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FINITE DIMENSIONAL SOLUBLE GROUPS

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ABSTRACT. The results for soluble groups of finite Morley rank are generalized to the finite dimensional context. In particular, finite dimensional connected soluble non-nilpotent groups interpret a field, and have a nilpotent derived subgroup.

INTRODUCTION

I

1. DIMENSION

Definition 1.1. A theory T is *finite-dimensional* if there is a dimension function \dim from the collection of all interpretable sets in models of T to $\{-\infty\} \cup \omega$, satisfying for a formula $\varphi(x, y)$ and interpretable sets X and Y :

- *Invariance:* If $a \equiv a'$ then $\dim(\varphi(x, a)) = \dim(\varphi(x, a'))$.
- *Algebraicity:* $\dim(\emptyset) = -\infty$, and $\dim X = 0$ iff X is finite.
- *Union:* $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$.
- *Fibration:* Let $f : X \rightarrow Y$ be an interpretable map.
 - If $\dim(f^{-1}(y)) \geq d$ for all $y \in Y$, then $\dim(X) \geq \dim(Y) + d$.
 - If $\dim(f^{-1}(y)) \leq d$ for all $y \in Y$, then $\dim(X) \leq \dim(Y) + d$.

Of course, fibration implies that

(†) If $\dim(f^{-1}(y)) = d$ for all $y \in Y$, then $\dim(X) = \dim(Y) + d$.

Note that we do not suppose that the dimension is definable, i.e. that all sets $\{y \in Y : \dim(f^{-1}(y)) = d\}$ are definable. If dimension is definable, then (†) (i.e. fibration with $=$) implies the two fibration axioms (with \geq and \leq).

Remark 1.2. In [6] a more general notion of dimension is defined. Our notion of finite-dimensionality would correspond to *fine finite-dimensional with lower fibration*.

Examples of finite-dimensional theories include theories of finite Lascar rank, finite SU-rank, finite U^b -rank and o -minimal theories.

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Fact 1.3. [6] *In a finite-dimensional theory, if $f : G \rightarrow H$ is a definable homomorphism of definable groups, then $\dim G = \dim H + \dim \ker f$. In particular, if $H \leq G$, then $\dim G = \dim H + \dim G/H$. So H has infinite index in G iff $\dim H < \dim G$.*

It follows that we have a descending and an ascending chain condition on definable subgroups up to finite index. Note however that there is no *a priori* assumption on the existence of connected components (although we usually assume connectedness of *one* of the groups involved). Part of the work is then overcome the lack of connectedness elsewhere.

Lemma 1.4. *Let G be a connected group acting on the infinite abelian group A in a finite-dimensional theory. Suppose that A has no infinite G -invariant definable subgroup of smaller dimension, and there is $g \in G$ with $(g - 1)A$ infinite. Then A has a G -invariant G -minimal definable subgroup of finite index.*

Proof. Suppose $A^* \leq A$ is a G -invariant definable subgroup of finite index. Then $C_G(A/A^*)$ has finite index in G and must be the whole of G . So $(g - 1)A \leq A^*$; it is thus contained in every G -invariant subgroup of finite index in A .

Now consider finitely many $g_i \in G$ such that $B = \sum_i (g_i - 1)A$ has maximal dimension possible. As $(g - 1)A$ is infinite for some $g \in G$, the group B is infinite; by maximality of $\dim B$ it must be commensurable with $gB \leq \sum_i [(gg_i - 1)A - (g - 1)A]$ for any $g \in G$. By compactness the commensurability is uniform, and by Schlichting's Theorem there is a G -invariant definable subgroup B_0 commensurable with B . Then $\dim A = \dim B_0 = \dim B$, so there is a maximal sum $A_0 = \sum_j (g_j - 1)A$ of finite index in A . Then A_0 is G -invariant, and contained in every G -invariant definable subgroup of finite index in A , whence G -minimal. \square

In order to preserve connectedness in nilpotent groups, we shall need the following Lemma.

Lemma 1.5. *Let G be a finite-dimensional nilpotent connected group. Then G has a definable connected lower central series.*

Proof. We use induction on the dimension. If G is abelian, the statement is clear. Otherwise choose $g \in \gamma_{n-1}G \setminus Z(G)$, where n is the nilpotency class of G . Then $x \mapsto [x, g]$ is a homomorphism from G to $\gamma_n G \leq Z(G)$; its image Z is definable, connected and infinite. Since $\dim(G/Z) < \dim G$ and $\gamma_i(G)$ is the pre-image of $\gamma_i(G/Z)$ with respect to the projection $G \rightarrow G/Z$, we finish by induction. \square

Lemma 1.6. *Let H_1, H_2 be normal subgroups of G . If G/H_1 and G/H_2 are connected-by-finite, so is $G/(H_1 \cap H_2)$.*

Proof. Let $(G_k)_k$ be an infinite strictly descending sequence of subgroups of G containing $H_1 \cap H_2$. Since G/H_1 is connected-by-finite, the sequence $(G_k H_1)_k$ will eventually be stationary, and we may assume that it is constant for all k . Similarly, the

sequence $(G_k \cap H_1)/(H_1 \cap H_2) \cong (G_k \cap H_1)H_2/H_2$ will eventually be stationary, and we may assume $G_k \cap H_1$ is constant for all k . But then $G_k/(G_k \cap H_1) \cong G_k H_1/H_1$ is stationary, as is G_k , a contradiction. \square

Lemma 1.7. *Let H be a normal subgroup of G and suppose G/H and H are connected-by-finite. Then G is connected-by-finite.*

Proof. Let $(G_k)_k$ be an infinite strictly descending sequence of subgroups of G . Then $G_k H$ and $G_k \cap H$ will eventually be stationary, and we may assume they are stationary for all k . But then $G_k/(G_k \cap H) \cong G_k H/H$ is stationary for all k , as is G_k , a contradiction. \square

Proposition 1.8. *Let G be a connected group acting faithfully on the abelian group A in a finite-dimensional theory. Suppose that $C_A(G)$, as well as $A[n]$ are finite for all $n < \omega$. Then A is connected-by-finite.*

Note that since the n -torsion is finite for all $n < \omega$, a connected subgroup of A must be divisible (and a divisible group is connected).

Proof. By induction on $\dim A$. If $(g-1)A$ were finite for all $g \in G$, then

$$\dim\{(g, a) : ga = a\} = \dim G + \dim A,$$

so $\dim C_G(a) = \dim G$ for $\dim A$ many $a \in A$, whence $C_G(a) = G$ by connectedness of G , and $C_A(G)$ has finite index in A , a contradiction. As in Lemma 1.4, there is a G -invariant subgroup $B \leq A$ commensurable with a finite sum of the form $\sum_i (g_i - 1)A$ of maximal dimension possible.

If $\dim B = \dim A$, then as in Lemma 1.4 there is a G -invariant subgroup A_0 of finite index in A of the form $A_0 = \sum_j (g_j - 1)A$, which is contained in all G -invariant subgroups of finite index in A . Since $A_0[n]$ is finite, nA_0 has finite index in A , so $A_0 \leq nA_0$ for all $n > 0$ and A_0 is divisible, whence connected.

If $\dim B < \dim A$, then B is connected-by-finite by induction hypothesis. Choose $n < \omega$ minimal such that there are $g_i \in G$ for $i \leq n$ such that $\sum_{i \leq n} (g_i - 1)A$ is commensurable with B , and put $N = \sum_{i < n} (g_i - 1)A$. Then $(g_n - 1)A/((g_n - 1)A \cap N)$ is infinite and commensurable with B/N , whence connected-by-finite. Put $K = \{a \in A : (g_n - 1)a \in N\}$. Then $A/K \cong (g_n - 1)A/((g_n - 1)A \cap N)$ is infinite, and connected-by-finite. Choose finitely many $h_j \in G$ such that $K_0 = \bigcap_j h_j K$ has minimal dimension possible. Then K_0 is uniformly commensurable with all its G -conjugates; by Schlichting's Theorem there is a G -invariant K_1 commensurable with K_0 . By Lemma 1.6, A/K_0 is connected-by-finite; since $\dim K_1 < \dim A$ also K_1 is connected-by-finite by inductive hypothesis, as is K_0 ; finally A is connected-by-finite by Lemma 1.7. \square

Corollary 1.9. *Let K be a field definable in a finite-dimensional theory. Then K does not have a connected infinite group of automorphisms.*

Proof. If $C_K(G)$ is infinite, then $[K : C_K(G)]$ is a field extension of degree $\dim K / \dim C_K(G)$, which is finite. Hence G embeds into $\text{Gal}(K/C_K(G))$, which is finite. Thus G is trivial by connectedness.

If $C_K(G)$ is finite, we consider G as a connected group of automorphisms of K^\times . Then K is divisible-by-finite by Proposition 1.8. However, the subfield K^{abs} of absolutely algebraic elements is contained in $C_K(G)$ by connectivity of G , and must be finite. But then K^{abs} has a cyclic extension of every prime degree q , as does K , and $(K^\times)^q$ has index at least q in K^\times , a contradiction. \square

2. SOLUBLE GROUPS

Theorem 2.1. *Let M be a connected abelian group acting faithfully on the infinite abelian group A in a finite-dimensional theory. Suppose that A has no infinite M -invariant definable subgroup of smaller dimension and $C_A(M)$ is trivial. Then A is M -minimal, and there is a definable field K such that $A \cong K^+$ and $M \hookrightarrow K^\times$.*

Proof. We consider the ring R of definable endomorphisms of A generated by M . Then R is commutative, and contains M as a multiplicative subgroup. Moreover, for any $r \in R$ both $\ker r$ and $\text{im } r$ are definable M -invariant, whence either finite or of finite index in A . But the centralizer of any finite M -invariant subgroup F has finite index in M , and must equal M by connectedness. Since $C_A(M)$ is trivial, so is F . But then, if $\ker r$ has finite index, $\text{im } r$ is finite, whence trivial, and $r = 0$. Thus $\ker r = \{0\}$ for all $r \in R \setminus \{0\}$, and R is an integral domain.

Let A_0 be the M -invariant M -minimal subgroup of finite index in A given by Lemma 1.4. Then $(m - 1)A = A_0 = (m - 1)^2 A$ for any $m \in M \setminus \{1\}$, since these groups are all M -invariant of finite index in A and M centralises A/A_0 . Hence $A = A_0 \oplus \ker(m - 1)$; as $\ker(m - 1)$ is trivial, $A_0 = A$. But then the image of any $r \in R \setminus \{0\}$ is infinite M -invariant, and must be equal to A by M -minimality, so R acts on A by automorphisms.

It follows that any $r \in R$ is determined by the couple (a, ra) for any $r \in R$ and non-zero $a \in A$. Thus $\dim X \leq 2 \dim A$ for any type-definable subset X of R . By [7, Proposition 3.6 and Proposition 3.3] the field of fractions K of R is definable, and A is a finite-dimensional vector space over K . By minimality of $\dim A$ it is 1-dimensional, so $A \cong K^+$; clearly $M \hookrightarrow R^\times = K^\times$. \square

Theorem 2.2. *Let G be a nilpotent connected group acting on an infinite abelian group A , in a dimensional theory. Then either the action is nilpotent, or there is a definable centreless 2-soluble section which naturally interprets a field.*

Proof. By induction on $\dim G + \dim A$. If $C_A(G)$ is infinite, $\dim(A/C_A(G)) < \dim A$ and we may use the induction hypothesis for G acting on $A/C_A(G)$. So we may assume that $C_A(G)$ is finite. Then a^G is finite for any $a \in A$ centralised by G

modulo $C_A(G)$, so $C_G(a)$ has finite index and $g \in C_A(G)$ by connectedness. Thus $C_{A/C_A(G)}(G)$ is trivial, and we may assume $C_A(G)$ is trivial. In particular, a finite G -invariant subgroup of A must be contained in $C_A(G)$, and be trivial.

By Lemma 1.5 the derived subgroup G' is connected and definable, and $\dim(G') < \dim(G)$. By induction hypothesis either the action of G' on A is nilpotent, or a field is naturally definable. So we may assume that we are in the former case, and we consider the definable series $A_0 = \{0\}$, $A_{n+1} = \{a \in A : [a, G'] \subseteq A_n\}$ which terminates in A . Note that G/G' acts on all the quotients A_{n+1}/A_n . Let Q be such a quotient. If it is finite, it is centralized by G by connectivity. If $C_Q(G)$ is finite, again by connectivity of G the action of G on $Q/C_Q(G)$ is centreless; if $Q_0 \leq Q$ is definable, infinite and G -invariant of minimal dimension possible, Theorem 2.1 yields an interpretable field K such that $Q_0 \cong K^+$ and $G/C_G(Q_0) \hookrightarrow K^\times$. Finally, if $C_Q(G)$ is infinite, we divide out and finish by induction. \square

Corollary 2.3. *A connected soluble non-nilpotent dimensional group interprets naturally a field.*

Proof. If $Z(G)$ is infinite, we divide by $Z(G)$ and use induction on $\dim G$. If $Z(G)$ is finite, then $G/Z(G)$ is centreless by connectivity, and we may assume that G is centreless. Then for any non-trivial a in the last non-trivial derived group, $A = Z(C_G(a^G))$ is a definable infinite abelian group. Either G/A is non-nilpotent and we finish by induction on $\dim G$, or G/A is nilpotent and we finish by Theorem 2.2. \square

Theorem 2.4. *Let G be a connected group acting faithfully and without fixed points on the infinite abelian group A in a finite-dimensional theory. Suppose that A is G -minimal, and that G has a definable infinite finite-by-abelian normal subgroup M . Then A is definably a vector space of finite linear dimension over a definable field K , such that the action of G is K -linear and M has a G -invariant subgroup M_0 of finite index acting by scalars (so M_0 is central in G and embeds into K^\times).*

Proof. Let \mathcal{C} be the G -invariant family of centralisers in M of finite tuples $\bar{m} \in M$, and note that they all have finite index in M . Consider an infinite definable subgroup $B \leq A$ of minimal possible dimension and invariant under some subgroup $N_B \in \mathcal{C}$. Choose finitely many $g_i \in G$ such that $S = \sum_i g_i B \leq A$ has maximal dimension possible. Then $\dim S + gS = \dim S = \dim gS$ for any $g \in G$, so the family of translates $\{gS : g \in G\}$ is commensurable, and uniformly so by compactness. By Schlichting's Theorem there is a G -invariant group S_0 commensurable with S ; by minimality of $\dim A$ we have $\dim A = \dim S_0 = \dim S$ and S has finite index in A . It follows that there is a maximal finite sum of G -translates of B containing S ; it must be G -invariant, whence equal to A by G -minimality. Renaming, we may assume that $S = A = \sum_i g_i B$.

Suppose $C_M(B)$ has finite index in M . Then so does $C_M(g_j B) = C_M(B)^{g_j^{-1}}$, as well as the intersection

$$C = \bigcap_j C_M(B)^{g_j^{-1}} = C_M\left(\sum_j g_j B\right) = C_M(A) = \{1\},$$

a contradiction.

Let R be the group of endomorphisms of A generated by M , and for any subgroup $N \in \mathcal{C}$ let R_N be the group of endomorphisms of A generated by N . Put $I_B = \{r \in R_{N_B} : rB \text{ finite}\}$. Now B is R_{N_B} -invariant, and for any $r \in R$ generated using elements $\bar{m} \in M$, rB is invariant under $N_B \cap C_M(\bar{m}) \in \mathcal{C}$. It follows by minimality that rB is either finite or has dimension $\dim rB = \dim B$, so $B \cap \ker r$ is finite. In particular I is a prime ideal in R_{M_B} . Note that $N_{gB} = N_B^{g^{-1}}$, $R_{N_{gB}} = R_{N_B}^{g^{-1}}$ and $I_{gB} = I_B^{g^{-1}}$.

We shall call g and h equivalent if there is a subgroup $N \in \mathcal{C}$ containing $N_{gB} \cap N_{hB}$ such that $I_{gB} \cap R_N = I_{hB} \cap R_N$. This clearly is a G -invariant equivalence relation. Suppose $(h_j : j \leq k)$ are inequivalent for some $h_0, \dots, h_k \in G$, and put $N = \bigcap_{j \leq k} N_{h_j B}$, again a group in \mathcal{C} . Then all $I_{h_j B} \cap R_N$ are distinct prime ideals in R_N , and one of them is minimal, say $I_{h_k B}$. So there is $r \in R_N \cap \bigcap_{j < k} I_{h_j B} \setminus I_{h_k B}$. Hence $rh_j B$ is finite for $j < k$ and $rh_k B$ has finite index in $h_k B$. Thus $h_k B \cap \sum_{j < k} h_j B$ is finite; iterating we find that the sum $\sum_{j \leq k} h_j B$ is almost direct. So $(k+1)\dim B = \dim \sum_{j \leq k} h_j B \leq \dim A$, and $k < \dim A$, as $\dim B > 0$. Thus there are only finitely many equivalence classes among the elements of G , and we may choose representatives $(e_i : i < c)$.

For any $i < c$ let S_i be a sum $\sum_{j,i} h_{j,i} B$ of maximal dimension, where all $h_{j,i}$ are equivalent to g_i . Then as above we see that the sum $\sum_{i < c} S_i$ is almost direct, and h is equivalent to g_i if and only if hB is almost contained in S_i . It follows that the action of G on the equivalence classes is definable, and trivial by connectivity. So there is just one equivalence class, since the action is transitive.

Recall that $A = \sum_i g_i B$. Take $N \in \mathcal{C}$ containing $\bigcap_i N_{g_i B}$ such that all $I_{g_i B} \cap R_N$ are the same prime ideal I_N of R_N . Now any $r \in R_N$ has finite image or image of finite index on every $g_i B$, whence on A . It follows that $I_N = \{r \in R_N : rA \text{ finite}\}$. Now for $m, m' \in N$ we have $m + I_N = m' + I_N$ iff $m - m' \in I_N$ iff $\text{im}(m - m')$ is finite iff $\text{im}(m'^{-1}m - 1)$ is finite iff $C_A(m'^{-1}m)$ has finite index in A iff $mM_0 = m'M_0$, where $M_0 = \{m \in M : |A : C_A(m)| \text{ finite}\}$. Note that the finiteness is definable by compactness, so M_0 is definable.

Suppose almost all $m \in N$ are equivalent modulo I . Then M_0 has finite index in M . Consider the set

$$\{(a, m) \in A \times M : ma = a\}.$$

It has dimension at least $\dim M_0 + \dim A = \dim M + \dim A$. It follows that the definable subgroup $A_0 = \{a \in A : Ma \text{ finite}\}$ has finite index in A , and both are G -invariant. Then $A_0 \rtimes M_0$ has finite derived group, which is a finite G -invariant subgroup of A , whence contained in $C_A(G) = \{1\}$. So M_0 acts trivially on A_0 . But $A_0 = A$ by G -minimality, contradicting faithfulness. Hence M/M_0 is infinite.

Suppose that there is no maximal finite subgroup of A invariant under some subgroup in \mathcal{C} . Then take a countably infinite restricted sum S of such subgroups; they are invariant under a subgroup N_0 of M which is a countable intersection of subgroups in \mathcal{C} . Since M_0 has infinite index in M , it has infinite index in $N \cap N_0$, and there are $m, m' \in N \cap N_0$ with $m'^{-1}m \notin M_0$ with the same action on S . Hence $\ker(m - m') \supseteq S$ is infinite, so $\text{im}(m - m')$ is finite, contradicting $m'^{-1}m \notin M_0$. It follows that there is a unique maximal finite subgroup invariant under some subgroup in \mathcal{C} . By uniqueness it is G -invariant, whence centralized by G by connectedness. But $C_A(G)$ is trivial, so there is no non-trivial finite subgroup of A invariant under some subgroup in \mathcal{C} . Thus $R_N \setminus \{0\}$ consists of injective endomorphisms whose image has finite index in A . In particular I_N is trivial.

Choose $a \in A \setminus \{0\}$. For any $r \in R_N$ the couple (a, ra) determines r , so we get a bound on $\dim R_N$, and the skew field of fractions K_N is definable by [7,].

The derived subgroup M' is finite, say of size n . It is characteristic in M , whence normal in G , and must be centralized by G by connectivity. It follows that

$$[x, y^n] = [x, y]^n = 1$$

for all $x, y \in M$. Moreover,

$$(xy)^{2n} = x^{2n}y^{2n}[y, x]^{(2n-1)2n/2} = x^{2n}y^{2n}.$$

So the map $x \mapsto x^{2n}$ is an endomorphism of M with characteristic abelian image; we want to show that the image M_0 has finite index. So suppose otherwise. Then the kernel $M[2n]$ is infinite, as is $N[2n] = M[2n] \cap N$. Since centralisers of finite tuples have finite index in M , we can find an abelian subgroup A of exponent $2n$ and size $> 2n$ in N . But A generates a commutative subfield of K_N , a contradiction.

It follows that M_0 has finite index in M ; clearly it is G -invariant. We repeat the above argument with M_0 instead of M , noting that this time $\mathcal{C} = \{M_0\}$. So R_{M_0} is a commutative ring of injective endomorphisms of A with finite kernels, whose field of fractions K_0 is definable. Then G acts on K_0 definably, and trivially by Corollary 1.9. Hence $m = m^g$ for all $m \in M_0$ and $g \in G$, and M_0 is central in G . Now $\text{im } r$ is G -invariant for any $0 \neq r \in R_{M_0}$, so $\text{im } r = A$ by G -minimality. It follows that A is a K_0 -vector space of finite linear dimension, G acts K_0 -linearly and M_0 acts K_0 -scalarly. \square

Theorem 2.5. *A finite-dimensional connected soluble group G has a nilpotent derived subgroup.*

Proof. Dividing out by the hypercentre, we may assume that G is centreless. In particular, it has no finite G -invariant subgroups. If $a \neq 1$ is in the last non-trivial derived subgroup of G , then $Z(C_G(a^G))$ is a definable G -invariant abelian subgroup of G . So we can choose a non-trivial definable G -invariant abelian subgroup $A \leq G$ such that $C_G(A)$ has maximal dimension possible. Replacing A by a definable subgroup, we may assume that $\dim A$ is minimal possible. Note that A is infinite.

If $[g, A]$ is finite for all $g \in G$, consider the set

$$\{(g, a) \in G \times A : ga = a\}$$

and note that it has dimension $\dim G + \dim A$. Hence $\dim\{a \in A : [G, a] \text{ finite}\} = \dim A$. But by connectivity $[G, a]$ is finite if and only if it is trivial, contradicting the fact that G is centreless. By Lemma 1.4 we may assume that A is G -minimal. We claim that $G' \leq C_G(A)$. So suppose not.

If $C_{G'}(A)$ has finite index in G' , then for every $g \in G$ the index $|G : C_G(g/C_G(A))|$ is finite, whence trivial by connectivity. It follows that $G' \leq C_G(A)$, a contradiction. Hence $C_{G'}(A)$ has infinite index in G' and there is maximal k such that $(G')^{(k)}/C_G(A)$ is infinite. Put $N = C_G(A)(G')^{(k+1)}$, a definable finite extension of $C_G(A)$. Then $Z(C_{G/N}((G')^{(k)}/N))$ is an infinite definable normal abelian subgroup of G/N containing $(G')^{(k)}/N$ (it is definable since the centralizer is the centraliser of the set of basic commutators generating $(G')^{(k)}$). Its preimage M in $G/C_G(A)$ is finite-by-abelian. By Theorem 2.4 there is a definable G -invariant subgroup M_0 of finite index in M centralized by G ; moreover A is definably a finite-dimensional vector space over a definable field, the action of G is linear and the action of M_0 is scalar. But $M_0 \cap (G')^{(k)}$ consists of scalar matrices of determinant 1 and must be finite, again a contradiction. Hence G' centralizes A . We divide out by A and finish by induction on the dimension. \square

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