - b^{-1})v \leq u$ with equality on the contact set

$$\mathcal{C} := \{\phi = bu - (b-1)v\}.$$

By Lemma 1.10 we have

$$\mathbf{1}_{\mathcal{C}} \left(b^{-k} \omega_{\phi}^k \wedge \omega_X^{n-k} + (1 - b^{-1})^k \omega_v^k \wedge \omega_X^{n-k} \right) \leq \mathbf{1}_{\mathcal{C}} \omega_u^k \wedge \omega_X^{n-k}.$$

Multiplying with $\mathbf{1}_{\{u < v\}}$ and using the assumption $\omega_u^k \wedge \omega^{n-k} \leq c\omega_v^k \wedge \omega^{n-k}$ on $\{u < v\}$ and $(1 - b^{-1})^k \geq c$, we then see that

$$\mathbf{1}_{\mathcal{C} \cap \{u < v\}} \omega_{\phi}^k \wedge \omega^{n-k} = 0.$$

Since $\omega_{\phi}^k \wedge \omega^{n-k}$ is supported on \mathcal{C} , we thus have

$$\int_{\{\phi < v\}} \omega_{\phi}^k \wedge \omega^{n-k} = \int_{\{bu - (b-1)v < v\}} \omega_{\phi}^k \wedge \omega^{n-k} = \int_{\{u < v\}} \omega_{\phi}^k \wedge \omega^{n-k} = 0.$$

Invoking the domination principle (Theorem 2.2) we then get $\phi \geq v$. It thus follows that $bu - (b-1)v \geq \phi \geq v$, hence $u \geq v$. \square

Corollary 2.4. *Assume ω is hermitian and u and v are continuous (ω, ω_X, k) -subharmonic functions. Then*

$$e^{-v}(\omega + dd^c v)^k \wedge \omega_X^{n-k} \geq e^{-u}(\omega + dd^c u)^k \wedge \omega_X^{n-k} \implies u \geq v.$$

Proof. Fix $a > 0$. On the set $\{u < v - a\}$ we have $H_k(u) \leq e^{-a}H_k(v)$. By Corollary 2.3, $u \geq v - a$. Since it holds for all $a > 0$, we obtain $u \geq v$. \square

3. PROOF OF THE MAIN THEOREMS

3.1. A priori estimates. In this section we prove Theorem B, which is an extension of the main result of Kołodziej-Nguyen [KN16]:

Theorem 3.1. *Let ω_0, ω be semi-positive and big forms such that $\omega_0 \leq \omega$. Let φ be a smooth (ω, ω_X, k) -subharmonic function satisfying*

$$(\omega + dd^c \varphi)^k \wedge \omega_X^{n-k} \leq f dV_X,$$

where $f \in L^p(dV_X)$, $p > n/k$. Then

$$\text{Osc}_X(\varphi) \leq C,$$

where $C > 0$ depends on ω_0, n, k, p and an upper bound for $\|f\|_{L^p}$.

Proof. We can assume that $\sup_X \varphi = 0$. Let $u := P_\omega(\varphi)$ be the largest ω -psh function lying below φ . Let \mathcal{C} denote the contact set $\mathcal{C} := \{u = \varphi\}$. Then u is (ω, ω_X, k) -sh with $u \leq \varphi$ hence Lemma 1.10 yields

$$\mathbf{1}_{\mathcal{C}}(\omega + dd^c u)^k \wedge \omega_X^{n-k} \leq \mathbf{1}_{\mathcal{C}}(\omega + dd^c \varphi)^k \wedge \omega_X^{n-k} \leq f dV_X.$$

Observe that the smooth function $h := \omega_X^n / dV_X \geq \delta_0$ is positive on X . Recall that the Monge-Ampère measure $(\omega + dd^c u)^n$ of u is concentrated on the contact set \mathcal{C} . We set $g := (\omega + dd^c u)^n / dV_X$. It follows from Gårding's inequalities (1.1) that

$$\delta_0^{1-k/n} g^{k/n} \leq h^{1-k/n} g^{k/n} \leq f,$$

hence $g \leq C_0 f^{n/k}$. Since $f \in L^p$ with $p > n/k$, we obtain $g \in L^{1+\delta}$ with $\delta > 0$. Theorem 1.12 therefore provides a uniform bound

$$\text{Osc}_X(u) \leq M.$$

This provides a uniform bound on $u \leq 0$ if we can bound $\sup_X u$ from below.

Let q' be the conjugate exponent to $p' = pk/n > 1$. We obtain a uniform lower bound on $\sup_X u$ by using [GL22, Proposition 3.4] and Hölder inequality, via

$$\begin{aligned} v_M^-(\omega)(-\sup_X u)^{1/q'} &\leq \int_X |u|^{1/q'} (\omega + dd^c u)^n \\ &\leq \left(\int_X |\varphi| \omega^n \right)^{1/q'} \left(\int_X f^{p'} \omega^n \right)^{1/p'} \\ &\leq C, \end{aligned}$$

as $\int_X |\varphi| \omega^n$ is bounded from above independently of φ by Proposition 1.6. Together with the uniform bound for $\text{Osc}_X(u)$, this yields a uniform bound for $|u|$. This implies a uniform bound for φ since $u \leq \varphi$ and $\sup_X \varphi = 0$. \square

Remark 3.2. We also have a uniform a priori bound when f satisfies the weaker integrability condition

$$\int_X f^{n/k} |\log f|^n |h \circ \log \circ \log f|^n dt < +\infty$$

where h is an increasing continuous function such that $\int^{+\infty} dt/h(t) < +\infty$. It follows indeed that g satisfies then Kołodziej's optimal integrability condition

$$\int_X g |\log g|^n |h \circ \log \circ \log g|^n dt < +\infty,$$

which ensures a uniform bound on $\text{Osc}_X(u)$ by [Kol98, Theorem 2.5.2]. We actually need here the extension of Kołodziej's result to the case of semi-positive and big forms [EGZ09], as well as to the hermitian setting [DK12, GL21].

The same proof thus applies with a minor modification: to obtain a lower bound for $\sup_X u$, we replace Hölder inequality by the additive Hölder-Young inequality. If w denotes the convex weight $w(t) \sim t(\log t)^n(h \circ \log \circ \log t)^n$, w^* is its conjugate convex weight, and $(w^*)^{-1}$ denotes the inverse of w^* , we obtain this way a uniform upper bound

$$\begin{aligned} v_M^-(\omega)(w^*)^{-1}(-\sup_X u) &\leq \int_X (w^*)^{-1}(u)(\omega + dd^c u)^n \\ &\leq \int_X w \circ f dV_X + \int_X (-u) dV_X \leq C. \end{aligned}$$

3.2. Stability estimates. We strengthen Theorem B by establishing the following stability estimate:

Theorem 3.3. *Assume ω is hermitian and $\omega \geq \omega_0$ for some semi-positive and big form ω_0 . Assume $0 \leq f, g \in L^p(dV_X)$ with $p > n/k$, φ, ψ are continuous (ω, ω_X, k) -subharmonic functions such that*

$$(\omega + dd^c \varphi)^k \wedge \omega_X^{n-k} \geq e^\varphi f \omega_X^n, \quad \text{and} \quad (\omega + dd^c \psi)^k \wedge \omega_X^{n-k} \leq e^\psi g \omega_X^n.$$

Then

$$(3.1) \quad \varphi \leq \psi + C \|f - g\|_p^{1/k}$$

for some $C > 0$ depending on ω_0, n, k and upper bounds for $\|f\|_p, \|\varphi\|_\infty, |\inf_X \psi|$.

In particular if φ, ψ are solutions of the corresponding equations, then

$$\|\varphi - \psi\|_\infty \leq C \|f - g\|_p^{1/k},$$

so that $\varphi - \psi$ is uniformly small if f is close to g in L^p .

Proof. Our proof uses a perturbation argument from [GLZ18, LPT21], which goes back to [Kol96]. We solve the complex Monge-Ampère equation

$$(\omega + dd^c u)^n = b h \omega_X^n, \quad \text{where} \quad h = \frac{|f - g|^{n/k}}{\|f - g\|_p^{n/k}} + 1,$$

$\sup_X u = 0$ and $b > 0$ is a constant. The existence of a continuous solution $u \in PSH(X, \omega)$ follows from [KN15, KN19].

We can assume without loss of generality that ω_X is a Gauduchon metric (i.e. $dd^c(\omega_X^{n-1}) = 0$) which dominates ω . Thus u is ω_X -psh with

$$(\omega_X + dd^c u)^n \geq (\omega + dd^c u)^n \geq b \omega_X^n$$

since $h \geq 1$. The mixed Monge-Ampère inequalities (see [Ng16]) yield

$$(\omega_X + dd^c u) \wedge \omega_X^{n-1} \geq b^{1/n} \omega_X^n,$$

hence $b \leq 1$ since $\int_X (\omega_X + dd^c u) \wedge \omega_X^{n-1} = \int_X \omega_X^n$ by the Gauduchon condition.

Since $h \in L^q$ with $q = pk/n > 1$ and $b \leq 1$, Theorem 3.1 ensures that $\|u\|_\infty$ is uniformly bounded. It thus follows from [GL22, Proposition 3.4] that $b \geq \delta_0 > 0$ is uniformly bounded from below.

If $\|f - g\|_p^{1/k}$ is not small, the stability inequality (3.1) is trivially obtained by adjusting the value of C . In the remainder of the proof, we can thus assume without loss of generality that

$$(3.2) \quad \varepsilon := b^{-1/n} \|f - g\|_p^{1/k} e^{\sup_X \varphi/k} \leq 1/2.$$

By the mixed Monge-Ampère inequalities (1.1) we have

$$(\omega + dd^c u)^k \wedge \omega_X^{n-k} \geq b^{k/n} \frac{|f - g|}{\|f - g\|_p} \omega_X^n.$$

We set $C_1 = 2k - \inf_X \varphi$ and

$$\psi_\varepsilon := (1 - \varepsilon)\varphi + \varepsilon u - C_1 \varepsilon,$$

The function ψ_ε belongs to $SH(X, \omega, k) \cap \mathcal{C}^0(X)$; its Hessian measure satisfies

$$\begin{aligned} (\omega + dd^c \psi_\varepsilon)^k \wedge \omega_X^{n-k} &\geq (1 - \varepsilon)^k e^\varphi f \omega_X^n + \varepsilon^k b^{k/n} \frac{|f - g|}{\|f - g\|_p} \omega_X^n \\ &= \left[(1 - \varepsilon)^k e^\varphi f + e^{\sup_X \varphi} |f - g| \right] \omega_X^n \\ &\geq e^{\varphi - 2k\varepsilon} g \omega_X^n, \end{aligned}$$

using (3.2), $|f - g| + f \geq g$ and the inequality $\log(1 - \varepsilon) \geq -2\varepsilon$ valid on $[0, 1/2]$.

Observe now that $\psi_\varepsilon \leq \varphi - \varepsilon\varphi - C_1\varepsilon \leq \varphi - 2k\varepsilon$ by the definition of C_1 and $u \leq 0$. We infer

$$(\omega + dd^c \psi_\varepsilon)^k \wedge \omega_X^{n-k} \geq e^{\psi_\varepsilon} g \omega_X^n,$$

hence $\psi_\varepsilon \leq \psi$ by Corollary 2.4. This yields

$$\varphi \leq \psi + \varepsilon(\varphi - u) + C_1\varepsilon \leq \psi + C_2\|f - g\|_p^{1/k},$$

for some uniform constant C_2 , as desired. \square

3.3. Proof of Theorem A. We approximate f in L^p by smooth and positive densities f_j . For $j \geq 1$ we consider the hermitian forms $\omega_j = \omega + 2^{-j}\omega_X$. It follows from [Zh17, Szek18] that there exist unique constants $c_j > 0$ and smooth (ω_j, k) -sh functions φ_j such that

$$(\omega_j + dd^c \varphi_j)^k \wedge \omega_X^{n-k} = c_j f_j \omega_X^n,$$

and $\sup_X \varphi_j = 0$.

We claim that the constants c_j stay uniformly bounded, as j increases to $+\infty$. To prove the claim we use the solutions to the Monge-Ampère equations by Tosatti-Weinkove [TW10]:

$$(\omega_j + dd^c u_j)^n = b_j f_j^{n/k} \omega_X^n.$$

Set $p' = pk/n > 1$. Since the $L^{p'}$ norm of $f_j^{n/k}$ is uniformly bounded from above, by [GL21, Lemma 3.3], there exists a constant $c > 0$ independent of j and a bounded ω -psh function u such that $\omega_u^n \geq c f_j^{n/k}$. By [GL21, Corollary 1.13], $b_j \geq c$. It then follows from Gårding's inequality that

$$(\omega_j + dd^c u_j)^k \wedge \omega_X^{n-k} \geq c^{k/n} f_j \omega_X^n.$$

Corollary 2.3 thus ensures that $c_j \geq c^{k/n}$ is uniformly bounded from below by a positive constant.

We next bound c_j from above. By Gårding's inequality we have

$$(\omega_j + dd^c \varphi_j) \wedge \omega_X^{n-1} \geq c_j^{1/k} f_j^{1/k} \omega_X^n.$$

The functions φ_j belong to $SH(X, \omega_1)$ and are relatively compact in L^1 , hence integrating over X and using that $\int_X f_j^{1/k} \omega_X^n \rightarrow \int_X f^{1/k} \omega_X^n > 0$, we obtain a uniform upper bound for c_j .

Since $\omega \leq \omega_j$ and $\|c_j f_j\|_{L^p}$ is uniformly bounded from above, it follows from Theorem 3.1 that the φ_j 's are uniformly bounded:

$$-C_1 \leq \varphi_j \leq 0,$$

for a uniform constant C_1 . Extracting and using compactness in $L^1(X)$ of normalized (ω, ω_X, k) -sh functions, we can thus assume that $c_j \rightarrow c > 0$ and φ_j converges in $L^1(X)$ and almost everywhere to a bounded function φ which is (ω, ω_X, k) -sh, as $j \rightarrow +\infty$.

We next prove that the convergence $\varphi_j \rightarrow \varphi$ is locally uniform in Ω . We set $g_j := c_j f_j e^{-\varphi_j}$ and $g = c f e^{-\varphi}$. By Hölder inequality, $\|g_j - g\|_q \rightarrow 0$ for some $q > n/k$. Fix $j < l$ large enough. Since $\omega_j \geq \omega_l$, we have

$$(\omega_j + dd^c \varphi_l)^k \wedge \omega_X^{n-k} \geq e^{\varphi_l} g_l \omega_X^n.$$

By Theorem 3.3, we infer

$$(3.3) \quad \varphi_l \leq \varphi_j + C_2 \|g_j - g_l\|_q^{1/k},$$

for a uniform constant C_2 . We next consider

$$\psi_j := (1 - \varepsilon)\varphi_j + \varepsilon\rho - (2k + C_1)\varepsilon,$$

where $\varepsilon = \delta^{-1}2^{-j} < 1/2$ when j is large enough. Since $\omega + dd^c \rho \geq \delta\omega_X$, we have

$$\begin{aligned} \omega_l + dd^c \psi_j &\geq \omega + \varepsilon dd^c \rho + (1 - \varepsilon) dd^c \varphi_j \\ &\geq \varepsilon \delta \omega_X + (1 - \varepsilon)(\omega_j + dd^c \varphi_j) - 2^{-j} \omega_X \\ &= (1 - \varepsilon)(\omega_j + dd^c \varphi_j). \end{aligned}$$

It follows that $\psi_j \in SH(X, \omega_l, k)$ and

$$(\omega_l + dd^c \psi_j)^k \wedge \omega_X^{n-k} \geq e^{\varphi_j + k \log(1-\varepsilon)} g_j \omega_X^n \geq e^{\varphi_j - 2k\varepsilon} g_j \omega_X^n \geq e^{\psi_j} g_j \omega_X^n$$

holds in Ω . The function $\psi_{j,l} := \max(\psi_j, \varphi_l)$ is (ω_l, k) -subharmonic and continuous on X . Lemma 1.10 yields

$$(\omega_l + dd^c \psi_{j,l})^k \wedge \omega_X^{n-k} \geq e^{\psi_{j,l}} \min(g_j, g_l) \omega_X^n.$$

Applying Theorem 3.3 we then get

$$\psi_{j,l} \leq \varphi_l + C_3 \|\min(g_j, g_l) - g_l\|_q^{1/k} \leq \varphi_l + C_3 \|g_j - g_l\|_q^{1/k},$$

for a uniform constant C_3 . Together with (3.3), this implies

$$\begin{aligned} \varphi_j + \delta^{-1}2^{-j}\rho - (2k + C_1)\delta^{-1}2^{-j} &\leq \varphi_l + C_3 \|g_j - g_l\|_q^{1/k} \\ &\leq \varphi_j + C_1 \|g_j - g_l\|_q^{1/k}. \end{aligned}$$

Therefore, φ_j converges locally uniformly in Ω to φ . Thus φ is continuous in Ω where it solves the complex Hessian equation

$$(\omega + dd^c \varphi)^k \wedge \omega_X^{n-k} = c f \omega_X^n.$$

Remark 3.4. When the form ω is hermitian, the solution φ that we construct is continuous on the whole of X . Continuity of φ at points where ω vanishes is a delicate issue, even in the case of complex Monge-Ampère equations. We refer the interested reader to [GGZ23] for a recent account.

The uniqueness of φ is also a subtle problem in the hermitian setting, see [KN19] for a partial result in the case of the complex Monge-Ampère equation.

REFERENCES

- [BT76] E. Bedford, B.A. Taylor, *The Dirichlet problem for the complex Monge-Ampère operator*. Invent. Math., 37 (1976), 1–44.
- [BT82] E. Bedford, B.A. Taylor, *A new capacity for plurisubharmonic functions*. Acta Math. **149** (1982), no. 1-2, 1–40.
- [BZ20] A. Benali, A. Zeriahi, *The Hölder continuous subsolution theorem for complex Hessian equations*. J. Éc. polytech. Math. 7 (2020), 981–1007.
- [Ber19] R. J. Berman, *From Monge-Ampère equations to envelopes and geodesic rays in the zero temperature limit*. Math. Z. **291** (2019), no. 1-2, 365–394.
- [Blo05] Z. Błocki, *Weak solutions to the complex Hessian equation*. Ann. Inst. Fourier (Grenoble) 55 (2005), no. 5, 1735–1756.
- [BEGZ10] S. Boucksom, P. Eyssidieux, V. Guedj, A. Zeriahi, *Monge-Ampère equations in big cohomology classes*. Acta Math. **205** (2010), no. 2, 199–262.
- [Char16] M. Charabati, *Modulus of continuity of solutions to complex Hessian equations*. Internat. J. Math. 27 (2016), no. 1, 1650003, 24 pp.
- [CZ19] J. Chu, B. Zhou, *Optimal regularity of plurisubharmonic envelopes on compact Hermitian manifolds*. Sci. China Math. 62 (2019), no. 2, 371–380.
- [CM21] J. Chu, N. McCleerey, *Fully non-linear degenerate elliptic equations in complex geometry*. J. Funct. Anal. 281 (2021), no. 9, Paper No. 109176.
- [Dem94] J.P. Demailly, *Regularization of closed positive currents of type $(1,1)$ by the flow of a Chern connection*. Contributions to complex analysis and analytic geometry, 105–126, Friedr. Vieweg, Braunschweig (1994).
- [DDT19] S. Dinew, H.S. Do, T.D. Tô, *A viscosity approach to the Dirichlet problem for degenerate complex Hessian-type equations*. Anal. PDE 12 (2019), no. 2, 505–535.
- [DK12] S. Dinew, S. Kołodziej, *Pluripotential estimates on compact Hermitian manifolds*. Advances in geometric analysis, 69–86, Adv. Lect. Math. (ALM), 21, Int. Press, 2012.
- [DK14] S. Dinew, S. Kołodziej, *A priori estimates for complex Hessian equations*. Anal. PDE 7 (2014), no. 1, 227–244.
- [DK17] S. Dinew, S. Kołodziej, *Liouville and Calabi-Yau type theorems for complex Hessian equations*. Amer. J. Math. 139 (2017), no. 2, 403–415.
- [DL15] S. Dinew, C.H. Lu, *Mixed Hessian inequalities and uniqueness in the class $\mathcal{E}(X, \omega, m)$* . Math. Z. 279 (2015), no. 3-4, 753–766.
- [EGZ09] P. Eyssidieux, V. Guedj, A. Zeriahi, *Singular Kähler-Einstein metrics*. J. Amer. Math. Soc. **22** (2009), no. 3, 607–639.
- [Gar59] L. Gårding, *An inequality for hyperbolic polynomials*. J. Math. Mech. 8 (1959) 957–965.
- [GGZ23] V. Guedj, H. Guenancia, A. Zeriahi, *Continuity of singular Kähler-Einstein potentials*. Int. Math. Res. Not. (2023) no. 2, 1355–1377.
- [GL22] V. Guedj, C. H. Lu, *Quasi-plurisubharmonic envelopes 2: bounds on Monge-Ampère volumes*. Algebr. Geom. 9 (2022), no. 6, 688–713.
- [GL21] V. Guedj, C. H. Lu, *Quasi-plurisubharmonic envelopes 3: Solving Monge-Ampère equations on hermitian manifolds*. Preprint (2021).
- [GLZ18] V. Guedj, C. H. Lu, A. Zeriahi, *Stability of solutions to complex Monge-Ampère flows*. Ann. Inst. Fourier (Grenoble) 68 (2018), no. 7, p. 2819–2836.
- [GLZ19] V. Guedj, C. H. Lu, A. Zeriahi, *Plurisubharmonic envelopes and supersolutions*. J. Differential Geom. **113** (2019), no. 2, 273–313.
- [GZ17] V. Guedj, A. Zeriahi, *Degenerate complex Monge-Ampère equations*. EMS Tracts in Mathematics, vol. 26, European Mathematical Society (EMS), Zürich, 2017.
- [GN18] D. Gu, N.C. Nguyen, *The Dirichlet problem for a complex Hessian equation on compact Hermitian manifolds with boundary*. Ann. Sc. Norm. Sup. Pisa (5) 18 (2018), 1189–1248.
- [GP22] B. Guo, D.H. Phong, *On L^∞ estimates for fully nonlinear partial differential equations on Hermitian manifolds*. Preprint arXiv (2022).
- [GPT21] B. Guo, D.H. Phong, F. Tong, *On L^∞ estimates for complex Monge-Ampère equations*. Preprint arXiv (2021).
- [GPTW21] B. Guo, D.H. Phong, F. Tong, C. Wang, *On L^∞ estimates for complex Monge-Ampère equations and Hessian equations on nef classes*. Preprint arXiv (2021).
- [HH72] M. Hervé, R.-M. Hervé, *Les fonctions surharmoniques dans l’axiomatique de M. Brelot associées à un opérateur elliptique dégénéré*. Ann. Inst. Fourier, tome 22 (1972), 131–145.

- [HL13] R. Harvey, B. Lawson, *The equivalence of viscosity and distributional subsolutions for convex subequations, a strong Bellman principle*. Bull. Braz. Math. Soc. 44 (2013), 621–652.
- [Kol96] S. Kołodziej, *Some sufficient conditions for solvability of the Dirichlet problem for the complex Monge-Ampère operator*. Ann. Polon. Math. 65 (1996), no. 1, p. 11–21.
- [Kol98] S. Kołodziej, *The complex Monge-Ampère equation*. Acta Math. **180** (1998), 69–117.
- [KN15] S. Kołodziej, N.C.Nguyen, *Weak solutions to the complex Monge-Ampère equation on compact Hermitian manifolds*. Contemp. Math. 644 (2015), 141–158.
- [KN16] S. Kołodziej, N.C.Nguyen, *Weak solutions of complex Hessian equations on compact Hermitian manifolds*. Compos. Math. 152 (2016), no. 11, 2221–2248.
- [KN19] S. Kołodziej, N.C.Nguyen, *Stability and regularity of solutions of the Monge-Ampère equation on Hermitian manifolds*. Adv. Math. 346 (2019), 264–304.
- [Lit83] W. Littman, *Generalized subharmonic functions: Monotonic approximations and an improved maximum principle*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) 17 (1963), 207–222.
- [Lu13] C. H. Lu, *Viscosity solutions to complex Hessian equations*. J. Funct. Anal. 264 (2013), no. 6, 1355–1379.
- [Lu15] C. H. Lu, *A variational approach to complex Hessian equations in \mathbb{C}^n* . J. Math. Anal. Appl. **431** (2015), no. 1, 228–259.
- [LN15] C. H. Lu, V.D. Nguyen, *Degenerate complex Hessian equations on compact Kähler manifolds*. Indiana Univ. Math. J. 64 (2015), no. 6, 1721–1745.
- [LPT21] C. H. Lu, T.T.Phung, T.D.To, *Stability and Hölder regularity of solutions to complex Monge-Ampère equations on compact hermitian manifolds*. Ann. Inst. Fourier 71 (2021), no. 5, 2019–2045.
- [Mic82] M.L. Michelsohn, *On the existence of special metrics in complex geometry*. Acta Math. 149 (1982), 261–295.
- [Ng16] N.-C. Nguyen, *The complex Monge-Ampère type equation on compact Hermitian manifolds and applications*. Adv. Math. 286 (2016), p. 240–285.
- [Szek18] G. Székelyhidi, *Fully non-linear elliptic equations on compact Hermitian manifolds*. J. Differential Geom. 109 (2018), no. 2, 337–378.
- [Tos18] V. Tosatti, *Regularity of envelopes in Kähler classes*. Math. Res. Lett. 25 (2018), no. 1, 281–289.
- [TW10] V. Tosatti, B. Weinkove, *The complex Monge-Ampère equation on compact Hermitian manifolds*. J. Amer. Math. Soc. 23 (2010), no. 4, 1187–1195.
- [Yau78] S. T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*. Comm. Pure Appl. Math. **31** (1978), no. 3, 339–411.
- [Zh17] D. Zhang, *Hessian equations on closed Hermitian manifolds*. Pacific J. Math. 291 (2017), no. 2, 485–510.

INSTITUT DE MATHÉMATIQUES DE TOULOUSE, UNIVERSITÉ DE TOULOUSE, 118 ROUTE DE NARBONNE, 31400 TOULOUSE, FRANCE,

Email address: vincent.guedj@math.univ-toulouse.fr

URL: <https://www.math.univ-toulouse.fr/~guedj/>

UNIV ANGERS, CNRS, LAREMA, SFR MATHSTIC, F-49000 ANGERS, FRANCE.

Email address: hoangchinh.lu@univ-angers.fr

URL: <https://math.univ-angers.fr/~lu/>