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# Optimal influence budget allocation for viral marketing using a multiple virus SIS model\*

Olivier Lindamulage De Silva, Vineeth S. Varma, Irinel-Constantin Morărescu, and Samson Lasaulce<sup>1</sup>

**Abstract**—In the problem addressed in this paper, a firm proposes a service/product to customers that are spread over several geographical regions (e.g., countries, provinces, or states). Each region has its own characteristics in terms of service propagation and is subject to the influence of several firms. The service propagation is modeled by a (marketing) virus process whose effective spreading rate depends on the influence exerted by the firms. The considered overall dynamical system thus involves multiple competing firms whose individual influence over their customers is modeled by a virus and multiple regions. One of the goals of the paper is to know how a firm should allocate its budget among the regions to maximize its market share. The corresponding optimization problem is studied by assuming a Susceptible-Infected-Susceptible (SIS) multiple virus model. In this setting, it is shown under which conditions the so-called “winner takes all” viral marketing strategy is optimal. More generally, the best strategy is derived and its performance is shown to outperform the uniform allocation strategy and advanced strategies like those derived from a Colonel Blotto approach.

**Index Terms**—SIS model, viral marketing, resource allocation.

## I. INTRODUCTION

Viral marketing (VM) is a popular method of market outreach and customer communication that can reach a huge audience quickly and efficiently. Indeed, consumers’ purchasing decisions are heavily influenced by suggestions and recommendations from family, friends, and colleagues rather than popular companies. One of the most famous examples of a very simple VM comes from Hotmail. The company provides free email addresses and services to its users, and each email sent by a Hotmail user promotes the firm’s services. As a result, Hotmail’s subscribers grew from zero to 12 million users in just eighteen months, with an advertising budget of just 50,000 \$ [1]. In the literature, it becomes more and more common to use epidemic models to describe the VM phenomenon. Indeed, when an advertising message goes viral, it is very similar to an epidemic, as it is a human-to-human transmission that spreads through a population. When several products or services are competitively trying to spread over a network, a competitive SIS model can be applied to study the spread of these products. Such a model has been proposed and studied in [2]. The main result of that paper is the so called “winner takes all”, which has also been verified on real data with some popular examples like Facebook and Myspace.

Marketing campaigns that rely on viral approaches to dis-

semination are generally considered to have advantages over conventional mass campaigns, including cost-effectiveness issues and the ability to reach specific customer groups. In [3], authors deal with the problem of opinion dynamics, in which social network agents (such as consumers) are influenced not only by their neighbors but also by external influencing entities called marketers. Marketers allocate a budget to campaigns over social networks to try to influence public opinion. Similarly, in [4], the authors introduced a duopoly model which accounts for the knowledge of opinion dynamics through a social network and shows that firms should influence users with a significant enough influence on other users to capture a larger market share. Unlike these works that take advantage of the node centrality (network topology) but rely on linear opinion dynamic models for the service spreading, we assume here a uniform spreading using epidemiological nonlinear models. This setup is more suitable for certain types of products/services such as video streaming platforms for instance.

Our goal is to determine the optimal budget allocation policy allowing a given firm to maximize its market share. The methodology builds on existing multi-virus epidemic results [2], [5], [6]. We consider a relatively simple model in which the budget allocated by a firm to a certain region modifies the spreading rate of the associated service in that region. Additionally, we assume that the spreading in different regions is uncorrelated (or weakly correlated) and each region has a similar contribution to the overall revenue of the firm. Under this assumption, the “winner takes all” result holds true yielding a decoupled investment analysis for each region. It is worth noting that the firm has an overall fixed budget that must be allocated to each region. From the mathematical point of view, the problem is formulated as the determination of a constrained best-response to given strategies for its competitors.

The optimal budget allocation problem introduced and addressed in this paper differs from the existing ones for several key reasons. In [7], a dynamic optimization problem under budget capacity is formulated to control a single-virus SIS model. In [8], optimal control of joint multi-virus infection and information dissemination is considered for the sensitive-warned-infected-recovered-susceptible (SWIRS) model, without budget constraints. The differences between the present work and the existing results on epidemic control (e.g., [9], [10], [11], [12], [13]) are mainly related to the fact that we handle a multi-virus SIS epidemic model and we consider budget constraints. The closest work to the present one is [14] in which the authors formulate

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a static and strategic form game to deal with a bi-virus SIS epidemic model over a single region without a budget constraint.

The key contributions of this paper can be summarized as follows:

- We propose a revenue model for a company that employs VM to spread its product or service over a network comprised of several disconnected regions, while in competition with other products. The company must allocate a finite budget across regions to improve the product quality/diversity in that region.
- We convert the original optimization problem (OP) related to a discontinuous revenue function into a convex OP over several mutually exclusive subsets of the optimization domain and provide an algorithm that solves the OP.

This paper is structured as follows. In Sec. II, the considered multi-virus SIS model is described. The proposed optimal allocation problem to implement by a given firm is provided. In Sec. III, the characterization of the optimal allocation problem is established. In Sec. IV, we conduct a numerical performance analysis for our theoretical results for a bi-virus case and we compare the optimal allocation strategy to conventional strategies like uniform allocation and the best responses for a classical Colonel Blotto game.

## II. PROBLEM STATEMENT

We consider a set of  $M > 1$  firms competing over a set of  $K > 1$  regions (e.g., countries, provinces, or states) to get a larger market share. Let  $\mathcal{K} := \{1, \dots, K\}$  and  $\mathcal{M} := \{1, \dots, M\}$  be the set of regions and firms, respectively. For a given region  $k \in \mathcal{K}$  and a firm  $m \in \mathcal{M}$ , we respectively denote by (i)  $\gamma_k^m$  the spreading rate of the service of provider  $m$  in the region  $k$ ; (ii) the value  $\delta_k^m$  is the churn rate at which the individual from Region  $k$  decides to dispense with the services of provider  $m$ . We assume that  $\gamma_k^m$  is a control variable since it can be modified by the firm  $m$  through new technological updates or price control. On top of this, we assume that firm  $m$  has a given budget  $B^m$  to allocate between the  $K$  regions in order to maximize the number of its subscribers.

We denote by  $\gamma^m := (\gamma_1^m, \dots, \gamma_K^m) \in \mathbb{R}_{>0}^K$  the action profile of the firm  $m$  and  $\gamma := (\gamma^1, \gamma^2, \dots, \gamma^M) \in \mathbb{R}_{>0}^{MK}$  the vector collecting the action profiles for all the firms. We will also use the notation  $\gamma^{-m}$  to refer the reduced action profile  $\gamma^{-m} := (\gamma^1, \dots, \gamma^{m-1}, \gamma^{m+1}, \dots, \gamma^M)$ .

**Problem:** Find the optimal resource allocation  $\gamma^{m,*}$  of the firm  $m$  that maximizes the revenue  $u^m(\gamma^m, \gamma^{-m})$  defined latter in (2).

The rest of this section provides both the model that we consider for the viral marketing dynamics (Sec. II-A) and the optimisation problem that each firm would solve (Sec. II-B).

### A. Viral marketing model

The control action  $\gamma_k^m$  is assumed to be constant on the working phase  $[0, +\infty]$ . In practice, it is sufficient to consider a working interval  $[0, T]$  where  $T$  is a sufficiently

large horizon to ensure the practical convergence of the VM dynamics.

The fractions of individuals in Region  $k \in \mathcal{K}$  who subscribe to the services of firm  $m \in \mathcal{M}$  are denoted by  $x_k^m \in [0, 1]$ . In contrast, the proportion of individuals in Region  $k \in \mathcal{K}$  who have not subscribed to any services is denoted by  $s_k \in [0, 1]$ . With these notations, the continuous-time VM dynamic in Region  $k \in \mathcal{K}$  is assumed to be given by:

$$\begin{cases} \frac{ds_k}{dt} = -s_k(t) \sum_{m=1}^M [\gamma_k^m x_k^m(t)] + \sum_{m=1}^M \delta_k^m x_k^m(t), \\ \frac{dx_k^1}{dt} = s_k(t) \gamma_k^1 x_k^1(t) - \delta_k^1 x_k^1(t), \\ \vdots \\ \frac{dx_k^M}{dt} = s_k(t) \gamma_k^M x_k^M(t) - \delta_k^M x_k^M(t), \\ s_k(t) + \sum_{m=1}^M x_k^m(t) = 1, \end{cases} \quad (1)$$

where the initial values verify  $s_k^0 > 0$ ,  $x_k^m \geq 0 \forall m \in \mathcal{M}$ .

For the sake of simplicity, we assume that the control imposed in Region  $k$  by firm  $m$  (namely,  $\gamma_k^m$ ) affects the subscribers of firm  $m$  in Region  $k$  who are in contact with all susceptibles of Region  $k$ , i.e.,  $\gamma_k^m$  is the controlled rate at which the subscribers of firm  $m$  in Region  $k$  influence the susceptibles in Region  $k$ . In practice, it would be quite difficult to measure its value or to assign it a prescribed value. Then, we assume that each firm would apply a strategy close enough to the abstract quantity  $\gamma^m$ . In what follows we denote by  $x_k^{m,\infty} := \lim_{t \rightarrow \infty} x_k^m(t) \forall k \in \mathcal{K}$  and  $\forall m \in \mathcal{M}$ . Note that in the literature, VM dynamics without a network formulation has been conducted in [15] where influence effects are incorporated into the (Susceptible-Infected-Recovered) SIR model when all consumers in a region are all interconnected with each other. Thus the approximation of decoupled dynamics between regions is reasonable.

### B. Revenue definition and optimization problem

As previously stated, we want to solve the allocation problem for a given firm  $m \in \mathcal{M}$  that maximizes a revenue under the global budget (denoted by  $B^m$ ) constraint when the parameters  $\gamma^{-m}$  of its competitors are fixed and known. This is formalized as the following optimization problem (OP). Let  $m \in \mathcal{M}$  and  $\gamma^{-m} \in \mathbb{R}_{>0}^{(M-1)K}$ :

$$\begin{aligned} \max_{\gamma^m \in \Gamma^m} u^m(\gamma^m, \gamma^{-m}) &:= \max_{\gamma^m} \sum_{k=1}^K p_k^m x_k^{m,\infty}(\gamma) \\ \text{s.t. } \gamma_k^m &\geq 0, \forall k \in \mathcal{K} \\ \sum_{k=1}^K \gamma_k^m &\leq B^m \end{aligned} \quad (2)$$

where:  $p_k^m$  is the winning price for firm  $m$  concerning Region  $k$  and  $B^m$  is the budget of firm  $m$ . The reasoning behind this choice is that: the quantity  $p_k^m x_k^{m,\infty}$  measures the final revenue of firm  $m$  concerning Region  $k$  where the competition (i.e., the action of each firm) appears implicitly (w.r.t. the strategy profile  $\gamma$ ) through the term  $x_k^{m,\infty}$ .

The term  $p_k^m$  represents the total potential revenue when all individuals in Region  $k$  subscribe to the services of firm  $m$ , whereas the budget  $B^m$  is imposed for economical reasons

(e.g., a firm has a given budget for investing). Hence, the aim of this paper is to solve an optimal budget allocation problem for a given firm  $m \in \mathcal{M}$  in response to the strategies  $\gamma^{-m}$  of other competitors and under a limited capacity of investment.

### III. PRELIMINARY RESULTS

In order to solve the problem stated previously, we first need to characterize the equilibrium points of the VM dynamics. The following Lemma provides the expression of  $x_k^{m,\infty}$ .

*Lemma 1:* For any  $\varepsilon > 0$  and for all  $k \in \mathcal{K}$ :

(i) If for all  $m \in \mathcal{M}$ ,  $\frac{\gamma_k^m}{\delta_k^m} < 1$ , then  $x_k^{m,\infty} = 0$ .

(ii) If there exists  $m \in \mathcal{M}$  such that  $\frac{\gamma_k^m}{\delta_k^m} \geq \max \left( \max_{n \in \mathcal{M}, n \neq m} \left[ \frac{\gamma_k^n}{\delta_k^n} + \varepsilon \right], 1 + \varepsilon \right)$ , then  $x_k^{m,\infty} = 1 - \frac{\delta_k^m}{\gamma_k^m}$  and  $\forall n \neq m, x_k^{n,\infty} = 0$ .  $\square$

**Proof.** The system in (1) can be reformulated such as:

$$\begin{cases} \frac{dx_k^1}{dt} = \left(1 - \sum_{n=1}^M x_k^n\right) \gamma_k^1 x_k^1(t) - \delta_k^1 x_k^1(t), \\ \vdots \\ \frac{dx_k^M}{dt} = \left(1 - \sum_{n=1}^M x_k^n\right) \gamma_k^M x_k^M(t) - \delta_k^M x_k^M(t). \end{cases} \quad (3)$$

Hence  $\forall m \in \mathcal{M}$  and  $k \in \mathcal{K}$ , the steady state of system (3) is given by:

**Case 1** -  $x_k^{m,\infty} = 0, \quad \forall m \in \mathcal{M}$

**Case 2** - There exists  $m \in \mathcal{M}$  such that

$$x_k^{m,\infty} = 1 - \frac{\delta_k^m}{\gamma_k^m} \text{ and } x_k^{n,\infty} = 0 \quad \forall n \neq m.$$

The component  $(m, n)$  of the Jacobian matrix  $J_k$  of system (3) at the steady state point is  $[J_k]_{m,n} :=$ ,

$$\begin{cases} \left(1 - \sum_{i=1}^M x_k^{i,\infty}\right) \gamma_k^m - \gamma_k^m x_k^{m,\infty} - \delta_k^m & \text{if } m = n \\ -\gamma_k^m x_k^{m,\infty} & \text{otherwise.} \end{cases}$$

To ensure the stability of the equilibrium points, we use a well-known result in [16]. Precisely, the equilibrium is stable if and only if the real parts of the eigenvalues of the Jacobian matrix  $J_k$  are all negative.

In **Case 1**, the eigenvalues of  $J_k$  are

$$\{\gamma_k^1 - \delta_k^1, \dots, \gamma_k^M - \delta_k^M\}.$$

Hence, the steady state

$$\forall m \in \mathcal{M}, x_k^{m,\infty} = 0$$

is stable if and only if  $\forall m \in \mathcal{M}, \frac{\gamma_k^m}{\delta_k^m} < 1$ .

In **Case 2**, the eigenvalues of  $J_k$  are

$$\begin{aligned} & \left\{ \gamma_k^1 \left( \frac{\delta_k^m}{\gamma_k^{m^*}} - \frac{\delta_k^1}{\gamma_k^1} \right), \dots, \gamma_k^{m-1} \left( \frac{\delta_k^m}{\gamma_k^m} - \frac{\delta_k^{m-1}}{\gamma_k^{m-1}} \right), \right. \\ & \left. -\gamma_k^m + \delta_k^m, \gamma_k^{m+1} \left( \frac{\delta_k^m}{\gamma_k^m} - \frac{\delta_k^{m+1}}{\gamma_k^{m+1}} \right), \right. \\ & \left. \dots, \gamma_k^M \left( \frac{\delta_k^m}{\gamma_k^m} - \frac{\delta_k^M}{\gamma_k^M} \right) \right\}. \end{aligned}$$

Hence, the steady state

$$x_k^{m,\infty} = 1 - \frac{\delta_k^m}{\gamma_k^m} \text{ and } \forall n \neq m, x_k^{n,\infty} = 0$$

is stable if and only if

$$\frac{\gamma_k^m}{\delta_k^m} \geq \max \left( \max_{n \in \mathcal{M}, n \neq m} \left[ \frac{\gamma_k^n}{\delta_k^n} + \varepsilon \right], 1 + \varepsilon \right).$$

### IV. BUDGET ALLOCATION PROBLEM FOR VM $\blacksquare$

The analysis of the optimization problem in (2) appears to be non-trivial since it is non-convex. In this section, we exploit the steady state analysis given in the Lemma 1 and propose a convex reformulation of the problem in (2) that helps us to obtain a characterization of the optimal budget allocation solution for a given firm.

#### A. Convex reformulation

Let  $\varepsilon > 0$ . In what follows, for a given  $m \in \mathcal{M}$  and  $\mathcal{K}^m \in 2^{\mathcal{K}}$ , we denote by  $\tilde{\Gamma}_\varepsilon^m(\mathcal{K}^m) :=$

$$\begin{aligned} & \left\{ \gamma^m \in \mathbb{R}_{\geq 0}^K : \forall k \in \mathcal{K}, \gamma_k^m \geq 0 \right. \\ & \left. \forall k \in \mathcal{K}^m, \frac{\gamma_k^m}{\delta_k^m} \geq \max \left( \max_{n \in \mathcal{M}, n \neq m} \left[ \frac{\gamma_k^n}{\delta_k^n} + \varepsilon \right], 1 + \varepsilon \right), \right. \\ & \left. \forall k \in \mathcal{K} \setminus \mathcal{K}^m, \frac{\gamma_k^m}{\delta_k^m} \leq \max \left( \max_{n \in \mathcal{M}, n \neq m} \frac{\gamma_k^n}{\delta_k^n}, 1 \right) \right\}. \end{aligned}$$

The following proposition reformulates the original OP in (2) into a convex OP.

*Proposition 1:* Let  $m \in \mathcal{M}, \gamma^{-m} \in \mathbb{R}_{\geq 0}^{K(M-1)}$  and  $\varepsilon > 0$ . The initial OP in (2) can be reformulated such as

$$\max_{\mathcal{K}^m \in 2^{\mathcal{K}}} F_\varepsilon^m(\mathcal{K}^m) \quad (P^*)$$

$$\text{s.t. } \exists \gamma^m \in \tilde{\Gamma}_\varepsilon^m(\mathcal{K}^m) : \sum_{k=1}^K \gamma_k^m \leq \mathbf{B}^m$$

where

$$F_\varepsilon^m(\mathcal{K}^m) := \max_{\gamma^m \in \tilde{\Gamma}_\varepsilon^m(\mathcal{K}^m)} \sum_{k \in \mathcal{K}^m} p_k^m \left( 1 - \frac{\delta_k^m}{\gamma_k^m} \right) \quad (4)$$

$$\text{s.t. } \sum_{k=1}^K \gamma_k^m \leq \mathbf{B}^m.$$

$\square$

*Remark 1:* It appears that, the condition

$$\forall \gamma \in \tilde{\Gamma}_\varepsilon^m(\mathcal{K}) : \sum_{k=1}^K \gamma_k^m \leq \mathbf{B}^{m''}$$

is verified, if and only if

$$\sum_{k \in \mathcal{K}^m} \delta_k^m \max \left( \max_{n \in \mathcal{M}, n \neq m} \left[ \frac{\gamma_k^n}{\delta_k^n} + \varepsilon \right], 1 + \varepsilon \right) \leq \mathbf{B}^m.$$

$\square$

**Proof.** Let us denote by  $\mathcal{H}^m(k) :=$

$$\left\{ \gamma_k^m \in \mathbb{R}_{\geq 0} : \frac{\gamma_k^m}{\delta_k^m} \geq \max \left( \max_{n \in \mathcal{M}, n \neq m} \left[ \frac{\gamma_k^n}{\delta_k^n} + \varepsilon \right], 1 + \varepsilon \right) \right\}.$$

According to Lemma 1, the original OP in (2) can be reformulated such as

$$\max_{\gamma^m} \sum_{k=1}^K p_k^m \left( 1 - \frac{\delta_k^m}{\gamma_k^m} \right) \mathbb{1}_{\mathcal{H}^m(k)}(\gamma_k^m) \quad (P_0)$$

$$\text{s.t. } \sum_{k=1}^K \gamma_k^m \leq \mathbf{B}^m,$$

where  $\mathbb{1}_{\mathcal{H}^m(k)}(\gamma_k^m) = 1$  if  $\gamma_k^m \in \mathcal{H}^m(k)$  and 0 otherwise. Next, we can rewrite (P<sub>0</sub>) in the following manner by considering all possible cases of firm m winning in region k

or not.

$$\max_{\gamma^m} \sum_{\mathcal{K}^m \in 2^{\mathcal{K}}} \left[ \sum_{k \in \mathcal{K}^m} p_k^m \left( 1 - \frac{\delta_k^m}{\gamma_k^m} \right) \right] \mathbb{1}_{\tilde{\Gamma}_\varepsilon^m(\mathcal{K}^m)}(\gamma^m) \quad (P_1)$$

$$\text{s.t. } \sum_{k=1}^K \gamma_k^m \leq \mathbf{B}^m,$$

where  $\mathbb{1}_{\tilde{\Gamma}_\varepsilon^m(\mathcal{K}^m)}(\gamma^m) = 1$  if  $\gamma^m \in \tilde{\Gamma}_\varepsilon^m(\mathcal{K}^m)$ , and 0 otherwise.

Let  $\gamma^m, \hat{\gamma}^i \in \Gamma^m$ ,  $\mathcal{K}^m, \hat{\mathcal{K}}^m \in 2^{\mathcal{K}}$  such that  $\mathcal{K}^m \neq \hat{\mathcal{K}}^m$  and  $\mathbb{1}_{\tilde{\Gamma}_\varepsilon^m(\mathcal{K}^m)}(\gamma^m) = \mathbb{1}_{\tilde{\Gamma}_\varepsilon^m(\hat{\mathcal{K}}^m)}(\hat{\gamma}^m) = 1$ . Then,  $\tilde{\Gamma}_\varepsilon^m(\mathcal{K}^m) \cap \tilde{\Gamma}_\varepsilon^m(\hat{\mathcal{K}}^m) = \emptyset$  and  $(P_1)$  can be reformulated such as

$$\max_{\mathcal{K}^m \in 2^{\mathcal{K}}} \left[ \max_{\gamma^m} \left[ \sum_{k \in \mathcal{K}^m} p_k^m \left( 1 - \frac{\delta_k^m}{\gamma_k^m} \right) \right] \mathbb{1}_{\tilde{\Gamma}_\varepsilon^m(\mathcal{K}^m)}(\gamma^m) \right] \quad (P_2)$$

$$\text{s.t. } \sum_{k=1}^K \gamma_k^m \leq \mathbf{B}^m.$$

Furthermore, for a given  $\mathcal{K}^m$  there exists a strategy  $\gamma^m \in \tilde{\Gamma}_\varepsilon^m(\mathcal{K}^m)$  that verifies the budget constraint if and only if the set  $\mathcal{K}^m$  verifies

$$\sum_{k \in \mathcal{K}^m} \delta_k^m \max \left( \max_{n \in \mathcal{M}, n \neq m} \left[ \frac{\gamma_k^n}{\delta_k^n} + \varepsilon \right], 1 + \varepsilon \right) \leq \mathbf{B}^m \quad (5)$$

i.e., the less restrictive action from  $\tilde{\Gamma}_\varepsilon^m(\mathcal{K}^m)$  verifies the budget constraint. Hence, by putting the indicator function into the constraint set and by adding the new constraint (5) among the feasibility sets of  $\mathcal{K}^m$ , the problem  $(P_2)$  can be rewritten such as in  $(P^*)$ . ■

In what follows, we establish a characterization of the solutions of the OP in (4).

### B. Computation of $F_\varepsilon^m$

It appears that, for a given  $\mathcal{K}^m \in 2^{\mathcal{K}}$ , the OP given in (4) corresponds to a multi-item inventory system with limited capacity [17]. In what follows, we consider  $\mathcal{K}^m \in 2^{\mathcal{K}}$  as feasible w.r.t. the OP in  $(P^*)$  i.e., we take  $\mathcal{K}^m$  that verifies the condition in Remark 1

$$\sum_{k \in \mathcal{K}^m} \delta_k^m \max \left( \max_{n \in \mathcal{M}, n \neq m} \left[ \frac{\gamma_k^n}{\delta_k^n} + \varepsilon \right], 1 + \varepsilon \right) \leq \mathbf{B}^m.$$

The goal of the next proposition is to characterize the solution of (4).

**Proposition 2:** Let  $m \in \mathcal{M}$ ,  $\mathcal{K}^m \in 2^{\mathcal{K}}$ ,  $\gamma^{-m} \in \mathbb{R}_{\geq 0}^{(M-1)K}$  and  $\varepsilon > 0$ . The solution  $\gamma^{m,*}$  that maximizes the OP in (4) is given by:  $\gamma_k^{m,*} =$

$$\begin{cases} 0, & \text{if } k \in \mathcal{K} \setminus \mathcal{K}^m, \\ \delta_k^m \max \left( \max_{n \in \mathcal{M}, n \neq m} \left[ \frac{\gamma_k^n}{\delta_k^n} + \varepsilon \right], 1 + \varepsilon \right) & \text{if } k \in \mathcal{K}^m \setminus \tilde{\mathcal{K}}^m, \\ \frac{\sqrt{p_k^m \delta_k^m} \left[ \mathbf{B}^m - \sum_{\ell \in \mathcal{K}^m \setminus \tilde{\mathcal{K}}^m} \delta_\ell^m \max \left( \max_{n \in \mathcal{M}, n \neq m} \left( \frac{\gamma_\ell^n}{\delta_\ell^n} + \varepsilon \right), 1 + \varepsilon \right) \right]}{\sum_{\ell \in \tilde{\mathcal{K}}^m} \sqrt{p_\ell^m \delta_\ell^m}}, & \text{if } k \in \tilde{\mathcal{K}}^m, \end{cases}$$

where

$$\tilde{\mathcal{K}}^m \in \operatorname{argmax}_{\tilde{\mathcal{K}}} \sum_{k \in \mathcal{K}^m} p_k^m \left( 1 - \frac{\delta_k^m}{\gamma_k^{m,*}} \right)$$

$$\text{s.t. } \left\{ \tilde{\mathcal{K}} \in 2^{\mathcal{K}^m} : \forall k \in \tilde{\mathcal{K}}, 0 \leq \frac{\sqrt{p_k^m \delta_k^m}}{\sum_{\ell \in \tilde{\mathcal{K}}} \sqrt{p_\ell^m \delta_\ell^m}} \times \right.$$

$$\left[ \mathbf{B}^m - \sum_{\ell \in \mathcal{K}^m \setminus \tilde{\mathcal{K}}} \delta_\ell^m \min \left( \max_{n \in \mathcal{M}, n \neq m} \left( \frac{\gamma_\ell^n}{\delta_\ell^n} + \varepsilon \right), 1 + \varepsilon \right) \right] \leq$$

$$\delta_k^m \max \left( \max_{n \in \mathcal{M}, n \neq m} \left[ \frac{\gamma_k^n}{\delta_k^n} + \varepsilon \right], 1 + \varepsilon \right)$$

$$\text{and } \sum_{k \in \mathcal{K}^m} \gamma_k^{m,*} = \mathbf{B}^m. \left. \right\} \quad (6)$$

□

**Proof.** Since the Optimization problem is convex with constraints that satisfy the Slater's conditions, then the OP has a unique solution. Let us exploit the KKT conditions by first defining the Lagrangian

$$\mathcal{L}_{\mathcal{K}^m}(\gamma^m, \mu^m, \underline{\mu}^m, \lambda^m) := \sum_{k \in \mathcal{K}^m} p_k^m \left( 1 - \frac{\delta_k^m}{\gamma_k^m} \right)$$

$$- \sum_{k \in \mathcal{K}^m} \mu_k^m \left( \gamma_k^m - \delta_k^m \max \left( \max_{n \in \mathcal{M}, n \neq m} \left[ \frac{\gamma_k^n}{\delta_k^n} + \varepsilon \right], 1 + \varepsilon \right) \right)$$

$$+ \sum_{k \in \mathcal{K} \setminus \mathcal{K}^m} \mu_k^m \left( \gamma_k^m - \delta_k^m \max \left( \max_{n \in \mathcal{M}, n \neq m} \frac{\gamma_k^n}{\delta_k^n}, 1 \right) \right)$$

$$- \lambda^m \left( \sum_{k=1}^K \gamma_k^m - \mathbf{B}^m \right)$$

$$+ \sum_{k \in \mathcal{K} \setminus \mathcal{K}^m} \underline{\mu}_k^m \gamma_k^m.$$

Let us denote by  $\gamma_k^{m,*}$ ,  $\mu_k^{m,*}$ ,  $\lambda^{m,*}$  and  $\underline{\mu}_k^{m,*}$  the variables that verify the first-order optimality condition.

For all  $k \in \mathcal{K} \setminus \mathcal{K}^m$ ,

$$\frac{\partial \mathcal{L}_{\mathcal{K}^m}}{\partial \gamma_k^m} = -\lambda^{m,*} - \mu_k^{m,*} + \underline{\mu}_k^{m,*} = 0.$$

$$\iff \mu_k^{m,*} = 0, \underline{\mu}_k^{m,*} = \lambda^{m,*} > 0 \text{ and } \gamma_k^{m,*} = 0.$$

For all  $k \in \mathcal{K}^m$ ,

$$\frac{\partial \mathcal{L}_{\mathcal{K}^m}}{\partial \gamma_k^m} = \frac{p_k^m \delta_k^m}{(\gamma_k^{m,*})^2} - \lambda^{m,*} - \mu_k^{m,*} = 0$$

$$\Rightarrow \gamma_k^{m,*} = \sqrt{\frac{p_k^m \delta_k^m}{(\lambda^{m,*} + \mu_k^{m,*})}}.$$

Let  $\tilde{\mathcal{K}}^m \in 2^{\mathcal{K}^m}$  (that may be empty) such that,

$$\tilde{\mathcal{K}}^m \in \{ \tilde{\mathcal{K}} \in 2^{\mathcal{K}^m} : \forall k \in \tilde{\mathcal{K}}, \mu_k^{m,*} = 0 \}.$$

For all  $k \in \mathcal{K}^m$ , the first-order optimality condition is verified when:

$$\forall k \in \tilde{\mathcal{K}}^m, \gamma_k^{m,*} = \sqrt{\frac{p_k^m \delta_k^m}{\lambda^{m,*}}} \leq \delta_k^m \max \left( \max_{n \in \mathcal{M}, n \neq m} \left[ \frac{\gamma_k^n}{\delta_k^n} + \varepsilon \right], 1 + \varepsilon \right)$$

and  $\forall k \in \mathcal{K}^m \setminus \tilde{\mathcal{K}}^m$ ,

$$\gamma_k^{m,*} = \delta_k^m \max \left( \max_{n \in \mathcal{M}, n \neq m} \left[ \frac{\gamma_k^n}{\delta_k^n} + \varepsilon \right], 1 + \varepsilon \right).$$

Since  $\lambda^{m,*} > 0$ , it follows that,

$$\sum_{k=1}^K \gamma_k^{m,*} = B^m.$$

Hence,

$$\begin{aligned} \sum_{\ell \in \tilde{\mathcal{K}}^m} \gamma_\ell^{m,*} &= B^m - \sum_{\ell \in \mathcal{K}^m \setminus \tilde{\mathcal{K}}^m} \gamma_\ell^{m,*}, \\ \Rightarrow \sum_{\ell \in \tilde{\mathcal{K}}^m} \frac{\sqrt{p_\ell^m \delta_\ell^m}}{\sqrt{\lambda^{m,*}}} &= B^m - \sum_{\ell \in \mathcal{K}^m \setminus \tilde{\mathcal{K}}^m} \gamma_\ell^{m,*} \\ \Rightarrow \sqrt{\lambda^{m,*}} &= \frac{\sum_{\ell \in \tilde{\mathcal{K}}^m} \sqrt{p_\ell^m \delta_\ell^m}}{B^m - \sum_{\ell \in \mathcal{K}^m \setminus \tilde{\mathcal{K}}^m} \gamma_\ell^{m,*}}. \end{aligned}$$

Finally, the optimal solution of  $F_\varepsilon^m(\mathcal{K}^m)$  in (4) is obtained w.r.t. the KKT conditions by taking the best part  $\tilde{\mathcal{K}}^m \in 2^{\mathcal{K}^m}$  that maximise the final revenue i.e.,  $\tilde{\mathcal{K}}^m$  verifies (6). ■

Having characterized the solutions of  $F_\varepsilon^m(\mathcal{K}^m)$  in (4), we can combine the results of the propositions 1 and 2 to get the optimal solution of the initial allocation problem stated in (2).

### C. Computational discussion

In Proposition 1 we showed that the initial OP in (2)

In view of the characterization of the solutions of (4) given in Proposition 2, the OP in (4) can be solved numerically with an exhaustive search algorithm with low cost computation. Indeed, to solve the OP in (4) (for a given  $\mathcal{K}^m \in 2^{\mathcal{K}}$ ), with an exhaustive search algorithm it requires

$$\sum_{\#\tilde{\mathcal{K}}^m=0}^{\#\mathcal{K}^m} \binom{\#\mathcal{K}^m}{\#\tilde{\mathcal{K}}^m} = 2^{\#\mathcal{K}^m} \text{ elementary operations.}$$

Hence, the total computational cost of an exhaustive search algorithm to solve the OP in ( $P^*$ ) is given by

$$C := \sum_{\#\mathcal{K}^m=0}^K \left[ \binom{K}{\#\mathcal{K}^m} \sum_{\#\tilde{\mathcal{K}}^m=0}^{\#\mathcal{K}^m} \binom{\#\mathcal{K}^m}{\#\tilde{\mathcal{K}}^m} \right] = 3^K.$$

## V. NUMERICAL PERFORMANCE ANALYSIS

k	1	2	3	4	5	6	6	8
$p_k$	$2 \cdot 10^8$	$2.6 \cdot 10^8$	$2.7 \cdot 10^8$	$3.5 \cdot 10^8$	$6 \cdot 10^8$	$7 \cdot 10^8$	$8 \cdot 10^8$	$9 \cdot 10^8$
$\delta_k$	0.8	0.5	0.4	0.3	0.15	0.1	0.08	0.05

TABLE I  
PARAMETERS

We consider  $M = 2$  competing firms over  $K = 8$  communities. For the numerical illustration we set the following parameters values. The budget of firm 2 is  $B^2 = 4$ ; the small quantity  $\varepsilon$  in ( $P^*$ ) is equal to  $10^{-5}$ ; the local awards for both firms in each region  $k$  are  $p_k^1 = p_k^2 = p_k$  and the churn rates are  $\delta_k^1 = \delta_k^2 = \delta_k$ , where the values of  $p_k$  and  $\delta_k$  are given in Table I.

In Fig. 1 we evaluate the potential revenue of firm 1 in response to a uniform strategy from firm 2 (i.e.,  $\forall k \in \mathcal{K}$ ,  $\gamma_k^{2,\text{unif}} = 1/2$ ) by varying the budget  $B^1$  and by applying:

(i) The best reply strategy that maximizes (2);

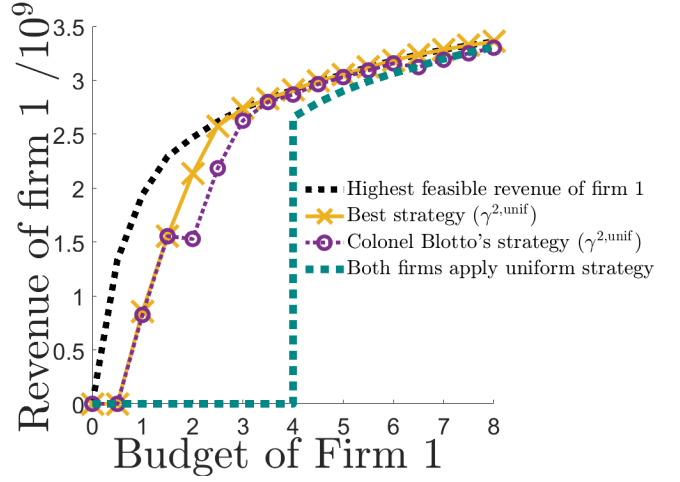


Fig. 1. Interpolation of Firm's 1 revenue by varying the budget  $B^1$  (with  $B^2 = 4$ ). We observe that: The upper bound of firm 1's revenue (solid line)  $\geq$  Firm 1's best revenue in response to firm 2's uniform strategy (cross-marked curve)  $\geq$  The revenue of firm 1 by applying the best response of a classical Colonel Blotto game (e.g., see [18]) (circled-marked line)  $\geq$  Firm's 1 revenue by applying uniform strategy in response to uniform strategy from firm 2 (dotted line).

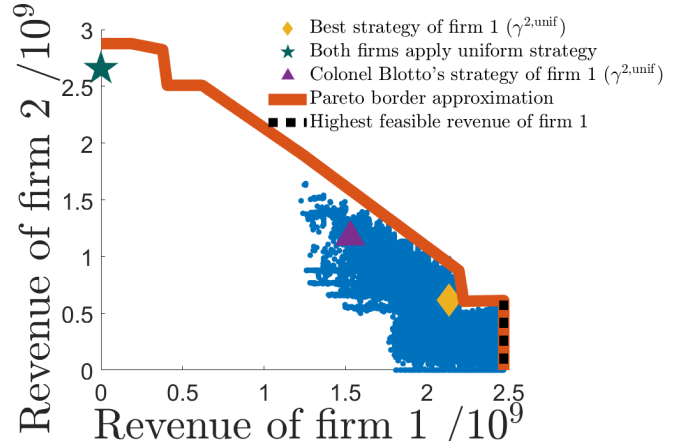


Fig. 2. By exploring 1.3 million feasible strategies  $\gamma = (\gamma^1, \gamma^2)$  (that verifies the budget constraint for both firms), the Pareto border of  $(u^1(\gamma), u^2(\gamma))$  is represented (with  $\forall m \in \{1, 2\}$ ,  $\sum_{k=1}^K \gamma_k^m \leq B^m$  where  $B^1 = 2$  and  $B^2 = 4$ ). First, we observe that, the best revenue of firm 1 in response to a strategy  $\gamma^2$  are at the bottom right of the figure (blue cloud). This means that firm 1 will tend to have a higher revenue than firm 2 if it decides to optimize its strategy given the optimization problem in (2).

(ii) The uniform strategy (i.e.,  $\forall k \in \mathcal{K}$ ,  $\gamma_k^{1,\text{unif}} = B^1/K$ );  
(iii) The Blotto's strategy (see [18]) that is solution of

$$\begin{aligned} \max_{\gamma^1} \sum_{k=1}^K p_k^1 \mathbb{1}_{\frac{\gamma_k^1}{\delta_k^1} \geq \max \left( 1 + \varepsilon, \frac{\gamma_k^2}{\delta_k^2} + \varepsilon \right)} \\ \text{s.t. } \sum_{k=1}^K \gamma_k^1 \leq B^1 \\ \gamma_k^1 \geq 0. \end{aligned}$$

(iv) The upper-bound revenue of firm 1 (i.e.,  $\max_{\gamma^2} \max_{\gamma^1} u^1(\gamma^1, \gamma^2)$  s.t.  $\forall m \in \{1, 2\}$ ,  $\sum_{k=1}^K \gamma_k^m \leq B^m$ ).

When firm 1's budget is greater than 3, we observe that firm 1's best revenue in response to firm 2's uniform

strategy tends toward the upper bound of firm 1's revenue. Furthermore, when the firm 1's budget is greater than the half budget of firm 2, the revenue of firm 1 by applying the best response of a classical Colonel Blotto game is strictly lower than the firm's 1 revenue when it applies the best strategy in response to a uniform strategy from firm 2. Finally, when the firm's 1 budget is greater than  $B^2$ , applying the uniform strategy by firm 1 is better (in terms of revenue for firm 1) but still worse than other strategies.

In Fig. 2, we set the budget of firm 1 as the half budget of firm 2. By exploring 1.3 million feasible strategies  $\gamma = (\gamma^1, \gamma^2)$  (i.e., by discretizing the action space into 1.3 million strategies verifying  $\forall m \in \{1, 2\}, \sum_{k=1}^K \gamma_k^m \leq B^m$ ) we approximate the Pareto border [19] of the couple  $(u^1(\gamma), u^2(\gamma))$ . When firm 2 applies the uniform strategy, we observe that the best reply strategy solution of (2) in response to a uniform strategy from firm 2, is more Pareto efficient (closer strategy from the Pareto border) than the other strategies listed in Fig. 1.

## VI. CONCLUSION

In this paper, we propose a characterization of the optimal allocation for a given firm in response to a strategy of its competitors. We use an SIS model to capture the spread of a firm's product or service across various regions and the competition of its products with other firms. We transform the resulting dis-continuous cost and its related OP into a convex continuous OP with additional constraints that must be solved for several possible cases. Numerical results show that the resulting optimal policy outperforms classical strategies such as those from the Colonel Blotto games literature. Future works will study the resulting game when all firms make rational decisions and try to maximize their individual revenues.

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